OPTIMAL SELECTION
FROM A FINITE SEQUENCE
WITH SAMPLING COST

by

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Abstract:
Two variations of the problem of choosing the largest of N independent and identically distributed random variables with sampling cost are studied. In the first case it is assumed that the underlying distribution is continuous and known, but the information obtained by sampling is whether the sampled variable is larger or smaller than some given level. In the second case it is assumed that the distribution of the random variables is continuous but unknown, and the information obtained is the rank of the sampled variable relative to the other variables already in the sample. In each case both the optimal strategy and the distribution of the stopping variable are discussed.
ABSTRACT

Two variations of the problem of choosing the largest of $N$ independent and identically distributed random variables with sampling cost are studied. In the first case it is assumed that the underlying distribution is continuous and known, but the information obtained by sampling is whether the sampled variable is larger or smaller than some given level. In the second case it is assumed that the distribution of the random variables is continuous but unknown, and the information obtained is the rank of the sampled variable relative to the other variables already in the sample. In each case both the optimal strategy and the distribution of the stopping variable are discussed.

KEY WORDS AND PHRASES

Sequential decision problem, Sampling cost, Stopping variable, Maximum of a Sequence.

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I. INTRODUCTION

Let $X_1, \ldots, X_N$ be independent and identically distributed continuous random variables which are to be sampled sequentially, where $N$ is a known fixed positive integer. The aim is to stop and choose the largest one. Exactly one random variable is to be selected and if, after any draw, a random variable is rejected, it cannot be recalled at a later stage. A large number of variations are possible in framing this, the so-called "Secretary Problem", some of which can be found in the references listed at the end of this article. Our aim in this paper is to study the following two variations of the above problem with a decision-theoretic approach.

**PROBLEM I.** The random variables are not observed directly. Rather, for each $X_i$ we observe whether $X_i < L_i$ or $X_i > L_i$, where $L_i$ is a level set by the experimenter, $1 \leq i \leq N$, and we stop experimentation the first time we find an $X_j > L_j$ (and we then select $X_j$). With certain gain (negative loss) and cost functions defined later (Sections 2 and 3 below), the aim is to find the optimal values of $L_1, \ldots, L_N$, that is, the levels that maximize the expected gain. It will be assumed that the distribution of $X_i$ is known and continuous.

Problem I is discussed in Section 2, where the form of the optimal strategy, the distribution of the stopping variable, and the optimum levels are defined. Optimal levels are numerically calculated for several different costs per observation and gain structures, for $N = 2(1)10$. Enns [3] studied this problem when the sampling cost is zero. Leonardz [6] studied it when one observes the random variables directly. When one wishes to choose the best of $N$ items from available stock (e.g. for use in a military or space mission), testing may well have associated cost (e.g. $\$ c. per test). In some applications
X₁ may be a life-length such that the gain due to functioning for X₁ time units is aX₁ + b (e.g., communications satellites or other equipment). Then Table II below would be used in practice.

Note that in this problem the L₁, ..., Lₙ are levels fixed in advance, and not set sequentially. However since (e.g.) we select X₁ if X₁ > L₁ (and hence do not then need to use L₂), thus needing L₂ iff X₁ ≤ L₁, the situation when L₂ will be needed is fully clear in advance of experimentation and it is also clear (since X₁ is not observed directly, but only whether it exceeds L₁ or not) that no gain can be realized by setting levels sequentially.
**PROBLEM II.** The random variables are observed directly but it is assumed that the distribution function is completely unknown. Also as each random variable is observed, only its rank relative to its predecessors is noted, or is able to be noted.

Problem II is discussed in Section 3, where the form of the optimal strategy and the distribution of the stopping variable are given. Optimal values are tabulated for several different costs per observation for values of \( N = 3(1)50 \), for two gain functions: gain \( b \) if the maximum of \( X_1, \ldots, X_N \) is selected (0 gain otherwise); and, gain \( b \chi(X_j) + a \) if \( X_j \) is selected, where \( \chi(X_j) \) is the rank of \( X_j \) among \( X_1, \ldots, X_N \). Gilbert and Mosteller [5] studied this problem, when the sampling cost is zero, for our first gain function. When one wishes to choose the best of several candidates for a position (e.g. a faculty or managerial position), interview cost is often measured in thousands of dollars.

Table III allows one to rationally choose the number of interviewees in such settings. Similarly for a seller evaluating multivariate bids on a depreciating or appreciating asset.

2. CASE OF KNOWN DISTRIBUTION:
RANDOM VARIABLES NOT OBSERVED DIRECTLY

The Optimal Strategy

When the distribution of \( X_i \) is known and continuous, it suffices to consider the sample as coming from a uniform distribution on the \([0,1]\) interval (\( X_i \) is \( U[0,1] \)) because, if \( F(x) \) is the distribution function of \( X_i \), then

\[ Y_i = F(X_i) \]

is distributed uniformly on \([0,1]\). So if \( L_i \) is the level used for \( Y_i \), then \( F^{-1}(L_i) \), the \( L_i \)-th quantile of the distribution of \( X_i \), is an equivalent level for \( X_i \). Therefore, suppose that \( X_i \) are independent and identically distributed as \( U[0,1] \), \( i = 1, \ldots, N \).
Let us call a particular sequence of levels \( L = (L_1, \ldots, L_N) \), used for making the selection, a **strategy**. Not all strategies are equally good. A strategy will be called **optimal** if it maximizes the expected gain (taking into account sampling cost and terminal decision gain) of the statistical decision problem.

Recall that a sequential decision problem consists of five elements: \( \Theta \), the space of the unknown parameter; \( \mathcal{A} \), the space of terminal actions available to the statistician; \( L \), the real-valued loss function on \( \Theta \times \mathcal{A} \); \( X = (X_1, X_2, \ldots) \), the random variables available to the statistician for observation; and \( \{c_j(\Theta, x_1, \ldots, x_j) : j = 1, 2, \ldots \} \), the cost function, a sequence of real-valued functions with \( c_j \) defined on \( \Theta \times X_1 \times \cdots \times X_j \), where \( X_i \) is the sample space of \( X_i \), \( i = 1, \ldots, j \), and \( c_j(\Theta, x_1, \ldots, x_j) \) represents the cost of taking observations \( X_1 = x_1, \ldots, X_j = x_j \) and then stopping, when \( \theta \) is the true value of the parameter.

Here \( \theta = \max(X_1, \ldots, X_N) \) and \( \theta \in [0, 1] = \Theta \). Also, since we are interested in selecting one of the random variables, let \( \mathcal{A} = \{X_1, \ldots, X_N\} \). Let the cost per observation be \( c \) and let the loss function be \( L(\theta, a) = -g_0(a) \), where \( g_0(a) \) (henceforth denoted \( g(a) \) for simplicity of notation), the gain function, is a non-decreasing function of \( a \) for each \( \theta \). Let the decision rule be

\[
 d_N(L, S) = \{d_j(X_1, \ldots, X_j), S(j), j = 1, \ldots, N \}
\]

where

\[
 S(j) = \begin{cases} 
 j, & \text{if } X_j > L_j \text{ and } X_i \leq L_i \ (i=1, \ldots, j-1) \\
 0, & \text{otherwise},
\end{cases}
\]

and

\[
 d_j(X_1, \ldots, X_j) = X_j, \text{ when } S(j) = j.
\]
Thus the expected gain conditional on stopping after the \( j \)-th draw is
\[
E(g(X_j) | S = j) - c_j.
\]

Therefore, the expected gain in employing levels \( \mathcal{L} \) is
\[
G_N(d \mathcal{L}) = \sum_{j=1}^{N} \{E(g(X_j) | S = j) - c_j\} \Pr(S = j) = \sum_{j=1}^{N} E(g(X_j) | S = j) \Pr(S = j) - c \Pr(S).
\]

(2.1)

We now show that the optimal strategy must consist of a non-increasing sequence of levels.

**PROPOSITION 2.1.** For the sequential decision problem outlined above, the optimal strategy consists of a non-increasing sequence of levels, \( L_1 \geq L_2 \geq \ldots \geq L_{N-1} \geq L_N \).

**PROOF:** Let \( a_1, \ldots, a_N \) be any levels for Problem I,
\[
0 \leq a_i \leq 1, \quad i = 1, \ldots, N.
\]

Since one of the random variables has to be accepted, one of the \( a_i \)'s is zero. Let \( M_v \) denote the event that the random variable chosen is \( \geq v \), and let \( S \) be the number of random variables sampled. Then
\[
\Pr(M_v, S = s | a_1, \ldots, a_s) = \Pr(X_i \leq a_i, i=1, \ldots, s-1; X_s > a_s; X_s \geq v)
\]
\[
= \int_{\max(a_s, v)}^{1} \left[ \prod_{i=1}^{s-1} \int_{0}^{a_i} dx_i \right] dx_s
\]
\[
\leq \int_{\max(a_{[s]}, v)}^{1} \left[ \prod_{i=1}^{s-1} \int_{0}^{a_{[i]}} dx_i \right] dx_s
\]
\[
= \Pr(M_v, S = s | a_{[1]} \geq a_{[2]} \geq \ldots \geq a_{[s]}) \tag{2.2}
\]

where \( a_{[s]} \leq \ldots \leq a_{[1]} \) denote the ordered \( a_i \)'s. Since (2.2) is true for all \( v \) and \( s \), it follows that the risk (2.1) will be minimized when the strategy consists of non-increasing levels.
Thus, we may without loss of optimality consider only the strategies which form a monotone sequence. We compare \( X_i \) with \( L_i, i = 1, \ldots, N \). If \( X_i > L_i \), we stop sampling and accept \( X_i \); if \( X_i \leq L_i \), we sample \( X_{i+1} \) and compare it with \( L_{i+1} \). Since one random variable has to be accepted, it must necessarily be true that \( L_N = 0 \). Let \( L_0 = 1 \). Then the optimal strategy forms a monotone sequence,

\[
0 = L_N \leq L_{N-1} \leq \ldots \leq L_1 \leq L_0 = 1.
\]

The Stopping Variable

Let \( S \) denote the stopping variable, that is, the number of random variables sampled before one is accepted. Then \( S \in \{1,2,\ldots,N\} \), and

\[
\Pr(S=j) = \Pr(X_1 \leq L_1, \ldots, X_{j-1} \leq L_{j-1}, X_j > L_j) \\
= \left\{ \prod_{k=0}^{j-1} \int_0^{L_k} dx_k \right\} \int_{L_j}^{1} dx_j = (1-L_j) \prod_{k=0}^{j-1} L_k, \tag{2.3}
\]

for \( j = 1, \ldots, N \), and

\[
E(S) = \sum_{j=1}^{N} j \Pr(S=j) = \sum_{j=1}^{N} j(1-L_j) \prod_{k=0}^{j-1} L_k = \sum_{j=1}^{N} \sum_{k=0}^{N-j} L_k = \sum_{j=1}^{N} \sum_{k=0}^{N-j} L_k. \tag{2.4}
\]

The Optimal Levels

We consider two different gain functions \( g \).

(i) Suppose that, for some constant \( b > 0 \),

\[
g(X_j) = \begin{cases} 
  b & \text{if } X_j \text{ is maximum} \\
  0 & \text{otherwise}.
\end{cases}
\]

Then

\[
G_N(dL) = b \sum_{j=1}^{N} \Pr(X_j \text{ is maximum and } S=j) - cE(S). \tag{2.5}
\]

Now, from Enns [3] we have, denoting by \( P_N(L) \) the probability that the maximum is actually attained using levels \( L = (L_1, \ldots, L_N) \),
\[ P_N(l) = \sum_{j=1}^{N} \Pr(X_j \text{ is maximum and } S=j) \]

\[ = \sum_{j=1}^{N} \frac{1}{3} \prod_{r=0}^{N-1} L_r - \frac{1}{3} \sum_{j=1}^{N} \frac{1}{j(j+1)} \sum_{j=1}^{N-2} \frac{1}{j} L_j^{j+1} \prod_{k=r+1}^{N-j+r-1} L_k \quad (N \geq 3), (2.6) \]

\[ P_2(L) = \frac{1}{2} + L_1 - L_1^2, \]

\[ P_1(L) = 1. \]

Therefore

\[ G_N(d|L) = \sum_{j=1}^{N} \left\{ \frac{b}{j} - c \right\} \prod_{k=0}^{N} L_k - \left( \frac{b}{N-1} \right) \sum_{j=1}^{N} L_j^N \]

\[ = b \left( \frac{1}{2} + L_1 - L_1^2 \right) - c(L_1+N), \quad (2.7) \]

\[ G_2(d|L) = \frac{b}{2} + L_1 - L_1^2, \quad G_1(d|L) = b - c. \]

(ii) Next, suppose that, for some constant \( b > 0 \),

\[ g(X_j) = \begin{cases} 0, & \text{if } S \neq j \\ bX_j + a, & \text{if } S = j, \ j = 1, \ldots, N. \end{cases} \]

Since the gain is now linear in the observations, it is more appropriate to consider the linear gain in the original observations, rather than in the transformed observations, because if the gain of accepting \( Y_j \) is taken as \( bY_j + a \), then the gain of accepting \( X_j = F(Y_j) \) is \( bF^{-1}(X_j) + a \), which is linear in \( X_j \) if and only if \( Y_j \) has a uniform distribution. Let \( Y_1, \ldots, Y_N \) be the original independent and identically distributed random variables with distribution function \( F(\cdot) \). Let the corresponding set of levels be

\[ Q_N \leq Q_{N-1} \leq \cdots \leq Q_1 \leq Q_0, \]

where \( Q_0 \) is the smallest \( x \) such that \( F(x) = 1 \) for every \( x > Q_0 \) and \( Q_N \) is the largest \( x \) such that \( F(x) = 0 \) for every \( x < Q_N \). The gain function is (with \( b > 0 \))
PROPOSITION 2.2. If $Y_j$, $j = 1, \ldots, N$, are independent and identically distributed as $F(\cdot)$, and if $S$ is the stopping variable for the strategy consisting of levels $Q_N \leq Q_{N-1} \leq \cdots \leq Q_1 \leq Q_0$, then

$$
Pr(S = j) = (1 - F(Q_j)) \prod_{k=0}^{j-1} F(Q_k), \quad j = 1, \ldots, N
$$

and

$$
E(S) = \sum_{j=1}^{N-1} \prod_{k=0}^{j-1} F(Q_k).
$$

This proposition's proof is trivial. Now to find the corresponding expected gain, note that the conditional distribution of $Y_j$ given $Y_j > Q_j$ is

$$
F^c(y) = Pr(Y_j \leq y | Y_j > Q_j)
$$

$$
= \begin{cases} 
0, & \text{if } y \leq Q_j \\
\frac{F(y) - F(Q_j)}{1 - F(Q_j)}, & \text{if } y > Q_j
\end{cases}
$$

Therefore,

$$
E(Y_j | Y_j > Q_j) = \int_{Q_j}^{Q_0} y dF^c(y) = \frac{1}{1 - F(Q_j)} \int_{Q_j}^{Q_0} y dF(y).
$$

Let $L_j = F(Q_j)$. Then the expected gain in employing levels

$$
Q = (Q_1 = F^{-1}(L_1), \ldots, Q_N = F^{-1}(L_N))
$$

is

$$
G_N(d | L) = \sum_{j=1}^{N} E(bY_j + a | Y_j > Q_j) Pr(S = j) - c E(S)
$$

$$
= a + b \sum_{j=1}^{N} E(Y_j | Y_j > Q_j) Pr(S = j) - c E(S)
$$

$$
= a + b \left[ \prod_{k=0}^{N-j} L_k - c \prod_{k=0}^{N-j} L_k \right].
$$
Special Cases

(1) (Uniform) Let $Y_j$ be independent and identically distributed as $U[0,1]$, $j = 1, ..., N$. Then $Q_N = 0$, $Q_0 = 1$, and $Q_j = L_j$. Also

$$Q_0 = 1,$$

$$\int_{Q_j} y_j dF(y_j) = \int_{L_j} y_j dy_j = \frac{1}{2} - \frac{L_j^2}{2}.$$

Therefore

$$G_N^{U}(d|L) = a + \sum_{j=1}^{N} \left[ \frac{(b(1-L_j)/2)}{L_k - c \sum_{k=0}^{N-j} L_k} \right],$$

(2.11)

where $G_N^{U}(d|L)$ denotes the expected gain when the underlying distribution is $U[0,1]$.

(2) (Exponential) Let $Y_j$ be independent and identically distributed as exponential ($\lambda = 1$). Then $Q_N = 0$, $Q_0 = \infty$, and $Q_j = -\log(1-L_j)$. Also

$$Q_0 = \int_{Q_j} y_j dF(y_j) = [1 - \log(1-L_j)](1-L_j).$$

Therefore

$$G_N^{Ex}(d|L) = a + \sum_{j=1}^{N} \left[ \frac{b(1-L_j)[1 - \log(1-L_j)]}{L_k - c \sum_{k=0}^{N-j} L_k} \right],$$

(2.12)

where $G_N^{Ex}(d|L)$ denotes the expected gain when the underlying distribution is exponential.

(3) (Normal) Let $Y_j$ be independent and identically distributed as $N(0,1)$. Then $Q_N = -\infty$, $Q_0 = \infty$, and $Q_j = \Phi^{-1}(L_j)$. Also

$$Q_0 = \int_{Q_j} y_j dF(y_j) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\Phi^{-1}(L_j))^2}.$$

Therefore

$$G_N^{N}(d|L) = a + \sum_{j=1}^{N} \left[ \frac{b}{\sqrt{2\pi}} e^{-\frac{1}{2}(\Phi^{-1}(L_j))^2} \frac{1}{L_k - c \sum_{k=0}^{N-j} L_k} \right],$$

(2.13)

where $G_N^{N}(d|L)$ denotes the risk when the underlying distribution is normal.
NOTE: We do not lose generality by assuming \( \lambda = 1 \) in case (2) or \((u = 0, \sigma^2 = 1)\) in case (3), since the gain function is linear in observations and a location and scale transformation does not change the linearity. The corresponding levels when \( \lambda \neq 1 \) or \( \mu \neq 0 \) or \( \sigma^2 \neq 1 \) can be obtained by suitable location and scale transformations.

**Numerical Results**

Table I below gives the optimum levels, \( L^* \), and the corresponding maximum expected gains \( G_N(d|L^*) \) for \( N = 2(1)10 \) in the case of (2.7). Tables II give the above quantities in case of (2.11), (2.12), and (2.13). [Note that all of the tables in this paper were obtained using the sequential simplex program for solving minimization problems which was developed by Olsson [7].] These tables show that for a given \( N \) and \( b \) (respectively, \( c \)) as \( c \) decreases (respectively, as \( b \) increases) the optimal levels \( L^* \) increase componentwise. Therefore, if the gain is not much as compared to the cost, we stop and make the selection earlier.
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**Table I**

Table showing the optimum levels \( L^2 \) and the corresponding maximum expected gain \( G_N(d|L^2) \) for the gain function

\[
g(X_j) = \begin{cases} 
    b, & \text{if } X_j \text{ is maximum} \\
    0, & \text{otherwise}.
\end{cases}
\]
### TABLE II

Table showing the optimum levels $\mathbf{L}^*_k$ and the corresponding maximum expected gain $G_N(d|\mathbf{L}^*_k)$ for the gain function $g(X_j) = \begin{cases} bX_j + a, & \text{if select } X_j \\ 0, & \text{otherwise.} \end{cases}$

(a) UNIFORM DISTRIBUTION

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<th>$N$</th>
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| $(G_N(d|\mathbf{L}^*_k)-a)/c$ | 6.48500 | 6.93376 | 7.39385 | 7.72170 | 7.90007 | 8.14605 | 8.31469 | 8.44611 | 8.56349 | 8.56349 |
(b) EXPONENTIAL DISTRIBUTION

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\( (G_{N}(d|l_{\alpha})-a)/c \)

\( (G_{N}(d|l_{\alpha})-a)/c \) values for different values of \( b/c \) are provided in the table. The values range from 0.05233 to 6.81658.
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<th>(C/N(dL^2_n-a)/c)</th>
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<th>(C/N(dL^2_n-a)/c)</th>
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3. CASE OF UNKNOWN DISTRIBUTION:
RANDOM VARIABLES OBSERVED DIRECTLY

Now we consider the case when the distribution function is continuous but unknown. The random variables are observed directly. As each random variable is observed, only its rank relative to its predecessors is noted or able to be noted.

The Optimal Strategy

For choosing the maximum of a sequence of $N$ random variables in this case the derivation of the form of the optimal strategy and terminology are well-known from [5]. Call $X_i$, the random variable drawn at the $i$-th draw, a "candidate" if $X_j < X_i$, $j = 1,...,i-1$. The optimal strategy is to pass, say, $r-1$ random variables and then choose the first candidate. Thus we want to find the optimal value of $r$. (It is known that this strategy, optimal for gain functions as in (i) below, is not optimal for gain functions such as that in (ii) below. However the optimal $r$ for this strategy is of interest in (ii), as is the effect of sampling cost, and these are studied below.)
The Stopping Variable

Let $S$ denote the draw at which we stop after passing $r - 1$ random variables. Then $S \in \{1, 2, \ldots, N-r+1\}$. For $s = 1, \ldots, N-r$, we have

$$Pr(S=s) = \frac{1}{s+r-1} \cdot \frac{r+1}{s+r-2}.$$

The Optimal Value of $r$

Suppose that the cost per observation is $c$ and that the gain is $g(X_j)$ if we accept $X_j$. Then the expected gain in employing the optimal strategy conditional on stopping at $S = s$ is $E(g(X_{s+r-1})|S=s) - (r-1)c-sc$, and therefore the expected gain in using the above strategy is $E(g(X_{s+r-1})|S=s) - (r-1)c-sc$, and therefore the expected gain in using the above strategy is

$$G_N(r) = \frac{N-r+1}{\sum_{s=1}^{N-r+1} E(g(X_{s+r-1})|S=s)Pr(S=s) - c(r-1)-cE(S)}.$$

We now consider two different gain functions $g$.

(i) Suppose that, for some constant $b > 0$,

$$g(X_{s+r-1}) = \begin{cases} b, & \text{if } X_{s+r-1} \text{ is maximum} \\ 0, & \text{otherwise} \end{cases}.$$
Here it is well-known that

$$E(g(X_{s+r-1})|S=s) = b \cdot \frac{r-1}{N(s+r-2)} ,$$

(3.3)

hence in this case

$$G_N(r) = \binom{b}{N-c} + (r-1) \binom{b}{N-c} \left( \frac{1}{r} + \frac{1}{r+1} + \ldots + \frac{1}{N-1} \right) - c(r-1).$$

(3.4)

Therefore, the optimal value of $r$ is the smallest $r^*$ such that

$$G_N(r^*) > G_N(r^*-1) \text{ and } G_N(r^*) > G_N(r^*+1):$$

$$\frac{1}{r^*} + \frac{1}{r^*+1} + \ldots + \frac{1}{N-1} < \frac{b}{b-Nc} < \frac{1}{r^*-1} + \frac{1}{r^*} + \ldots + \frac{1}{N-1} , \quad b > Nc .$$

(3.5)

(ii) Since the distribution of the random variables is unknown, let us consider the gain function

$$g(X_{s+r-1}) = \begin{cases} b\bar{R}(X_{s+r-1}) + a , & \text{if } S=s \\ 0 , & \text{if } S \neq s , \; s = 1, \ldots, N-r+1 , \end{cases}$$

where $\bar{R}(X_{s+r-1})$ is the rank of $X_{s+r-1}$ among $X_1, \ldots, X_N$ , and $b > 0$. (For $c=0$ this reduces to the problem of maximizing expected rank, which has been studied by Chow, Moriguti, Robbins, and Samuels [1] and De Groot [2]. More general functions of rank, but with $c=0$ also, have been studied by Rasmussen [8].)
Here it is well-known that

\[ E[\mathcal{H}_r | S=s] = \begin{cases} \frac{N(N+1)}{2(N-s-r+2)} - \frac{(s+r-2)(s+r-1)}{2(N-s-r+2)}, & s = 1, \ldots, N-r \\ \frac{N+1}{2}, & s = N-r+1, \end{cases} \]

hence in this case

\[ G_N(r) = a + (\frac{b}{2} - c)(r-1)\left[ 1 + \frac{1}{r} + \ldots + \frac{1}{N-1} \right] + \frac{b(N+1)}{2} - \frac{b(r-1)}{2} - c. \quad (3.6) \]

Therefore, the optimal value of \( r \) is the smallest \( r^* \) such that

\[ G_N(r^*) > G_N(r^* - 1) \quad \text{and} \quad G_N(r^* - 1) > G_N(r^* + 1); \]

\[ \frac{1}{r^*} + \frac{1}{r^* + 1} + \ldots + \frac{1}{N-1} < \frac{b}{b-2c} < \frac{1}{r^* - 1} + \frac{1}{r^*} + \ldots + \frac{1}{N-1}, \quad b > 2c. \quad (3.7) \]

It is interesting to compare (3.5) and (3.7). Since \( \frac{b}{b-2c} < \frac{b}{b-nc}, \quad N > 3 \), (3.7) yields a smaller value of \( r^* \). This is as one could expect on comparing the two gain functions.

**Numerical Results**

Table III below gives the optimum values \( r^* \) and the corresponding maximum expected gains \( G_N(r^*) \) for \( N = 3(1)50 \) in case of (3.4) and (3.6). The table shows that if the gain is much more than the cost of sampling one should observe a larger number of random variables before making a final selection.
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Table showing optimum \( m \) and the corresponding maximum expected gain.
Acknowledgments

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