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ON \( \Gamma \)-MINIMAX, MINIMAX, AND BAYES PROCEDURES FOR SELECTING POPULATIONS CLOSE TO A CONTROL

by

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ON Γ-MINIMAX, MINIMAX, AND BAYES PROCEDURES FOR SELECTING POPULATIONS CLOSE TO A CONTROL

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ABSTRACT

Let \( \Pi_0, \Pi_1, \ldots, \Pi_k \) be \((k+1)\) normally distributed populations and let \( \Pi_0 \) be a control population. Our goal is to select those populations which are sufficiently close to the control in terms of the (unknown) means of the populations. A zero-one type loss function is defined. \( \Gamma \)-minimax rules, Bayes rules and minimax rules are derived for this problem and compared. Some optimal properties of \( \Gamma \)-minimax rules are shown; also, \( \Gamma \)-minimax rules are derived for distributions other than the normal.
ON $\tau$-MINIMAX, MINIMAX, AND BAYES PROCEDURES FOR SELECTING POPULATIONS CLOSE TO A CONTROL*

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1. Introduction and summary

Problems of selecting populations close to a control arise frequently in industrial production, in situations such as for matching parts. Assume that we have \((k+1)\) populations and one of them is the control or standard population. Our goal is to select those populations which are sufficiently close to the control. Many authors have considered problems of comparing populations with a control under different types of formulations. Paulson (1952), Bechhofer and Turnbull (1974) discussed problems of selecting the best population if the best population is better than the control. Dunnett (1955), Gupta and Sobel (1958) considered problems of selecting a subset containing all populations better than the control. Lehmann (1961), Tong (1969), Randles and Hollander (1971) dealt with problems of selecting populations better than control. For problems of classifying a set of populations into three groups which are 'superior', 'inferior' and 'equivalent' to a control, see Kim (1979) and Gupta and Kim (1980) and related references therein. However, not much work has been done for the problem of selecting populations close to a control.

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Singh (1977) and Gupta and Singh (1979) considered this problem and derived Bayes (and empirical Bayes) rules for various distributions. In this paper, ι-minimax rules for selecting populations close to a control are derived, and these are compared with minimax rules and Bayes rules for robustness against the prior information. In Section 2, definitions and notations are introduced and a decision theoretic formulation of the problem is given. Results in Section 3 and Section 4, deal with the cases when all populations are assumed to be normally distributed. ι-minimax decision rules are derived when the control parameter θ0 is known, and restricted ι-minimax rules are derived when θ0 is unknown. In Section 5, some optimal properties of ι-minimax rules are found. In Section 6, results of Section 3 are generalized and ι-minimax rules are derived for distributions other than the normal. ι-minimax rules for selecting binomial populations with large entropy are also discussed. In Section 7, θ0 is treated as an unknown. Bayes rules are found under the assumptions that θ_i has a normal prior distribution with mean α_i and variance β_i^2, i = 0, 1, ..., k, which are assumed known. Minimax rules are also derived in this section. And Bayes rules, ι-minimax rules and minimax rules are compared in Section 8 in terms of the Bayes risk, the maximum risk over ι and the maximum risk over all the possible prior distributions.

2. Notations and formulation of the problem.

Let Π_0, Π_1, ..., Π_k be (k+1) independent normal populations with means μ_0, μ_1, ..., μ_k and known variances σ_0^2, σ_1^2, ..., σ_k^2, respectively. Assume that Π_0 is the control population, with mean μ_0 which may
be known or unknown. For \( \pi_1, \ldots, \pi_k \), the treatment populations, \( \theta_1, \ldots, \theta_k \) are all assumed to be unknown. When \( \theta_0 \) is unknown, let \( \theta = (\theta_0, \theta_1, \ldots, \theta_k) \) and \( X = (X_0, X_1, \ldots, X_k) \) where \( X_i \) is an observation from \( \pi_i \), \( i = 0, 1, \ldots, k \). When \( \theta_0 \) is known, no observation from \( \pi_0 \) is taken, and \( \theta_0, X_0 \) are deleted from \( \theta \) and \( X \), respectively. When there is no confusion, \( \theta \) and \( X \) are used to represent either case. Let \( \Theta \) be the parameter space and \( \chi \) be the sample space. For \( i = 1, 2, \ldots, k \), define \( G_i = \{ \theta \in \Theta \mid \theta_i - \theta_0 \leq \lambda + \epsilon \} \) and \( B_i = \{ \theta \in \Theta \mid \theta_i - \theta_0 \geq \lambda + \epsilon \} \) where \( \lambda \) and \( \epsilon \) are given positive constants. \( \pi_i \) is said to be good (or acceptable) if \( \theta \in G_i \) and bad (not acceptable) if \( \theta \in B_i \). We consider decision rules of the form \( \delta (x) = (\delta_1 (x), \ldots, \delta_k (x)) \), where \( \delta_i (x) \) denotes the conditional probability of selecting \( \pi_i \) as a good population given \( X = x \). The objective is to select all the good populations while rejecting all the bad ones. Let \( L_1 \) be the loss incurred when we fail to select a good population and \( L_2 \) the loss for each bad population selected. The loss function is defined by

\[
L(\theta, \delta) = \sum_{i=1}^{k} L_1 (1 - \delta_i) I_{G_i} (\theta) + L_2 \delta_i I_{B_i} (\theta) = \sum_{i=1}^{k} L_i (\theta, \delta_i). \tag{2.1}
\]

Where \( I_A \) denotes the indicator function of \( A \). We assume that the partial information available is of the form: \( \pi_i \) has probability \( \lambda_i \) to be good and probability \( \lambda_i' \) to be bad. Let \( \lambda = (\lambda_1, \ldots, \lambda_k) \) and \( \lambda' = (\lambda_1', \ldots, \lambda_k') \). We define \( \sigma^* \) as \( \{ \pi \mid \pi \text{ is a prior distribution on } \Theta \} \) and \( \Gamma = \{ (\lambda, \lambda') \mid \pi \in \sigma^*, \pi (G_i) = \lambda_i, \pi (B_i) = \lambda_i' \} \) for \( i = 1, \ldots, k \), where \( 0 \leq \lambda_i + \lambda_i' \leq 1 \) and \( \pi (A) = \int_{A} d\pi (\theta) \). Then, \( \Gamma (\lambda, \lambda') \) denotes the class of all the prior distributions which
summarizes our information about \( \theta \). We restrict our investigation to this class. Let \( R(\theta, \delta) = E[0] \{ L(\theta, \delta(X)) \} \) and \( r(\tau, \delta) = E[1] \{ R(\theta, \delta) \} \).

In this framework, an ith component problem is concerned with the selection or rejection of \( \Pi_i \). Then \( R^{(i)}(\theta, \delta_i) = E[0] \{ L^{(i)}(\theta, \delta_i) \} \) and \( r^{(i)}(\tau, \delta_i) = E[1] \{ R^{(i)}(\theta, \delta_i) \} \) denote the risk function and the Bayes risk function of the ith component problem, respectively. It is found that

\[
R(\theta, \delta) = \sum_{i=1}^{k} R^{(i)}(\theta, \delta_i) \quad \text{and} \quad r(\tau, \delta) = \sum_{i=1}^{k} r^{(i)}(\tau, \delta_i).
\]

A rule \( \delta^* \) is called a \( \Gamma \)-minimax rule in \( D \) if

\[
\sup_{\tau \in \Gamma} r(\tau, \delta^*) = \inf_{\delta \in D} \sup_{\tau \in \Gamma} r(\tau, \delta)
\]

where \( D \) is a class of decision rules.

3. Derivation of a \( \Gamma \)-minimax rule when \( \nu_0 \) is known.

In this section, \( \theta_0 \) is assumed to be known. We define

\[
G_{i1} = \{ \theta | \theta = \theta_0 + a \}, \quad G_{i2} = \{ \theta | \theta = \theta_0 - b \}, \quad B_{i1} = \{ \theta | \theta = \theta_0 + \Delta \}
\]

and \( B_{i2} = \{ \theta | \theta = \theta_0 - \Delta \} \). Let \( \delta_i(x) = \delta_i(x_i) \) be an ith component decision rule and let \( \nu_1 \) be an ith component decision rule and let \( g_1(\nu_1) = E_{\Pi_i}[\delta_1(X_i)] \), then we have

Lemma 3.1. For any fixed \( i \), if \( \inf_{\theta \in G_i} g_1(\theta) = g_1(\theta_0 + \Delta) = g_1(\theta_0 - \Delta) \)

and

\[
\sup_{\theta \in B_i} g_1(\theta) = g_1(\theta_0 + \Delta) = g_1(\theta_0 - \Delta),
\]

then

\[
\sup_{\tau \in \Gamma} r^{(i)}(\tau, \delta_i) = r^{(i)}(\nu_0, \delta_i) \quad \text{for all} \quad \tau \in \Gamma.
\]

where

\[
\Gamma_0(i) = \{ \tau \in \Gamma | P_1(G_{i1}) + P_1(G_{i2}) = \nu_1, P_1(B_{i1}) + P_1(B_{i2}) = \nu_i \}.
\]
Proof: \[
\begin{align*}
r^{(i)}(\tau, \delta_i) &= \int_{G_i} E_0 [L_1 (1 - \delta_i(X))] d \theta + \int_{B_i} E_0 [L_2 \delta_i(x)] d \theta \\
&\leq L_1 \lambda_i - L_1 \lambda_i \inf_{\theta \in G_i} q_i(\theta, i) + L_2 \lambda_i \sup_{\theta \in B_i} q_i(\theta, i) \\
&= L_1 \lambda_i - L_1 \lambda_i \int_{0}^{1} (G_{i1}) q_i(\theta, 0 + \Lambda) + \int_{0}^{1} (G_{i2}) q_i(\theta, 0 - \Lambda) \\
&+ L_2 \int_{0}^{1} (B_{i1}) q_i(\theta, 0 + \Lambda + \Gamma) + \int_{0}^{1} (B_{i2}) q_i(\theta, 0 - \Lambda - \Gamma) \\
&= r^{(i)}(\tau, \delta_i) \quad \text{for all } \tau \in \Gamma(i).
\end{align*}
\]

This completes the proof.

Theorem 3.1. If there exists a \( \tau^* \in \cap_{i=1}^{k} \Gamma(i) \) such that \( \lambda^*(X) = \lambda^* (x_i) \) is a Bayes rule wrt \( \tau^* \) for the ith component problem and if (3.1) is satisfied for \( g_i(\theta, i) = E_0 [\lambda^* (X_i)] \) for all \( i = 1, 2, \ldots, k \), then \( \tau^* = (\lambda^*_{i1}, \ldots, \lambda^*_{ik}) \) is a \( \Gamma \)-minimax rule.

Proof: \[
\sup_{\tau \in \Gamma} r^{(i)}(\tau, \delta_i^*) \leq \sum_{i=1}^{k} \sup_{\tau \in \Gamma} r^{(i)}(\tau, \delta_i^*)
\]

by Lemma 3.1

\[
= \sum_{i=1}^{k} r^{(i)}(\tau, \delta_i^*)
\]

for all \( \tau \). This completes the proof.

Lemma 3.2. Let the pdf \( f(x\mid \eta) \) of \( X \) be TP_3 (Totally Positive of order 3). If \( q(\eta) = E_i [1_{(a, b)}(X)] \), and for some \( \eta_0, q(\eta_0 + \eta) - q(\eta_0 - \eta_0) \), then \( q \) is increasing for \( \eta + \eta_0 \) and hence decreasing for \( \eta - \eta_0 \).
Proof: Let $h_c(x) = I(a, b)(x) - c$ for $c \in (0, 1)$, then $q_c(0) = q(0) - c$ where $q_c(0) = E_0[h_c(X)]$. Let $S(h_c)$ denote the number of sign changes of the function $h_c$, then $S(h_c) = 2$. Now by Variation Diminishing Property (VDP) (Karlin (1968), see p. 21) it is seen that $S(g_c) \leq 2$ for all $c \in (0, 1)$. If $g$ is not increasing for $\theta < \theta_0$, then there exist $\theta_1 < \theta_2 < \theta_0$ and $g(\theta_1) - g(\theta_2)$. Let $\theta'_1 = 2\theta_0 - \theta_1$ and $\theta'_2 = 2\theta_0 - \theta_2$, then $g(\theta'_1) - g(\theta'_2)$. We find that $S(g_{c_0}) \geq 2$ for $c_0 = 1/2[q(\theta_1) + q(\theta_2)]$, so $S(g_{c_0}) = 2$. But $g_{c_0}$ does not change signs in the same way as $h_c$ does which contradicts VDP. This completes the proof.

Now let

$$q_{\alpha i}^c(x) = \frac{1}{\sigma i} \left[\phi\left(\frac{x + \Delta}{\sigma i}\right) + \phi\left(\frac{x - \Delta}{\sigma i}\right)\right]$$

(3.2)

$$f_{\alpha i}(x) = \frac{1}{\sigma i} \left[\phi\left(\frac{x + \Delta + \epsilon}{\sigma i}\right) + \phi\left(\frac{x - \Delta - \epsilon}{\sigma i}\right)\right].$$

where $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$. Then we have

Theorem 3.2. If $\delta_{\alpha i}^*(x) = \delta_{\alpha i}^*(x_i) = I[-t_i, t_i](x_i - \theta_0)$ and $t_i > 0$ satisfies

$$L_{2\sigma i} f_{\alpha i}(t_i) = L_{1\sigma i} q_{\alpha i}^c(t_i) \text{ for } i = 1, \ldots, k,$$

(3.3)

then $\delta^* = (\delta_{\alpha 1}^*, \ldots, \delta_{\alpha k}^*)$ is a $\alpha$-minimax rule.

Proof: Let $\epsilon^* \in \Gamma$ be a prior distribution on $\alpha$ such that $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{k}$ are independent under $\epsilon^*$, and $P_{\epsilon^*}(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{k}) = 1^{\alpha_1 - 1} - 1^\alpha_{1}$, $P_{\epsilon^*}(G_{11}) = \frac{1}{2}$, $P_{\epsilon^*}(R_{11}) = P_{\epsilon^*}(B_{12}) = \frac{1}{2}$ for $i = 1, 2, \ldots, k$. Let $f(x|\alpha) = \frac{1}{\sigma i} \phi\left(\frac{x_i - \theta_i}{\sigma i}\right)$, then we have

$$f(x|\alpha) = \frac{1}{\sigma i} \phi\left(\frac{x_i - \theta_i}{\sigma i}\right),$$
\[ r(i)(*, \delta_i) = \int_{x_1} L_1(x) \sum_{\theta \in G_i} f(x|\theta) P_{\theta}(i) \]
\[ + L_2 \int_{x \in B_i} f(x|\theta) P_{\theta}(i) \, dx. \]

If we let \( C(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k) = \sum_{\theta \in G_i} f(x|\theta) P_{\theta}(i) \)
\[ = \frac{\lambda_i}{2} \frac{1}{\sigma_i} \phi\left( \frac{x_i - \theta_i^0}{\sigma_i} \right) \]
\[ = \frac{\lambda_i}{2} \frac{1}{\sigma_i} \phi\left( \frac{x_i - \theta_i^0}{\sigma_i} \right), \]
then
\[ r(i)(*, \delta_i) = \int_{x_1} L_1(x) g_{\delta_i}(x) C(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k) \, dx \]
\[ + 1/2 \int_{x_1} \delta_i(x) [L_2 g_{\delta_i}(x) - L_1 g_{\delta_i}(x)] C(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k) \, dx. \]

Hence the Bayes rule for the ith component problem wrt \( \star \) is
\[ \delta_i^*(x) = \delta_i(x) = \begin{cases} 
1 & \text{if } L_1 g_{\delta_i}(x) > L_2 g_{\delta_i}(x) \\
0 & \text{otherwise} \end{cases} \]

Let \( h_i(x) = \frac{L_2 g_{\delta_i}(x)}{L_1 g_{\delta_i}(x)} = k_i \frac{c_{\delta_i}(x)}{c_{\delta_i}(x)}, \)
where \( k_i = \frac{L_2}{L_1} \frac{g_{\delta_i}(x)}{g_{\delta_i}(x)} \exp\left[ -\frac{1}{2} (2^i - 1) \right], \)
then \( h_i(x) = h_i(-x) \) and \( h_i(x) \) is increasing for \( x < 0, \) hence \( h_i(x) < 1 \) if and only if \( |x| < t_i, \)
where \( t_i \) satisfies \( h_i(t_i) = 1. \) So, \( \delta_i^*(x) = 1_{[-t_i, t_i]}(x) \).
Now, if $g_i(\theta_i) = E_{\theta_i} [\delta_i(X_i)]$, we find $g_i(\theta_1 + \theta_0) = g_i(\theta_0 - \theta_1)$.

Also, $X_i \sim N(\theta_i, \sigma^2_i)$, so the pdf of $X_i$ is $TP$, hence $TP_3$ from Karlin (1968) (see p. 18). Now, by Lemma 3.2, (3.1) is satisfied, then Theorem 3.1 shows that $\delta^*$ is a $\Gamma$-minimax rule. This finishes the proof.

Let $\lambda/\lambda'$ be defined as $\left(\frac{\lambda_1}{\lambda'}\ldots\frac{\lambda_k}{\lambda'}\right)$. If $\Gamma(\gamma) = \{\tau|\theta_i|P_i|G_i_1/P_i|B_i_1 \}$ $\gamma_i$ for $i=1,2,\ldots,k$ where $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k)$, then we have $\Gamma(\gamma) = \frac{\lambda}{\lambda'} \Gamma(\lambda, \lambda')$. Since $\delta^*$ depends on $\lambda, \lambda'$ only through $\lambda/\lambda'$, we find

$$\sup_{\tau \in \Gamma(\gamma)} r(\tau, \delta) = \sup_{\lambda/\lambda' = \gamma} \sup_{\tau \in \Gamma(\lambda, \lambda')} r(\tau, \delta)$$

$$\geq \sup_{\lambda/\lambda' = \gamma} \sup_{\tau \in \Gamma(\lambda, \lambda')} r(\tau, \delta^*)$$

$$= \sup_{\tau \in \Gamma(\gamma)} r(\tau, \delta^*)$$

hence $\delta^*$ is a $\Gamma$-minimax rule for $\Gamma = \Gamma(\gamma)$.

It is possible that (3.3) does not have a solution. In this case, the $\Gamma$-minimax rules imply that all populations are bad.

4. A restricted $\Gamma$-minimax rule for $\theta_0$ unknown.

When $\theta_0$ is unknown, decision rules are restricted in a subclass $D'$, where $D' = \{\delta = (\delta_1, \ldots, \delta_k) | \delta_i(x) = \delta_i(x_0, x_i) \text{ for } i=1,\ldots,k\}$. That such a restriction is needed was first pointed out by Randles and Hollander (14). The following lemma has been used by Miescke (1979). The original idea of this lemma is due to Ferguson (1967) and Lehmann (1959).

**Lemma 4.1.** Let $\tau_n$ be a sequence of prior distributions on $\Theta$, and let $\tau_n$ be a Bayes rule wrt $\tau_n$ for the $i$th component problem. If
\[
\lim \inf_{n \to \infty} \gamma(i)(t_n, \delta^0_{in}) \leq \sup_{\mathbf{t} \in \mathcal{T}} \gamma(i)(t, \delta^0_i)
\]
for all \(i = 1, 2, \ldots, k\), then \(\delta^0 = (\delta^0_1, \delta^0_2, \ldots, \delta^0_k)\) is a \(\Gamma\)-minimax rule.

A prior distribution \(\varphi\) on \(\Theta_0 \times \Theta_1 \times \cdots \times \Theta_k\) can be specified by the marginal distribution \(\tau\) on \(\Theta_0\) and the conditional distribution \(\omega^0_{\Theta_1, \ldots, \Theta_k}\) given \(\theta^0 = 0\). We use \(\tau = (\tau, \omega^0_0)\) to denote such prior distributions. Let \(\tau_n = (\tau_n, \omega^0_0)\), where \(\tau_n\) is uniformly distributed over the interval \([-n, n]\) and \(\theta^0_1, \theta^0_2, \ldots, \theta^0_k\) are conditionally independent under \(\omega^0_0\), and

\[
P_{\omega^0_0}(B_1 | \theta^0_0) = \frac{\lambda^*_1}{2} \\
P_{\omega^0_0}(G_1 | \theta^0_0) = \frac{\lambda^*_1}{2} \\
P_{\omega^0_0}(0 | \theta^0_0 = \theta^*_0 + \xi_0 \frac{\lambda^*_1}{2}) = 1 - \lambda^*_1 - \lambda^*_1 \text{ for } i = 1, \ldots, k.
\]

Let \(\omega^*_0\) denote the conditional marginal distribution of \(\theta^*_1\) under \(\omega^*_0\). Then, we have

**Theorem 4.1.** When \(\theta^0\) is unknown, a \(\Gamma\)-minimax rule in \(D'\) is given by \(\delta^0 = (\delta^0_1, \ldots, \delta^0_k)\), where \(\delta^0_i(x) = I_{[-t^*_i, t^*_i]}(x_i - x_0)\) and \(t^*_i = 0\) satisfies

\[
L_1 \lambda^*_1 q^*_1(t^*_i) = L_2 \lambda^*_1 f^*_1(t^*_i), \text{ with } \alpha^*_1 = (\alpha^*_0 + \gamma^*_0) \text{ and } \gamma^*_1 = (3.2)
\]

\(q^*_1\) and \(f^*_1\) are defined in (3.2).

**Proof:** For \(\tau_n\) defined above, let \(h_n(x) = \frac{1}{\sqrt{n}} : (\frac{x}{n})\), then

\[
r^*(i)(t^*_n, \lambda^*_i) = \int \int \int L_1 (1 - E_{\theta^0_0}) \lambda^*_1 (X_0, X_1) \, d \omega^*_0, \lambda^*_i (X_0, X_1) \, d \theta^*_0, \lambda^*_i \, (\cdot) \, d T_n (\cdot)
\]
Hence, the Bayes rule wrt for the $i$th component problem is

$$\delta^0_{in}(x_i, x_0) = \begin{cases} 
1 & \text{if } \frac{L_1 \lambda_i}{2} \int g_{i} (x_i - \theta_0) h_{\sigma_0} (x_0 - \theta_0) \frac{1}{\sqrt{2\pi}} \, d\theta_0 \, dx_i \, dx_0 \\
\frac{L_1 \lambda_i}{2} \int g_{i} (x_i - \theta_0) h_{\sigma_0} (x_0 - \theta_0) \frac{1}{\sqrt{2\pi}} \, d\theta_0 \, dx_i \, dx_0 & \text{otherwise}
\end{cases}$$

and the Bayes risk of $\delta^0_{in}$ is given by

$$r(i)(\tau_n, \delta^0_{in}) = \int \int \min\left(\frac{L_1 \lambda_i}{4n} \int g_{i} (x_i - \theta_0) h_{\sigma_0} (x_0 - \theta_0) \frac{1}{\sqrt{2\pi}} \, d\theta_0 \, dx_i \, dx_0, \frac{L_2 \lambda_i}{4n} \int f_{\sigma_0} (x_i - \theta_0) h_{\sigma_0} (x_0 - \theta_0) \frac{1}{\sqrt{2\pi}} \, d\theta_0 \, dx_i \, dx_0\right) \, dx_i \, dx_0.$$ 

Now consider the change of variables

$$\begin{align*}
x_i = ny_i + y_0 \\
x_0 = ny_i - y_0
\end{align*}$$

for the outside two integrals, then let $\theta_0 = ny_i - y_0$ for the inside integral.

Since $\frac{\partial (x_i, x_0)}{\partial (y_i, y_0)} = 2n$ and $h_0(x) = h_0(-x)$, we find

$$r(i)(\tau_n, \delta^0_{in}) = \int \int \min\left(\frac{L_1 \lambda_i}{2} \int g_{i} (y_0 + n_0) h_{\sigma_0} (n_0 - y_0) \, d\theta_0, \frac{L_2 \lambda_i}{2} \int f_{\sigma_0} (y_0 + n_0) h_{\sigma_0} (n_0 - y_0) \, d\theta_0\right) \, dy_i \, dy_0.$$
It is known that
\[
\int_{-\infty}^{\infty} \frac{1}{\sigma^n} \phi\left(\frac{x-a}{\sigma}\right) \phi\left(\frac{x-b}{\sigma}\right) dx = \frac{1}{\sqrt{\sigma^2 + \eta^2}} \phi\left(\frac{b-a}{\sqrt{\sigma^2 + \eta^2}}\right),
\]
hence
\[
\int_{-\infty}^{\infty} g_{\theta_i}(\eta_0-a) h_{\theta_0}(\eta_0-b) d\eta_0 = g_{\theta_i}(a-b)
\]
and
\[
\int_{-\infty}^{\infty} f_{\theta_i}(\eta_0-a) h_{\theta_0}(\eta_0-b) d\eta_0 = f_{\theta_i}(a-b),
\]
where
\[
\omega_1 = (\omega_1^2 + \omega_0^2)^{1/2}. \text{ Now by Fatou's lemma and (4.3), we get}
\]
\[
\lim \inf_{n \to \infty} (\tau_n, \delta_0) \geq \int_{-\infty}^{\infty} \min\left(\frac{L_1 \lambda_i}{2} g_{\theta_i}(2y_0), \frac{L_2 \lambda_i}{2} f_{\theta_i}(2y_0)\right) dy_0
\]
\[
= \int_{-\infty}^{\infty} \min\left(\frac{L_1 \lambda_i}{2} g_{\theta_i}(x), \frac{L_2 \lambda_i}{2} f_{\theta_i}(x)\right) dx.
\]
On the other hand, for all \(i=(T, \omega_0, 0) \in \Gamma\)
\[
\gamma(i)(\tau_i, \delta_0) = \int_{-\infty}^{\infty} \left( L_1 (1-E_{\theta_i} - \theta_i) \delta_0 (X_i-X_0) \right) d\omega_0, i \in (0, 1)
\]
\[
+ \int_{-\infty}^{\infty} \left( L_2 E_{\theta_i} - \theta_i \delta_0 (X_i-X_0) \right) d\omega_0, i \in (0, 1) dT(i_0)
\]
\[
\geq L_1 \lambda_i \left[ 1 - \inf_{|\eta_i| \leq \Lambda} q_i(\eta_i) \right] + L_2 \lambda_i \sup_{|\eta_i| \geq \Lambda + \epsilon} q_i(\eta_i),
\]
where \(\eta_i = \theta_i - \theta_0, g_{\theta_i}(\eta_i) = E_{\eta_i} \delta_0 (Y_i)\) and \(Y_i = X_i - X_0\).

Since \(Y_i \sim N(\eta_i, \omega_1^2)\), so as was shown in the proof of Theorem 3.2, we have
\[
\sup_{|\eta_i| \geq \Lambda + \epsilon} g_i(\eta_i) = g_i(\Lambda + \epsilon) = g_i(-\Lambda - \epsilon)
\]
and
\[
\inf_{|\eta_i| \leq \Lambda} g_i(\eta_i) = g_i(\Lambda) = g_i(-\Lambda).
\]
Thus,
\begin{align*}
\mathbf{r}(i) & \leq L_1 \lambda_i \left[ 1 - \frac{g_1(\lambda) + g_1(-\lambda)}{2} \right] + L_2 \lambda_i \left[ 1 - \frac{g_1(\Lambda + \epsilon) + g_1(-\Lambda - \epsilon)}{2} \right] \\
& = \int_{-\infty}^{\infty} L_1 \lambda_i \left[ 1 - \delta_0(y_i) \right] g_{y_i}(y_i) + \frac{L_2 \lambda_i}{2} \delta_1(y_i) f_{y_i}(y_i) dy_i \\
& = \int_{-\infty}^{\infty} \min \left\{ \frac{L_1 \lambda_i}{2} g_{y_i}(x), \frac{L_2 \lambda_i}{2} f_{y_i}(x) \right\} dx.
\end{align*}

By (4.4), \( \sup_{\tau \in \Gamma} r^{(i)}(\tau, \delta^0_0) \leq \lim_{n \to \infty} \inf_{\tau \in \Gamma} r(\tau_n, \delta_n^0) \)

for all \( i = 1, 2, \ldots, k \). Lemma 4.1 now implies that \( \delta^0 = (\delta^0_1, \ldots, \delta^0_k) \)
is a \( \Gamma \)-minimax rule in \( D' \). This completes the proof.

## 5. Optimal properties of the \( \Gamma \)-minimax rule.

Suppose that we have \( n_i \) independent observations \( X_i \), \( X_i \in D_i \), \( i = 1, 2, \ldots, k \).

Let \( \bar{X}_{in_i} = \frac{1}{n_i} \sum_{j=1}^{n_i} x_{ij} \), then the \( \Gamma \)-minimax rule is of the form

\[
\delta_1^*(\bar{X}_{in_i}) = \left[ -t_1(n_i), t_1(n_i) \right] (\bar{X}_{in_i} - \delta_0)
\]

where \( t_1(n_i) \) is the positive root of the equation

\[
h_1(x) = k_1' \cosh(n_1(\Lambda + \epsilon)x/\sigma_1^2) / \cosh(n_1 \Lambda x/\sigma_1^2) = 1
\]

with \( k_1' = L_2 \lambda_i / (L_1 \lambda_i) \cdot \exp(-n_1(2\Lambda + \epsilon)\nu/2\sigma_i^2) \).

Consider

\[
f_i(x) = k_1' \exp(n_1(\Lambda + \epsilon)x/\sigma_1^2) / \exp(n_1 \Lambda x/\sigma_1^2)
\]

and

\[
g_i(x) = \frac{1}{2} k_1' \exp(n_1(\Lambda + \epsilon)x/\sigma_1^2) / \exp(n_1 \Lambda x/\sigma_1^2).
\]

Then, \( g_1(x) < h_1(x) < f_1(x) \), for \( x > 0 \).
Let $r_i(n_i)$ and $s_i(n_i)$ be the only positive root of $g_i(x) = 0$ and $f_i(x) = 0$ respectively, then $r_i(n_i) > s_i(n_i)$.

Now, $r_i(n_i) = \Lambda + \frac{c}{2} - \frac{\sigma_i^2 \ln(L_2 \lambda_i / 2L_1 \lambda_i)}{n_i}$

and

$s_i(n_i) = \Lambda + \frac{c}{2} - \frac{\sigma_i^2 \ln(L_2 \lambda_i / L_1 \lambda_i)}{n_i}$,

hence $\lim_{n_i \to \infty} r_i(n_i) = \lim_{n_i \to \infty} t_i(n_i) = \lim_{n_i \to \infty} s_i(n_i) = \Lambda + \frac{c}{2}$.

Then,

$$\lim_{n_i \to \infty} \inf_{\|\theta - \hat{\theta}\| \leq \Delta} E_{n_i}[\delta_i^* (\hat{X}_{n_i})] = \lim_{n_i \to \infty} \{ \frac{t_i(n_i) - \Lambda}{\sigma_i \sqrt{n_i}} - \frac{s_i(n_i) - \Lambda}{\sigma_i \sqrt{n_i}} \} = 1$$

(5.2)

and

$$\lim_{n_i \to \infty} \sup_{\|\theta - \hat{\theta}\| \geq \Delta + \epsilon} E_{n_i}[\delta_i^* (\hat{X}_{n_i})] = \lim_{n_i \to \infty} \{ \frac{t_i(n_i) - (\Lambda + \epsilon)}{\sigma_i \sqrt{n_i}} - \frac{s_i(n_i) - (\Lambda + \epsilon)}{\sigma_i \sqrt{n_i}} \} = 0,$$

(5.3)

for all $i = 1, 2, \ldots, k$.

**Theorem 5.1.** $\lim_{\min(n_1, \ldots, n_k) \to \infty} \sup_{i \in \Gamma} r(i, \delta^*) = 0$,

where $\delta^* = (\delta_1^*, \ldots, \delta_k^*)$ is the $l$-minimax rule with $\delta_i^*$ defined by (5.1), for $i = 1, 2, \ldots, k$.

**Proof:** $\sup_{i \in \Gamma} r(i, \delta^*) \leq \sum_{i=1}^{k} \sup_{i \in \Gamma} r(i, \delta_i^*)$
Hence, by (5.2) and (5.3)

\[
\lim_{\min(n_1, \ldots, n_k) \to \infty} \sup \tau(\xi, \delta^*) = 0.
\]

This completes the proof.

When \( \theta_0 \) is unknown, let \( \delta^0 = (\delta^0_1, \ldots, \delta^0_k) \) and

\[
\delta_i^0(\bar{x}_{i,n_1}, \bar{x}_{0,n_0}) = \max \left(-t_i'(n_i, n_0), t_i'(n_i, n_0) \right) \left( \bar{x}_{i,n_1} - \bar{x}_{0,n_0} \right),
\]

where \( t_i'(n_i, n_0) \) is defined in (4.2) with \( \sigma_i^2 \) and \( \sigma_0^2 \) replaced by \( \sigma_i^2/n_i \) and \( \sigma_0^2/n_0 \) respectively. Then

\[
\lim_{\min(n_0, \ldots, n_k) \to \infty} \sup \gamma(\xi, \delta^0) = 0
\]

also holds. The proof is thus similar to that of Theorem 5.1 and is omitted.

In deriving the \( r \)-minimax rule \( \delta^* \), we have proved that \( \delta^* \) is a Bayes rule wrt \( r^* \). It is easily seen from (3.4) that \( \delta^* \) is the unique Bayes rule wrt \( r^* \), and hence it is admissible.

**Theorem 5.2.** When \( \theta_0 \) is unknown, the \( r \)-minimax rule \( \delta^0 = (\delta^0_1, \ldots, \delta^0_k) \) is admissible in \( D' \).

**Proof:** Let \( \tau = (T_0, \omega^*_0) \) be a measure on \( \Theta \) such that \( T_0 \) is Lebesgue measure on \( \Theta_0 \) and \( \omega^*_0 \) is defined by (4.1). Then for all \( \delta \in D' \),

\[
\tau(\xi, \delta) = \int_{i=1}^{k} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2} L_1(1-\frac{x_0-x_i}{\sigma_i})u_{\omega_i}(x_i)h_0(x_0-x_i) f_{\omega_i}(x_i)dx_0dx_i
\]

\[
+ \frac{1}{2} L_2 \int_{i=1}^{k} g_{\omega_i}(x_i-x_0)dx_0dx_i
\]

\[
= \int_{i=1}^{k} \int_{-\infty}^{\infty} \frac{1}{2} L_1 u_{\omega_i}(x_i)h_0(x_0-x_i) f_{\omega_i}(x_i)dx_0dx_i
\]

\[
+ \frac{1}{2} L_2 \int_{i=1}^{k} g_{\omega_i}(x_i-x_0)dx_0dx_i
\]
\[ + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho_i(x_1, x_0) \left[ \frac{L_2}{2} f_{0_1}^i(x_1 - x_0) - \frac{L_1}{2} g_{0_1}^i(x_1 - x_0) \right] dx_1 dx_0 \]

by (4.3).

Hence, the generalized Bayes rule is given by \( \delta^0(0) = (\delta^0_1, \ldots, \delta^0_k) \) where

\[
\delta^0_i(x_1 - x_0) = \begin{cases} 
1 & \text{if } \lambda_i^0 L_2 f_{0_1}^i(x_1 - x_0) \leq \lambda_i^0 L_1 g_{0_1}^i(x_1 - x_0) \\
0 & \text{otherwise}
\end{cases}
\]

which is exactly the rule we defined in (4.2). Also, \( \delta^0 \) is the unique (up to equivalence) generalized Bayes rule wrt \( \tau_0 \) in \( D' \), and \( r(\tau_0, \delta^0) < \infty \). Hence \( \delta^0 \) is admissible in \( D' \). This completes the proof of Theorem 5.2.

6. Relaxing the assumption of normality.

In this section, \( \Pi_i \)'s are not limited to be normal populations. Let \( X_i \) be an observation from \( \Pi_i \) with pdf \( f_i(x|\theta_i) \) and let \( A_i = (x|L_2 \lambda_i^0 f_{0_1}^i(x|\theta_0+\Lambda) + f_i(x|\theta_0-\Lambda)) \leq L_1 \lambda_i^0 [f_i(x|\theta_0+\Lambda) + f_i(x|\theta_0-\Lambda)] \) for \( i = 1, 2, \ldots, k \). (6.1)

**Theorem 6.1.** Let \( q_i(\theta_1) = E_{i_1} [I_{A_i} (X_i)] \) where \( A_i \) is defined by (6.1). If \( q_i(\theta_1, \theta_0) = q_i(\theta_0-\theta_1) \) and \( q_i \) is increasing for \( \theta_1 < \theta_0 \) for all \( i = 1, 2, \ldots, k \), then \( \Lambda = (\Lambda_1, \ldots, \Lambda_k) \) is a \( \Lambda \)-minimax rule where \( \Lambda_i(x) = I_{A_i} (X_i) \).

**Proof:** Let \( \tau^* \) be defined as in the proof of Theorem 3.2, then the Bayes rule wrt \( \tau^* \) for the \( i \)-th component problem is given by
\( \delta_1(x) = I_{A_1}(x) \). Now, since \( g_i \) is symmetric about \( \theta_0 \) and \( g_i(\theta_i) \) is increasing for \( \theta_i > \theta_0 \), so

\[
\sup_{|\theta_i - \theta_0| \geq \Delta + \varepsilon} g_i(\theta_i) = g_i(\theta_0 + \Delta + \varepsilon) = g_i(\theta_0 - \varepsilon)
\]

and

\[
\inf_{|\theta_i - \theta_0| \leq \Delta} g_i(\theta_i) = g_i(\theta_0 - \Delta) = g_i(\theta_0 + \Delta).
\]

Then by Theorem 3.1, we conclude that \( \delta \) is a \( \Gamma \)-minimax rule.

As an example of this theorem, we consider the problem of selecting binomial populations with entropy larger than a given constant. For \( i = 1, 2, \ldots, k \), let \( H_0 \) be \( \Pi_i \) and \( \Psi(n_i, \theta_i) = -n_i \log_2 (1 - \theta_i) \) \( \theta_i \) known and \( \theta_i \) unknown. \( \Psi(n_i, \theta_i) \) is the entropy associated with \( \Pi_i \). Define \( H_1 \) to be good if \( \Psi(n_i, \theta_i) > \frac{1}{2} \) and bad if \( \Psi(n_i, \theta_i) \leq \frac{1}{2} \). This is equivalent to saying that \( H_1 \) is good if \( \frac{1}{2} - \frac{1}{2} \leq \Delta \) and \( H_1 \) is bad if \( \frac{1}{2} - \frac{1}{2} \leq \Delta + \varepsilon \).

where \( \Delta \) and \( \varepsilon \) satisfy

\[
\frac{1}{2} \leq \varepsilon, \quad \Psi\left(\frac{1}{2} + \Delta + \varepsilon\right) = \beta + \varepsilon', \quad \text{and} \quad \Psi\left(\frac{1}{2} + \Delta + \varepsilon\right) = \beta.
\]

Let

\[
L_1 = \frac{L_2^{x_1} \left(1 + \Delta + \varepsilon) \right)^x \frac{1}{2} - \lambda \right) \frac{\left(1 + \Delta + \varepsilon\right)^x \frac{1}{2} - \lambda \right)}{\left(1 + \Delta + \varepsilon\right)^x \frac{1}{2} - \lambda \right)}
\]

we find

\[
\frac{h_1(x + 1)}{h_1(x)} = h_1(x + 1 - x) = h_1(x - x) = h_1(x - x)
\]

and

\[
\frac{h_1(x + 1)}{h_1(x)} = \frac{n_i - 1}{x - x}.
\]

Hence \( h_1(x) \) is decreasing for \( x < \frac{n_i}{2} \) and increasing for \( x > \frac{n_i}{2} \). Now, in view of (6.2), we find \( \theta_0 = \frac{1}{2} \), so that

\[
A_1 = \{x | L_2^{x_1} \left[f_i(x, \frac{1}{2} + \Delta + \varepsilon) + f_i(x, \frac{1}{2} - \Delta - \varepsilon)\right]
\]

\[
L_1 \left[f_i(x, \frac{1}{2} + \Delta) + f_i(x, \frac{1}{2} - \Delta)\right] = \{x | h_1(x) = 1\}
\]
= \{x \mid \frac{n_i}{2} - m_i \leq x \leq \frac{n_i}{2} + m_i\}, where \( f_i(x|0) = (x^{n_i-2} + m_i)^{n_i/2} \) and

\[
\frac{n_i}{2} + m_i \text{ satisfies } h_i(\frac{n_i}{2} + m_i) = 1. \text{ Then, }
\]

\[
g_i(0, i) = E_{\delta_i} [I_{\delta_i}(X_i)] = E_{\delta_i} [I_{\delta_i}(n_i - X_i)]
\]

\[
= E_{\delta_i} [I_{\delta_i}(X_i)] = q_i(1 - \delta_i), \text{ so } q_i(\frac{1}{2} + \delta_i) = q_i(\frac{1}{2} - \delta_i). \text{ Now, by }
\]

Lemma 3.2 and Theorem 6.1, \( \delta = (\delta_1, \ldots, \delta_K) \) with \( \delta_i(x_i) = I_{n_i/2 - m_i, n_i(x_i) + m_i} \) is a \( \delta \)-minimax rule.

A density function \( f(x|0) \) is said to be a PF (Polya-Frequency) function if \( \alpha \) is a location parameter and \( f(x|\alpha) \) is TP. It is known that if \( X \) has a PF density \( f(x|\alpha) \) and \( f(x) = f(-x) \), then \( |X| \) has a TP density (see Karlin (1968) p. 738).

Hence,

\[
f(x + \alpha) + f(x - \alpha)
f(x + \alpha) + f(x - \alpha)
\]

is symmetric about 0 and is increasing for \( x > 0 \) when \( \alpha_2 \alpha_1 = 0 \).

\textbf{Theorem 6.2.} If \( X \) has a PF density \( f_i(x|0) = f_i(x - \alpha) \) and \( f_i(x) = f_i(-x) \), then the assumptions of Theorem 6.1 are satisfied.

\textbf{Proof:} Now \( A_i \) defined in (6.1) reduces to

\[
A_i = [x^{\alpha} - t_i - x : t_i]
\]

by the monotonicity of (6.3)

Then \( q_i(\alpha) = E_{\delta_i} [I_{\delta_i}(X_i)] = P[t_i + \alpha \cdot Z_i + t_i - \alpha \cdot 0] \]

where \( Z_i = X_i - \alpha \). Since \( Z_i \) and \( \alpha \cdot Z_i \) have the same distribution, it follows that so \( q_i(\alpha) = q_i(\alpha) \). By Lemma 3.2, the assumptions of Theorem 6.1 are satisfied.
An example where Theorem 6.2 is applied is when $f_1$ has a double exponential density $f_i(x|\nu_i) = \frac{c_i}{2} e^{-c_i|x_i-\nu_i|}$ for $i=1,2,\ldots,k$. In this case the $\Gamma$-minimax rule is $\delta = (\delta_1, \ldots, \delta_k)$ with

$$\delta_i(x_i) = \begin{cases} 1 \text{ if } \frac{\lambda_i L_2}{\lambda_i L_1} e^{-c_i|x_i-\nu_i|} + e^{-c_i|x_i-\nu_i|} > 0, \\ 0 \end{cases}$$


In Section 2, we assumed that partial information about $\nu$ is known and is summarized in the class $\mathcal{G}$. In this section, we consider two extreme cases, namely, either complete information or no information about $\nu$ is known. Then we are interested in the Bayes rules and minimax rules respectively. The problem will be treated under the assumption that $\nu_0$ is unknown and $X_i \sim N(\nu_i, \sigma_i^2)$ for $i=0,1,\ldots,k$. Assume that $\nu_i$ has a normal prior distribution with mean $a_i$ and variance $b_i^2$, then $\nu_i | X_i \sim N(a_i, b_i^2)$ where

$$a_i = \frac{a_i^2 + x_i \sigma_i^2}{\sigma_i^2 + b_i^2} \quad \text{and} \quad b_i = \frac{\sigma_i^2}{\sigma_i^2 + b_i^2}. \quad \text{With the same loss function as defined in (2.1), it can be easily shown that the Bayes rule is}$$

$$\delta^B = (\delta_1^B, \ldots, \delta_k^B)$$

where

$$\delta_i^B(x) = \begin{cases} 1 \text{ if } L_2 P[\nu_i - \nu_0 | x_i, x_0] L_1 P[\nu_i - \nu_0 | x_i, x_0] > 0, \\ 0 \end{cases} \quad \text{and} \quad \frac{L_2 \Phi(-x_i + \nu_i | x_0) + \Phi(\nu_i - x_i | x_0)}{L_1 \Phi(\nu_i | x_0) + \Phi(x_i | x_0)} = 1 \quad (7.1)$$
Theorem 7.1. Let \( a=(a_1, a_2, \ldots, a_k) \) and \( l=(1,1,\ldots, 1) \) and let \( \delta^M=(\delta^M_1, \ldots, \delta^M_k) \) be the \( i \)-minimax rule in \( \Delta^* \) for \( l=(a,1-a) \) (see (3.3)). If \( a_i \) is chosen such that \( q_i(a)=1 \) for all \( i=1,2,\ldots,k \), where

\[
\begin{align*}
q_i(a) &= \frac{\alpha_i(a)+\lambda_i+1}{\beta_i(a)+\lambda_i+1} - \frac{\alpha_i(a)+\lambda_i-1}{\beta_i(a)+\lambda_i-1} \\
&= \frac{\alpha_i(a)+\lambda_i+1}{\beta_i(a)+\lambda_i+1} - \frac{\alpha_i(a)+\lambda_i-1}{\beta_i(a)+\lambda_i-1},
\end{align*}
\]

(7.2)

then \( \delta^M \) is a minimax rule.

Proof: For \( \theta \in G_i \), \( R(i)(\theta, M_i) = L_i P[|X_i - X_0| \leq \alpha_i(a_i)| 0, \alpha_i] \)

\[
= L_i [\mu(\frac{\alpha_i(a_i) - (\lambda_i - 1)}{\alpha_i}) + \mu(\frac{\alpha_i(a_i) + (\lambda_i - 1)}{\alpha_i})]
\]

Similarly, for \( \theta \in B_i \),

\[
R(i)(\theta, M_i) = L_i [\mu(\frac{\alpha_i(a_i) + \lambda_i}{\alpha_i}) + \mu(\frac{\alpha_i(a_i) - \lambda_i}{\alpha_i})]
\]

If \( \theta \in G_i \), then \( R(i)(\theta, M_i) = 0 \). Now from (4.4) and (4.6), we get
\[
\lim_{n \to \infty} \inf_r r(n, \delta^0_{in}) = L_1 \left[ \Phi\left( \frac{t_i(a_i) + \Delta}{\sigma_i} \right) + \Phi\left( -\frac{t_i(a_i) - \Delta}{\sigma_i} \right) \right]
\]
\[
+ L_2 (1-a) \left[ \Phi\left( \frac{t_i(a_i) + \Delta + \xi_i}{\sigma_i} \right) - \Phi\left( -\frac{t_i(a_i) + \Delta + \xi_i}{\sigma_i} \right) \right]
\]
\[
= L_1 \left[ \Phi\left( \frac{t_i(a_i) + \Delta}{\sigma_i} \right) + \Phi\left( -\frac{t_i(a_i) - \Delta}{\sigma_i} \right) \right]
\]
\[
= L_2 \left[ \Phi\left( \frac{t_i(a_i) + \Delta + \xi_i}{\sigma_i} \right) - \Phi\left( -\frac{t_i(a_i) + \Delta + \xi_i}{\sigma_i} \right) \right]
\]
\]
\[
> \sup_{v \in \Theta} R(i)(v, \delta^M_i) \text{ for all } i = 1, \ldots, k.
\]

Then,
\[
\lim_{n \to \infty} \inf_r r(n, \delta^0_{in}) \geq \frac{1}{k} \lim_{n \to \infty} \inf_i r(n, \delta^0_{in})
\]
\[
> \frac{1}{k} \sup_{i=1}^k R(i)(v, \delta^M_i) \geq \sup_{v \in \Theta} R(v, \delta^M) .
\]

It follows that \( \delta^M \) is a minimax rule.

Let \( \gamma_i(a, x) = \frac{L_2 (1-a) f_{\delta^0_i, i}(x)}{L_1 a q_{\delta^0_i, i}(x)} \) \hspace{1cm} (7.3)

then \( \gamma_i(a, t_i(a)) = 1 \) by (4.2), which implies that \( t_i(a) \) is a continuous function of \( a \) by the implicit function theorem.

Hence \( q_i(a) \) is a continuous function of \( a \), for \( 0 \leq a \leq 1 \). Now,
\[
\lim_{a \to 1} q_i(a) = \lim_{a \to 1} q_i(a) = \infty \text{ by (7.1) and (7.2). Also, for } a \neq 1
\]
\[
a_i^0 = \frac{L_2 \exp[-\epsilon(2\Delta + \xi_i)/2\sigma_i^2]}{L_1 + L_2 \exp[-\epsilon(2\Delta + \xi_i)/2\sigma_i^2]}, \quad t_i(a_i^0) = 0, \text{ so } q_i(a_i^0) = 0. \text{ Then, by the continuity of } q_i \text{ there exists an } a_i(a_i^0, a_i^0, 1) \text{ such that } q_i(a_i) = 1. \text{ This shows that a minimax rule always exists.}
8. Comparison among Bayes, \( \tau \)-minimax and minimax rules

When one faces a decision problem, the choice of the optimal rules depends on the prior information one has. In general, one would use Bayes rules if the prior distribution is known exactly, use \( \tau \)-minimax rule for incomplete prior information and use minimax rule if no prior information is available. Hence one is interested in studying the robustness of these rules against the assumption of the prior information. In this section, we compare these rules in terms of the Bayes risk, maximum risk over \( \Gamma \) and the maximum risk over \( \Omega^* \). Since the loss function is assumed to be additive, the comparison is made for the first component problem only. In this section, \( x=(x_0,x_1) \), \( \theta=(\theta_0,\theta_1) \) and \( d_1B(\theta)=d_1(\theta_0)d_1(\theta_1) \), where \( \tau_i \sim N(\mu_i,\beta_i^2) \) for \( i=0,1 \). Let \( \hat{\delta}_B(x)=I_{[-t_1B,t_1B]}(a_1-a_0) \) be the Bayes rule wrt \( \tau_B \) (see Section 7), where \( a_i=\frac{\mu_i x_i^2}{\frac{\beta_i^2}{1}+\beta_i^2} \) for \( i=0,1 \).

Also, let \( \delta_G(x)=I_{[-t_G,t_G]}(x_1-x_0) \) be the \( \tau \)-minimax rule in \( \Omega^* \) and \( \delta_M(x)=I_{[-t_M,t_M]}(x_1-x_0) \) be the minimax rule. Define \( r_1(\cdot)=r^{(1)}(\cdot,\cdot) \), \( r_2(\cdot)=\sup_{\tau \in \Gamma} r^{(1)}(\cdot,\cdot) \) and \( r_3(\cdot)=\sup_{\tau \in \Omega^*} r^{(1)}(\cdot,\cdot) \).

Then, \( r_1(\theta_B)=L_1p_{B} \left[ |a_1-a_0| t_B, |\theta_1-\theta_0| t_1 \right]+L_2p_{B} \left[ |a_1-a_0| t_B, |\theta_1-\theta_0| t_2 \right] \).

Let \( d=\tau_1-\tau_0 \), \( w_1^2=\frac{\mu_1^2}{\frac{\beta_1^2}{1}+\beta_1^2} \), \( i=0,1 \)

\( u^2=w_0^2+w_1^2 \) and \( v^2=w_0^2+w_1^2 \). Then, we find that
\[
\begin{pmatrix}
\frac{a_1-a_0-d}{v} \\
\frac{g_1-g_0-d}{u}
\end{pmatrix}

\sim \mathcal{N}
\begin{pmatrix}
0 \\
0
\end{pmatrix},
\begin{pmatrix}
1 & \rho \\
0 & 1
\end{pmatrix}
\]

where \( \rho = \frac{v}{u} \).

Hence \( r_1(\delta_B) = \)

\[
L_1\{F(-B_1,C; -\rho) + F(-B_1,D; -\rho) + F(B_2,C; \rho) + F(B_2,D; \rho) +
\]

\[
+ L_2\{F(B_1,D-A; \rho) + F(B_2,D-A; \rho) + F(B_1,C-A; \rho) + F(B_2,C-A; \rho) \},
\]

where \( B_1 = \frac{t_B-d}{v}, \quad B_2 = \frac{-t_B-d}{v}, \quad A = \frac{\epsilon}{u}, \quad C = \frac{\Lambda-d}{u}, \quad D = \frac{-\Lambda-d}{u} \) and

\[
F(x_0,Y_0; \rho) = P\left[ Z_1 \leq x_0, Z_2 \leq y_0 \right] \text{ with }\frac{Z_1}{Z_2} \sim \mathcal{N}\left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right).
\]

Similarly, \( r_1(\delta_G) = L_1\{F(-G_1,C; -\rho) + F(-G_1,D; -\rho) + F(G_2,C; \rho) + F(G_2,D; \rho) \}
\]

\[
+ L_2\{F(G_1,D-A; \rho) + F(G_2,D-A; \rho) + F(G_1,C-A; \rho) + F(G_2,C-A; \rho) \},
\]

where \( G_1 = \frac{t_G-d}{\gamma}, \quad G_2 = \frac{t_G-d}{\gamma}, \quad \gamma = \sigma_0^2 + \sigma_1^2 + \mu^2 \) and \( \rho' = \frac{u}{\gamma} \).

Since \( \delta_G \) and \( \delta_M \) have the same form except for the constant \( t_G \)

\[
\text{and} \quad t_M, \quad \text{so if we let} \quad M_1 = \frac{t_M-d}{\gamma} \quad \text{and} \quad M_2 = \frac{-t_M-d}{\gamma} \quad \text{and replaces} \quad G_1, \quad G_2 \quad \text{by} \quad M_1, \quad M_2, \quad \text{respectively, in the above formula, we get} \quad r_1(\lambda_M).
\]

The following lemma is used to compute the maximum risk over \( \Gamma \).

\textbf{Lemma 8.1.} \( r_2(\epsilon) = L_1 \{ 1 - \inf_{\theta_1-\theta_0 \leq \Lambda} E_{\theta_1}[\delta(X)] \} + L_2 \{ 1 - \sup_{\theta_1-\theta_0 \leq \Lambda^+} E_{\theta}[\delta(X)] \}. \)

\textbf{Proof:} \( \epsilon \) is trivial. To prove the other inequality, let \( \{ \theta_n \} \)

\( \text{and} \quad \{ \theta_n \} \quad \text{be two sequences such that} \quad \epsilon_n \subset \{ \theta_n \}, \quad \theta_n \subset \{ \theta_n \}, \quad \epsilon_n \subset \{ \theta_n \} \quad \text{and} \quad E_{\theta}[\delta(X)] = \inf_{\theta \in \epsilon_n} E_{\theta}[\delta(X)]. \)

\( \epsilon_n \subset \{ \theta_n \} \quad \text{and} \quad E_{\theta}[\delta(X)] = \inf_{\theta \in \epsilon_n} E_{\theta}[\delta(X)]. \)
Let $\tilde{\eta} = \max \{\eta, 0\}$ be defined by $p_n [\tilde{\eta} - \tilde{\eta} + 0] = \lambda^1_1$, then

$$r_2(\delta) = \sup_{\tilde{\eta}} \max \{\eta - \tilde{\eta} + 0\} \cdot \chi_{\tilde{\eta}} \cdot E_0[\eta - \tilde{\eta} + 0]$$

$$+ L_2 \lambda^1_1 [\eta - \tilde{\eta} + 0] \cdot \chi_{\tilde{\eta}} \cdot E_0[\eta - \tilde{\eta} + 0] \cdot E_0[\tilde{\eta} - \tilde{\eta} + 0]$$

This finishes the proof.

From Lemma 8.1 and (4.5), we get $r_2(\delta_G) = L_1 \lambda^1_1 \left[ \frac{t_G - \lambda^1_1}{\tilde{\eta} - \tilde{\eta} + 0} \right] - \frac{t_G - \lambda^1_1}{\tilde{\eta} - \tilde{\eta} + 0} \cdot e \cdot (\frac{t_G - \lambda^1_1}{\tilde{\eta} - \tilde{\eta} + 0})$.

When $t_G$ is replaced by $t_M$, we get $r_2(\delta_M) = (\lambda^1_1 + \lambda^1_2) \cdot L_1 \left[ \frac{t_M - \lambda^1_2}{\tilde{\eta} - \tilde{\eta} + 0} \right] - \frac{t_M - \lambda^1_2}{\tilde{\eta} - \tilde{\eta} + 0} \cdot e \cdot (\frac{t_M - \lambda^1_2}{\tilde{\eta} - \tilde{\eta} + 0})$.

To find $r_2(\delta_B)$, first note that $a_1 = a_0 |_{\tilde{\eta} - \tilde{\eta} + 0} \cdot N(0, t^2)$ where

$$\mu = \frac{1}{t^2 + r^2 \cdot \tilde{\eta} - \tilde{\eta} + 0} - \frac{t^2 \cdot \tilde{\eta} - \tilde{\eta} + 0}{t^2 + r^2 \cdot \tilde{\eta} - \tilde{\eta} + 0}$$

and $\zeta^2 = \frac{t^2 - \tilde{\eta} + 0}{t^2 + r^2 \cdot \tilde{\eta} - \tilde{\eta} + 0}$. Let $u = \frac{t^2 \cdot \tilde{\eta} - \tilde{\eta} + 0}{t^2 + r^2 \cdot \tilde{\eta} - \tilde{\eta} + 0}$

$$q_B(\mu) = E_0[\delta_B(X)] = P_{\mu} \left[ \frac{t_B - \tilde{\eta} + 0}{t_B - \tilde{\eta} + 0} \right] - \frac{t_B - \tilde{\eta} + 0}{t_B - \tilde{\eta} + 0}$$

(8.1)

then $q_B(\mu) = q_B(-\mu)$ and $q_B(\mu)$ is decreasing in $|\mu|$

We consider the following two cases:

(a) if $\frac{t^2 \cdot \tilde{\eta} - \tilde{\eta} + 0}{t^2 + r^2 \cdot \tilde{\eta} - \tilde{\eta} + 0} \neq \frac{t^2 \cdot \tilde{\eta} - \tilde{\eta} + 0}{t^2 + r^2 \cdot \tilde{\eta} - \tilde{\eta} + 0}$. Let $n = n^1_1 = \cdots = n^1_{m_1}$ then $|\mu| = \frac{1}{n} \cdot \mu$.

So, $\inf_{\mu} E_{\mu}[\tilde{\eta} - \tilde{\eta} + 0] = \lim_{n \to \infty} q_B(\mu) = 0$ and $\sup_{\mu} E_{\mu}[\tilde{\eta} - \tilde{\eta} + 0] = q_B(\mu) = \frac{1}{n} \cdot \mu$.

(b) if $\frac{t^2 \cdot \tilde{\eta} - \tilde{\eta} + 0}{t^2 + r^2 \cdot \tilde{\eta} - \tilde{\eta} + 0} = \frac{t^2 \cdot \tilde{\eta} - \tilde{\eta} + 0}{t^2 + r^2 \cdot \tilde{\eta} - \tilde{\eta} + 0}$. Then $u = \frac{1}{1 + e^2} \left[ \frac{t_B - \tilde{\eta} + 0}{t_B - \tilde{\eta} + 0} \right]$. So, when $|\mu| = \frac{1}{1 + e^2} \left[ \frac{t_B - \tilde{\eta} + 0}{t_B - \tilde{\eta} + 0} \right]$, the maximum value of $|\mu|$ is $|\mu| = \frac{1}{1 + e^2} \left[ \frac{t_B - \tilde{\eta} + 0}{t_B - \tilde{\eta} + 0} \right]$.
When $| \theta_1 - \theta_0 | \geq \Delta + \varepsilon$, the minimum value of $|u|$ is 0 if $e^2 |a_1 - a_0| \geq \Delta + \varepsilon$; and is

$$
\mu_1 = \frac{1}{1 + e^2} \left[ (\Delta + \varepsilon) - e^2 |a_1 - a_0| \right] \text{ if } e^2 |a_1 - a_0| \leq \Delta + \varepsilon. \tag{8.3}
$$

Hence, we get

$$
r_2(\delta_B) = \begin{cases}
L_1 \lambda_1 [1 - g_B(\mu_0)] + L_2 \lambda_2 g_B(0) & \text{if } e^2 |a_1 - a_0| \geq \Delta + \varepsilon \\
L_1 \lambda_1 [1 - g_B(\mu_0)] + L_2 \lambda_2 g_B(\mu_1) & \text{if } e^2 |a_1 - a_0| \leq \Delta + \varepsilon,
\end{cases}
$$

where $g_B$ is defined in (8.1). To find $r_3(\cdot)$, we need the following lemma.

**Lemma 8.2.** $r_3(\delta) = \max \{ L_1 \lambda_1 (1 - \inf_{|\theta_1 - \theta_0| \leq \Delta} E_\delta(\cdot)), L_2 \sup_{|\theta_1 - \theta_0| \leq \Delta + \varepsilon} E_\delta(\cdot) \}$.

**Proof:** The proof is similar to that of Lemma 8.1.

Now, from Theorem 7.1, $r_3(\delta_M) = L_1 \left[ \frac{-t_{M-\Delta}}{\sigma_1} + \frac{-t_{M+\Delta}}{\sigma_1} \right] =
L_2 \left[ \frac{t_{M-\Delta}}{\sigma_1} - \frac{t_{M-\Delta}}{\sigma_1} \right]$. From (4.5), $r_3(\delta_G) = \max \{ L_1 \left[ \frac{-t_{G-\Delta}}{\sigma_1} + \frac{-t_{G+\Delta}}{\sigma_1} \right] \}$.

We also find that $r_3(\delta_H) = \max \{ L_1, L_2 g_B(0) \}$ if $\frac{\sigma_2}{\sigma_1} \neq \frac{\sigma_1}{\sigma_2}$. For $\frac{\sigma_2}{\sigma_1} = \frac{\sigma_1}{\sigma_2} = e^2$,

$$
r_3(\delta_B) = \begin{cases}
\max \{ L_1 (1 - g_B(\mu_0)), L_2 g_B(0) \} & \text{if } e^2 |a_1 - a_0| \geq \Delta + \varepsilon, \\
\max \{ L_1 (1 - g_B(\mu_0)), L_2 g_B(\mu_1) \} & \text{if } < \end{cases}
$$

where $\mu_0$ and $\mu_1$ are defined in (8.2) and (8.3). Thus we have all the formulas needed to compute the tables for comparison.
Table I, II exhibit \( t_B, t_G, t_M \) and \( r_i(\delta) \) for \( \delta = \delta_B, \delta_G, \) and \( \delta_M \), \( i=1,2,3 \). They are arranged in the following manner:

\[
\begin{array}{c|ccc}
\delta_B & r_1(\delta_B) & r_2(\delta_B) & r_3(\delta_B) \\
\hline
\delta_G & r_1(\delta_G) & r_2(\delta_G) & r_3(\delta_G) \\
\delta_M & r_1(\delta_M) & r_2(\delta_M) & r_3(\delta_M)
\end{array}
\]

The tables are computed with \( n_1, n_0 = (0,1) \). The selected values of the variables are:

1. \( \sigma^2 \) is .2 in Table I and is .5 in Table II.
2. \((a_1, B_1^2)\) is chosen as (0, .5), (0, 1), or (0, 2).
3. \( \Lambda \) is chosen as 1. or 1.5.
4. For \( \Lambda=1. \), \( \epsilon \) is chosen as .3 or .8.
   For \( \Lambda=1.5 \), \( \epsilon \) is chosen as .5 or 1.
5. For \((a_1, B_1^2), \Lambda, \epsilon, \) and \( \sigma^2 \) fixed, \( \lambda_1 \) and \( \lambda_1^1 \) are computed so that \( t_B \in \Gamma \).
<table>
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<tr>
<th>( \Delta = 1 )</th>
<th>( (\alpha_1, \beta_1^2) = (0, 0.5) )</th>
<th>( \Delta = 1.5 )</th>
<th>( (\alpha_1, \beta_1^2) = (0, 0.5) )</th>
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<td>( \epsilon = 0.3 )</td>
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TABLE II. $\theta^2 = .5$

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| $\lambda_1 = .4363 \quad \lambda_1' = .2987$ | | $\lambda_1 = .4363 \quad \lambda_1' = .2987$ |
| 1.1396 | .1205 | .7177 | 1.0 | 2.0000 | .0679 | .7615 | 1.0 |
| 1.8999 | .1230 | .2423 | .5397 | 3.4159 | .0878 | .1391 | .8204 |
| 1.4117 | .1409 | .2559 | .3482 | 2.0003 | .0967 | .2353 | .3087 |
Discussion of the Tables

It is seen from Table I and II that:

1. Minimax rules compare favorably with \( \bar{\gamma} \)-minimax rules in terms of \( r_2(.) \), and with Bayes rules in terms of the risk \( v_1(.) \).

2. The Bayes risk of the \( \bar{\gamma} \)-minimax rules is only a little more than that of Bayes rules.

3. When \((a_1, b_1) = (0, 1)\), the performance of Bayes rules is close to \( \bar{\gamma} \)-minimax rule in terms of \( r_2(.) \) and close to that of the minimax rule in terms of \( r_3(.) \). If \((a_1, b_1) \neq (0, 1)\), Bayes rules show some large increase of risks \( y_2(.) \) and \( y_3(.) \) when compared with \( \bar{\gamma} \)-minimax rules and minimax rules, respectively.

To illustrate the use of the tables, let us look at the following example:

**Example 8.1.** Type \( \Pi_0 \) (control) machines produce part \( P(p) \) where \( p \) is the diameter of \( P \), and \( p|\Pi_0 \sim N(\theta_0, 1) \). Type \( \Pi_1, \Pi_2 \), and \( \Pi_3 \) machines produce part \( Q_i(q_i) \), and \( q_i|\Pi_i \sim N(\theta_i, 1) \) for \( i=1, 2, 3 \).

Let us assume that when \( |\theta_i - \theta_0| < 1.5 \), part \( P \) and part \( Q_i \) can be matched, and when \( |\theta_i - \theta_0| \geq 2.5 \) they cannot be matched. Assume that the partial prior information \( \Gamma \) is as follows:

\[
\begin{align*}
P[|\theta_1 - \theta_0| \leq 1.5] &= .78 & P[|\theta_2 - \theta_0| \leq 2.5] &= .04 \\
P[|\theta_2 - \theta_0| \leq 1.5] &= .71 & P[|\theta_2 - \theta_0| \leq 2.5] &= .08 \\
P[|\theta_3 - \theta_0| \leq 1.5] &= .61 & P[|\theta_3 - \theta_0| \leq 2.5] &= .15
\end{align*}
\]

Now, there are machines \( a_0, a_1, a_2, a_3 \) for sale where \( a_j|\Pi_j \) for \( j=0, 1, 2, 3 \). Suppose we can take 5 samples from each machine and let \( \bar{x}_1 \) be the mean diameter of the samples from machine \( a_1(i=0, 1, 2, 3) \).

Since \( \Delta=1.5, \quad \epsilon = 1.0 \), from Table I, the \( \bar{\gamma} \)-minimax rule is:
If we feel the claim regarding the partial prior may not be correct and we would rather assume that there is no prior information, then we might use the following minimax rule: $a_1$ is good for $a_0$ iff $|\bar{x}_1 - \bar{x}_0| \leq 3.1757$.

If from some other source, we know that $\theta_0 \sim N(0, 1)$, $\theta_1 \sim N(0, 5)$, $\theta_2 \sim N(0, 1)$ and $\theta_3 \sim N(0, 2)$. Then, we might use the Bayes rules, from Table I we get

- $a_1$ is good for $a_0$ if $|\frac{5}{7} \bar{x}_1 - \frac{5}{6} \bar{x}_0| \leq 2.0$
- $a_2$ is good for $a_0$ if $|\bar{x}_2 - \bar{x}_0| \leq 2.4$
- $a_3$ is good for $a_0$ if $|\frac{10}{11} \bar{x}_3 - \frac{5}{6} \bar{x}| \leq 2.0$

If we are not sure about the definiteness of any prior information, we may then use the rule which is most robust to the assumption of the prior distribution. So from Table I, we may use $\Gamma$-minimax rule for $a_1$, use Bayes rule for $a_2$ and use minimax rule for $a_3$. 

- $a_1$ is good for $a_0$ iff $|\bar{x}_1 - \bar{x}_0| \leq 3.1757$
- $a_2$ is good for $a_0$ iff $|\bar{x}_2 - \bar{x}_0| \leq 2.8887$
- $a_3$ is good for $a_0$ iff $|\bar{x}_3 - \bar{x}_0| \leq 2.5663$
REFERENCES


On $\Gamma$-Minimax, Minimax, and Bayes Procedures for Selecting Populations Close to a Control.

Let $\Pi_0, \Pi_1, \ldots, \Pi_k$ be (k+1) normally distributed populations and let $\Pi_0$ be a control population. Our goal is to select those populations which are sufficiently close to the control in terms of the (unknown) means of the populations. A zero-one type loss function is defined. $\Gamma$-minimax rules, Bayes rules and minimax rules are derived for this problem and compared. Some optimal properties of $\Gamma$-minimax rules are shown; also $\Gamma$-minimax rules are
derived for distributions other than the normal.