USE OF A SUBJECTIVE PRIOR DISTRIBUTION FOR THE RELIABILITY OF C-ETC(U)

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1980

N00014-75-C-0733
Use of a Subjective Prior Distribution for the Reliability of Computer Software*

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In the development of large-scale computer software and in the management of the development process, it is often useful to model the reliability and the cost of development of these software packages. There have been many papers that develop models and show their usefulness as management tools. The models that use Bayesian methodology assume that a prior distribution is given.

Our paper offers a methodology of assessing a prior distribution subjectively. Two computer programs have been developed for this particular purpose: One assesses a subjective prior distribution and the other suggests a family of probability functions.

The importance of consistent prior distributions is twofold. First, these distributions reflect consistent initial predictions because they are developed by a structured process. Second, these distributions are the starting point for applying Bayes' theorem to develop the posterior distribution by modifying the prior distribution with actual data available later.

INTRODUCTION

During the past decade, several probability distributions have been used in modeling the reliability of computer software. Among the models proposed are the exponential distribution, the Rayleigh distribution, and the Poisson distribution [1-5]. Recently, Bayesian methodology has been proposed [6-9]. To apply this method, a prior distribution is necessary.

In this paper, we offer a structured approach in subjective assessment of the probability distribution. Once this has been done, the general shape of the distribution can be ascertained. Then, the search for the mathematical form is greatly simplified. For instance, the probability distribution may be skewed, not exist for negative values of the random variable. This would eliminate a class of probability models like the normal distribution and give rise to a host of others. Although it is still possible to select a model from many available basic models, the selection process is at least based upon some evidence, namely, the opinion of the experts in charge of developing the software package.

Two computer programs were written:

1. The first assesses a subjective prior distribution by eliciting answers to questions on a cathode ray tube (crt). The answers to these questions are used to plot the distribution function as well as the density function.

2. From the general shape of these functions, the second suggests a family of probability functions. For instance, an inverted gamma distribution, a beta distribution, or a lognormal distribution might be hypothesized. Some of the summary outputs of the first program become inputs for finding the parameters of the assumed distributions.

An example of a lognormal distribution is used, but other families of distributions could have been selected as well.

This paper does not deal explicitly with the derivation of the posterior distribution which is found via Bayes' theorem in conjunction with incoming data. The prior distribution, however, is an essential part of finding the posterior distribution. If the prior distribution found is integrated with test information as data become available, then obviously this is more complete information than test information alone.

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*This research was supported by the Office of Naval Research under Contract N00014-75-C-0733. Task No. 042-323, Code 434. Reproduction in whole or part is permitted for any purpose of the United States Government.

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PREVIOUS WORK IN PROBABILITY ASSESSMENT

More recently, decision theory has been considered as a general framework for logical analysis of a decision problem under uncertainty. As such, considerable attention has been given to problem formulation and methods for the assessment of a prior distribution. For example, Schlaifer's book [10], is largely devoted to the formulation and prior analysis of decision problems: posterior analysis is discussed only in the last part of the text. Howard and his associates (see, for example, [11,12]), have emphasized the application of decision theory to complex, dynamic, and uncertain decision problems. In dealing with these problems, they have explicitly included the problem formulation phase in the decision analysis cycle.

Decision theory, either concerned with specific models or general frameworks, treats uncertainty through subjective probability and treats attitude toward risk through utility theory. Regardless of whether the decision maker is concerned with prior or posterior analysis, the prior probability distribution, reflecting his quantified judgments about uncertainty, is an indispensable input to the analysis.

One difficulty associated with probability assessment is the assessor's inconsistencies which often occur in formulating a prior distribution. The question of how to discover and remove inconsistencies is of general interest to decision analysts. Another question of interest is how to fit a probability distribution using the assessed fractile in order to make the subsequent analysis more tractable. Both of these questions are addressed in this paper. The paper offers two computer programs. The first program allows a person to interact with the computer via a graphical device (cathode ray tube [crt]) during the course of establishing a subjective distribution. The second program fits a lognormal distribution to the subjective distribution.

During recent years, subjective probability has been studied by researchers in various disciplines such as psychology, mathematics, statistics, engineering, and business administration (as evidenced by the references at the end of the paper). While some of these studies are mainly theoretical or philosophical, others are experimental.

In their text [13], Pratt et al. present the method of equally likely subintervals. Subsequently, Raiffa [14] illustrates this method in detail by providing a dialogue between a decision analyst and his client. Schlaifer [10] advocates this method and offers a computer program for fitting a cumulative function through assessed fractiles.

For his experimental study, Winkler [15] developed a questionnaire using four assessment techniques:
1. cumulative distribution function—assessment of fractiles by means of equally likely subintervals or direct questions regarding fractiles.
2. hypothetical future samples.
3. equivalent prior sample information, and
4. probability density function.

He used this questionnaire to elicit prior distributions from 38 selected subjects involved in his study.

The use of penalty functions, or scoring methods, has been discussed by several researchers as means of encouraging honest assessments. Specifically, de Finetti [16] presents the quadratic scoring rule. Savage [17] derives the general class of strictly proper scoring rules by considering probabilities as special cases of rates of substitutions. Winkler discusses the use of scoring rules and other payoff schemes [18] and reports his experimental results [19].

Staël von Holstein and his associates [12,20] focus on the subject of eliciting the opinions of experts in practical situations rather than laboratory experiments. They discuss probability encoding in the context of decision analysis and propose the use of a probability wheel to facilitate the encoding process.

At the Reliability Conference in 1970, Lin and Schick [1] presented the use of an on-line computer system to assist a person in developing a prior distribution to represent his beliefs. Although the console-aided procedure is illustrated by a problem in the reliability field, this procedure is applicable to assessment of any prior distribution. Since then, considerable experience with this procedure has been gained from experiments involving students in several statistics and decision theory classes at the University of Southern California.

The present paper results from the authors' continued effort in making the probability assessment more practical by using modern electronic computers. This paper offers a newly designed computer program which has incorporated the experience gained from the use of the previous program. To simplify the assessment procedure, the new program
1. reduces the number of questions significantly (from 12 to 6).
2. is highly conversational and interactive.
3. checks for consistency as the user answers questions by question.
4. uses graphical display rather than the typewriter terminal to help the user visualize the assessment process as well as to greatly increase the speed of drawing the assessed probability curves, and
5. plots not only the cumulative function but also the
density function.

Once a subjective distribution has been deter-
moved, a second computer program will fit a lognor-
mal distribution to the subjective distribution to make
the subsequent analysis of maintainability problems
more tractable mathematically.

METHOD OF ASSESSMENT
Several methods have been suggested for estimating
prior distributions (see, for example [13.15.21]). Our
computer program makes use of the method of
equally likely subintervals, which perhaps is the most
commonly used approach. The basic idea of this
method is to ask the decision maker, at any stage, to
divide a given interval into two judgmentally equally
likely subintervals.

To begin with, the interval covering all possible
values of an uncertain quantity (usually called a ran-
dom variable) is split into two subintervals and the
decision maker is asked to choose which subinterval
to bet on. The dividing point is then changed until a
point of indifference as to betting on one or the other
subinterval is reached. When this point is reached,
the decision maker feels that it is equally likely that
the actual value of the uncertain quantity will fall above
(to the right of) or below (to the left of) this point. The
indifference point, which divides the entire interval
into two subintervals with equal probabilities, is the
median. Next, the decision maker is asked to specify
a point that will further divide the subinterval to the
left of the median into two equally likely parts. This
new point is the first quartile. Similarly, the subinter-
val to the right of the median may be further divided
into two equally likely parts. The decision maker may
proceed in this manner to divide any given interval
generated previously) into two equally likely subin-
tervals.

Suppose we let \( x_k \) designate the \( k \)th fractile of
the uncertain quantity \( x \), i.e.,

\[
P(x < x_k) = k, \quad 0 \leq k \leq 1.
\]

Then, using the method of equally likely subintervals,
the decision maker is asked to respond to a series of
questions that will lead to a determination of \( x_k \)
values for such \( k \) as 0.5, 0.25, 0.75, etc.

COMPUTER PROGRAM
The program stores a set of questions for the method
of equally likely subintervals. These questions are
displayed successively on a CRT: the user responds to
the questions by typing answers on a teletype. The
response to each of the questions is processed im-
mmediately and checked for logical consistency.

Assuming you are the user of the program, the first
question calls for the lower limit of the probability
distribution by asking you to:

Specify the largest value such that you feel virtually cer-
tain that the actual value of the uncertain quantity will
fall above this value.

The second question, on the other hand, calls for the
upper limit of the distribution by asking you to:

Specify the smallest value such that you feel virtually
certain that the actual value of the uncertain quantity will
fall below this value.

In terms of the fractile notation described earlier, the
first question asks for \( x_0 \) and the second question asks
for \( x_1 \). The program will check to see if \( x_0 \) is less than
\( x_1 \) and if you feel virtually certain that the actual value
of the uncertain quantity will lie in between \( x_0 \) and \( x_1 \).

The third question asks you to divide the interval de-
defined by the limits \( x_0 \) and \( x_1 \) into two equally likely
subintervals. The question says:

Specify the value such that it is equally likely that the
actual value of the uncertain quantity
is either above or below this value.

The answer to this question yields \( x_{0.5} \), which should
lie in between \( x_0 \) and \( x_1 \).

The fourth question, which calls for \( x_{0.25} \), is as
follows:

Suppose you were told that actual value is less than \( x_{0.5} \).
Specify the value such that it is equally likely that the
actual value of the uncertain quantity is either above or
below this value.

The program will check to see if this answer lies in
between \( x_0 \) and \( x_{0.5} \).

The fifth question, which calls for \( x_{0.75} \), is the
following:

Suppose you were told the actual value is greater than
\( x_{0.5} \). Specify the value such that it is equally likely that the
actual value of the uncertain quantity is either above or
below this value.

This answer is checked to see if it lies in between \( x_{0.5} \)
and \( x_1 \).

At this point, the program further checks for con-
sistency. Specifically, it asks:

Now, do you feel it is equally likely that the actual value
of the uncertain quantity will lie within the interval between $x_{0,15}$ and $x_{0,15}$ or outside of this interval?

If the check is not met, the program will direct you to review and revise each of your previous answers. Otherwise, the program will proceed to ask you to specify the most likely value (the mode).

The assessments thus obtained are summarized on the crt. The program then fits a smooth cumulative distribution function through the assessed fractiles. At your request, it will plot the cumulative curve and the corresponding density curve. If these graphs do not seem to reflect your judgments about the uncertain quantity, you will be guided by the program to revise your previous responses. Whenever you are satisfied with the assessed distribution, the mean and the standard deviation are computed. In addition, you may ask for 0.005, 0.015, 0.025, ..., 0.995 fractiles of the distribution.

**COMPUTER OUTPUT**

To illustrate the computerized method of probability assessment discussed above, the computer output of an example is presented. In this example, the expert (italics) is asked to quantify judgments concerning the debugging hours for a particular job. As we can see from this output, the expert violates some of the probability axioms and is asked to revise his responses several times.

**THIS PROGRAM IS DESIGNED TO ASSIST YOU IN**

(A) QUANTIFYING YOUR PROBABILITY JUDGMENTS CONCERNING AN UNCERTAIN QUANTITY. (B) CALCULATING THE MEAN AND STANDARD DEVIATION OF THE PROBABILITY DISTRIBUTION OBTAINED FROM THIS QUANTIFICATION, AND (C) FITTING THE ASSESSED DISTRIBUTION TO A THEORETICAL DISTRIBUTION.

WHAT IS THE UNCERTAIN QUANTITY OF YOUR CONCERN NOW?

**NUMBER OF DEBUGGING HOURS**

PLEASE RESPOND TO THE FOLLOWING QUESTIONS WITH YOUR CAREFUL JUDGMENTS:

(1) SPECIFY THE LARGEST VALUE SUCH THAT YOU FEEL VIRTUALLY CERTAIN THAT THE ACTUAL VALUE OF THE UNCERTAIN QUANTITY WILL FALL ABOVE THIS VALUE.

650

(2) SPECIFY THE SMALLEST VALUE SUCH THAT YOU FEEL VIRTUALLY CERTAIN THAT THE ACTUAL VALUE OF THE UNCERTAIN QUANTITY WILL FALL BELOW THIS VALUE.

100

SORRY, YOU HAVE MISINTERPRETED THESE TWO QUESTIONS. QUESTION (1) ASKS FOR THE LOWER LIMIT OF THE UNCERTAIN QUANTITY. PLEASE GIVE YOUR NEW ANSWER TO (1).

100

QUESTION (2) ASKS FOR THE UPPER LIMIT OF THE UNCERTAIN QUANTITY. PLEASE GIVE YOUR NEW ANSWER TO (2).

650

YOUR ANSWERS TO THESE TWO QUESTIONS IMPLY THAT YOU FEEL VIRTUALLY CERTAIN THAT THE ACTUAL VALUE OF THE UNCERTAIN QUANTITY WILL LIE IN BETWEEN 100 AND 650. DO YOU AGREE?

NO

THEN YOU MUST REVISE YOUR ANSWER(S). PLEASE TYPE YOUR NEW ANSWER TO (1).

50

PLEASE TYPE YOUR NEW ANSWER TO (2).

650

(3) SPECIFY THE VALUE SUCH THAT YOU FEEL IT IS EQUALLY LIKELY THAT THE ACTUAL VALUE OF THE UNCERTAIN QUANTITY WILL FALL ABOVE OR BELOW THIS VALUE.

250

(4) SUPPOSE YOU WERE TOLD THAT THE ACTUAL VALUE IS LESS THAN 250. SPECIFY THE VALUE SUCH THAT IT IS EQUALLY LIKELY THAT THE ACTUAL VALUE OF THE UNCERTAIN QUANTITY IS EITHER ABOVE OR BELOW THIS VALUE.

310

I BELIEVE YOU DIDN'T ANSWER THIS QUESTION CAREFULLY. PLEASE GIVE IT SOME THOUGHT. DO YOU NOW REALIZE WHAT YOU DID WRONG?

NO

YOUR ANSWER TO (4) SHOULD HAVE BEEN IN BETWEEN 50 AND 250. PLEASE TYPE YOUR NEW ANSWER.

130

(5) SUPPOSE YOU WERE TOLD THAT THE ACTUAL VALUE IS GREATER THAN 250. SPECIFY THE VALUE SUCH THAT IT IS EQUALLY LIKELY THAT THE ACTUAL VALUE OF THE UNCERTAIN QUANTITY IS EITHER ABOVE OR BELOW THIS VALUE.

400
(6) NOW, DO YOU FEEL IT IS EQUALLY LIKELY
 THAT THE ACTUAL VALUE OF THE UNCERTAIN QUANTITY WILL LIE WITHIN THE INTERVAL BETWEEN 130 AND 400 OR OUTSIDE OF THIS INTERVAL?

NO

SORRY. YOUR NO ANSWER INDICATES THAT YOUR ANSWERS TO QUESTIONS (3) THROUGH (5) MAY NOT REFLECT YOUR PROBABILITY JUGMENTS. SO, LET'S REVIEW EACH OF THESE ANSWERS.

(3A) ACCORDING TO YOUR ANSWER TO QUESTION (3), YOU FEEL IT IS EQUALLY LIKELY THAT THE ACTUAL VALUE OF THE UNCERTAIN QUANTITY WILL FALL ABOVE OR BELOW 250. DO YOU AGREE?

YES

(4A) WE NOW LOOK AT YOUR ANSWER TO QUESTION (4). IF THE ACTUAL VALUE OF THE UNCERTAIN QUANTITY WERE LESS THAN 250, WOULD YOU FEEL EQUALLY LIKELY THAT THE ACTUAL VALUE IS EITHER ABOVE OR BELOW 130?

NO

PLEASE TYPE YOUR NEW VALUE FOR WHICH YOU WOULD FEEL EQUALLY LIKELY THAT THE ACTUAL VALUE OF THE UNCERTAIN QUANTITY WILL FALL ABOVE OR BELOW THIS VALUE.

160

(5A) FINALLY, YOUR ANSWER TO QUESTION (5) IMPLIED THAT IF THE ACTUAL VALUE OF THE UNCERTAIN QUANTITY WERE GREATER THAN 250 YOU WOULD FEEL EQUALLY LIKELY THAT THE ACTUAL VALUE IS EITHER ABOVE OR BELOW 400. DO YOU STILL AGREE?

YES

(6A) NOW, LET'S CHECK THE CONSISTENCY OF YOUR REVISED ANSWERS. THE LOGICAL CONSISTENCY REQUIRES YOU TO FEEL EQUALLY LIKELY THAT THE ACTUAL VALUE OF THE UNCERTAIN QUANTITY WILL LIE WITHIN THE INTERVAL BETWEEN 160 AND 400 OR OUTSIDE OF THIS INTERVAL. DO YOU FEEL THAT WAY?

YES

(7) SPECIFY THE MOST LIKELY VALUE (THE MODE). [SEE TABLE 1].

200

---------

Table 1

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<th>Cum. pros.</th>
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</tr>
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<td>2</td>
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<tr>
<td>Mode</td>
<td>200</td>
<td>7</td>
</tr>
</tbody>
</table>

(8) GREAT. YOU NOW HAVE DONE YOUR ASSESSMENTS AS SUMMARIZED.

DO YOU WISH TO REVISE YOUR ANSWERS?

NO

FROM THESE ASSESSED POINTS, A SMOOTHED CUMULATIVE DISTRIBUTION WILL BE OBTAINED. WOULD YOU LIKE TO SEE THE GRAPHS FOR THE DISTRIBUTION FUNCTION AND THE DENSITY FUNCTION?

YES [SEE FIGURES 1 AND 2]

(9) WOULD YOU ACCEPT THESE GRAPHS AS REASONABLY ACCURATE EXPRESSIONS OF YOUR JUDGMENTS CONCERNING THE UNCERTAIN QUANTITY?

YES

(10) SUMMARY MEASURES OF THIS DISTRIBUTION ARE

MEAN 284.9094
STANDARD DEVIATION 155.4427

(11) DO YOU WANT TO SEE THE VARIOUS FRACILES?

YES [SEE TABLE 2]

(12) DO YOU WANT TO FIT THE ASSESSED DISTRIBUTION TO A THEORETICAL DISTRIBUTION?

NO

---------

Figure 1. Distribution and density functions for debugging hours.
program was developed that allows some 20 different input combination pairs in the procedure for determining the parameters of the lognormal distribution. The density function of the lognormal distribution is given by:

$$f(x) = \frac{1}{\beta \sqrt{2\pi}} x^{-1} \exp \left[ -\frac{1}{2} \left( \frac{\ln x - \alpha}{\beta} \right)^2 \right], \quad x > 0$$  (1)

where $\alpha$ and $\beta$ are the parameters of the lognormal distribution.

It is well known that the mean $E(x)$ and the variance $V(x)$ are given by:

$$E(x) = \mu = \exp(\alpha + \frac{1}{2}\beta^2),$$
$$V(x) = \sigma^2 = \mu^2 \left( \exp(\beta^2) - 1 \right).$$

The mode of this distribution is at

$$\text{mode} = \exp(\alpha - \beta^2),$$

whereas the median or 50th percentile $P_{50}$ is at

$$P_{50} = e^\mu.$$

By letting $y = \ln x - \alpha/\beta$ in (1) and using standard normal tables, the 90th percentile was found to be

$$P_{0.9} = \exp(1.282(\beta + \alpha)).$$

Other fractile points can be found in a similar fashion.

As we have seen the lognormal distribution has two parameters $\alpha$ and $\beta$. Thus to fit a lognormal distribution to the subjectively derived distribution we only

<table>
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Use of a Subjective Prior Distribution for the Reliability of Computer Software

have to specify two values such as \( P_{50} \) and \( P_{90} \), or the mean and the standard deviation. For the following example the mode = 200 and the median = 250 are used. The program output includes a distribution function and a density function. The latter is given in Figure 3.

LOG NORMAL DISTRIBUTION

DO YOU NEED THE COMBINATION PRINTOUT?
YES=1. NO=0.70

WHAT IS THE INPUT COMBINATION NUMBER ?13
MEDIAN = 2250
MODE = 200

ALPHA BETA MEDIAN MEAN
5.5215 0.4724 250.0000 279.5085

90TH
STD DEV MODE PCTLE TIME
139.7542 200.0000 458.6842

DO YOU WISH TO INTEGRATE-NO=0. YES=1.
RETURN = 2.70

DO YOU WISH TO PRINT X AND Y-NO=0. YES=1.
WHAT IS XMIN. XMAX. DELX
5100.650...20

X-VALUES Y-VALUES X-VALUES Y-VALUES
100 1.28704E-03 400 1.28704E-03
120 2.10476E-03 420 1.10046E-03
140 2.84011E-03 440 9.37937E-04
160 3.37145E-03 460 7.90084E-04
180 3.84005E-03 480 6.78084E-04
200 3.77808E-03 500 5.75580E-04
220 3.70775E-03 520 4.88276E-04
240 3.50578E-03 540 4.16090E-04
260 3.23764E-03 560 3.51159E-04
280 2.93064E-03 580 2.97842E-04
300 2.61306E-03 600 2.32707E-04
320 2.30232E-03 620 1.84516E-04
340 1.74150E-03 640 1.82094E-04
360
380 1.50045E-03

DO YOU WISH TO PLOT X AND Y-NO=0. YES=1 ? [See Figure 3.]

Now the distribution function or density function can be visually compared with the subjectively derived prior distribution using the questionnaire involving the debugging hours. If "reasonable" agreement has been achieved, the mathematical form of the density has been found. Several combinations of input values might have to be examined in order to achieve the "best" fit. This form is important in order to establish the posterior distribution using incoming data and the likelihood function according to Bayes' theorem. On the other hand, if "reasonable" agreement between the two distribution functions has not been achieved, a new family of distributions may be tried and/or the empirical distribution might be ques-

REFERENCES

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(continued next page)
Our paper offers a methodology of assessing a prior distribution subjectively. Two computer programs have been developed for this particular purpose: One assesses a subjective prior distribution and the other suggests a family of probability functions.

The importance of consistent prior distributions is twofold. First, these distributions reflect consistent initial predictions because they are developed by a structured process. Second, these distributions are the starting point for applying Bayes' theorem to develop the posterior distribution by modifying the prior distribution with actual data available later.