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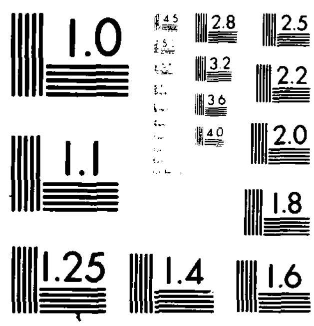

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BY

D. SIEGMUND and YIH-SHYH YUH

TECHNICAL REPORT NO. 13
DECEMBER 15, 1980

PREPARED UNDER CONTRACT
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BROWNIAN APPROXIMATION TO FIRST PASSAGE PROBABILITIES

1. Introduction and Summary.

Let $x_1, x_2, \ldots$ be independent and identically distributed with mean $E(x_1) = \mu$. Let $s_n = x_1 + \ldots + x_n$, and for $a < 0 \le b$ define the stopping times

$$\tau = \tau(b) = \inf\{n: s_n > b\} \qquad (\tau_+ = \tau(0))$$

and

$$T = T(a,b) = \inf\{n: s_n \notin [a,b]\} .$$

The probabilities

$$\text{(1)} \qquad P\{\tau \le m\}$$

and

$$\text{(2)} \qquad P\{T \le m,\ s_T > b\}$$

arise in a variety of probability models. They are difficult to compute exactly, but under certain conditions may be approximated by the corresponding probabilities for the Brownian motion process. Siegmund (1979) gave a heuristic argument based on Laplace transforms to show that this approximation can be improved considerably by obtaining what amounts to the first term in an Edgeworth type expansion of (1) and (2). This method has been extended by Yuh (1980) for studying joint probabilities of the form

$$\text{(3)} \qquad P\{\tau < m,\ s_m < b - x\}$$

and

$$\text{(4)} \qquad P\{T < m,\ s_T > b,\ s_m < b - x\}$$

as well as related conditional probabilities, e.g. $P\{\tau < m \mid s_m = b - x\}$, which arise in the study of Kolmogorov-Smirnov statistics.

The purpose of this paper is to give a direct probabilistic calculation of a one-term Edgeworth expansion to probabilities like (3). The method is in principle applicable to (4) although the computations are much more involved, and no details are given in this case. An example of our results is as follows.

<u>Theorem 1.</u> *Suppose* $\mu = 0$, $Ex_1^2 = 1$, *and* $\gamma = Ex_1^3$ *is finite. Let* $b = \zeta m^{1/2}$. *If the distribution of* x_1 *is strongly non-lattice in the sense that* $\limsup_{|t|\to\infty} |E\exp(itx_1)| < 1$, *then for each* $x > 0$ *as* $m \to \infty$

$$P\{\tau < m,\ s_m < (\zeta - x)m^{1/2}\} = 1 - \Phi(\zeta + x)$$
$$(5) \qquad - m^{-1/2}\varphi(\zeta+x)[2\beta + (\gamma/6)(x^2 - \zeta^2 - 1)] + o(m^{-1/2}) \quad .$$

Here $\beta = E(s_{\tau_+}^2 / 2Es_{\tau_+})$ *if* $\zeta > 0$ *and* $\beta = Es_{\tau_+}$ *if* $\zeta = 0$; φ *and* Φ *denote the standard normal density and distribution functions.*

<u>Remarks.</u>

(a) Since $P\{\tau \le m\} = P\{s_m > b\} + P\{\tau < m,\ s_m \le b\}$, if $P\{\tau < m,\ s_m < b-x\}$ were known exactly for all $x > 0$, then (at least for continuous distributions) one would obtain $P\{\tau \le m\}$ by letting $x \to 0$. Although it seems plausible that (5) should hold uniformly in x for x near 0 , and in fact the right hand side of (5) with $x = 0$ agrees with the result obtained heuristically by Siegmund (1979), we have been unable to prove this uniformity.

(b) In the case $\zeta = 0$, Theorem 1 is equivalent to a result of Iglehart (1974). In this case the asymptotic behavior of $P\{\tau_+ > m\}$ is known from fluctuation theory, e.g. Feller (1966), p.399, which shows that (5) is true with $x = 0$, although a completely different proof is involved.

(c) Siegmund (1979) has given a method for calculating β numerically.

(d) Under the stronger assumptions that the x's have a finite moment generating function, Borovkov (1962) gave a complete asymptotic expansion of (3). His methods use complex analysis, and the results are not given in a form which permits simple comparisons with (5). Also Borovkov's methods appear to handle the case $x = 0$ without difficulty, although they do not seem to adapt readily to two-sided stopping rules.

(e) Analogous results may be obtained for arithmetic distributions.

(f) If for some n the characteristic function of s_n is integrable, one can obtain a similar expansion for the density $-\frac{d}{dx} P\{\tau < m,\ s_m < (\zeta-x)m^{\frac{1}{2}}\}$, which can be formally calculated by differentiating (5). As an application one can improve the limiting distribution of the one sample Kolmogorov-Smirnov statistic - see Yuh (1980) for details.

The remainder of this paper is arranged as follows. Theorem 1 is proved in Section 2. Section 3 discusses the case $Ex_1 \neq 0$. In Section 4, in the much simpler context of Brownian motion we describe an alternative approach to these problems and use it to rederive the results of Anderson (1960).

2. Proof of Theorem 1

Let F_n denote the distribution function of s_n, $n = 0,1,\ldots$. It is easy to see that

$$\begin{aligned} P\{\tau < m,\ s_m < (\zeta-x)m^{\frac{1}{2}}\} &= \int_{\{\tau<m\}} P\{s_m < (\zeta-x)m^{\frac{1}{2}} \mid \tau,\ s_\tau\}\, dP \\ &= \int_{\{\tau<m\}} F_{m-\tau}\left((\zeta-x)m^{\frac{1}{2}} - s_\tau\right) dP = \int_{\{\tau\leq m\}} F_{m-\tau}(-x\,m^{\frac{1}{2}} - R_m)\, dP , \end{aligned} \tag{6}$$

where $R_m = s_\tau - \zeta m^{1/2}$. Similarly

$$P\{s_m > (\zeta+x)m^{\frac{1}{2}}\} = P\{\tau \le m,\ s_m > (\zeta+x)m^{\frac{1}{2}}\}$$

$$(7) \qquad = \int_{\{\tau\le m\}} [1 - F_{m-\tau}(x m^{\frac{1}{2}} - R_m)]\, dP ,$$

so

$$(8) \quad P\{\tau < m,\ s_m < (\zeta-x)m^{\frac{1}{2}}\} = P\{s_m > (\zeta+x)m^{\frac{1}{2}}\}$$

$$- \int_{\{\tau\le m\}} [1 - F_{m-\tau}(x m^{\frac{1}{2}} - R_m) - F_{m-\tau}(-x m^{\frac{1}{2}} - R_m)]\, dP .$$

The customary Edgeworth expansion applies to the first term on the right hand side of (8); hence the remainder of the proof is a detailed expansion of the integrand in (8) and an asymptotic evaluation of the resulting integral.

To carry out the following analysis it is technically useful to modify (8) to insure that in the integrand $m-\tau$ is not too small and R_m is not too large. Let $m_1 = m(1 - (\log m)^{-2})$. A consequence of Lemmas 2 and 3 below is that for some $\varepsilon_m \to 0$

$$P\{\tau < m,\ s_m < (\zeta-x)m^{\frac{1}{2}}\} = P\{\tau < m_1,\ R_m < m^{\frac{1}{2}}\varepsilon_m,\ s_m < (\zeta-x)m^{\frac{1}{2}}\} + o(m^{-\frac{1}{2}}) ,$$

and hence by an argument similar to that leading to (8)

$$P\{\tau < m,\ s_m < (\zeta-x)m^{\frac{1}{2}}\} = P\{s_m > (\zeta+x)m^{\frac{1}{2}}\}$$

$$(9) \qquad - \int_{\{\tau<m_1, R_m < m^{\frac{1}{2}}\varepsilon_m\}} [1 - F_{m-\tau}(x m^{\frac{1}{2}} - R_m) - F_{m-\tau}(-x m^{\frac{1}{2}} - R_m)]\, dP .$$

According to Petrov (1972) VI.3 Theorem 3

$$F_n(x n^{\frac{1}{2}}) = \Phi(x) - (\gamma/6 n^{\frac{1}{2}})(x^2-1)\,\varphi(x) + (1+|x|^3)^{-1}\, o(n^{-\frac{1}{2}}) ,$$

where $o(\cdot)$ is uniform in x. This may be used to expand the integrand in (9), and a subsequent expansion of the normal distribution function Φ by Taylor's theorem show that uniformly on $\{\tau < m_1, R_m < m^{1/2}\varepsilon_m\}$ the integrand in (9) equals

$$m^{-\frac{1}{2}}(1-\tau/m)^{-\frac{1}{2}}\varphi\left(x/(1-\tau/m)^{\frac{1}{2}}\right)\{2R_m + (\gamma/3)[x^2/(1-\tau/m) - 1]\}$$

$$+ O\left(m^{-\frac{1}{2}}\varepsilon_m R_m\right) + o(m^{-\frac{1}{2}}) .$$

Suppose initially that $\zeta > 0$. By Donsker's theorem τ/m converges in law to a distribution with density function $\zeta t^{-3/2}\varphi(\zeta/t^{1/2})$. By the renewal theorem $R_m = s_\tau - \zeta m^{1/2}$ converges in law to a distribution with density function $P\{s_{\tau_+} > y\}/E\, s_{\tau_+}$, and according to Siegmund (1975) it is asymptotically independent of τ/m. Moreover, by Lemma 1 below $\{R_m\}$ is uniformly integrable, so these limits may be taken under the integral. Hence the integral in (9) is asymptotically

$$m^{-\frac{1}{2}}\int_0^1 [2\beta + (\gamma/3)\{x^2/(1-t) - 1\}](1-t)^{-\frac{1}{2}}\varphi\left(x/(1-t)^{\frac{1}{2}}\right)\zeta t^{-\frac{3}{2}}\varphi(\zeta/t^{\frac{1}{2}})\, dt ,$$

which after some calculus yields (5).

In the case $\zeta = 0$ the argument is similar but much simpler. It is obvious that $\tau/m \to 0$ in probability and $R_m = s_{\tau_+}$.

<u>Lemma 1.</u> $E\, s_{\tau_+}^2 < \infty$ *and* $\{s_{\tau(b)} - b,\ b > 0\}$ *is uniformly integrable.*

<u>Proof.</u> That $E\, s_{\tau_+}^2 < \infty$ is known from random walk theory, e.g. problem 6, p.232 of Spitzer (1976). From renewal theory it is known that $P\{s_\tau - b \le x\} \to (E s_{\tau_+})^{-1}\int_0^x P\{s_{\tau_+} > y\}dy$; and since $E s_{\tau_+}^2 < \infty$, renewal theory also yields $E(s_\tau - b) \to E\, s_{\tau_+}^2/2\ E\, s_{\tau_+}$, which proves uniform integrability. (The uniform integrability may alternatively be proved

from first principles by an elaboration of the indicated idea for proving $Es_{\tau_+}^2 < \infty$. This may be the best approach to use in Lemma 4 below.)

__Lemma 2.__ *For each* $\varepsilon > 0$, $P\{R_m > \varepsilon m^{\frac{1}{2}}\} = o(m^{-\frac{1}{2}})$.

__Proof.__ By the Markov inequality

$$P\{R_m > \varepsilon m^{\frac{1}{2}}\} \leq \varepsilon^{-1} m^{-\frac{1}{2}} \int_{\{R_m > \varepsilon m^{1/2}\}} R_m \, dP ,$$

which is $o(m^{-\frac{1}{2}})$ by Lemma 1.

__Lemma 3.__ *Let* $m_1 = m(1 - (\log m)^{-2})$ *as in the proof of Theorem 1. Then as* $m \to \infty$

$$P\{m_1 < \tau < m,\ s_m < (\zeta - x)m^{\frac{1}{2}}\} = o(m^{-\frac{1}{2}})$$

and

$$P\{m_1 < \tau \leq m,\ s_m > (\zeta + x)m^{\frac{1}{2}}\} = o(m^{-\frac{1}{2}}) \quad .$$

__Proof.__ By Lemma 2

$$P\{m_1 < \tau \leq m,\ s_m > (\zeta+x)m^{\frac{1}{2}}\} = P\{m_1 < \tau \leq m,\ R_m < \tfrac{1}{2} x m^{\frac{1}{2}},\ s_m > (\zeta+x)m^{\frac{1}{2}}\}$$

$$+ o(m^{-\frac{1}{2}}) \leq \sup_{m_1 < n \leq m_1} P\{s_{m-n} > \tfrac{1}{2} x m^{\frac{1}{2}}\} + o(m^{-\frac{1}{2}})$$

$$\leq \sup_{m_1 < n \leq m} P\{(m-n)^{-\frac{1}{2}} s_{m-n} > \tfrac{1}{2} x \log m\} + o(m^{-\frac{1}{2}}) ,$$

which is easily seen to be $o(m^{-\frac{1}{2}})$ by Nagaev's (1965) improvement of the Berry-Esseen theorem or by the related result of Petrov quoted earlier. A similar but easier argument shows that the first probability in Lemma 3 is also $o(m^{-1/2})$.

3. The case $Ex_1 \neq 0$.

When $Ex_1 \neq 0$, results analogous to Theorem 1 are more complicated technically. Although it is probably possible to formulate a comprehensive theorem, it would be extremely cumbersome. Therefore, in this section we briefly discuss two important special cases: (i) when the distribution of x_1 can be imbedded in an exponential family, and (ii) when Ex_1 is a location parameter. Although the treatment of these two cases is slightly different, it should be apparent that modulo certain technicalities the methods can be applied to other similar problems.

To consider briefly the simpler case of an exponential family, suppose that the distribution of x_1 is given by $F_\theta(dx) = \exp[\theta x - \psi(\theta)]F_o(dx)$, where F_o is a strongly non-lattice distribution having mean 0 and variance 1. It is easily seen that $\psi(0) = 0$, $\psi'(\theta) = E_\theta x_1$, and $\psi''(\theta) = \mathrm{var}_\theta(x_1)$. By taking F_o to have mean 0, ψ has been standardized so that $\psi'(0) = 0$, and thus $\operatorname{sgn} E_\theta x_1 = \operatorname{sgn}\theta$. For $\theta_o < 0$ $(\theta_1 > 0)$ it will be assumed that there exists a $\theta_1 > 0$ $(\theta_o < 0)$, necessarily unique for which $\psi(\theta_o) = \psi(\theta_1)$.

The basic identity (6) remains true and in the obvious notation becomes

$$(10) \qquad P_\theta\{\tau < m,\ s_m < (\zeta - x)m^{\frac{1}{2}}\} = \int_{\{\tau<m\}} F_{\theta,m-\tau}(-x m^{\frac{1}{2}} - R_m)\, dP_\theta .$$

However, (7) must now be altered. Assume to be specific that $\theta > 0$ and write θ_1 for θ. It is easy to see from the exponential family structure that for every function $h \geq 0$ such that $h\, I_{\{\tau=n\}}$ is $\mathcal{B}(x_1, \ldots, x_n)$ measurable for all n,

$$\int_{\{\tau<\infty\}} h\, dP_{\theta_o} = \int_{\{\tau<\infty\}} h \exp\{-(\theta_1-\theta_o)s_\tau\} dP_{\theta_1} ,$$

and hence

$$e^{(\theta_1-\theta_o)b} P_{\theta_o}\{s_m > (\zeta+x)m^{\frac{1}{2}}\} = e^{(\theta_1-\theta_o)b} \int_{\{\tau\leq m\}} [1-F_{\theta_o,m-\tau}(xm^{\frac{1}{2}}-R_m)]dP_{\theta_o}$$

(11)

$$= \int_{\{\tau\leq m\}} [1-F_{\theta_o,m-\tau}(xm^{\frac{1}{2}}-R_m)]\exp[-(\theta_1-\theta_o)R_m]dP_{\theta_1} \quad .$$

From (10) and (11) one obtains the following analogue of (8):

$$P_{\theta_1}\{\tau < m,\ s_m < (\zeta-x)m^{\frac{1}{2}}\} = e^{(\theta_1-\theta_o)b} P_{\theta_o}\{s_m > (\zeta+x)m^{\frac{1}{2}}\}$$

$$- \int_{\{\tau\leq m\}} \{e^{-(\theta_1-\theta_o)R_m}[1-F_{\theta_o,m-\tau}(xm^{\frac{1}{2}}-R_m)] - F_{\theta_1,m-\tau}(-xm^{\frac{1}{2}}-R_m)\}dP_{\theta_1} \quad .$$

A similar identity holds for $P_{\theta_o}\{\tau < m,\ s_m < (\zeta-x)m^{\frac{1}{2}}\}$.

Modification of these identities and subsequent expansion as in the proof of Theorem 1 shows that if $(\theta_1-\theta_o) = \xi m^{-1/2}$, then

(12)
$$P_{\theta_1}\{\tau < m,\ s_m < (\zeta-x)m^{\frac{1}{2}}\} = \exp[2\xi(\zeta+\beta m^{-\frac{1}{2}})][1 - \Phi(\zeta+x+\xi)]$$
$$- e^{2\xi\zeta} m^{-\frac{1}{2}} \varphi(\zeta+x+\xi)\{2\beta + (\gamma/6)[\xi(\zeta-x) + x^2 - \zeta^2 - 1]\} + o(m^{-\frac{1}{2}}) \quad .$$

For θ_o the result is formally identical provided ξ is defined as $-m^{+1/2}(\theta_1-\theta_o)$. The details of these calculations have been omitted. (For the ideas justifying a version of Lemma 1 in this context, see Siegmund, 1979.)

Since the right hand side of (12) makes sense provided only that the x's have a finite third moment, the exponential family assumption appears to be much too strong. However, the likelihood ratio of the exponential family plays an important role in the derivation of (12), and avoiding its use raises some additional technical problems.

To minimize the number of unpleasant technicalities it will be assumed that θ is a location parameter (which is further restricted below to be non-negative). Hence let F_0 denote a continuous strongly non-lattice distribution function having mean 0, variance 1, and finite third moment γ. Let $F_\theta(x) = F_0(x-\theta)$, and let $F_{\theta,n}$ be the n-fold convolution of F_θ with itself. Let P_θ denote the probability measure under which $x_1, x_2, \ldots$ are independent with common probability distribution F_θ. Except for the indicated convergence in distribution, the following generalization of Lemma 1 may be proved by the method suggested in the parenthetical remark at the end of the proof of Lemma 1. The details are omitted.

Lemma 4. *As* $\theta \to 0$, $\int_{\{\tau_+<\infty\}} s_{\tau_+}^{\lambda} dP_\theta \to E_0 s_{\tau_+}^{\lambda}$ *for all* $0 < \lambda \leq 2$.

Let $\zeta > 0$ *and* $\theta = \xi\, m^{-1/2}$ *for some fixed* $\xi \geq 0$. *Then the* P_θ *distributions of* $(s_\tau - \zeta m^{1/2})$ *converge to the distribution with density function* $(E_0 s_{\tau_+})^{-1} P_0\{s_{\tau_+} > y\}$ *and have uniformly integrable first moments.*

Assume now that $\theta = \xi\, m^{-1/2}$ for some fixed $\xi \geq 0$. The identity (10) remains true in the present context. However, instead of (11) consider

$$(13) \quad \int_{(x,\infty)} \exp(-2\xi y) P_\theta\{s_m \in (\zeta+dy)m^{\frac{1}{2}}\} = \int_{\{\tau\leq m\}} \Big[\int_{(x,\infty)} \exp(-2\xi y) F_{\theta,m-\tau}(m^{\frac{1}{2}}dy - R_m) \Big] dP_\theta ,$$

which in the exponential family model is actually equivalent to (11). From (10) and (13) one obtains the following analogue of (8):

$$(14) \quad \begin{aligned} P_\theta\{\tau < m,\ s_m < (\zeta-x)m^{\frac{1}{2}}\} &= \int_{(x,\infty)} \exp(-2\xi y) P_\theta\{s_m \in (\zeta+dy)m^{\frac{1}{2}}\} \\ &- \int_{\{\tau\leq m\}} \Big[\int_{(x,\infty)} \exp(-2\xi y) F_{\theta,m-\tau}(m^{\frac{1}{2}}dy - R_m) - F_{\theta,m-\tau}(-x\, m^{\frac{1}{2}} - R_m) \Big] dP_\theta . \end{aligned}$$

With the aid of Lemma 4, this identity may now be expanded along established lines. (Expansion of the integral on the left hand side of (13) and the inner integral on the right hand side is facilitated by integration by parts, application of Petrov's theorem, and integration back by parts, which has the formal effect of expanding $F_{\theta,n}$ as if it had a density which had the appropriate local expansion.) In this case the resulting asymptotic expansion is as $m \to \infty$

$$P_\theta\{\tau < m,\ s_m < (\zeta - x)m^{\frac{1}{2}}\} = \exp\{2\xi[\zeta + m^{-\frac{1}{2}}(\beta + 2\gamma\xi\zeta/3)]\}[1 - \Phi(\zeta + x + \xi)]$$
$$- m^{-\frac{1}{2}} e^{2\xi\zeta} \varphi(\zeta + x + \xi)\{2\beta + (\gamma/6)[(x+\xi)^2 - \zeta^2 - 1 + 4\xi^2]\} + o(m^{-\frac{1}{2}}) .$$

4. Brownian motion

Our first approach to studying the problems of this paper involved a different method, which unfortunately seems to be more complicated and to require stronger assumptions than the method of Sections 2 and 3. On the other hand, it seems conceptually simpler and can be used formally to construct the answers to problems where none is known in advance. By way of contrast, one must know what the dominant term in (5) is in order to add and subtract it in (6) to implement the proof of Theorem 1. If one proceeds formally to expand (6), he gets the wrong answer.

To illustrate our original method in a very simple setting, we apply it in this section to give a simple derivation of the principal probabilistic result of Anderson (1960). The argument is constructive in the sense that the process of successive reflection, which is usually the difficult part of two-boundary problems, is accomplished in a purely mechanical way by means of a simple recursion.

Let $\{X(t),\ 0 \le t \le 1\}$ denote standard Brownian motion and for $\zeta_1 < 0 < \zeta_2$ and $\zeta_1 + \eta_1 \le \zeta_2 + \eta_2$ define $T = \inf\{t\colon X(t) = \zeta_i + \eta_i t$ for $i = 1$ or $2\}$. We shall calculate

(15) $\quad P\{T < 1,\ X(T) = \zeta_2 + \eta_2 T \mid X(1) = \mu\}$

(except for the case $\mu = \zeta_1 + \eta_1 = \zeta_2 + \eta_2$) . There is no loss of generality in assuming $\mu < \zeta_2 + \eta_2$, because in the contrary case one can use the argument to calculate the complementary probability, namely

$$P\{T < 1,\ X(T) = \zeta_1 + \eta_1 T \mid X(1) = \mu\} .$$

Let P_μ denote the conditional distribution of $X(t),\ 0 \le t \le 1$ given that $X(1) = \mu$. Then the probabulity in (15) equals

$$P_\mu\{T < 1,\ X(T) = \zeta_2 + \eta_2 T\} .$$

Let $\mathcal{F}(t) = \mathcal{B}(X(s),\ s \le t)$. For any $t < 1$ and $\mu_1 \neq \mu$ the measures P_μ and P_{μ_1} contracted to $\mathcal{F}(t)$ are mutually absolutely continuous with an easily computed likelihood ratio:

(16) $$\frac{dP_{\mu,t}}{dP_{\mu_1,t}} = \exp\{(\mu-\mu_1)[X(t) - \tfrac{1}{2}t(\mu+\mu_1)]/(1-t)\} = L(X(t),t;\ \mu,\mu_1) ,$$

say. Hence by standard likelihood ratio (or martingale) arguments

$$P_\mu\{T < 1,\ X(T) = \zeta_2+\eta_2 T\} = \int_{\{T<1,X(T)=\zeta_2+\eta_2 T\}} L(\zeta_2+\eta_2 T,T;\mu,\mu_1)dP_{\mu_1} .$$

From (16) one easily sees that the choice of μ_1 for which $\zeta_2 + \eta_2 T - \frac{1}{2}T(\mu+\mu_1) = \zeta_2(1-T)$, i.e. $\mu_1 = 2(\zeta_2 + \eta_2) - \mu$ leads to

(17) $$P_\mu\{T < 1,\ X(T) = \zeta_2+\eta_2 T\} = \exp[-2\zeta_2(\zeta_2+\eta_2-\mu)]P_{\mu_1}\{T < 1,\ X(T) = \zeta_2+\eta_2 T\} .$$

Since this choice of μ_1 exceeds $\zeta_2 + \eta_2$ (by virtue of the assumption $\mu < \zeta_2 + \eta_2$) , $P_{\mu_1}\{T < 1\} = 1$; and hence (17) may be rewritten

$$(18) \quad P_{\mu}\{T < 1,\ X(T) = \zeta_2+\eta_2 T\} = \exp[-2\zeta_2(\zeta_2+\eta_2-\mu)][1 - P_{\mu_1}\{T < 1,\ X(T) = \zeta_1+\eta_1 T\}] \ .$$

The identity (18) may now be used recursively to calculate (15), since by the same argument

$$(19) \quad P_{\mu_1}\{T < 1,\ X(T) = \zeta_1+\eta_1 T\} = \exp[-2\zeta_1(\zeta_1+\eta_1+\mu-2(\zeta_2+\eta_2))]$$

$$[1 - P_{\mu_2}\{T < 1,\ X(T) = \zeta_2 + \eta_2 T\}] \ ,$$

where $\mu_2 = 2(\zeta_1 + \eta_1 - \zeta_2 - \eta_2) + \mu < (\zeta_1 + \eta_1)$, etc.

In general the result of carrying out this computation is an infinite series of terms which alternate in sign. For the very special case $\zeta_1 + \eta_1 = \zeta_2 + \eta_2 \neq \mu$, it turns out that $\mu_2 = \mu$; and it is only necessary to solve the two equation (18) and (19) simultaneously to obtain

$$P_{\mu}\{T < 1,\ X(T) = \zeta_2+\eta_2 T\} = \frac{\exp\{-2\zeta_1(\zeta_2+\eta_2-\mu)\}-1}{\exp\{2(\zeta_2-\zeta_1)(\zeta_2+\eta_2-\mu)\}-1}$$

(cf. equation (4.24) of Anderson, 1960 for this result as well as the answer in the general case).

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By direct probabilistic argument one term of an Edgeworth type asymptotic expansion is obtained for certain first passage distributions for random walks. These results provide partial justification for and extensions of approximations suggested earlier as a heuristic consequence of Laplace transform calculations.

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