Fluidic Heading System Study: Summary Report

By Abraham Finkel

Prepared by
The Johns Hopkins University
Applied Physics Laboratory
Laurel, MD 20810

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Harry Diamond Laboratories
Adelphi, MD 20783

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The feasibility of incorporating a fluidic angular rate sensor into a vehicle heading sensor system is investigated. A heading sensor of this type would find application in armored vehicles where a magnetic compass cannot be used. The study covers system considerations, microprocessor-based computational requirements, and the effects of rate sensor errors.
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1. INTRODUCTION

1.1 Statement of Task

The purpose of this study was to investigate the feasibility of incorporating a fluidic angular rate sensor into a heading sensor system. The fluidic sensor may offer a cheap and rugged alternative to the rate gyro. A heading sensor of this type would find application in armored vehicles where a magnetic compass cannot be used. Goals for the system are low production cost (< $6,000) and a drift rate of 0.8 deg per hour, although a higher drift rate may be acceptable in some applications.

1.2 Principal Conclusions and Recommendations

(a) The fluidic sensor (including necessary fluidic amplifiers) demonstrates sensitivity of the required order of magnitude.

(b) Drift characteristics under laboratory conditions are encouraging.

(c) A key element (a transducer which converts pressure to an electrical signal with the necessary stringent requirements) is yet to be demonstrated and developed.

(d) Many important environment characteristics are yet to be determined with precision.

(e) Essential environmental tests should be applied to an integrated sensor-transducer package.

(f) Certain effects such as the long-term equivalent drift under vibration are best measured by integration of the output. A high-quality continuously rebalanced integrator should be developed. An integrator of this type will likely be required to furnish angular incremental data to the computer.

(g) Modern microprocessors can handle the computation burden. Thirty-two bit precision will be required with a sampling frequency of about 100 per second.

(h) The main thrust of further effort should be directed toward the construction of an integrated sensor-transducer package before conducting precision environmental tests.
2. SYSTEM CONSIDERATIONS

2.1 Description

For a vehicle whose orientation is perfectly horizontal (no pitch or roll), a heading may be obtained by simply integrating the angular rate of the vehicle's vertical axis. Hence, for such a single axis system, a single angular rate sensor is sufficient. With pitch and roll present, large errors would result and it is necessary to measure angular rates about all three vehicle axes. The required computations are, of course, much more complex and are briefly outlined in Appendix A.

At some point, digitizing is required so that computation may proceed with the necessary accuracy. It would be possible to sample the angular rate sensors directly and enter rates into the computer. Noise and vibration are apt to make this approach unworkable unless strong filtering is employed, in which case vital rate information may be lost. Furthermore, a rate resolution equal to the desired drift rate of 0.8 deg/hr (0.0002 deg/s) is required. If the vehicle's maximum angular rate is 40 deg/s, an 18-bit analog-to-digital conversion is implied for each sample.

A more practical approach utilizes angle increments as the basic unit. Angle increments are obtained by integrating the rate sensor output and sampling the integrator output. Rapid sampling is still required to keep the increments small, since knowledge of varying velocity between samples will be lost. The computational algorithm assumes constant velocity between samples, although it is possible, with added complexity, to estimate acceleration and higher-order terms utilizing more than one sample interval. The integration which takes place prior to digital computation effectively filters noise and properly reacts to high-frequency motion, thus giving correct angle increments. A carefully constructed pulse-balanced integrator can circumvent the need for 18-bit readout while maintaining 18-bit sensitivity to any accumulated angular increments by virtue of the memory aspect of the integrator capacitor. The features of this type of circuit will be described at the end of this section.

A system would therefore take the form illustrated in Figure 1.
2.2 Operation

The operator will put the computer into one of two modes: ALIGN or NAVIGATE. In the align mode, start-up information must be fed into the computer so that it can calculate initial values for the elements of the attitude matrix and earth-rate components. Entry could be made via either a simple keyboard or a number of switches. The required input data are latitude and pitch, roll, and heading angles.

Latitude must be known so that the effect of earth rate may be properly accounted for. Pitch and roll angles may be read from gravity-sensitive devices (e.g., a gimballed platform with level indicators). Initial heading must be determined with reference to a known direction. Note that the computer will not properly calculate departure from an arbitrarily defined initial heading. The absolute heading must be known to enable calculation of earth rate effects. The gyrocompassing technique of determining initial alignment, conventionally used in inertial navigation systems, requires very accurate sensing of earth rate and does not appear to be within reach for the present fluidic sensors.
With the initial alignment procedure complete, the stationary vehicle should sense no motion with respect to the ground. Nonzero outputs are due to sensor bias, and an observation period is required to determine the magnitudes (and rate of change, if uniform). Bias (and bias rate) are thus known corrections to be applied during the NAVIGATE mode. Advantage could be taken of any subsequent vehicle stops to obtain updates on the sensor bias.

The computer is switched to NAVIGATE prior to the start of vehicle travel. Updating of the attitude matrix and display of heading then proceeds automatically.

### 2.3 Microprocessor

The computational requirements for one particular algorithm are outlined in Appendix A. Thirty-two bit precision is required, with updating at a rate of about 100/s. A 16-bit microprocessor is a logical choice with the possible use of a hardware multiplier if execution time must be improved. Variations from a basic configuration to provide various compensations will not have major impact on either cost or execution time (e.g., look-up tables to modify the transfer function). A significant simplification could be made if no compensation at all were required, for then power-of-two angular increments could be used as multipliers, resulting in simple shift operations. Some simplification of the algorithm may also be possible by changing the reference coordinate system, as described in Appendix B.

For high-volume production, development and programming costs need not be considered. The cost of a fully packaged microprocessor with the program contained in masked read-only memory (ROM) will probably lie in the $1- to $2-K range. As an example, a prewired and assembled Texas Instruments module (TM 990/101M) containing memory is available for $500 to $600 in production quantities. The microprocessor component alone (TMS 9900) is available for $30. A full-temperature military version (SBP 9900AM) costs $400.

### 2.4 Integrator

Considerable dynamic range must be exhibited at the output of the angular rate sensor. The sensor must be capable of responding to both the maximum angular rate ($\approx 40 \text{ deg/s}$) and the allowable drift rate ($\approx 0.0002 \text{ deg/s}$), a range equivalent to 18 bits. However, if integration precedes the computer, sampling of the integrator output requires only that the readout resolution be sufficient for accurate updating. Thus, if 10 ms
sampling is used, the maximum angular step is 0.4 deg, while a resolution of 0.04 deg may be sufficient provided that the remaining angle increment less than 0.04 deg is retained and added to the next increment. Despite the coarse angle readout, there is no accumulation of errors. These properties are exhibited by the pulse-rebalanced integrator shown in simple form in Figure 2.

![Diagram of Pulse-rebalanced integrator](image)

**Figure 2 Pulse-rebalanced integrator.**

The integrator output represents angle. When a positive or negative threshold equal to the least significant readout bit is reached, a precision pulse is generated to rebalance the integrator. The change of integrator output voltage due to the pulse must equal this resolution angle. The pulses drive a counter which is read out periodically by the processor.

While the readout need only consist of a few relatively coarse bits, the bias and drift at the integrator input must still satisfy the 18 bit requirement. This is readily achievable with good quality modern operational amplifiers and careful design. Changes of components or charge quantities result in scale factor errors. Scale factor stability of 0.1 to 1% is tolerable provided that such changes apply uniformly to all segments of the transfer function. Innovative circuit design will be required to guarantee equal scale factors for positive and negative inputs (≤ 0.01%).

This circuit is in reality a voltage-to-frequency converter with the resulting pulses counted to develop angle increments and will be necessary to accommodate a pressure-to-voltage transducer. If a pressure-to-frequency transducer of sufficient accuracy is developed, then, of course, only a counter would be required.
2.5 System Limitations

The heading sensor system cannot calculate position unless linear velocity (or acceleration) is also measured. Hence, the change of attitude due to travel over the earth's curved surface must be neglected.

Travel over level ground of 70 miles from the starting point results in an apparent tilt of 1 deg. Bias errors also develop, since the effect of earth rate is not properly compensated. In particular, a north-south travel of 70 miles from the equator will cause the vertical axis to sense an uncompensated rotation rate of 0.25 deg/hr.

For missions of up to three hours, therefore, these errors will have little impact on system accuracy.
3. COMPUTER STUDIES OF SENSOR REQUIREMENTS

3.1 Purpose

It is relatively easy to see the effects of sensor errors in a single-axis system. A scenario of angular motion is multiplied by the normalized transfer function of the sensor (output vs. angular rate). The product is integrated over time to obtain the angle output. This may be compared to the precise angular position which is given by the integral of the input angular motion. Hence, for example, a perfectly linear, stable transfer function with an unanticipated offset of 1 deg/hr will simply accumulate an angular error at the rate of 1 deg/hr, regardless of the input scenario. Nonlinear transfer functions produce additional errors which are scenario-dependent due to a rectification effect which is equivalent to adding additional bias or offset.

For the full 3-axis system, there will be additional errors due to cross-coupling. It was always felt that, given a system which is not allowed to produce errors greater than, say 3 deg (1 deg/hr for a 3 hr mission), the 3-axis effects would not be drastic and that the simple one-axis system could be the basis of rule-of-thumb requirements on the sensors. However, to demonstrate the correctness of the intuitive conclusions, a computer simulation was performed. This simulation allowed for sensor bias and scale factor errors only. "Bias" should be considered to be due to all contributory factors: unpredictable offset, the effect of motion scenario upon an asymmetrical or nonlinear transfer function, and the rectification effects of vibration (discussed further in Section 6).

The following requirements for a successful computer analysis are somewhat incompatible:

3.1.1 Long simulated (vehicle) time

If 3-hr missions are contemplated, the full 3 hr should be simulated. Heading error may grow nonlinearly, and it would be incorrect to extrapolate the results of a shorter run.

3.1.2 Reasonably short computer time

If a multitude of runs is to be made so that many factors may be varied, a short run time is essential.
3.1.3 Highly accurate attitude updating

Typical similar computer studies derive "system" errors rather than errors due solely to sensor deficiencies. Contributing to these system errors are also the approximation errors inherent in the attitude-updating procedure. Accuracy is obtained by high-frequency updating which, of course, requires long computer runs and, even then, roundoff errors may become a factor.

3.2 Approach

The foregoing incompatible requirements may be satisfied through a slight subterfuge. The underlying vehicle scenario is described in terms of sine waves (a convenient mathematical representation is needed to enter 3-hr worth of motion into the computer). These sinusoidal waveforms are then quantized at sampling intervals which are short enough to preserve the basic nature of the motion; i.e., many samples per cycle. It is this quantized motion, a series of angular steps, which then becomes the assumed angular motion applied to the vehicle. Computer updating need be performed only when the angular steps occur. The three principal sources of computation errors are either eliminated or minimized:

(1) There is no varying velocity between updates since there is no vehicle motion between steps.

(2) The solution of the matrix differential equation required by the updating procedure involves a matrix exponential. In the normal algorithm, one or more terms of the series approximation will be used. With a large-scale computer and the relatively infrequent updating required by the quantized motion, an exact closed-form solution may be used.

(3) Computer roundoff error is considerably reduced due to the reduced frequency of updating.

To further reduce the total number of updates, only angular steps exceeding a prescribed minimum are allowed. If, at a sampling instant, the angular step were less than the minimum, no step at all would be allowed. Thus, if the scenario includes long periods of little or no motion, fewer computations are made, thereby decreasing both the errors and computer time.
As described elsewhere, a nine-element direction cosine (or "C") matrix describes the vehicle attitude. In this computer program, two such C matrices are updated at each angular step. The reference matrix, CREF, describes the actual vehicle attitude. The computed matrix, CCMP, describes the attitude obtained based on erroneous data from the rate sensors. Since heading error is of primary interest, heading is calculated from each of the matrices with the difference tabulated as the heading error. Whenever an angular step is encountered, the following three C-matrix updates are performed:

(1) CCMP is updated for any rotation sensed since the previous angular step. A sensor bias will, of course, result in a fictitious rotation. Also, since the matrix is misaligned, earth rotation is not properly compensated for so that the computer thinks there has been some net motion with respect to the ground.

(2) CREF is updated for the angular step, involving one, two, or all three axes.

(3) CCMP is similarly updated for the angular step. In this case, the angular steps reflect the erroneous scale factors assumed for the angular rate sensors.

The exact, closed-form solution of the matrix differential equation is shown in Appendix C, along with an equivalent formulation convenient for computation.

3.3 Program Inputs and Outputs

To allow flexibility of vehicle scenario, angular motion of each of the three axes is specified in terms of angular amplitude and period for each of four successive intervals. The time at the end of each interval may also be selected. Any number of iterations of the four time intervals may be chosen to simulate any total vehicle time. Additionally, in one version, a steering instability (or Z-axis fluctuation) was superimposed.

The program as written also allows selection of latitude, initial heading, sampling interval, minimum angular step, and printout interval.

Output quantities tabulated are time, heading, tilt, and heading error. Since pitch, roll, and yaw are independently specified, the resulting attitude is fully determined. As
further discussed in Appendix D, symmetrical motion about the vehicle axes can result in a considerable attitude "creep." The tilt readout shows whether this effect is getting out of hand, resulting in an unrealistic scenario although, in any event, the ability of the computer to follow with acceptable error is the concern of these studies.

The final two columns of the readout are labelled DETCREF and DETCCMP. These list at each time the value of the determinants of the respective matrices. For many of the C-matrix updating procedures, a prime concern is the development of "skewness" or lack of orthogonality as updating proceeds. A property of an orthogonal matrix is that the value of its determinant is unity. The printout of DETCREF and DETCCMP is one way of checking that the matrix is being updated properly. A listing for this program is given in Appendix E. A selected number of computer printouts is reproduced in Appendix F.

3.4 Tabulation of Results

A quick look at the results under various conditions may be obtained by examining the heading at the end of 3 hr. For a fair comparison, we wish to exclude that portion of the error which is simply the final heading excursion multiplied by the scale factor error. Thus, if at a heading of 100 deg a system with a 2% scale factor error shows a heading error of 2 deg (of comparable sign), then there has been no error accumulation. The latitude is 45 deg unless otherwise indicated.

3.4.1 Z-axis motion only: continuous half sine waves of 90 deg amplitude repeating every 10 minutes

(a) Z sensor scale factor 1.02; 1 deg/hr bias
   accumulated heading error after 3 hr = 2.08 deg

(b) Z sensor bias error only of 1 deg/hr
   heading error = 2.85 deg

(c) Z sensor bias error only of 1 deg/hr; latitude = 0 deg
   heading error = 2.70 deg

(d) Z sensor bias error of 1 deg/hr; latitude = 90 deg
   heading error = 3.00 deg
[shows that at 90 deg LAT, where the Z axis is aligned to the earth's spin axis, a misalignment due to Z-axis rotation does not pick up additional errors due to earth rate. Also, the computation is exact.]

(e) Z sensor bias error of 1 deg/hr; sampling interval reduced from 1.0 to 0.5 s
    heading error = 2.85 deg (same as (b))

(f) All sensors with bias of 1 deg/hr and scale factors of 1.02
    heading error = 1.06 deg
    (change of ≈ 2 deg due to cross axis effects)

(g) Reduced Z amplitude of 5 deg; 0.1 deg angle no sensor errors
    heading error = 0

(h) Reduced Z amplitude of 5 deg; 0.1 deg steps; Z sensor bias of 1 deg/hr
    heading error = 2.85 deg (same as (b))

(i) Reduced Z amplitude of 5 deg; 0.1 deg steps; all sensors with 1 deg/hr bias and scale factors of 1.02
    heading error = 1.30 deg

(j) Full Z amplitude; X, Y, Z scale factors 1.02; X, Y, Z biases of -1, -1, +1 deg/hr
    heading error = 2.91 deg
    (compare to (f); this choice of signs has increased error)

3.4.2 Full Scenario: Three Axes Motion

1st 100 s - X=10 deg amplitude, 10 s period
    Y=10 deg amplitude, 11 s period
    Z=90 deg amplitude, 1200 s period

next 100 s - no Y motion
next 100 s - no X motion
next 300 s - no X nor Y motion
    (pattern repeats every 600 s)

1.0 s sampling
1.0 deg minimum angle step
(a) Z sensor bias of 1 deg/hr
    heading error = 2.83 deg
    (almost identical to 3.4.1(b); no cross-axis effect with perfect X,Y sensors)

(b) All sensors with 1 deg/hr bias
    heading error = 1.84 deg

(c) Bias errors (X, Y, Z) of -1, -1, +1 deg/hr
    heading error = 3.71 deg

(d) Scale factors of 1.02 on all sensors; no bias
    heading error = -1.07 deg

(e) Scale factors of 1.05 on all sensors; no bias
    heading error = -2.73 deg

(f) X, Y, Z scale factors of 0.95, 0.95, 1.05; no bias
    heading error = -2.60 deg

(g) Combined sensor errors: scale factor 1.02 and bias of 1 deg/hr on all sensors
    heading error = 0.78 deg
    (Note: (b) + (d) = 0.77 deg)

(h) Combined sensor errors: 1.02 scale factor on all sensors; X, Y, Z bias of -1, -1, +1 deg/hr
    heading error = 2.65 deg

(i) Repeat of (g) with minimum step of 0.5 deg
    heading error = 0.97 deg

(j) Repeat of (g) with minimum step of 0.5 deg, 0.5 s sampling
    heading error = 0.97 deg

3.4.3 Full scenario changed to eliminate simultaneous pitch and roll (X motion removed during 1st 100 s).
   Therefore, no Z axis coning motion. Sensor errors as in 3.4.2(g)
   heading error = 0.47 deg
3.4.4 Full scenario changed to reflect higher frequencies as shown on charts of measured pitch and roll

A chart for one particular vehicle course shows high pitch and roll amplitudes of 10 deg and 5 deg, respectively. Those motions are in phase with a period of 2.5 s. A worst-case is simulated where this motion continues for the entire 3 hr. A single half-sine wave of yaw having 90 deg amplitude and a 200-s half-period is assumed every 600 s. In addition, a yaw fluctuation or steering wander is assumed to be continuously present with 3 deg amplitude and a 3 s period. The sampling interval is reduced to 0.3 s to accommodate these higher frequencies.

(a) \( Z \) sensor bias of 1 deg/hr

heading error = 2.90 deg

(compare to 3.4.2(a) = 2.83 deg)

(b) \( Z \) sensor scale factor 1.02

heading error = -1.48 deg

(c) \( X, Y, Z \) bias errors of \(-1, -1, +1\) deg/hr

heading error = 3.63 deg

(compare to 3.4.2(c) = 3.71 deg)

(d) Combined sensors errors: scale factor 1.02 and bias of 1 deg/hr on all sensors

heading error = 2.35 deg

(compare to 3.4.2(g) = 0.78 deg)

3.4.5 Full scenario changed to include coning motion

Since in-phase motion does not produce the serious coning effects of quadrature motion, the roll (\( Y \)) period was changed to 2.6 s, thereby giving a continuously varying phase difference between \( X \) and \( Y \). There is now considerably greater creep of the vehicle both in heading and tilt. (See discussion of this effect in Appendix D). While somewhat unrealistic, the heading sensor system should still be able to follow with reasonably small error.

(a) \( Z \) sensor bias of 1 deg/hr

heading error = 2.89 deg

(b) Scale factor 1.02, bias of 1 deg/hr on all sensors

heading error = 4.58 deg
(c) Repeat of (b) but with steering wander removed

heading error = 4.70 deg

3.4.6 Full scenario changed to test for highest frequency pitch and roll

Most of the measured vehicle motion shows pitch and roll motion of 3 deg amplitude, 1.5 s period. An additional program was run with continuous X axis and Y axis motion of 3 deg amplitude and periods of 1.5 and 1.6 s, respectively. Yaw motion remained as in 3.4.4. The sampling interval was further reduced to 0.2 s. With scale factors of 1.02 and biases of 1 deg/hr on all axes:

heading error = 1.80 deg

3.5 Conclusions

The data substantiated the tentative conclusion stated earlier; namely, that for systems with allowable errors of only a few degrees, the contributions of cross-axis effects will be minor. In none of the cases where scale factor errors were limited to 2% did the heading error after 3 hr exceed the simple single-axis calculation by more than 2 deg (single-axis heading error = Z-axis bias times 3 hr). A case of 5% scale factor errors (2e) did produce a heading error of 2.73 deg, but a 5% scale factor error may already be considered intolerable, since the heading will always be in error by 5% X angular excursion. Thus, anytime the vehicle has turned 180 deg, an error of 9 deg will be encountered.

The foregoing analyses included only two of many types of sensor errors: bias and scale factor. Some other types of sensor errors such as asymmetry and nonlinearity may be examined in terms of bias and scale factor (see Section 6). The effects of g-sensitivity could be entered into this program if a realistic profile of vehicle acceleration could be modeled.
4. COMPUTER STUDIES OF MICROPROCESSOR BASED ALGORITHMS

4.1 Purpose

The results of Section 3 indicate heading errors due to various combinations of sensor errors and scenarios, processed by near-perfect computations. This section addresses the corollary question: Assuming perfect sensors, will a practical microprocessor achieve the desired accuracy? The algorithm utilized requires periodic updating of the nine-element direction cosine matrix as described in Appendix A. Alternate computational techniques were not considered since feasibility rather than optimization was the objective.

4.2 Approach

4.2.1 Reference Program

As in Section 3, an appropriate scenario must be selected since errors will be scenario-dependent. Following somewhat the procedures used in that section, a basic 600 s time span was divided into intervals of 192 and 408 s. A yaw motion of a full half sine wave (amplitude = 1.5 radian and period = 384 s) is imposed during the first 192 s, followed by no yaw during the balance of the 600-s span. Throughout the 600-s span, sinusoidal pitch and roll motions of 0.06 radian amplitude are assumed. The pitch and roll periods are 1.5 and 1.6 s, respectively. As noted in Section 3, a choice of slightly different pitch and roll frequencies allows varying phase relationships.

A reference program, referred to in the program listing as the tank simulation program (TNKSIM), accepts the foregoing scenario and generates heading, tilt, and a record of angular increments to be sensed by the physical sensors, with earth rate accounted for. These angular increments are subsequently used as the inputs to the "test" program. Due to the lengthy time required, the program allows for follow-on runs to previous records by reading the last attitude matrix as a starting point. A succession of such runs provided a record of 30-min of vehicle travel.

Since maximum accuracy is an objective of the reference program, calculations proceed with full precision and with 2 ms updating. A second-order algorithm (see Appendix A) was utilized with parallel updating (wherein all values of the
present instant are obtained solely from values of the previous instant). Skewness of the attitude matrix was monitored by printing the value of its determinant. After 30 min, a departure from unity of $2 \times 10^{-8}$ was obtained. A further test of the reference algorithm accuracy is described in Appendix G.

4.2.2 Test Program

In this program (TNKTST), the reference record is the source of sensor inputs. The program models the pulse-balanced integrator to develop periodic angular increments which are then used as the source of updating. A second-order algorithm identical to that used for the reference program is used, except that now computational limitations are intentionally inserted. For the two sets of runs completed, precision of 32 and 24 bits were selected. In both cases, a 10-ms update interval was used, coupled with an integrator threshold (least significant bit) of 0.06 deg. Heading and tilt are calculated and compared to the reference to obtain errors.

4.2.3 Plots of Outputs

Plots of heading, tilt, heading error, and tilt error were obtained by further sampling of the computed outputs. To plot 30 min of data requires a sampling frequency well below the frequency of the fine structure. Therefore, care must be exercised in interpreting fine structure of the plots (as well as the more frequently sampled data available in printouts). The low-frequency components are of much more significance.

Figures 3 through 11 were obtained with 32 bit computation. Heading (as obtained by the 32 bit processor) and heading error for the entire 30 min are shown in 3 and 4. Expanded plots of part of the error record are shown in 5 and 6. The effect of the 0.06 deg integrator threshold is clearly seen. Calculated tilt angle for the entire period is shown in 7. Tilt error for the entire period is shown in 8, with expanded plots in 9 and 10.

Figures 11 through 19 are similar to the previous plots, except for 24-bit computation. Figure 16 (as well as Figure 8) seems to show a discontinuity at 200 s. The expanded plot of 19 demonstrates that this is merely a quirk of insufficient resolution coupled with the piecewise construction of the scenario. The apparently missing 2-deg fluctuations are now made visible.
Figure 4. Heading error obtained with 32 bit computation.
Figure 5. Heading error obtained with 32 bit computation (expanded time scale).
Figure 6. Heading error obtained with 32 bit computation (expanded time scale).
Figure 7. Tilt angle obtained with 32 bit computation.
Figure 8. Tilt error obtained with 32 bit computation.
Figure 9. Tilt error obtained with 32 bit computation (expanded time scale).
Figure 10. Tilt error obtained with 32 bit computation (expanded time scale).
Figure 11. Heading obtained with 24 bit computation.
Figure 12. Heading error obtained with 24 bit computation.
Figure 13. Heading error obtained with 24 bit computation (expanded time scale).
Figure 16. Tilt error obtained with 24 bit computation.
Figure 17. Tilt error obtained with 24 bit computation (expanded time scale).
Figure 18. Tilt error obtained with 24 bit computation (expanded time scale).
Figure 19. Tilt error obtained with 24 bit computation (expanded time scale).
4.3 Conclusions

The choice of 10 ms, second-order updating with 32-bit precision is demonstrated to be quite adequate. No significant long-term trend in heading error can be detected. The more sensitive tilt error shows a gradual growth to 0.2 deg after 30 min. The sine wave pattern of heading excursions during periods of no yaw motion as seen in Figure 3 is a result of the coning phenomenon (see Appendix D). The direction and rate of change of heading continuously change as the pitch and roll phase relationship varies.

A reduction to 24 bits shows a significant deterioration. In addition to no longer accurately following the 90 deg yaw excursions, a steady error accumulation to 1.5 deg at 30 min is apparent. The tilt error is a full 16 deg at 30 min.

A further comparison may be made of the determinants for the two cases. For the 32-bit case, the determinant of the attitude matrix at the end of 30 min was 0.99935. For the 24-bit case, the determinant equalled 0.8529, a considerable departure from the ideal unity.

Ten minutes worth of data was also obtained from a test of 32-bit, first-order processing. While no serious heading errors had yet developed, the determinant at the end of 10 min was a horrendous 1.45. It is safe to conclude that 10 ms. first-order processing is unsatisfactory.

Computer program listings for Section 4 are given in Appendix H.
5. MEASUREMENTS OF SENSOR CHARACTERISTICS

In addition to the studies of a heading sensor system which would incorporate a fluidics angular rate sensor, a concurrent measurement effort was undertaken with joint participation of APL and HDL personnel. These measurements were aimed at key fluidics sensor characteristics.

The tests were performed under admittedly benign laboratory conditions, but demonstration of feasibility of the existing sensor design was the immediate objective. As noted elsewhere in this report, the successful development of a heading sensor with a 1 deg/hr drift requires extremely accurate measurement of angular rate inputs. Whether or not the angular rate sensor will meet these objectives must await the construction of a combined sensor-transducer package which can then be subjected to precision testing under a variety of environmental conditions.

Measurement of angular rate was made via three stages of fluidic amplification and a Barocel pressure transducer. Data were recorded on an X-Y plotter. Angular rates were obtained with a rate table. Standard laboratory air was used as the power source with different venting arrangements. (A single output port reduces the ambient noise effects.)

Pertinent data from one set of tests are reproduced as Figure 20. An operating point of 13 mm Hg was selected as being least sensitive to supply pressure fluctuations.

The transfer function of output pressure change versus angular rate was 0.05 mm Hg per deg/s. The response to angular rate excursions of ±0.02 deg/s is shown on Plot 2 of Figure 20. Plot 3 shows a very discernible response to ±0.002 deg/s. Thus, the equivalent of earth rate (0.004 deg/s) can be seen by the sensor. To achieve the desired goal of 0.8 deg/hr, a sensitivity of 0.0002 deg/s is required. Since it is expected that rate signals will drive an integrator, it is the long-time integral which must be sensitive to rates of 0.0002 deg/s. Hence, even if such signal levels at the X-Y plotter would appear to be submerged in noise, conclusions about sensitivity for this application are premature unless suitable instrumentation is employed.

Plot 4 shows one measurement of a second key characteristic; namely, offset drift. The offset appears to have changed by the equivalent of a rate change of 0.001 deg/s over 24 min or a drift rate of 9 deg/hr/hr. In a simple
single-axis system, the angle is simply the integral of the rate. Hence, the error would be 4.5 deg after 1 hr and 40.5 deg after 3 hr. If, in a final system, constant drift rates are experienced and can be accurately determined during an alignment period, suitable compensation can be obtained within the processor. If offset changes cannot be predicted, their magnitudes would have to be further reduced.

It should be noted that an available sensor was used as the basis for these tests. A potential for improvement exists by optimizing the sensor design for the specific application at hand. (For example, the present high angular rate capability may be relaxed.)
6. CONTRIBUTIONS TO SENSOR ERRORS

Results of the computer simulations given in Section 3 demonstrate that the limitations on rate sensor errors can be estimated by considering only a single-axis system. Hence, for a 1 deg/hr heading drift, the angular sensor should have an unpredictable (and hence uncompensated) bias of approximately the same magnitude or less. A general requirement may be stated as follows: the total effect of all sensor deficiencies should produce an output which, on the average, tracks the input angular rate with an error no greater than the desired heading drift. Many of the contributory factors will be scenario dependent. The "bias" of Section 3 should be recognized as the effective sum of many error sources.

6.1 Offset

Any unpredictable or unexpected shift of the null point is, of course, a bias in the sense of Section 3. This offset, in turn, may be the result of uncompensated temperature or pressure changes within the angular rate sensor or transducer.

6.2 Asymmetry

In Section 3, it was shown that scale factor errors of one percent or so would be tolerable. However, should there be a lack of symmetry of the transfer function, rectification of input motion results in a bias, as shown in Figure 21.

![Figure 21 Nonsymmetrical transfer function.](image-url)
If the input motion is sinusoidal with a 6 deg peak-to-peak amplitude, then 6 deg represents the integral of a half sine wave of the corresponding angular rate waveforms. If positive and negative scale factors differ by 1%, then a 0.06 deg error is generated every cycle. For motion with a 1.5 s period, the equivalent "bias" is 145 deg/hr. Therefore, considerably better matching of positive and negative characteristics (of the order of 0.02%) is required (after compensation).

6.3 Nonlinearity

A similar rectification effect can be produced from a nonlinear transfer function, even if symmetry is maintained, because of nonsymmetrical input motion.

![Nonlinear Transfer Function](image)

Figure 22 Nonlinear transfer function.

One can readily visualize a vehicle drifting slowly in yaw which is periodically corrected by the driver as illustrated in Figure 22. The drift is long and slow; the correction short and fast. For the vehicle to maintain its average direction, the integration of the input waveform must equal zero. But the integral of the output waveform delivers a "bias" to the computer by virtue of the unequal positive and negative scale factors. If a 5 deg drift were corrected every five seconds, a nonlinearity of 1% gives an angular error of 0.05 deg every five seconds or an equivalent "bias" of 36 deg/hr.

It will be noted that, insofar as certain characteristics such as non-linearity, hysteresis, g-sensitivity are concerned, the specification for an acceptable sensor is intimately related to the profile of vehicle motion. For the sensor to operate acceptably under all possible conditions, the angular rate error would have to be limited to the desired
drift rate independent of input motion. Hence, for a 1 deg/hr sensor (≈ 0.0002 deg/s), and a maximum rate of 40 deg/s, the nonlinearity must be limited to 0.0005% of full scale. Since maximum rate is not encountered frequently, such a requirement is much too stringent. (The maximum angular rate corresponding to the example of 3 deg, 2.5 Hz. motion is 12 deg/s). The simple examples given here to illustrate the effects of asymmetry and nonlinearity indicate that 0.01% would be more realistic.

6.4 Hysteresis

Any memory exhibited by the device can result in an effective bias, again depending on the vehicle scenario. Ideally, the output of the sensor as a function of past history should vary by less than the desired drift rate.

6.5 Cross-Axis Sensitivity

Obviously, the sensor should not produce outputs for angular motion about any but its designated input axis. To the extent that this effect can be measured and is repeatable, compensation can be introduced into the processor.

6.6 Sensitivity to Linear Acceleration

If the sensor is g-sensitive, errors due to gravity can be compensated for since the attitude is always known in the computer (not without penalty, however; such additional computations will take additional time). Without accelerometers as part of the system, there would be no way to compensate for errors due to vehicle acceleration. A detailed computer run of the type of Section 3 could be made for a vehicle scenario of three-axis acceleration.

6.7 Two-Axis Quadrature Vibration

If two axes of the vehicle undergo quadrature motion, the third axis will show an effective rotation which is not sensed by the third axis sensor. This is the coning effect which must be properly handled by the computational procedures. Now, should the two axes experience quadrature vibration which is not true vehicle motion, the computer will erroneously calculate a third-axis rotation which never takes place. This vibration must be slow enough to be seen at the sampling intervals (≈10 msec).

6.8 Rectification of Vibration

Another vibration effect occurs when vibrational motion is accompanied by vibrational tilt, resulting in a rectification effect.
The effects of vibration may be reduced via shock-mounting of an integrated 3-axis sensor package. The attitude of the package rather than the vehicle is tracked, allowing momentary insignificant vehicle heading errors with no error accumulation. Small angular differences between the sensor package and the vehicle would be of some concern if an attempt were made to combine a vehicle speedometer with the heading sensor to obtain position.
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REFERENCES


APPENDIX A.--COMPUTATIONAL ALGORITHM

This section briefly outlines the computational algorithm needed to continuously track a vehicle heading. No attempt was made in this study to determine an optimum approach. Rather, the purpose of this phase of the study was to demonstrate feasibility. Accordingly, the popular direction-cosine matrix technique was used, neglecting other possibilities (such as the use of quaternions) described in the literature. With such a feasibility demonstrated, development efforts could be concentrated upon the input sensor and transducer. Feasibility consists of demonstrating acceptably small errors when the algorithms are limited to the capabilities of a modern microprocessor. Any marginal speed problems can be safely neglected since it is a small matter to add a hardware multiplier.

The direction cosine matrix is a nine element matrix which fully describes the attitude of the vehicle with respect to the local (or navigation) frame. Each element is the cosine of the angle between a pair of axes.

<table>
<thead>
<tr>
<th>Vehicle Axes</th>
</tr>
</thead>
<tbody>
<tr>
<td>X'</td>
</tr>
<tr>
<td>X</td>
</tr>
<tr>
<td>Y</td>
</tr>
<tr>
<td>Z</td>
</tr>
</tbody>
</table>

X, Y, and Z are the longitudinal, transverse, and vertical axes of the vehicle. For a right-hand system, positive directions are taken as forward, right, and down. X', Y', Z' are the corresponding local frame axes and are chosen to coincide with north, east, and down*, respectively. Any vector


*Toward the center of the earth.
in the vehicle frame may be transformed to a vector in the NAV frame by premultiplying by this direction cosine matrix \( \mathbf{C}_{v} \).

So defined, the elements \( C_{11} \) and \( C_{21} \) give the north and east components, respectively, of the vehicle's longitudinal axis. Therefore, heading may be extracted from the C-matrix by computing the arc tangent of \( C_{21}/C_{11} \). The more difficult task is the accurate updating of the C-matrix as angular increments over a sample period are developed from the angular rate sensors.

The rate of change of the C-matrix is given by

\[
\dot{\mathbf{C}} = \mathbf{CW},
\]

where \( \mathbf{W} \) is a matrix describing the angular rates of the vehicle's axes, \( \dot{x}, \dot{y}, \dot{z} \):

\[
\mathbf{W} = \begin{pmatrix}
0 & -\dot{z} & \dot{y} \\
\dot{z} & 0 & -\dot{x} \\
-\dot{y} & \dot{x} & 0
\end{pmatrix}
\]

(2)

For constant velocity, the solution for \( \mathbf{C} \) at any time \( t \) is

\[
\mathbf{C}(t) = \mathbf{C}(0)e^{\mathbf{W}t}
\]

or, in terms of samples every \( n \)th instant,

\[
\mathbf{C}_{n+1} = \mathbf{C}_{n}e^{\mathbf{W}At}.
\]

(3)

The matrix \( \mathbf{W}A \) is now a matrix of angular increments rather than angular rates.

The exponential term may be expressed as an infinite series, leading to
\[ C_{n+1} = C_n \left[ I + \frac{WAt}{2} + \frac{(WAt)^2}{2} + \ldots \right] \]  \hspace{1cm} (4)

where \( I \) is the identity matrix.

The microprocessor is thus required to update the most recent C-matrix after each sampling, with summation and matrix multiplication as indicated by equation (4). If only the terms \( I \) and \( WAt \) are used, the process is simple first-order updating. Greater accuracy is obtained by including the term \( \frac{(WAt)^2}{2} \), referred to as second-order updating.

Prior to updating per equation (4), a correction must be applied to \( WAt \) to reflect the effect of earth rate. This is necessary since the angular rate sensors measure the vehicle's motion with respect to inertial space, whereas the C-matrix gives the vehicle's attitude with respect to the local frame which, in turn, is rotating. For the earth spin rate \( \Omega \) and latitude \( L \), the NAV (local) frame components are \( \Omega \cos L \), 0, and \( -\Omega \sin L \). Premultiplying these components by the transpose of the \( C_N^V \) matrix then gives the earth rate effect as seen by the vehicle axes. Subtracting from the measured angular rates will give the net rates (or increments) with respect to the NAV frame.

Some 42 multiplications per sample are indicated for the foregoing procedures. Assuming 200 \( \mu \)sec per 32-bit multiplication, a 10 msec sampling period should allow sufficient time for computation. A hardware multiplier can always be used for faster execution, if necessary, at a small (< $200) additional cost.
APPENDIX B.—ALTERNATE ALGORITHMS

The computational procedure described in Appendix A is fairly conventional for "strap-down" inertial systems. When so desired, ground speed information may be combined with attitude to calculate position. If display of heading only is the ultimate objective, then computation of heading may be performed relatively inaccurately and infrequently. However, the underlying attitude matrix must be updated with precision. A tradeoff may be implemented wherein the chore of matrix updating is simplified at the expense of a more complex heading calculation.

The C-matrix is now chosen to describe the vehicle attitude with respect to inertial space, $C^I_V$. Updating of this matrix from the measured angular increments requires no prior correction for earth rate, thus eliminating the multiplication involving the transpose of $C^N_V$. Only when a heading calculation is to be performed is an additional coordinate rotation required.

In one version of this approach, the inertial axes are selected to coincide with the local navigation frame at the start of a mission, simplifying the initialization procedure. Accurate updating of the $C^I_V$ matrix from the angular increment readouts proceeds every sampling instant. When at any time $t$ it is desired to compute heading, it is first necessary to compute $C^N_I$ (inertial-to-navigational) matrix at that time.

The change of attitude of the navigational frame in inertial space over the time $t$ may be visualized as a succession of three rotations:

1. rotation $+L$ deg about Y axis (pitch up)
2. rotate $+\Omega t$ about new X axis
3. rotate $-L$ deg about new Y axis (pitch down)

($L$ is latitude and $\Omega$ is the earth spin rate.)

The product of the corresponding rotation matrices gives the matrix $C^N_I$. The elements $C_{11}$, $C_{21}$, and $C_{31}$ of the $C^I_V$ matrix
form a vector describing the direction of the vehicle's longitudinal axis with respect to the inertial frame. When pre-multiplying by $C_i^N$, the components with respect to the navigational frame are obtained, from which heading may be calculated in the usual manner.

An even greater simplification may be obtained at the cost of greater complexity in the initial alignment procedure. Earth-aligned inertial axes are chosen:

- **X** - directed toward and perpendicular to the earth's axis.
- **Y** - east
- **Z** - parallel to negative spin axis

(X and Y lie in a plane parallel to the equatorial plane.)

Only two rotations are needed to transform from this inertial frame to the local NAV frame:

1. rotate $-\alpha_t$ about the Z axis
2. rotate (90 deg-$\lambda$) about the new Y axis

The calculation of heading proceeds as before.

These procedures have merit only if heading is to be calculated infrequently. If position navigation is required, continuous updating of heading may be needed to minimize errors, in which case the use of inertial frames would be of no benefit.
APPENDIX C.-- "PRECISION" MATRIX UPDATING USED IN SECTION 3, BODY OF REPORT

The computer program described in Section 3 attempts to eliminate computational errors so that the heading errors obtained are due solely to angular rate sensor errors. One effect of the quantizing of input motion is to allow for a more time-consuming but highly precise updating.

Given the three angular increments between sampling instants, the microprocessor-based algorithm is described by equation (4), Appendix A:

\[ C_{n+1} = C_n \left[ I + Wt + \frac{(Wt)^2}{2} + \ldots \right] \]

With a large-scale computer available, a closed-form solution may be utilized for the above infinite series:

\[ C_{n+1} = C_n (\phi) \]

where the elements of \( \phi \) are

\[ \phi_{11} = \frac{x^2}{\beta^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\beta^2} \cos \theta \]

\[ \phi_{12} = \frac{-z}{\beta} \sin \theta + \frac{x}{\beta} \frac{\Delta y}{\beta^2} (1 - \cos \theta) \]

\[ \phi_{13} = \frac{\Delta y}{\beta} \sin \theta + \frac{x}{\beta} \frac{\Delta z}{\beta^2} (1 - \cos \theta) \]

\[ \phi_{21} = \frac{\Delta z}{\beta} \sin \theta + \frac{x}{\beta} \frac{\Delta y}{\beta^2} (1 - \cos \theta) \]

\[ \phi_{22} = \frac{y^2}{\beta^2} + \frac{z^2}{\beta^2} + \frac{x^2}{\beta^2} \cos \theta \]

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\[ \phi_{33} = -\frac{\Delta x}{\beta} \sin \beta + \frac{\Delta y \Delta z}{\beta^2} (1 - \cos \beta) \]

\[ \phi_{31} = \frac{-\Delta y}{\beta} + \frac{\Delta x \Delta z}{\beta^2} (1 - \cos \beta) \]

\[ \phi_{32} = \frac{\Delta x}{\beta} \sin \beta + \frac{\Delta y \Delta z}{\beta^2} (1 - \cos \beta) \]

\[ \phi_{33} = \frac{\Delta x^2}{\beta^2} + \frac{\Delta y^2 + \Delta x^2}{\beta^2} \cos \beta \]

\[ \beta^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 \]

A more convenient, equivalent form is obtained for the computer updating via the following steps:\[\]

\[ \beta^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 \]

\[ \beta = \sqrt{\beta^2} \]

\[ s = \frac{\sin \beta}{\beta} \]

\[ Q = \frac{1 - \cos \beta}{\beta^2} \]

\[ \phi_{11} = 1 - Q(\Delta y^2 + \Delta z^2) \]

\[ \phi_{12} = Q \Delta x \Delta z - S \Delta z \]

\[ \phi_{13} = Q \Delta x \Delta z + S \Delta y \]

---

\[ \phi_{21} = Q \Delta x \Delta y + S \Delta z \]

\[ \phi_{22} = 1 - Q(\Delta x^2 + \Delta z^2) \]

\[ \phi_{23} = Q \Delta x \Delta z - S \Delta x \]

\[ \phi_{31} = Q \Delta x \Delta z - S \Delta y \]

\[ \phi_{32} = Q \Delta y \Delta z + S \Delta x \]

\[ \phi_{33} = 1 - Q(\Delta x^2 + \Delta y^2) \]

For each update to be performed, the \( \phi \) matrix is developed from the three angular increments in a subroutine. The previous C-matrix is then multiplied by the \( \phi \) matrix to obtain the updated C-matrix.
APPENDIX D.--CONING MOTION AND ITS SIGNIFICANCE

This section discusses some aspects of coning motion with a view towards understanding its impact on attitude creep, computer simulation, and processing requirements.

The term "coning" derives from the fact that if two axes are subjected to sinusoidal motion of equal amplitude and frequency but 90 deg out of phase, the third axis describes a cone in space. In general, any combination of motion about two axes will result in changes of orientation about all three axes. The computer's function is to accurately track this changing attitude.

Sequential Pitch and Roll

Some insight may be obtained by examining the effect of sequential pitch and roll operations. For a rotation $\theta$ about any axis, the corresponding transformation matrix contains $1$, $\cos \theta$, $\cos \theta$ in the main diagonal. $\sin \theta$ and $-\sin \theta$ appear once in symmetrical off-diagonal positions, with zero in the remaining positions. Rotation in the opposite direction ($-\theta$) results in the identical matrix except for a change of sign in the sine terms, equivalent to the transpose of the original matrix. Hence, if $P$ is the rotation matrix for a pitch of $\theta$ and $R$ is the rotation matrix for a roll of $\theta$, the sequence of pitch $+\theta$, roll $+\theta$, roll $-\theta$, and pitch $-\theta$ is given by

$$P^T R^T R P = P^T I P = I$$

and we return to the starting position. This is motion where the Z axis retraces its path. If there is any misalignment in the C-matrix to begin with, its errors remain undisturbed.

If the Z axis does not retrace its path, such as in the sequence pitch $+\theta$, roll $+\theta$, pitch $-\theta$, roll $-\theta$, a net change in attitude occurs. (This should be intuitively obvious, since the positive and negative rotations are made about axes whose spatial orientation are different). The sequence of $+\text{pitch}$ and $+\text{roll}$ is given by
The sequence of -pitch and -roll is as above but with -θ substituted for +θ. The resulting matrix is post multiplied by RP to obtain the final transformation matrix. When this is done, the following terms of interest result:

\[
C_{11} = \cos^2 \theta + \sin^2 \theta \cos \theta
\]

\[
C_{12} = -\sin^2 \theta
\]

\[
C_{33} = 2 \sin^2 \theta \cos \theta + \cos^4 \theta
\]

For \( \theta = 5 \) deg, a change of heading (\( \tan^{-1} \frac{C_{12}}{C_{11}} \)) of 0.435 deg is obtained. A very small change of tilt (\( \cos^{-1} C_{33} \)) of 0.027 deg also results. Note that, in this case, if the C-matrix is misaligned to begin with, the computer will calculate net motion about an axis whose orientation is in error (even with perfect calculation). Hence, errors can grow as a result.

The motion about the third axis is real motion, the result of stipulated motion about the other axes. Much of the literature assumes, as a condition, that there is no cumulative motion about the third axis. Coning is then interpreted as that compensating axis rotation (\( z \) in this case) which would keep the heading constant.
Coning Theorem

A very useful theorem states that if some motion brings an axis (say Z) back into coincidence with its starting position, a Z rotation will have been experienced equal to the solid angle swept out by the Z axis. This rotation in addition to any rotation due to the integral of zdt. A simple demonstration is to allow a level vehicle to pitch up 90 deg, roll 90 deg, and pitch down 90 deg. The vehicle undergoes a heading change of 90 deg with no sensed Z axis rotation whatsoever. Applying the theorem, the Z axis returns to its original orientation after sweeping out 1/8 of a sphere or $\frac{47\pi}{3}$ radians.

For the coning motion test of Appendix G, the Z axis sweeps out a cone of half angle 36.85 deg five times, while the heading rotates 360 deg. The solid angle per sweep is $2\pi(1 - \cos 36.85 \text{ deg}) = 0.4\pi$. A full rotation of $2\pi$ therefore requires five such conical sweeps.

For the 5 deg pitch and roll example of this section, a 5 deg x 5 deg square is swept out. If the slight misalignment of the Z axis after one cycle is neglected, the approximate solid angle is $\frac{25}{3283} \times \frac{180}{\pi} = 0.435$ deg, in agreement with the previous calculation of heading change.

Significance for Computer Simulation

The "creep" of heading and tilt for the vehicle scenarios of Sections 3 and 4 may be understood as coning effects resulting from a stipulation of 3-axis motion. The input motion of Section 3 is due to irregular quantizing of the underlying sine wave inputs. This irregularity destroys the compensating effects of the scenarios of Section 4, leading to greater magnitudes of creep.

The received plots of pitch and roll for vehicle test runs show an in-phase relationship. Assuming that such a phase relationship is not universal and knowing that quadrature effects are more severe, some of the runs were changed to reflect pitch and roll inputs of slightly different frequencies. Hence, all phase relationships are experienced. The results indeed show greater perturbations for the latter inputs.

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The plots of heading for the simulations of Section 4 are based on pure sine wave inputs, again with slightly differing pitch and roll frequencies. The effect on heading is clearly seen during the periods of no yaw (e.g., 200 to 600 sec.). As the phase between pitch and roll changes steadily, the heading drift changes direction, producing the beat pattern seen.

It must be emphasized that the foregoing effects describe real motion which must result if the 3-axis inputs are as specified. What is principally of concern is the ability of the system's output to properly define the attitude.

Significance for Sampling Interval

It has previously been mentioned that the computation algorithm is a "constant velocity" algorithm. To the extent that velocity is not constant between updates, some error will be introduced. It naturally follows that the maximum acceptable sampling interval is dependent on vehicle angular acceleration. If it is assumed that continuous coning motion due to pitch and roll represents a severe test, a simple intuitive estimation of drift may be made with the aid of the coning theorem.

Assume a worst case of quadrature motion. For equal amplitude pitch and roll, the Z-axis describes a cone, as shown in Figure D-1. The period of this sweep equals the pitch roll period. The half-cone angle equals the pitch and roll amplitude.

![Figure D-1 Conical motion of Z-axis.](attachment://conical-motion.png)
For small angles, there is little error in considering the spherical area intercepted by the cone to be a plane. The solid angle swept out per cycle is then proportional to the area of the plane circle ($\pi R^2$). The processor, however, assumes that the Z axis describes segments of the circle according to the sampling interval. It is assumed that the angular rate sensors are not sampled. Their outputs are continuously fed into integrators and it is the integrators which are periodically sampled. Hence, the angular position read at each sample is correct but the constant velocity algorithm assumes that the new position was attained via constant angular rate of the X and Y axes. The vehicle is thus thought to have rotated about a fixed axis in the X-Y plane. With no additional computation errors, a vehicle rotation about the Z axis will be calculated proportional to the area represented by the sum of the segments of the circle rather than the circle itself.

The area per segment is

$$A = R \sin \alpha \cdot R \cos \alpha = \frac{R^2}{2} \sin 2\alpha .$$

For $N$ segments per circle,

$$\alpha = \frac{2\pi}{2N}$$

and the total area of $N$ segments is

$$NA = \frac{NR^2}{2} \sin \frac{2\pi}{N} \approx \frac{NR^2}{2} \left[ \frac{2\pi}{N} - 1/6 \left( \frac{2\pi}{N} \right)^3 \right].$$

The second term within the brackets represents the departure from the area of the circle. The fractional error is

$$\text{Fractional error} = \left( \frac{2\pi}{N} \right)^3 \times 1/6 \times \frac{N}{2\pi} = 1/6 \left( \frac{2\pi}{N} \right)^2$$
Now, according to the coning theorem, the Z-axis rotation equals the solid angle swept out by the Z axis $\approx \pi \theta^2$ per cycle. The processor error is then

$$\varepsilon = \pi \theta^2 \times \frac{1}{6} \frac{2\pi^2}{N^2} = \frac{2}{3} \frac{\pi^3 \theta^2}{N^2} \text{ per cycle}$$

If $T_p = \text{pitch and roll period}$ and $T_s = \text{sampling period}$:

$$N = \frac{T_p}{T_s}$$

error rate $= \frac{2}{3} \pi^3 \frac{\theta^2 T_s}{T_p^2} \times \frac{1}{T_p} \text{ radians/sec.}$

In terms of $\theta$ degrees and error rate in degrees/hour,

error rate $= 1290 \frac{\theta^2}{T_p^3} T_s^2 \text{ deg/hr.}$

Measured data indicate pitch and roll amplitudes of 3 deg with a period of 1.5 seconds. Then

error rate $= 3400 T_s^2 \text{ deg/hr.}$

A sampling interval of 0.1 s is clearly insufficient. At 0.01 s, the error is reduced to 0.34 deg per hour. The input motion is, of course, exceptionally severe since a constant 90 deg phase relationship between pitch and roll is assumed.
APPENDIX E.--PROGRAM LISTING FOR SECTION 3, BODY OF REPORT

The results of Section 3 were obtained with the following computer program, LNDNAV. The listing shows a particular choice of input parameters which, of course, are changed to accommodate the various vehicle scenarios. The notation AMP(3,2) refers to the Z-axis amplitude during the second interval. FREQ(1,4) is the period of the X-axis motion during the fourth interval. T1, T2, T3, and T4 give the time at the end of each interval.

In the listing shown, some of the constants were preselected even though the ability of the operator to insert these constants in response to computer prompting is preserved.
PROGRAM LAXNAV
REAL CREF,CMPH,XHAT,DHAT,DOSH,SHAT,XM,YM,PM,DEL,T
REAL EOUT,EPH,CMPH,VELR,DELX,VELE,DELZ,ELEL,XELN,ELCH,STEP
REAL PI,PIA,B
DIMENSION CHPF(3,3),CMPH(3,3),X(3),Z(3),AMP(3,4)
1 IFPAX(3,4)=X(3),Y(3),Z(3),X(3),Z(3)
2 SELECT CONSTANTS
3 AMP(1,1)=0.0
4 AMP(1,2)=10.0
5 AMP(1,3)=0.0
6 AMP(1,4)=0.0
7 AMP(2,1)=0.0
8 AMP(2,2)=10.0
9 AMP(2,3)=0.0
10 AMP(2,4)=0.0
11 NO.SYS=0.0
12 FNXU(1,1)=0.0
13 FNXU(2,2)=1.0
14 FNXU(3,3)=0.0
15 AMP(3,1)=0.0
250 FNXU(3,1)=1200.0
260 T1=300.0
270 T2=0.0
280 T3=300.0
290 T4=0.0
300 DLALS=999999.
310 DHOLS=0.0
2 VFUN=1 (546,2)
13 VFUN=1 (546,2)
15 VFUN=1 ('ENTER QUANTIZING INTERVAL IN TENTHS OF SECONDS')
33 RCHA (5,10) INT
34 10 FORMAT(13)
35 RCHA (5,10) LIMIT
36 RCHA (5,10) LIMIT
37 RCHA (5,10) LIMIT
38 RCHA (5,10) LIMIT
39 RCHA (5,10) LIMIT
40 RCHA (5,10) LIMIT
41 RCHA (5,10) LIMIT
42 RCHA (5,10) LIMIT
43 RCHA (5,10) LIMIT
44 RCHA (5,10) LIMIT
45 RCHA (5,10) LIMIT
46 RCHA (5,10) LIMIT
47 RCHA (5,10) LIMIT
48 RCHA (5,10) LIMIT
49 RCHA (5,10) LIMIT
50 VFUN=1 ('ENTER X,Y,Z SENSOR SCALE FACTORS')
51 RCHA (5,2) A(1), A(2), A(3)
52 RCHA (5,2) A(1), A(2), A(3)
53 RCHA (5,2) A(1), A(2), A(3)
54 RCHA (5,2) A(1), A(2), A(3)
55 RCHA (5,2) A(1), A(2), A(3)
56 RCHA (5,2) A(1), A(2), A(3)
57 RCHA (5,2) A(1), A(2), A(3)
58 RCHA (5,2) A(1), A(2), A(3)
59 RCHA (5,2) A(1), A(2), A(3)
60 RCHA (5,2) A(1), A(2), A(3)
61 RCHA (5,2) A(1), A(2), A(3)
62 RCHA (5,2) A(1), A(2), A(3)
63 RCHA (5,2) A(1), A(2), A(3)
64 RCHA (5,2) A(1), A(2), A(3)
65 RCHA (5,2) A(1), A(2), A(3)
66 RCHA (5,2) A(1), A(2), A(3)
67 RCHA (5,2) A(1), A(2), A(3)
68 RCHA (5,2) A(1), A(2), A(3)
69 RCHA (5,2) A(1), A(2), A(3)
70 RCHA (5,2) A(1), A(2), A(3)
71 RCHA (5,2) A(1), A(2), A(3)
72 RCHA (5,2) A(1), A(2), A(3)
73 RCHA (5,2) A(1), A(2), A(3)
74 RCHA (5,2) A(1), A(2), A(3)
75 RCHA (5,2) A(1), A(2), A(3)
76 RCHA (5,2) A(1), A(2), A(3)
77 RCHA (5,2) A(1), A(2), A(3)
78 RCHA (5,2) A(1), A(2), A(3)
79 RCHA (5,2) A(1), A(2), A(3)
80 RCHA (5,2) A(1), A(2), A(3)
81 RCHA (5,2) A(1), A(2), A(3)
82 RCHA (5,2) A(1), A(2), A(3)
83 RCHA (5,2) A(1), A(2), A(3)
84 RCHA (5,2) A(1), A(2), A(3)
85 RCHA (5,2) A(1), A(2), A(3)
86 RCHA (5,2) A(1), A(2), A(3)
87 RCHA (5,2) A(1), A(2), A(3)
88 RCHA (5,2) A(1), A(2), A(3)
89 RCHA (5,2) A(1), A(2), A(3)
90 RCHA (5,2) A(1), A(2), A(3)
91 RCHA (5,2) A(1), A(2), A(3)
92 RCHA (5,2) A(1), A(2), A(3)
93 RCHA (5,2) A(1), A(2), A(3)
94 RCHA (5,2) A(1), A(2), A(3)
95 RCHA (5,2) A(1), A(2), A(3)
96 RCHA (5,2) A(1), A(2), A(3)
97 RCHA (5,2) A(1), A(2), A(3)
98 RCHA (5,2) A(1), A(2), A(3)
99 RCHA (5,2) A(1), A(2), A(3)
100 RCHA (5,2) A(1), A(2), A(3)
101 RCHA (5,2) A(1), A(2), A(3)
102 RCHA (5,2) A(1), A(2), A(3)
103 RCHA (5,2) A(1), A(2), A(3)
104 RCHA (5,2) A(1), A(2), A(3)
105 RCHA (5,2) A(1), A(2), A(3)
106 RCHA (5,2) A(1), A(2), A(3)
107 RCHA (5,2) A(1), A(2), A(3)
108 RCHA (5,2) A(1), A(2), A(3)
109 RCHA (5,2) A(1), A(2), A(3)
110 RCHA (5,2) A(1), A(2), A(3)
SEL EXTENDED FORTRAN IV (REV-0) II D S E R V U D

Wednesday

30  \text{WHTE} (1,1) \text{ (FHEW,1,1)}, \text{J}=1,4
31  \text{FURNAI (} \text{F PERIOUS} = 1 \text{, FIV,3) \text{FWHITE} (1,17) FNP (1,1,1) \text{J}=1,4)
32  \text{FURNAI (Y AMPLI} \text{TIONS} = 1 \text{, FIV,3) \text{FWHITE} (1,13) FRED (2,2), \text{J}=1,4)
33  \text{FURNAI (Y PERIousy = 1 \text{, FIV,3) \text{FWHITE} (1,13) FNP (3,3), \text{J}=1,4)
34  \text{FURNAI (Y AMPLI} \text{TIONS} = 1 \text{, FIV,3) \text{FWHITE} (1,13) FRED (3,3), \text{J}=1,4)
35  \text{FURNAI (Y PERIousy = 1 \text{, FIV,3) \text{FWHITE} (1,13) FNP (3,3), \text{J}=1,4)
36  \text{FURNAI (} \text{F TIME AT END OF EAC} \text{H INTERVAL IN SECONDS = 1 \text{, FIV,3) \text{FWHITE} (1,13) FRED (3,3), \text{J}=1,4)
37  \text{FURNAI (} \text{F QUANTI} \text{ZING INTERVAL IN TE} \text{NSES OF SECONDS = 1 \text{, FIV,3) \text{FWHITE} (1,13) FRED (3,3), \text{J}=1,4)
38  \text{FURNAI (F INITIAL LATITUDE} = 1 \text{, FIV,3) \text{FWHITE} (1,13) FRED (3,3), \text{J}=1,4)
39  \text{FURNAI (F LATITUDE} = 1 \text{, FIV,3) \text{FWHITE} (1,13) FRED (3,3), \text{J}=1,4)
40  \text{FURNAI (} \text{F SECONDARY RADIUS} = 1 \text{, SFY,3) \text{FWHITE} (1,13) FRED (3,3), \text{J}=1,4)
41  \text{FURNAI (} \text{F SECONDARY DIA} \text{G IN UN} \text{EEDS DEG HOURS = 1 \text{, SFY,5}) \text{FWHITE} (1,13) FRED (3,3), \text{J}=1,4)
42  \text{FURNAI (F SECONDARY DIA} \text{G IN UN} \text{EEDS DEG HOURS = 1 \text{, SFY,5}) \text{FWHITE} (1,13) FRED (3,3), \text{J}=1,4)
43  \text{CALCULATE EARTHMA} \text{E OF NAV FRAME (FIV,3) \text{FWHITE} (1,13) FRED (3,3), \text{J}=1,4)
44  \text{INITIALIZE DIRECTION COSINE MATRICES}
45  \text{CALCULATE EARTHMA} \text{E OF NAV FRAME (FIV,3) \text{FWHITE} (1,13) FRED (3,3), \text{J}=1,4)
46  \text{INITIALIZE DIRECTION COSINE MATRICES}
47  \text{CALCULATE EARTHMA} \text{E OF NAV FRAME (FIV,3) \text{FWHITE} (1,13) FRED (3,3), \text{J}=1,4)
48  \text{INITIALIZE DIRECTION COSINE MATRICES}
49  \text{CALCULATE EARTHMA} \text{E OF NAV FRAME (FIV,3) \text{FWHITE} (1,13) FRED (3,3), \text{J}=1,4)
50  \text{INITIALIZE DIRECTION COSINE MATRICES}
51  \text{CALCULATE EARTHMA} \text{E OF NAV FRAME (FIV,3) \text{FWHITE} (1,13) FRED (3,3), \text{J}=1,4)
52  \text{INITIALIZE DIRECTION COSINE MATRICES}
53  \text{CALCULATE EARTHMA} \text{E OF NAV FRAME (FIV,3) \text{FWHITE} (1,13) FRED (3,3), \text{J}=1,4)
54  \text{INITIALIZE DIRECTION COSINE MATRICES}
55  \text{CALCULATE EARTHMA} \text{E OF NAV FRAME (FIV,3) \text{FWHITE} (1,13) FRED (3,3), \text{J}=1,4)
56  \text{INITIALIZE DIRECTION COSINE MATRICES}
57  \text{CALCULATE EARTHMA} \text{E OF NAV FRAME (FIV,3) \text{FWHITE} (1,13) FRED (3,3), \text{J}=1,4)
58  \text{INITIALIZE DIRECTION COSINE MATRICES}
59  \text{CALCULATE EARTHMA} \text{E OF NAV FRAME (FIV,3) \text{FWHITE} (1,13) FRED (3,3), \text{J}=1,4)
60  \text{INITIALIZE DIRECTION COSINE MATRICES}
CCMP(j,j) = 1.0 J+J
C
C CONVERT INPUT AMPLITUDE TO RADIAN
C
C N= 50 J= 2,2
C NO 50 K= 2,2
C 50 AMP(j,k)=AMP(3,k)+ PI/180,0
C   STEP=3*PI/180,0
C
C CONVERT INPUT PERIOD TO HORIZONTAL FREQUENCIES
C N= 60 J= 2,2
C NO 60 K= 2,2
C 60 FREQ(j,k)=2*PI/FREQ(j,k)
C
C INITIALIZE PRINT INTERNAL COUNTER IN TERMS OF SECONDS
C
C NPRINT=1
C
C BEGIN MAIN PROGRAM
C
P1=U,20+2
P2=U,20+2
P3=U,20+2
MAX=4*10,2
MAX
N=20 INT=1,101
N=20 NT=INT,MAX,INT
TANG
TANG
TAO=J+1
IF (T,J,J1) JU 10 110
IF (T,J,J2) JU 10 120
IF (T,J,J3) JU 10 130
JU 100 J=1,2
105 F(j)=AMP(j,4)*SIN(FREQ(j,4)+T)
JU 10 140
110 JU 111 J=1,2
111 F(j)=AMP(j,1)*SIN(FREQ(j,1)+T)
JU 10 140
120 JU 121 J=1,2
121 F(j)=AMP(j,2)*SIN(FREQ(j,2)+T)
JU 10 140
130 JU 131 J=1,2
131 F(j)=AMP(j,3)*SIN(FREQ(j,3)+T)
JU 10 140
NPRINT=NPINT+1
J=1
145 DEL1=AMP(1,2)*PI
DEL2=AMP(1,2)*PI
DEL3=AMP(1,2)*PI
IF (UABS(DEL1),L1,STEP) DEL1=U,20+2
IF (UABS(DEL2),L1,STEP) DEL2=U,20+2
IF (UABS(DEL3),L1,STEP) DEL3=U,20+2
103  IF (DNLX + UFLY + DNLZ ,EWU ) U0 TO 160
104  HAPRX + DNLX
105  PRTLY + DNLX
106  PFLZ + DNLZ
107  CALL MATRIX UPDATE ROUTINE
108  SUM Pi IF ANGLE INCREMENTS BETWEEN STEPS AS SEEN BY COMPUTER
110  NA (1) = A + 1
112  CNP(X) = CNP(X) + 1/6400000.WD*W
114  NA (1) = NA(1) + 1
115  CNP(X) = CNP(X) + (A(J) + CNP(X) + CNP(X) = CNP(X) = CNP(X) + ERN(A)
118  DLNL(J) = CNP(J) * DNL
120  SUMPi
121  UPDATE CCNP WITH PHI MATRIX
122  CALL MAUL(ELR(L) , UELR(2) , UELR(3) , PHI)
123  CALL UPDATE (CCNP, PHI)
128  UPDATE CREF WITH ANGLE STEP
127  CALL MAUL(UELY, DELT, UELU, PHI)
128  CALL UPDATE (CREF, PHI)
140  UPDATE CCNP WITH MEASURED ANGLE STEP
141  DNLX = A(J) * UFLX
142  DNLX = A(2) * UFLY
143  DNLX = A(3) * UFLZ
144  CALL MAUL(UELY, UELM, DELT, PHI)
145  CALL UPDATE (CCNP, PHI)
146  END OF MATRIX UPDATE ROUTINE
204  IF (NPC = 0 OR 1) OR (U) U0 TO 200
205  TIME = TIME / 1000000
206  SmH0 = MAUL(UFLX, CREF(2,1), CREF(1,1) + 1, 0, PHI)
207  CNP = CNP(CREF, 3, 3)
208  STM = CNP(CREF, 3, 3)
209  FNR = UFLX, T, CREF(2,1), CREF(1,1) + 1, 0, PHI = SMU
210  NE(1) = CREF(1,1) + CREF(2,1) + CREF(3,1) + CREF(2,3)
211  JER = CNP(2,1) * CREF(2,1) + CREF(3,1) + CREF(2,3)
212  S = CREF(2,1) + CREF(3,1)
213  DE(1) = CNP(2,1) * CREF(2,1) * CREF(3,1) * CREF(2,3)
214  C = CNP(2,1) * CREF(2,1) + CREF(3,1) + CREF(2,3)
215  J = CREF(2,1) * CREF(3,1)
216  WRITE (7,170) TIME, SmH0, STM, ERHON, UEIV, DELT
03/11/80 151515A AF SYSTEMS NFAL-TIME POSITION=1.0
SELECTED FORTRAN IV (REV-0/105-5-SEPT)

L amendments
217 110 FURN (9A,4A,2,F10.2,2F15.8)
218 410 FURN (3F12.0)
219
220
221
222
240 CONTINUE
210 CONTINUE
END
SUBROUTINE MAT (Y, Z, C)

IMPLICIT REAL*8(A-E)

N = 4 / 41 N T Y C (1, 3)

X = 0.024 X

Y = 0.024 Y

Z = 0.024 Z

X = X + Y + Z

S = 1.0 / S

IF (S .LT. 0.01) GO TO 350

R = COS(THETA) / S

COS = COS(THETA) / S

N = 1.0 / COS

Y = X - Y - Z

C(1, 1) = 1.0 * X

C(1, 2) = 1.0 * Y

C(1, 3) = 1.0 * Z

C(2, 1) = 0.024 X

C(2, 2) = 0.024 Y

C(2, 3) = 0.024 Z

C(3, 1) = 0.024 X

C(3, 2) = 0.024 Y

C(3, 3) = 0.024 Z

RETURN

END
SUBROUTINE UPDATE (C, PHI)
REAL * K, PHI, TEMP
INTEGER C(3,3), PHI(3,3), TEMP(3,3)
PU 306 J = 1, 3
PU 504 K = 1, 3
PU 700 TEMP(J, K) = 0, 3 + 0
PU 504 L = 1, 3
500 TEMP(J, L) = TEMP(J, K) * C(J, L) * PHI(L, K)
PU 304 J = 1, 3
PU 504 L = 1, 3
510 C(J, K) = TEMP(J, K)
RETURN
END
APPENDIX F.--PRINTOUTS FOR SECTION 3

Printouts for a number of cases discussed in Section 3 are shown. These correspond to cases 3.4.2e, 3.4.2g, 3.4.3, 3.4.4(d), 3.4.5(b), and 3.4.6 of that section.
<table>
<thead>
<tr>
<th>TIME (MINUTES)</th>
<th>HEADING</th>
<th>TAU</th>
<th>ERROR</th>
<th>DETLREF</th>
<th>WEICHER</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00</td>
<td>44.13</td>
<td>2.79</td>
<td>3.02</td>
<td>0.99999998</td>
<td>0.99999998</td>
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<tr>
<td>2.00</td>
<td>53.79</td>
<td>0.13</td>
<td>2.86</td>
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<td>0.99999998</td>
</tr>
<tr>
<td>3.00</td>
<td>63.46</td>
<td>0.14</td>
<td>3.42</td>
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<td>0.99999998</td>
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<tr>
<td>4.00</td>
<td>73.13</td>
<td>0.14</td>
<td>4.01</td>
<td>0.99999998</td>
<td>0.99999998</td>
</tr>
<tr>
<td>5.00</td>
<td>82.80</td>
<td>0.10</td>
<td>4.74</td>
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<td>0.99999998</td>
</tr>
<tr>
<td>7.00</td>
<td>75.75</td>
<td>0.10</td>
<td>3.51</td>
<td>0.99999998</td>
<td>0.99999998</td>
</tr>
<tr>
<td>8.00</td>
<td>53.97</td>
<td>0.10</td>
<td>2.54</td>
<td>0.99999998</td>
<td>0.99999998</td>
</tr>
<tr>
<td>9.00</td>
<td>27.96</td>
<td>0.10</td>
<td>1.78</td>
<td>0.99999998</td>
<td>0.99999998</td>
</tr>
<tr>
<td>10.00</td>
<td>7.46</td>
<td>0.12</td>
<td>-0.15</td>
<td>0.99999998</td>
<td>0.99999998</td>
</tr>
<tr>
<td>11.00</td>
<td>49.50</td>
<td>2.78</td>
<td>2.90</td>
<td>0.99999998</td>
<td>0.99999998</td>
</tr>
<tr>
<td>12.00</td>
<td>53.90</td>
<td>0.20</td>
<td>2.56</td>
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<td>0.99999998</td>
</tr>
<tr>
<td>13.00</td>
<td>53.17</td>
<td>0.24</td>
<td>3.34</td>
<td>0.99999998</td>
<td>0.99999998</td>
</tr>
<tr>
<td>14.00</td>
<td>55.91</td>
<td>0.30</td>
<td>4.04</td>
<td>0.99999998</td>
<td>0.99999998</td>
</tr>
<tr>
<td>15.00</td>
<td>50.58</td>
<td>0.20</td>
<td>3.87</td>
<td>0.99999998</td>
<td>0.99999998</td>
</tr>
<tr>
<td>16.00</td>
<td>49.02</td>
<td>0.20</td>
<td>3.55</td>
<td>0.99999998</td>
<td>0.99999998</td>
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97
APPENDIX G.--CHECK ON ACCURACY OF REFERENCE ALGORITHM

In the study of computation algorithms (Section 4), the "error" is simply the difference between a particular algorithm's output and that of the "reference" algorithm. A nagging question remains as to the sufficiency of the reference algorithm. A precise mathematical solution for any but extremely simple input motion is beyond reach.

One source of confidence is, of course, the small errors between the reference and the best of the test algorithms.

An additional check on the reference algorithm was made utilizing the mathematical formulation of attitude as a result of coning motion. This formulation is found in the literature. Coning motion is the result of quadrature motion on two of the axes, a motion which presents a most difficult test of computation adequacy.

For X and Y axis motion given by

\[ \begin{align*}
\dot{x} &= A \sin Bt \\
\dot{y} &= A \cos Bt \\
R^2 &= A^2 + B^2 ,
\end{align*} \]

the C-matrix components \( C_{11} \) and \( C_{21} \) at any time \( t \) may be expressed as

\[ \begin{align*}
C_{11} &= \frac{1}{2R} [(B+R) \cos (B-R)t - (B-R) \cos (B+R)t] \\
C_{21} &= \frac{A^2}{R^2} \sin Bt + \frac{B}{2R^2} [(B-R) \sin (B+R)t + (B+R) \sin (B-R)t] .
\end{align*} \]

A suitable choice of \( A \) and \( B \) which makes the arithmetic simple, while giving a severe but practical test is \( A = 0.15\pi \) and \( B = 0.2\pi \). Then \( C_{11} \) and \( C_{12} \) reduce to

\[ \begin{align*}
C_{11} &= 0.9 \cos 0.05 \pi t + 0.2 \cos 0.45 \pi t
\end{align*} \]

---

$C_{21} = 0.36 \sin 0.2 \pi t - 0.08 \sin 0.45 \pi t$

$$-0.72 \sin 0.05 \pi t$$

The period of this motion is 40 s. At $t = 10$ s, $C_{11} = 0$ for a heading of 90 deg, while at $t = 20$, $C_{21} = 0$ and the heading is 180 deg.

This motion was used to test the "reference" algorithm and confirmed precise agreement with the theoretical results.

**NOTE**

Since the program was written in terms of angle rather than rate, the rates $\dot{x}$ and $\dot{y}$ must be integrated to obtain the proper input.

\[ x = -\frac{A}{B} \cos Bt + \frac{A}{B} \]

\[ y = \frac{A}{B} \sin Bt \]

The constant of integration $\frac{A}{B}$ must be added to $x$ so that $x(0) = 0$, since the expressions for the C-matrix elements assume initial alignment (identify matrix) at the start. It is, in fact, this initial alignment which makes the equations complex, since coning takes place about an axis offset from the vertical.

As a matter of interest, the tilt angle $\theta$ of the Z axis is given by

\[ \cos \theta = C_{33} = 1 - \frac{A^2}{R^2} (1 - \cos Rt) \]

\[ = 0.64 + 0.36 \cos 0.25 \pi t \]

This describes conical motion with $\theta_{\text{max}} = 74$ deg and a period of 8 s. Hence, five complete cones are swept out by the Z axis during the 40 seconds it takes for the heading to make one complete revolution.
APPENDIX H.—PROGRAM LISTING FOR SECTION 4

The results of Section 4 were obtained with the following computer programs:

a) TNKSIM calculates the reference heading and tilt angles and develops the record of angle increments which would be measured by the vehicle-mounted sensors. Earth rotation is accounted for in these outputs.

b) TNKTST models the anticipated microprocessor-based system, including a pulse-balanced integrator as the analog-to-digital device.
PROGRAM SIMSIM
1 IMPLICIT REAL*(4), INTEGER
3 DIMENSION X(N, M), Y(N, M), SimulationValue(N, M)
4 COMMON /DUMMY/ SimulationValue, SimulationPeriod, SimulationTime
5 SimulationPeriod = 10.0
6 SimulationTime = 0.0
7 A = 1.0
8 B = 1.0
9 C = 1.0
10 D = 1.0
11 E = 1.0
12 F = 1.0
13 G = 1.0
14 H = 1.0
15 I = 1.0
16 J = 1.0
17 K = 1.0
18 M = 1.0
19 N = 1.0
20 DO I = 1, N
21 DO J = 1, M
22 SimulationValue(I, J) = 0.0
23 END DO
24 DO I = 1, N
25 DO J = 1, M
26 SimulationValue(I, J) = SimulationValue(I, J) + 1.0
27 END DO
28 END DO
29 SimulationTime = SimulationTime + SimulationPeriod
30 IF (SimulationTime .gt. 10.0) THEN
31 SimulationTime = SimulationTime - 10.0
32 END IF
33 WRITE (*, 100) SimulationTime, SimulationPeriod, SimulationValue
34 END
100 FORMAT (1X, 3F10.0)
35 END
/* SYSTEMS REAL-TIME MONITOR-7.0

SEL EXTENDED FURTHER IV (REV 07/085SEPUN)

IMAIN

55

WHITE(AY)

9 FIGHT('ENTER LATITUDE IN DEGREES')

11 READ(AY,91LAT)

60 FIGHT(2)

64 IF(KN,EU,006)GU TO 104

65 WHITE(AY,105)SIM

105 FIGHT('PRESAT' RUNNING TIME = 'F9.1', 'IN SECONDS')

1 'ENTER CONTINUE RUNNING TIME IN SECONDS')

GU TO 107

109 WHITE(AY,42)

11 FIGHT('ENTER RUNNING TIME IN SECONDS')

107 READ(AY,93TMAX)

112 WHITE(AY,94)

113 FIGHT(200,12)

114 WHITE(AY,200)

200 FIGHT('ENTER') TO INPUT VEHICLE AXIS MOTION')

70 READ(AY,203AX)

201 WHITE(AY)

73 PIN INPUT*+BSIM1

LPCU=25

96 FIGHT('X', 'Y', 'Z', 'W', 'SAMPLE', '10X', 'AMPLITUDE', '18X',

1 'PERIOD', '13X', 'PHASE(DOCS, E58)', '10X', 'MOTION(14V, 01N)')

99 FIGHT('17F9.3, 315, 002, 11')


2 'C51', 'AX', 'PITCHING', 'AX', 'TYL')

100 C

60 C

INITIALIZE C MATRIX

DO 100 K=1,3

100 DUMP(K),SUMP(K)

101 IF(KN,EU,106) TO 102

DO 10 J=1,3

102 DO V J=1,3

91 IF(KN,EU,106) TO 101

92 CVN(J,K)=0.000

103 GU TO 10

104 CVN(J,K)=1.000

105 CONTINUE

C

INITIALIZE ANGULAR INPUT MOTION AND EARTH KNOTIATION MATRIX

102 ENU=PI/45200.000

103 TSN=TS1=1

104 DO 11 K=1,3

105 IF(KN,EU,106)GU TO 112

106 PNAV(K)=0.0

107 GO TO 11

108 PPNAV(K),DUMP(K)

109 CONTINUE

110 SLAT=SLAT=PI/140.000

111 CONTINUE

112 CONTINUE

113 END*/
100 ENATE(1)=OCON(SLAT) * EDOY
110 ENATE(2)=U0, UO+0
111 ENATE(3)=U51, U51+O
112 HOP(1) = 81314
113 DU 111 K=13
114 HOP(4K)=AMM (K)
115 HOP(4++) = EPR (K)
116 DPNU(4)=4.09+0/PRU(K)
117 HOP(144) = HX(K)
118 MIN(14)=XV
119 HOM(7++) = ENATE(K)
120 HOM(14) = XA
121 HOM(1) = SLAT
122 HOM(12) = TMAX
123 WRITE (21) SHUR
124 DUU(41) = 0.0, UO+0
125 WRITE (21) CVNS
126 C TUP OF PROCESSING LOOP
127 C OBTAIN INCREMENTS OF INPUT ANGLES
128 C
130 DU 15 = 0.25
131 TSU = TSU+0.0TSU
132 DU 16 = 1/3
133 TMAX = TMAX
134 IF (KX(K), E0, 0.0) GO TO 122
135 PNU(K) = RAMP(K) = USIN(THETA)
136 GO TO 140
137 C PNU(K) = RAMP(K) = USIN(THETA)
138 DUU(K) = PNU(K) = PPHV(K)
139 C PNU(K) = RAMP(K)
140 IF (KX(K), E0, 0.0) GO TO 164
141 DU 165 = 1/3
142 C DUU(K) = PNU(K)
143 DU 166 = 166
144 C NOTATE INTO VEHICLE AXIS
145 C
146 DU 163 = 1/3
147 DUU(J) = 0.0, UO+0
148 DU 163 = 1/3
149 C DUU(J) = RPHV(J) = CVA(K, J) = UPNVI(K)
150 C C DETERMINE INCLINATION MATRIX AND MUTATE C MATRIX
151 C
152 C
153 C
154 C
155 C
156 C
157 C
158 C
159 C
160 C
161 C
162 C
DATA

161 CONTINUE

166 DO 12 K=1,3

167 TEMPR=0.000

168 DO 12 L=1,3

169 TEMP=TEMP+CVN(J,L)*(DPMVH(L,K)+UPHVN(L,K))

171 CVN(J,J)=CVN(J,J)+TEMP

172 DO 13 J=1,3

173 DO 13 K=1,3

174 CVN(J,K)=CVN(J,K)+TEMP

177 C DETECTIVE SECOND EARTH ROTATION ON VEHICLE AXIS

178 DO 1 J=1,3

179 EKM(J)=EKM(J)+F0

180 DO 14 J=1,3

181 EKM(J)+EXIN(J)+CVN(K,J)+EXIN(J)

182 OUIT(J,4)=EKM(J)+EXIN(J)+CVN(K,J)+EXIN(J)

183 OUIT(J,4)=-EATAN2(CVN(2,J),CVN(1,J))#100.00000000+PI

184 SCVN=CVN(3,J)

185 OUIT(J,5)-SCCSS(CVN(3,J))#100.00000000+PI

186 IF(TS1.LT.1.00000000-4) GO TO 15

187 IF(TS1.LT.1.00000000-6) GO TO 151

188 IF(LPCN(J,1.0E-4)) GO TO 151

189 CPCH=0.000

190 WRITE(7,152)

191 FORMAT(*17)

192 WRITE(7,946)

193 WRITE(7,945)AT1,0,TB1II,(AMP(K),K=1,3),(FRE(J),J=1,3)

194 1 (KK(L),L=1,5),RV

195 WRITE(7,947)

196 WRITE(7,155)TIME4((CVN(J,K),K=1,3),J=1,3),OUT(M,N)

197 1 OUIT(S,M)

198 FORMAT(11,F9.3,9F11.6,F11.4,F9.3)

199 DO 15 J=1,3

200 TEMPR=0.000

201 DO 15 K=1,3

202 TEMP=TEMP+CVN(J,K)*S2

203 CVN(J,J)=S2

204 DO 13 J=1,3

205 TEMPR=0.000

206 DO 13 K=1,3

207 TEMP=TEMP+CVN(J,K)*S2

208 CVN(J,J)=S2

209 1 CVN(J,J)=CVN(J,J)+CVN(2,J)*CVN(3,J)*S2

210 CVN(J,J)=CVN(J,J)+CVN(2,J)*CVN(3,J)*S2

211 CVN(J,J)=CVN(J,J)+CVN(2,J)*CVN(3,J)*S2

212 CVN(J,J)=CVN(J,J)+CVN(2,J)*CVN(3,J)*S2

213 CVN(J,J)=CVN(J,J)+CVN(2,J)*CVN(3,J)*S2

214 WRITE(7,139)CVN(J,J),S2

215 FORMAT(5X,13.6)
LPCu1=LPCu1+1
21 Cu+TIME
IF((131**2,001),LT.TMAX)GO TO 119
WRITE(7,152)
EMUFIL 21
WRITE (31) (21) ,Cu+6
EMUFIL 21
WRITE(KITY,01)
FORMAT('ENTER 1 TO CONTINUE RUN, 0 TO EXIT/')
READ(14,Y,20)X0
IF(KXY.EQ.0)GO TO 20
RETURN
END
GO TO 103
END
SUBROUTINE TAKIST

IMPLICIT REAL*8(A-H,O-Z)

REAL*8 SNOM,SI,O,SNUM,SNUM,SNUM,SNUM

IMPLICIT INTEGER(A-H,O-Z)

DIMENSION UCNV(5),UCNV(5),UCNV(5),UCNV(5),UCNV(5)

EXTERNAL UCNV(1),UCNV(1),UCNV(1),UCNV(1),UCNV(1)

DATA A4/2400000060/

105 READ(21)SNOM

WRITE(15,106)SNOM

1 FORMAT('INPUT SAMPLING TIME =',F10.4,2X,'SECONDS')

106 WRITE(15,107)

2 FORMAT('RUNNING TIME =',F10.4,2X,'SECONDS')

207 WRITE(15,208)

3 FORMAT('ENQUE THRESHOLD = I VALUE IN (-1) HAU,/')

308 READ(15,309)

4 FORMAT(12)

401 WRITE(15,402)

5 FORMAT(15)

502 WRITE(15,503)

6 FORMAT('ENTER PRINT TIME INTERVAL IN SECONDS/')

603 READ(15,604)

7 FORMAT('ENTER PLOT TIME INTERVAL IN SECONDS/')

704 READ(15,705)

8 FORMAT('ENTER NO. OF HITS FOR PROCESSING/')

805 READ(15,806)

9 FORMAT('SET SCALE FACTORS AND COMPUTE EARTH RATE MATRIX/')

906 READ(15,907)
BEGIN C

C WHITE OUTPUT MEASUREMENTS

C

N1 ASK DATETIME

C

N2 CONTINUE

C

N3 READ(1) = GETTIME

C

N4 WRITE(1) = datetime

C

END
SEL EXTENDED FORTRAN IV (HEV = 0 / 79B EPUB)

TNK1ST

163  PNVIC(1,1)=PNVI(3)
164  PNVIC(1,3)=PNVI(2)
165  PNVIC(2,1)=PNVI(3)
166  PNVIC(2,3)=PNVI(1)
167  PNVIC(3,1)=PNVI(2)
168  PNVIC(3,2)=PNVI(1)
169  C
170  C UPDATE C + AXIIX
171  IF(KU+LU+1)=GO TO 511
172     DU 506 K=1,L
173     DU 506 K=2,L
174     DO 507 K=1,L
175     PNVK=5(K,J)=0
176     DU 507 L=1,L
177     CALL NHLS2(PNVC(J,L),PNVC(L,K),TEMPE,BIT)
178     507  PNVMS(J,K)=PNVMS(J,K)+PNVM(J,K)
179     506  CONTINUE
180     DU 506 J=1,J
181     DU 506 J=1,J
182     510  PNVK=5(K,J)=PNVC(J,K)+PNVMS(J,K)
183     511  DU 509 J=1,J
184     DU 509 J=1,J
185     CVND(J,K)=0
186     DU 510 L=1,L
187     CALL NHLS2(CVNL,J,L),PNVC(L,K),TEMPE,BIT
188     510  CVND(J,K)=CVND(J,K)+TEMPE
189     510  CONTINUE
190     DU 2H=1,H
191     DU 2H=1,H
192     26  CVNL=CVNL+CVND(J,K)
193     C
194     C CHECK FOR OUTPUT DATA
195     C
196  530  TTIME=TTIME+1
197     IF (TTIME.NE.MAX) GO TO 501
198     TTIME=0
199     DCVN=CVNL(1,1)+ECSC
200     DCVN=CVNL(L,L)+ECSC
201     DOUU(1,M)=DATA2(DCVNL1)*100,0,0,0/UP1
202     DOUU(2,M)=DATA2(DCVNL1)*100,0,0,0/UP1
203     SCVN=CVNL(3,3)+ECSC
204     IF(SCVN.GT.1.U) SCVN=1.0
205     DOUU(3,M)=SCU5(SCVN)*100,0,0,0/UP1
206     DOUU(4,M)=DATA2(UCS)+DOUU(3,M)
207     530  IF(DOVU(2,M).GE.150.0)GO TO 531
208     IF(DOVU(2,M).LE.(-150.0))GO TO 532
209     GO TO 533
210     532  DOUU(2,M)=DOUU(2,M)+100,0
211     GO TO 534
212     531  DOUU(2,M)=DOUU(2,M)+100,0
213     GO TO 534
214     C
215     C CHECK FOR PRINTER DATA
216     C
217     C
**SRC**

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<th>Line</th>
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<tr>
<td>104</td>
<td>END FILE 22</td>
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<tr>
<td>106</td>
<td>WRITE (22) C API</td>
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<td>107</td>
<td>WRITE (22) TAPI</td>
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<td>109</td>
<td>WRITE (22) UAPI</td>
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<tr>
<td>111</td>
<td>ENDFILE 22</td>
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<tr>
<td>116</td>
<td>WRITE (24, 159)</td>
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<tr>
<td>150</td>
<td>FORMAT (' Enter 1 TO REMIND INPUT TAPE, 0 TO LEAVE AS IS')</td>
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<tr>
<td>151</td>
<td>READ (24, 151) SIND</td>
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<td>READ (21, 152)</td>
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<td>ENDFILE 22</td>
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<tr>
<td>154</td>
<td>READ (22) SODHUN</td>
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<td>155</td>
<td>READ (15) SODR</td>
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<td>156</td>
<td>CONTINUE</td>
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<tr>
<td>157</td>
<td>CALL EXIT</td>
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<td>201</td>
<td>END</td>
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**Notes**

- The code snippet appears to be a FORTRAN IV program, likely for file handling or data input.
- Lines 104 to 157 are involved in file operations and data reading.
- Lines 158 to 200 are part of the main program flow, including reading from tapes and file handling.

**References**

- The source code snippet is from a FORTRAN IV program, possibly used in a scientific or engineering context, given the context of the JOHNS HOPKINS UNIVERSITY APLIED PHYSICS LABORATORY.
04124140 FORTRAN SYSTEMS REAL-TIME MONITOR-7.0
SELF EXTENDED FORTRAN IV (REV-D/78SEP08)

MUL32

SUBROUTINE MUL32(I,J,K,L)
10 I=I+1
12 J=J+1
13 IF(I LE L)
14 I=M+1
15 M=M+1
16 RETURN
END
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