BOUNDARY INTEGRAL EQUATIONS FOR THE HELMHOLTZ EQUATION: THE THI—ETC(U)
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The third boundary value problem, or Robin problem, for the Helmholtz equation in an exterior domain in $\mathbb{R}^3$, may be cast with Green's theorem, as a boundary integral equation. This equation is not equivalent to the original problem because it has eigenvalues at wave numbers for which the exterior problem is solvable. However, a second boundary equation is derived and it is shown that the pair is equivalent to the original boundary value problem. The boundary condition is of the form $u + \nu \cdot \nabla u = g$ where both $o$ and $g$ are in $L^{2}$ on the boundary and $n$.
it is shown that the coupled boundary integral equations as well as the original boundary value problem have a unique solution for all real values of wave number. The second boundary integral equation involves the normal derivative of the double layer and in order to show that this operator is always well defined on the class of functions of interest, a trace theorem is proved for single and double layer potentials with $L^2$ densities.
Boundary Integral Equations for the Helmholtz Equation; The Third Boundary Value Problem*

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The present work extends boundary integral equation techniques, developed in an earlier paper for the Dirichlet and Neumann problems [6], to the boundary value problem of the third kind (also referred to as the Robin problem) associated with the Helmholtz equation in exterior domains. While the existence and uniqueness of solutions of the exterior Robin problem have been established previously (see e.g. Leis [9]; Danilova [2]), the application of the Helmholtz representation and the subsequent derivation of boundary integral equations does not appear to have been studied until now.

The considerations below will show that the familiar difficulties at interior eigenvalues arise in the case of the Robin boundary conditions. The non-uniqueness of solutions of the boundary integral equation at interior eigenvalues is, of course, an artifact of the method chosen to solve the original boundary value problem. As explained in [6], supplementary conditions of some form are needed. Either the fundamental solution may be modified, as was done by Danilova [2] in her layer theoretic treatment of the third boundary value problem with an oblique derivative, or an additional equation is needed if the fundamental solution is unaltered. In the present work we employ the Green's theorem approach to boundary integral equations with an unmodified fundamental solution and show that, unlike the situation for the exterior Dirichlet and Neumann problems, the supplementary condition appears as an operator equation of the second kind rather than one of the first kind.
Rather than repeat the bibliographic details reviewed by Kleinman and Roach [6], we refer the reader to the introduction of that paper. Moreover, we will adopt the notational conventions of that paper. Section 2, below, briefly reviews those conventions as they are appropriate to the present considerations. However, the boundary value problem discussed here, in contrast to those in [6] and, in the papers of Leis [9] and Danilova [2], is generalized so that the data and the multiplier which appears in the boundary condition may be specified in $L_\infty(\partial D)$ rather than $C_\infty(\partial D)$ where $\partial D$ is the boundary of the exterior domain.

This more general setting requires some extensions of theorems which appear in the literature, in particular extensions of a regularity theorem appearing in Mikhlin [10] and of a uniqueness theorem due to Leis [9]. More significantly, it requires a careful formulation of a trace theorem suitable for functions described by single and double layer potentials. Standard trace theorems, e.g. Miranda [11] were not sufficient to ensure existence of the normal derivatives of double layer potentials required for application in the present setting. These matters are discussed carefully in section 3 and in the appendix which contains some computational details collected there so as not to interrupt the flow of the main argument.

Having discussed these background matters, the main results concerning the boundary integral equation formulation of the Robin problem are presented in section 4 and the equivalence of this formulation with the original boundary value problem is proven. Section 4 concludes with a result pertaining to the scattering problem for imperfectly reflecting surfaces.
2. Formulation of boundary integral equations.

Let $D_-$ denote a bounded domain in $\mathbb{R}^3$ with boundary $\partial D$ which is a closed Lyapunov surface with index 1. We denote the exterior of $\partial D$ (i.e. the complement of $D_-\cup\partial D$ in $\mathbb{R}^3$) by $D_+$. Let $R=R(P,Q)$ denote the distance between two typical points $P$ and $Q$ in $\mathbb{R}^3$. A fundamental solution of the Helmholtz equation $(\Delta+k^2)u=0$, $\text{Im}k>0$, is a two-point function of position, $\gamma(P,Q)$, which for convenience we write in the form

$$\gamma(P,Q) := -e^{ikR/2\pi R}.$$  

Complex conjugates will be denoted by a bar, thus

$$\overline{\gamma}(P,Q) := -e^{-ikR/2\pi R}.$$  

We denote by $\partial/\partial n_P$ differentiation in the direction of the unit vector $\hat{n}_P$ normal to $\partial D$ at the point $P \in \partial D$. A point $P \in \partial D_+$ is assumed to have spherical polar coordinates $(r_P, \theta_P, \phi_P)$ relative to a Cartesian coordinate system erected with origin in $D_-$. We emphasize that throughout we shall assume that $\hat{n}_P$ is the outward drawn normal with respect to $\partial D$, that is $\hat{n}_P$ is directed from $\partial D$ into $D_+$. Further, we shall write $\partial/\partial n_P^-$ and $\partial/\partial n_P^+$ to denote the normal derivative when $P \in \partial D_-$ from $D_-$ and $D_+$ respectively.

Let $(S_u)(P)$ and $(D_u)(P)$ denote the effect at the point $P \in \mathbb{R}^3$ of single layer and double layer distributions respectively on $\partial D$ of density $u$ as follows:
\begin{equation}
(2.3) \quad (Su)(P) := \int_{\partial D} u(q) \gamma(P,q) dS_q, \quad P \in \mathbb{R}^3
\end{equation}

and

\begin{equation}
(2.4) \quad (Du)(P) := \int_{\partial D} u(q) \frac{\partial \gamma}{\partial n} (P,q) dS_q, \quad P \in \mathbb{R}^3.
\end{equation}

Further, for \(p \in \partial D\) we define

\begin{equation}
(2.5) \quad (Ku)(p) := \int_{\partial D} u(q) \frac{\partial \gamma}{\partial n} (p,q) dS_q.
\end{equation}

Notice that \(K\) is a completely continuous operator on \(L_2(\partial D)\) (Mikhlin [10]). Jump conditions for the single and double layer potentials on \(\partial D\) can be expressed in terms of \(K\) and its \(L_2(\partial D)\) adjoint \(K^*\) which is given by

\begin{equation}
(2.6) \quad (K^*u)(p) = \int_{\partial D} u(q) \frac{\partial \gamma}{\partial n} (p,q) dS_q.
\end{equation}

These jump conditions as well as continuity properties of the single and double layers depend on the smoothness of the density \(u\) and will be discussed in Section 3.

Finally, we remark that the single and double layer operators with \(P \in D_\) can be considered as operators with range in \(L_2(D_\) and, as such, are continuous operators from \(L_2(\partial D)\) to \(L_2(D_\) (see e.g. Miranda [11]).
With this notation, representations of solutions of the Helmholtz equation obtained by applying Green's Theorem or the Helmholtz representation have the following forms. For radiating wave functions $\phi_+$ satisfying

$$ \left( \Delta + k^2 \right) \phi_+(P) = 0, \quad P \in D_+, $$

\[ (2.7) \]

$$ \lim_{r \to \infty} \left\{ r P \left[ \frac{\partial \phi_+(P)}{\partial r} - i k \phi_+(P) \right] \right\} = 0, $$

we obtain

\[ (2.8) \]

$$ \int_{\partial D} \left\{ \frac{\partial \phi_+}{\partial n_q} (q) \gamma(P,q) - \phi_+ (q) \frac{\partial \gamma}{\partial n_q} (P,q) \right\} \, dS_q = \begin{cases} \frac{2 \phi_+(P)}{P}, & P \in D_+, \\ \phi_+(P), & P \in \partial D, \\ 0, & P \in D_- . \end{cases} $$

whilst for any solution $\phi_-$ of the Helmholtz equation in $D_-$, we have

$$ \int_{\partial D} \left\{ \phi_-(q) \frac{\partial \gamma}{\partial n} (P,q) - \frac{\partial \phi_-}{\partial n_q} (q) \gamma(P,q) \right\} \, dS_q $$

\[ (2.9) \]

$$ = (D\phi_-)(P) - \frac{\partial \phi_-}{\partial n}(P) = \begin{cases} 0, & P \in D_+, \\ \phi_-(P), & P \in \partial D, \\ 2 \phi_-(P), & P \in D_- . \end{cases} $$
We will be concerned with the following boundary value problem:

**Exterior Robin Problem:**

\[
(\nabla^2 + k^2) u_+(p) = 0, \quad \text{Pe} \Omega_+ \tag{2.10}
\]

\[
\frac{\partial u_+}{\partial n} + \sigma(p) u_+(p) = g_+(p), \quad \text{Pe} \partial \Omega_+ \tag{2.11}
\]

\[
\lim_{r \to \infty} \frac{1}{r} \left( \frac{\partial^2 u_+}{\partial r^2} (p) - ik u_+(p) \right) = 0 \tag{2.12}
\]

where \( g_+ \in L^\infty(\partial \Omega) \) and \( \sigma \in L^\infty(\partial \Omega) \). In the exterior scattering problem, \( q_+ \) is replaced by \( -\frac{\partial u_i}{\partial n} - \sigma u_i \) where \( u_i \) is the known incident field.

Proceeding formally to obtain boundary integral equations whose solutions will subsequently be used to construct solutions to the boundary value problem we apply the Robin condition to the representation formula (2.8) which we write as

\[
\begin{cases}
\frac{\partial^2 u_+}{\partial r^2} (p) - (D u_+)(p) = 2 u_+(p), & \text{Pe} \Omega_+ \\
\frac{\partial u_+}{\partial n} (p) = u_+(p), & \text{Pe} \partial \Omega_+ \\
0, & \text{Pe} \Omega_-
\end{cases} \tag{2.13}
\]

to obtain

\[
(I + \hat{F}^* + S \sigma) u_+(p) = (S g_+)(p), \quad \text{Pe} \partial \Omega \tag{2.14}
\]

where we have also used the fact that on \( \partial \Omega \)

\[
D u_+ = \hat{F}^* u_+.
\]
If this boundary integral equation has a solution \( u_+ \in L_2(\partial D) \), then equation (2.13) may be employed to define \( u_+ \) in the exterior domain \( D_+ \) as

\[
(2.15) \quad u_+(P) = \frac{1}{2} S(g_+ - \sigma u_+)(P) - \frac{1}{2} (Du_+)(P), \quad P \in D_+.
\]

This function is clearly a solution of the Helmholtz equation in \( D_+ \). To insure that this solution in fact satisfies the boundary conditions, we require that the function \( u_+ \) satisfy an additional relation which we derive from the Helmholtz representation by computing normal derivatives as follows:

\[
(2.16) \quad \lim_{P \to P} \left[ \frac{\partial}{\partial n_p} \left( S(\frac{\partial u_+}{\partial n_q}) - \frac{\partial}{\partial n_p} D(u_+) \right)(P) \right] = 0
\]

or, using the relation (3.16) below,

\[
(2.17) \quad -\frac{\partial u_+}{\partial n_p}(p) + K \left( \frac{\partial u_+}{\partial n_q}(p) \right) - \lim_{P \to P} \frac{\partial}{\partial n_p} D(u_+)(P) = 0.
\]

By utilizing the boundary condition we have

\[
(2.18) \quad (-g_+ + \sigma u_+ + K(g_+ - \sigma u_+) - D_n u_+)(p) = 0, \quad p \in \partial D,
\]

or, rearranging terms,

\[
(2.19) \quad (-\sigma + K \sigma + D_n) u_+ = (K-I)g_+.
\]
where

\[(2.20) \quad (D_n u_+)(p) = \lim_{p \to p} \frac{1}{\partial p} (Du_+)(p).\]

Care must be exercised to ensure existence of this limit. We deal with this question (which depends on both the smoothness of the density and the degree of regularity of \(\partial D\)) in Lemma 4.1.

As with the integral equations derived in \([6]\), we will see that the representation (2.15) affords a solution of the exterior boundary value problem (the Robin problem in the present case) provided \(u_+ \in L^2(\partial D)\) satisfies the pair of boundary integral equations. In fact, as we shall see below, solutions of the system (2.14) and (2.19) will necessarily be more regular than merely square-integrable.

Moreover we will see that, as in \([6]\), this pair of boundary integral equations is redundant unless the homogeneous adjoint of equation (2.14) has non-trivial solutions. In such a case, however, the pair of integral equations will have a unique solution; it is this result we wish to establish in the following pages. Note, however, that unlike the cases discussed in \([6]\), the second of these equations is an operator equation of the second kind.

The spectrum of \(K\) is clearly of vital importance in discussing the solutions of these equations and, since \(K\) may be considered as an operator-valued function of the parameter \(k\), it is convenient to treat separately the operators \((-K)\) and \((+K)\) rather than introduce another parameter, \(\lambda\), in order to obtain the standard form \((I-\lambda K)\). To this end, we adopt the following definition. Those
real values of \( k \) for which the equation \( w - Kw = 0 \) has nontrivial solutions, \( \tilde{w} \), will be called characteristic values of \( K \).

Similarly, those real values of \( k \) for which \( \tilde{w} + Kw = 0 \) has nontrivial solutions, \( \hat{w} \), will be called characteristic values of \((-K)\). We note that if \( k \) is a characteristic value of \( K \), then it is also a characteristic value of \( K^* \) ([5] Chap. XIII, §1.3), and trivially it is also a characteristic value of \( \tilde{K} \) and \( \hat{K^*} \). Similarly, \((-K), (-K^*), (-\tilde{K}), (-\hat{K^*})\) have the same characteristic values, which may differ from the preceding. Further, if \( \hat{w} \) is a nontrivial solution of \( \hat{w} + Kw = 0 \), then \( \tilde{w} \) is a nontrivial solution of \( \tilde{w} + \tilde{K}w = 0 \), and \( \tilde{w} + \tilde{K}w = 0 \) implies \( \hat{w} + \hat{K}w = 0 \). Hereafter, an elevated index zero, for example \( \tilde{w} \), will indicate that the function concerned is a solution of a homogeneous equation.

The values of \( k \) for which there exist nontrivial solutions of the homogeneous Helmholtz equation in \( D \) with vanishing normal derivative on \( \partial D \) will be called eigenvalues of the interior Neumann problem. Similarly, eigenvalues of the exterior Neumann and exterior and interior Dirichlet problems are those values of \( k \) for which there exist nontrivial solutions of the corresponding boundary value problems.

It is well known ([14] vol. IV, §229) that there are no eigenvalues of the exterior problems, but there are eigenvalues of the interior problems, and furthermore, these eigenvalues are real (Stakgold ([15] vol. II, p. 137)). The relationship between characteristic values of \( K \) and eigenvalues of the interior problems is clarified in the following theorem, which appears in this form in [5, p. 222].
Theorem 2.1:

(A) \( k \) is a characteristic value of \( K \) if and only if \( k \) is an eigenvalue of the interior Neumann problem.

(B) \( k \) is a characteristic value of \((-K)\) if and only if \( k \) is an eigenvalue of the interior Dirichlet problem.

The interested reader is referred to [6] for the proof of this result. We remark only that an immediate consequence is that if \( k \) is an eigenvalue of the interior Dirichlet problem, then \( k \) is a characteristic value of \((-K)\) and hence \((-K*)\).

To conclude this section, we wish to point out that the analysis presented here is applicable to the scattering problem for imperfectly reflecting surfaces. Under appropriate hypotheses, among which is one requiring the radius of curvature to be large relative to the penetration depth, the exact transmission conditions may be approximated by a boundary condition of the third kind with purely imaginary \( \sigma \) which may interpreted as an acoustic impedance [13]. The analogous problem in electromagnetics is discussed by Senior [12]. It is possible, then, to reduce the scattering problem to an equivalent pair of boundary integral equations. These equations differ from equations (2.14) and (2.19) only in their right hand members and, as we shall see, for sufficiently smooth incident fields the analysis of Section 4 will be valid for the scattering problem.

Specifically, we may proceed as follows. Denoting the incident and scattered fields, respectively by \( u^i \) and \( u^s \) and recognizing that the incident field must be a solution of the Helmholtz equation on the interior domain, we may use the Helmholtz representations (2.8) and (2.9) to show that, on the boundary 3D
(2.21) \[ u^s = S(\partial u^s / \partial n) - \bar{\bar{K}} u^s \]

(2.22) \[ u^i = \bar{\bar{K}} u^i - S(\partial u^i / \partial n). \]

Consequently,

\[ u = u^s + u^i = 2u^i + S(\partial u / \partial n) - \bar{\bar{K}} u. \]

Invoking the boundary condition \( \partial u / \partial n + \sigma u = 0 \), this last relation may be rewritten as

(2.23) \[ (I + S \sigma + \bar{\bar{K}}) u = 2u^i \]

where, we emphasize, \( u^i \) is the known incident field.

Likewise, since

(2.24) \[ u^s(P) = (1/2)[S(\partial u^s / \partial n) - Du^s](P), \text{ } P \in D_+; \]

and

(2.25) \[ u^i(P) = (1/2)[Du^i - S(\partial u^i / \partial n)](P), \text{ } P \in D_-; \]

we have, taking normal derivatives and taking account of the jump conditions, that for \( P \in \partial D \),

\[ \partial u^s / \partial n = (1/2)[\partial u^s / \partial n + K \partial u^s / \partial n] - (1/2) D_n u^s, \]

and

\[ \partial u^i / \partial n = (1/2)[\partial u^i / \partial n - K \partial u^i / \partial n] + (1/2) D_n u^i. \]
It is easy to check, from these last two relations, that since \( \partial u/\partial n_p = -\sigma u \) on \( \partial D \), the total field \( u \) must satisfy the equation

\[
(2.26) \quad (-\sigma + K \sigma + D_n) u = 2 (\partial u^i/\partial n_p).
\]

Equations (2.23) and (2.26) are thus the particular forms of (2.14) and (2.19) applicable to the scattering problem.

3. Auxillary Results

In this section we collect some known results and their extensions as well as some new results which will be used in the subsequent discussion. For more details, the interested reader should consult the cited references; we have attempted to select not original references to these particular theorems but rather those which are readily accessible.

We begin with the result alluded to above which we will refer to, in the sequel, as the regularity theorem. The particular version of the results presented here is, on the one hand, a specialization to weakly singular surface integrals of a theorem of Mikhlin [10] for closed, bounded sets in \( \mathbb{R}^n \) which, on the other hand, is a minor generalization of Mikhlin's theorem because of the occurrence of the factor \( \mu \) which lies in \( L_\infty(D) \).

**Theorem 3.1:**

Let \( D \) be a Lyapunov surface (or a finite number of disjoint Lyapunov surfaces). If \( A: \partial D \times \partial D \to \mathbb{C} \) and \( f: \partial D \to \mathbb{C} \) are continuous, \( u \in L_\infty(\partial D) \) and \( \alpha < 2 \), then any solution of
which belongs to $L_2(\Omega)$ is also continuous on $\Omega$.

**Proof:** For any $\varepsilon > 0$, choose $\eta(t)$ to be a continuous, real-valued, non-increasing function of $t > 0$, with the property that $\eta(t) = 1$, $0 < t < \varepsilon/2$, and $\eta(t) = 0$, $t \geq \varepsilon$. Then, setting

$$(3.2) \quad K_1(p, p_0) = \frac{A(p, p_0) \eta(R(p, p_0))}{R^3(p, p_0)}$$

and

$$(3.2) \quad K_2(p, p_0) = \frac{A(p, p_0)[1 - \eta(R(p, p_0))]}{R^3(p, p_0)}$$

where $K_2(p, p_0)$ is continuous on $\Omega \times \Omega$, the integral equation (3.1) may be written as

$$(3.4) \quad u(p_0) - (K_1 u)(p_0) = F(p_0)$$

where

$$(3.5) \quad (K_1 u)(p_0) = \int_{\Omega} K_1(p, p_0) u(p) dS_p.$$ 

and

$$(3.6) \quad F(p_0) = f(p_0) + \int_{\Omega} K_2(p, p_0) u(p) u(p) dS_p.$$
The function \( F \) is continuous since \( f \) is continuous and

\[
(3.7) \quad \left| \int_{\mathbb{R}^3} \left[ K_2(p,p_0) - K_2(p,p_1) \right] \cdot (p) u(p) \, ds_p \right| \leq \left\{ \int_{\mathbb{R}^3} \left| K_2(p,p_0) - K_2(p,p_1) \right|^2 \, ds_p \right\}^{1/2} \left\| u u \right\|_{L^2(\partial \Omega)}
\]

where \( K_2 \) is continuous in both variables. Note that \( uu \in L^2(\partial \Omega) \) since \( u \in L^2(\partial \Omega) \) and \( u \in L^\infty(\partial \Omega) \).

Now observe that, with the Schwartz inequality and the fact that \( n \leq 1 \), and writing \( R \) for \( R(p,p_0) \),

\[
(3.8) \quad \left\| K_1 uu \right\|_{L^2(\partial \Omega)}^2 = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} A(p,p_0) \eta(R) u(p) \frac{u(p)}{R^{n/2}} \, ds_p \, ds_{p_0} \leq \int_{\mathbb{R}^3} \left\{ \int_{\mathbb{R}^3} \frac{A(p,p_0)}{R^{n/2}} \eta^2(R) \left| u(p) \right|^2 \, ds_p \right\} \int_{\mathbb{R}^3} \frac{\left| u(p) \right|^2 \, ds_p}{R^{n/2}} \, ds_{p_0} \leq c^2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{ds_p}{R^{n/2}} \int_{\mathbb{R}^3} \frac{\left| u(p) \right|^2 \, ds_p}{R^{n/2}} \, ds_{p_0}
\]

where \( c^2 = ||A||^2_{C_0(\partial \Omega \times \partial \Omega)} \cdot ||u||^2_{L^\infty(\partial \Omega)} \) and \( B_\varepsilon(p_0) \) is a ball of radius \( \varepsilon \) centered at \( p_0 \), outside of which \( n(R(p,p_0)) = 0 \).
If $d$ is the radius of the Lyapunov sphere associated with $\partial D$ and $\varepsilon < d$, we have the following estimates:

\begin{equation}
(3.9) \int_{\partial D \cap B_{\varepsilon}(p_0)} \frac{dS_p}{\rho^a} \leq c' \varepsilon \int_0^{\varepsilon} \rho^{1-a} d\rho = \frac{c' \varepsilon^{2-a}}{2-a}
\end{equation}

for some $c'$ independent of $p_0$ and $\varepsilon$ and

\begin{equation}
(3.10) \int_{\partial D} \frac{dS_p}{\rho^a} = \int_{\partial D \cap B_{\varepsilon}(p_0)} \frac{dS_p}{\rho^a} + \int_{\partial D \setminus \partial D \cap B_{\varepsilon}(p_0)} \frac{dS_p}{\rho^a} \leq \frac{c' d^{2-a}}{2-a} + \frac{S}{d^a}
\end{equation}

where $S$ is the surface area of $\partial D$.

Employing these estimates in (3.8) we find that

\begin{equation}
(3.11) ||K_1 u||^2_{L_2(\partial D)} \leq c'^2 c' \varepsilon^{2-a} \frac{\varepsilon^{2-a} - \frac{c' d^{2-a}}{2-a} + \frac{S}{d^a}}{2-a} ||u||_{L_2(\partial D)}.
\end{equation}

Thus for $\varepsilon$ sufficiently small

\begin{equation}
(3.12) ||K_1 u||_{L_2(\partial D)} < 1
\end{equation}

and the solution of (3.4) in $L_2(\partial D)$ is given by

\begin{equation}
(3.13) u(p_0) = \sum_{n=0}^{\infty} (K_1 u)^n_F(p_0).
\end{equation}
In fact, this series is uniformly convergent hence \( u(p_0) \) is continuous. This follows because \( F \) is continuous, hence bounded on \( \partial \Omega \), and, using arguments similar to the preceding (in particular using (3.9)) we may establish

\[
| (K_1 u F)(p_0) | = | \int_{\partial \Omega} \frac{A(p, p_0)}{R^\alpha} \cdot n(R) \mu(p) \cdot F(p) \ dS_p | \\
\leq c | F |_{C_0(\partial \Omega)} \int_{\partial \Omega \cap B_\epsilon(p_0)} \frac{dS_p}{R^\alpha} \leq \frac{cc'}{2-\alpha} \epsilon^{2-\alpha} | F |_{C(\partial \Omega)}.
\]

Thus we have the estimate

\[
| | (K_1 u)^n F | |_{C_0(\partial \Omega)} \leq \frac{cc'}{2-\alpha} \epsilon^{2-\alpha} | | (K_1 u)^{n-1} F | |_{C(\partial \Omega)}
\]

\[
\leq \left( \frac{cc'}{2-\alpha} \epsilon^{2-\alpha} \right)^n | F |_{C(\partial \Omega)}
\]

which, for \( \epsilon \) sufficiently small, establishes the uniform convergence of (3.13) and hence the continuity of \( u \).

The next three results may be found in the book of Günter [4] and relate to the behavior of single and double layer potentials. The results on potentials with integrable densities, due originally to Fichera [3], are proven in [4] for the case \( k=0 \) but remain valid for \( k \neq 0 \) since the singular part of the kernels concerned remains unaltered. The first of these results establishes continuity properties of single and double layer potentials.
Theorem 3.2:

If $\partial D$ is a Lyapunov surface (or a finite number of disjoint Lyapunov surfaces) and if $u \in \mathcal{L}_m(\partial D)$ then the single layer potential with density $u$ is Hölder continuous in the entire space, while the double layer potential with the same density is Hölder continuous on $\partial D$. Moreover, both are infinitely differentiable off the boundary.

Remarks: The statement of this theorem may be summarized by the assertion: $u \in \mathcal{L}_m(\partial D)$ implies (a) $Su \in C^0(\mathbb{R}^3 \setminus \partial D)$ and (b) $\tilde{K}u \in C^0(\partial D)$, $Du \in C^0(\mathbb{R}^3 \setminus \partial D)$. The statement here differs from that appearing in Günter in that the assumption that $u \in \mathcal{L}_m(\partial D)$ replaces the assumption that $u$ is bounded on $\partial D$. The estimates necessary for the proof are identical in the present case to those appearing in Günter.

Turning our attention to the jump relation, we may state the following result. Recall that if $u$ is integrable, then almost every point of its domain is a Lebesgue point.

Theorem 3.3:

If $\partial D$ is a surface as described above, $u \in \mathcal{L}_1(\partial D)$ and if $p \in \partial D$ is a Lebesgue point of $u$, then

$$\lim_{P \to p^-} \frac{\partial}{\partial n_P} (Su)(P) = (-u + Ku)(p),$$

(3.16)

$$\lim_{P \to p^+} \frac{\partial}{\partial n_P} (Su)(P) = (u + Ku)(p),$$

(3.17)
and, for the double layer,

\[(3.18) \lim_{P \to P^+} (Du)(P) = (u + \bar{K} \ast u)(p),\]

and

\[(3.19) \lim_{P \to P^-} (Du)(P) = (-u + \bar{K} \ast u)(p).\]

Moreover, the limits in each of the four cases is integrable.

We complete our discussion of the properties of single and double layer potentials with the following:

**Theorem 3.4:**

Let \( \mathcal{D} \) be a Lyapunov surface (or a finite number of disjoint Lyapunov surfaces) with index 1 (i.e., \( \mathbf{n}_1 \cdot \mathbf{n}_2 < E \rho(p_1, p_2) \)) where \( \mathbf{n}_i \), \( i=1,2 \), is the exterior unit normal to \( \mathcal{D} \) at the point \( p_i \) and \( E \) is a constant independent of \( p_1, p_2 \) and let \( u \) be continuous on \( \mathcal{D} \). If the double layer potential (2.4) possesses one of the derivatives

\[(3.20) \frac{3}{3n^+_p} Du(p) = D_n^+ u(p) \quad \text{or} \quad \frac{3}{3n^-_p} Du(p) = D_n^- u(p)\]

then both derivatives exist and are equal at the point \( p \in \mathcal{D} \). Their common value will be denoted by \( D_n^u(p) \).
Next, we present a version of the divergence theorem which is essential to what follows. This treatment allows the development of an $L_2$ theory of surface potentials which is of interest for its own sake as well as the present application. It is closely allied to the treatment of the volume integral as an improper integral presented in Smirnov [14]. We remark that an alternative generalized divergence theorem involving potentials with continuous rather than square integrable densities and based on the interpretation of the normal derivative as a distribution or "boundary flow" is available [1], [8] and has been used in an integral equation approach to the Neumann problem for the Helmholtz equation [7]. That treatment does not address the non-uniqueness problem at characteristic values of the integral operator in contrast to the present approach.

**Theorem 3.5:**

Let $\partial D$ be a Lyapunov surface of index 1 and let \( \{ S_\delta \} \) be a family of "parallel" surfaces in $D_-$ such that (i) for fixed $\delta$, $S_\delta$ is the image of $\partial D$ under a continuously differentiable transformation $F_\delta^{-1} : \partial D \to S_\delta$; (ii) $F_\delta^{-1}(p) := P_\delta = p + O(\delta)$ and if $p(u,v)$ is a parameterization of $\partial D$, $\frac{\partial P_\delta}{\partial u} = \frac{\partial P}{\partial u} + O(\delta)$ and $\frac{\partial P}{\partial v} = O(\delta)$; (iii) $\int_{S_\delta} dS_P = \int_{S} dS_P + O(\delta)$ for every $S \subset \partial D$ where $\Sigma_\delta = F_\delta^{-1}(\Sigma)$.

Now, if $\Phi \in C_1(\partial D)$ is a vector-valued function with limiting values in $L_2(\partial D)$ by which is meant

\[
(3.21) \quad \left| \Phi(F_\delta^{-1}(p)) - \Phi(p) \right|^2 dS_P = O(\delta^0)
\]

then

\[
(3.22) \quad \int_{\partial D} \hat{n}_P \cdot \Phi(p) dS_P = \int_{D_-} \nabla \Phi(p) d\tau_P.
\]
Remark: The order symbol $O(\delta)$ denotes a vector whose magnitude decreases uniformly with respect to $\delta$. While it is not necessary to require that $|P_\delta - p|$ be independent of $p$ for fixed $\delta$, we do require that the surface $S_\delta$ not coincide with $\partial D$ at any point, i.e.,

$$\inf_{p,q \in \partial D} |P_\delta(p) - q| \geq \theta \delta$$

for some $\theta > 0$, independent of $\delta$.

Proof: Since $\phi \in C^1_\partial(D_\delta)$ and $S_\delta = \partial D_\delta$, the divergence theorem in its classical form yields

$$\int_{S_\delta} \hat{n}_P \cdot \phi(P_\delta) \, dS_P = \int_{D_\delta} \nabla \phi(P) \, d\tau.$$  \hspace{1cm} (3.24)

where $D_\delta$ denotes the interior of $S_\delta$. Consider, first, the integral on the left hand side. The surface $\partial D$ may be decomposed into a finite number of patches, $\partial D = \bigcup_{i=1}^{N} \Sigma_i$ where each $\Sigma_i$, $i=1,\ldots,n$, lies in a Lyapunov sphere. Let $\Sigma_{i\delta} = F_\delta^{-1}(\Sigma_i)$. Then

$$\int_{S_\delta} \hat{n}_P \cdot \phi(P_\delta) \, dS_P = \sum_{i=1}^{N} \int_{\Sigma_{i\delta}} \hat{n}_P \cdot \phi(P_\delta) \, dS_P.$$  \hspace{1cm} (3.25)

Points on $\Sigma_i$ may be represented in local coordinates $(\xi, \eta, \zeta(\xi, \eta))$ where $\Sigma_i$ is the image of some domain $D_{i\xi, \eta}$ in the $(\xi, \eta)$-plane. Similarly, $(\xi, \eta)$ may be chosen to parameterize $\Sigma_{i\delta}$. An element of area on $\Sigma_i$ is $dS_P = \sqrt{1 + \zeta^2 + \eta^2} \, d\xi d\eta$ whereas an element of area on $\Sigma_{i\delta}$ is, using condition (ii), $dS_{P\delta} = [\sqrt{1 + \zeta^2 + \eta^2} + O(\delta)] d\xi d\eta$ or $dS_P = dS_{P\delta} + O(\delta)$. 
Similarly, on $\Sigma_i$, $\hat{n}_p = \frac{(-\xi, -\eta, 1)}{\sqrt{1 + \xi^2 + \eta^2}}$, while on $\Sigma_i \delta$, $\hat{n}_p = \hat{n}_p + o(\delta)$. Hence, with $\Phi(P)$ and $\Phi(F^{-\delta}_\delta(P)) \in L_2(\partial D)$, we have

$$
(3.26) \quad \int_{\Sigma_i \delta} \hat{n}_p \cdot \Phi(F^{-\delta}_\delta(P)) dS_p = \int_{\Sigma_i} \hat{n}_p \cdot \Phi(F^{-\delta}_\delta(P)) dS_p + o(\delta).
$$

Also, in light of formula (3.21), we have the estimate

$$
(3.27) \quad \left| \int_{\Sigma_i} \hat{n}_p \cdot \Phi(F^{-\delta}_\delta(P)) dS_p - \int_{\Sigma_i} \hat{n}_p \cdot \Phi(P) dS_p \right| < \delta.
$$

Therefore

$$
(3.28) \quad \lim_{\delta \to 0} \int_{\Sigma_i \delta} \hat{n}_p \cdot \Phi(F^{-\delta}_\delta(P)) dS_p = \int_{\Sigma_i} \hat{n}_p \cdot \Phi(P) dS_p.
$$

Thus the limit on the left hand side of (3.24) exists as $\delta \to 0$, is equal to the left hand member of the relation (3.22) and hence the limit of the right hand side in relation (3.24) exists and equals $\int_{\partial D} \hat{n}_p \cdot \Phi(P) dS_p$. Thus the integral $\int_{\partial D} \nabla \cdot \Phi(P) dS_p$ has a meaning as an improper integral and, in this sense, we have established the equality (3.22).
Remark: The parallel surfaces used here are similar to, but more restrictive than those discussed by Burago, Maz'ja and Sapozhnikova in [1]. The differentiability of the transformation ensures the convergence of normals and surface elements and, while the assumption of such convergence may, in fact, insure differentiability of the transformation, we have not established this assertion. We also wish to point out that the construction of parallel surfaces given by Smirnov in [14] does not guarantee that the normals converge unless the surface $\partial D$ is in fact $C^2$, i.e. the normal is differentiable rather than merely Lipschitz continuous.

Theorem 3.5 has two immediate consequences which we state here.

Corollary 1: If $w \in C^2(D_\pm)$ and if $w$ and $\frac{\partial w}{\partial n}$ have limits in $L^2(\partial D)$ from $D_-$ in the sense of Theorem 3.5, then

\begin{equation}
(3.29) \int_{\partial D} w(p) \frac{\partial w}{\partial n}(p) dS_p = \int_{D_-} (|\nabla w(P)|^2 + w(P) \Delta w(P)) d\tau_P.
\end{equation}

Corollary 2: If $w \in C^2(D_\pm)$, if $w$ and $\frac{\partial w}{\partial n}$ have limits in $L^2(\partial D)$ from $D_+$ in the sense of Theorem 3.5, and if $B_R$ is a ball of radius $R$ containing $D_-$ in its interior, then

\begin{equation}
(3.30) \int_{\partial D} w(p) \frac{\partial w}{\partial n}(p) dS_p = -\int_{D_+ \cap B_R} (|\nabla w(P)|^2 + w(P) \Delta w(P)) d\tau_P + \int_{\partial B_R} w(p) \frac{\partial w}{\partial n}(p) dS_p.
\end{equation}

The first of these corollaries follows immediately from Theorem 3.5 by identifying $\phi$ in that theorem with the function $w\bar{w}$. The second follows from the first by applying the statement to the
interior domain bounded by $\partial D$ and $\partial B_R$.

We conclude this section with a uniqueness theorem for the exterior Robin problem for the Helmholtz equation. The proof is based on Rellich's lemma and was given by Leis [9] in the classical case for real, continuous $\sigma$. The extension presented here takes into account complex-valued $\sigma$ in $L^\infty(\partial D)$ and considers a more general class of solutions. Specifically, we consider solutions of the Helmholtz equation in $\Omega_+$ which, together with their normal derivatives take on boundary values, square-integrable on $\partial D$, in the sense of Theorem 3.5. This notion is precisely the notion of generalized boundary conditions described by Miranda [8; §29]. We refer the reader to that work for a discussion of the history of this interpretation of boundary conditions for elliptic equations. It is however vital to the results of section IV to know that single and double layer distributions assume their boundary values in this generalized sense which we state in the following:

**Theorem 3.6:** If $\partial D$ is a Lyapunov surface (or a finite number of disjoint Lyapunov surfaces) and $\mu \in L^2(\partial D)$ then $Su$, $Du$, and $\frac{\partial Su}{\partial n^+}$ assume boundary values in $L^2(\partial D)$ in the sense of Theorem 3.5.

The proof of this theorem is deferred to the appendix. Throughout the remainder of this paper, the boundary value problem will be understood in this generalized sense and the statement of the boundary value problem (2.10)-(2.12) should be so interpreted. With this understanding we state
Theorem 3.7: 
Let \( \partial D \) be a Lyapunov surface of index 1, let \( k \in \mathbb{C} \) with \( 0 < \arg k < \pi \) (\( \text{Im} k > 0 \)) and let \( \sigma \in \mathcal{L}_\infty (\partial D) \) with \( \arg k < \arg \sigma < \arg k + \pi \).

If \( w \in C^2_0(D^+) \) is such that

\[
(3.31) \quad (\Delta + k^2)w(P) = 0, \quad P \in D^+, \quad P \in D^+, \quad P \in D^+
\]

\[
(3.32) \quad \frac{\partial w}{\partial \mathbf{n}} - ikw = o(1/r), \quad r = |P|,
\]

and

\[
(3.33) \quad \frac{\partial w}{\partial \mathbf{n}}(p) + \sigma w(p) = 0, \quad p \in \partial D,
\]

then \( w = 0 \) almost everywhere in \( D^+ \cup \partial D \).

Remark: We repeat for emphasis that the boundary condition (3.33) is to be interpreted in the generalized sense and the equality, consequently, holds only almost everywhere on \( \partial D \).

Proof: Let \( B_R \) be a ball with boundary \( \partial B_R \), center in \( D^- \) and radius \( R \) sufficiently large so that \( D^- \subset B_R \). Assuming that \( w \) satisfies the hypothesis of the theorem, we may apply Green's theorem to the function \( w \) and its complex conjugate \( \bar{w} \) in the region \( D^+ \cap B_R \) obtaining, using Theorem 3.5 and the relation (3.30) the integral identity

\[
(3.34) \quad \int_{D^+ \cap B_R} (w \Delta \bar{w} - \bar{w} \Delta w) \, d\tau = \int_{\partial B_R} \left( \frac{\partial \bar{w}}{\partial \mathbf{n}} - \bar{w} \frac{\partial w}{\partial \mathbf{n}} \right) \, dS
\]

\[
- \int_{\partial D} (w \frac{\partial \bar{w}}{\partial \mathbf{n}} - \bar{w} \frac{\partial w}{\partial \mathbf{n}}) \, dS.
\]
The radiation condition (3.32) shows that

\[ (3.35) \quad \left( \frac{\partial w}{\partial n} - ikw \right) \left( \frac{\partial \bar{w}}{\partial n} + i\bar{k}\bar{w} \right) = o \left( \frac{1}{r^2} \right) \]

from which we may deduce that, on \( \partial B_R \),

\[ (3.36) \quad \frac{\partial \bar{w}}{\partial n} - \frac{\partial w}{n} = \frac{1}{1\text{Re}k} \left[ \left| \frac{\partial w}{\partial n} \right|^2 + \left|kw\right|^2 + (\text{Im}k) \left( \frac{\partial \bar{w}}{\partial n} + \frac{\partial w}{\partial n} \right) \right] + o \left( \frac{1}{r^2} \right). \]

Substituting this result in the identity (3.34) and making use of the fact that both \( w \) and \( \bar{w} \) are solutions of the Helmholtz equation and satisfy homogeneous boundary conditions (equations (3.31 and 3.33 and their conjugates) we obtain

\[ (3.37) \quad (k^2 - \bar{k}^2) \int_{D + \partial B_R} |w|^2 \, d\tau = \frac{1}{1\text{Re}k} \left[ \int_{\partial B_R} \left( \frac{\partial w}{\partial n} \right)^2 + \left|kw\right|^2 + (\text{Im}k) \left( \frac{\partial \bar{w}}{\partial n} + \frac{\partial w}{\partial n} \right) \right] dS \]

\[ + \int_{\partial D} (\bar{\sigma} - \sigma) |w|^2 dS + o(1). \]

However, Theorem 3.5 may be applied to \( \bar{w} \Delta w + w \Delta \bar{w} \) and, again using the boundary conditions, we obtain

\[ (3.38) \quad \int_{\partial B_R} \left( \frac{\partial w}{\partial n} \right)^2 \, dS = \int_{D + \partial B_R} \left( \frac{2 \left| \nabla w \right|^2 - (k^2 + \bar{k}^2) \left| w \right|^2 \right) \, d\tau - \int_{\partial D} (\sigma + \bar{\sigma}) |w|^2 dS. \]

Substituting this result in (3.37) and making use of the facts

\[ (3.39) \quad i(\text{Re}k) (k^2 - \bar{k}^2) - (\text{Im}k) (k^2 + \bar{k}^2) = 2 |k|^2 (\text{Im}k) \]

and
we obtain

\[ (3.41) \quad \int_{\partial B_R} ( |\frac{\partial w}{\partial n}|^2 + |kw|^2) \, d\Sigma + 2(\text{Im}k) \int_{D_+ \cap B_R} (|kw|^2 + |\nabla w|^2) \, d\tau + 2 \int_{\partial D} (\text{Im}\bar{k}w) |w|^2 \, d\Sigma = o(1). \]

All the terms on the left are easily seen to be non-negative if \((\text{Im}k)>0\) and \((\text{Im}\bar{k}w)>0\), this last condition being equivalent to the condition \(\arg k < \arg \sigma < \arg k + \pi\). It follows, then, from Rellich's Lemma that \(w=0\).

Note that the space in which \(w\) lies allows us to conclude that \(w=0\) in \(D_+\) whereas we can assert only that \(w=0\) almost everywhere on \(\partial D\). It should also be noted that the classical solution space, \(C^2(D_+) \cap C^1(D_+ \cup \partial D)\) is a subspace of the space of functions specified in the statement of the theorem hence in the classical case \(w=0\) everywhere in \(D_+ \cup \partial D\).

4. Existence and Uniqueness of Solutions

The formal calculations of section 2 have given us a pair of boundary integral equations for a function \(u \in L^2(\partial D)\), namely the equations

\[ (4.1) \quad (I + \bar{K}^* + S\bar{\sigma})u = Sg_+ \]

\[ (4.2) \quad (-\sigma + K\sigma + D_n) u = Kg_+ - g_+. \]
Our object in this section is to demonstrate the existence of a unique solution of this pair of equations and to establish the equivalence of this problem to the original boundary value problem. The proof will proceed by means of a series of lemmas. Throughout this section, unless explicitly mentioned to the contrary, (e.g. Lemma 4.3) the functions \( g_+ \) and \( \sigma \) will be assumed to lie in the space \( L_\infty(\mathbb{D}) \). Furthermore, the boundary \( \partial \mathbb{D} \) will be assumed to satisfy the hypothesis of theorem 3.5.

We remark, first, that by applying the regularity theorem (Theorem 3.1) to any solution of equation (4.1), we may conclude that any \( L_2 \)-solution of equation (4.1) must, in fact, be continuous. Note that the regularity theorem is applicable here since, under the hypotheses on \( g_+ \), the function \( Sg_+ \) is continuous everywhere.

(Theorem 3.2), and the kernels of the integral operators \( R^* \) and \( S_0 \) are products of weakly singular, essentially bounded and continuous functions. In fact it is to ensure the continuity of \( Sg_+ \) and the subsequent applicability of the regularity theorem that we restrict \( g_+ \) to lie in \( L_\infty \) rather than \( L_2 \). As we shall see below the fact that \( L_2 \) solutions of (4.1) must also be continuous will be of crucial importance in ensuring that certain functions used in the construction of the required solution of the system (4.1)-(4.2) will lie in the domain of the operator \( D_n \). Indeed, we will begin with the following basic result relating the set of solutions of equation (4.1) and the domain of this operator.

**Lemma 4.1:** Suppose that the wave number \( k \) is not an eigenvalue of the interior Dirichlet problem for the Helmholtz equation. Then, for any solution \( u \) of equation (4.1) in \( L_2(\mathbb{D}) \) \( D_n^-u \), and hence \( D_n^+u \), exists and, moreover, \( D_n^-u = D_n^+u = D_n u \).
**Proof:** Let \( u \) be any solution of the equation (4.1). In light of the theorem of Günter (Theorem 3.4 above) it will be sufficient to show that \( D_n^u \) exists, at least almost everywhere on \( \partial D \).

To this end, define the function \( v \) on \( D_\cdot \) by the equation

\[
(4.3) \quad v(P) := (Du + S\mu)(P) - Sg^+(P), \quad P \in D_\cdot.
\]

Then \( v \) takes on boundary values in the sense of Theorem 3.5.

Since the function \( u \) is a solution of the integral equation (4.1), the regularity theorem, 3.1, assures that \( u \) is continuous on \( \partial D \). Therefore \( u \) is bounded on \( \partial D \) and hence the function \( \sigma u \) is essentially bounded, measurable, and so integrable on \( \partial D \).

Likewise, the function \( \sigma u - g^+ \) is essentially bounded and integrable on \( \partial D \), and so the single layer with density \( \sigma u - g^+ \) is continuous everywhere by Theorem 3.2. Recalling the form of the jump conditions for double layer with continuous density (theorem 3.5), we have for all \( P \in \partial D \),

\[
(4.4) \quad \lim_{P \to P} v(P) = (u + \bar{K}u + S\sigma u)(P) - (Sg^+)(P).
\]

Since, by assumption, the function \( u \) is a solution of the equation (4.1) we may conclude that \( \lim_{P \to P} v(P) = 0 \) for all points \( P \in \partial D \).

The function \( v \), therefore, is a solution of the interior homogeneous Dirichlet problem with zero boundary data. Since \( k \) is not an eigenvalue of this problem, the function \( v \) must vanish identically.
in \( \bar{D}_- \) and so \( \partial v / \partial n = 0 \) on \( \partial D \).

We now rewrite the relation defining \( v \) in \( D_- \) as follows:

\[(4.5) \quad (D_P u)(P) = v(P) + S[g_+ - \sigma u](P), \quad P \in D_-.
\]

Again, since density of the single layer operator in this last relation is integrable, we may use Theorem 3.3 to conclude that

\[(4.6) \quad \lim_{P \to P} \frac{\partial}{\partial n_P} S[g_+ - \sigma u](P) = [-g_+ + Kg_+ + \sigma u - K\sigma u](P)
\]

almost everywhere on \( \partial D \). Since we have just shown that \( \partial v / \partial n = 0 \) for all \( P \in \partial D \), we may conclude that \( D_n^- u \) exists for almost all \( P \in \partial D \) which is what we wished to demonstrate. Moreover at such points \( P \),

\[(4.7) \quad D_n^- u(P) = (-g_+ + Kg_+ + \sigma u - K\sigma u)(P),
\]

which relates the solutions of equation (4.1) to those of equation (4.2). More precisely, we have established the following result.

**Lemma 4.2:** If \( k \) is not an eigenvalue of the interior homogeneous Dirichlet problem and if the function \( u \) is a solution of equation (4.1) in \( L_2(\partial D) \), then the function \( u \) is likewise a solution of equation (4.2).

We may now turn to a discussion of the existence and uniqueness of solutions.
Lemma 4.3: For all choices of $g_+ \in L_2(\partial D)$, the equation (4.1) has a solution in $L_2(\partial D)$.

Proof: The integral operators in (4.1), $\tilde{K}^*$ and $S\sigma$, are completely continuous on $L_2(\partial D)$ (Mikhlin [10]) hence Fredholm's Alternative applies to (4.1). Observe that the operator $S\sigma$ should be viewed as a composition of multiplication by the essentially bounded function $\sigma$ which maps $L_2(\partial D)$ into itself and the single layer which is weakly singular. Fredholm's Alternative implies that for any $g_+$ either equation (4.1) has a solution or the homogeneous problem has non-trivial solutions, in which case, equation (4.1) has a solution if and only if the function $Sg_+$ is orthogonal to all solutions of the homogeneous adjoint equation which may be written $(I + \tilde{K} + \sigma S^*)\hat{w} = 0$ or, equivalently, $(I + \tilde{K} + \sigma S)\hat{w} = 0$. In fact, we will show that, for all such $\hat{w}$, $(Sg_+ , \hat{w}) = 0$ for any choice of $g_+ \in L_2(\partial D)$.

To this end note that $(Sg_+ , \hat{w}) = (g_+ , S\hat{w})$ and define a function $v$ in $\mathcal{D}_+$ by

$$ (4.9) \quad v(P) := (S\hat{w})(P), \quad P \in \mathcal{D}_+. $$

Then, since $\hat{w} \in C(\partial D)$, the jump relations yield, for almost all $P \in \partial D$,

$$ (4.9) \quad \partial v/\partial n^+_P (p) = (\hat{w} + \tilde{K}\hat{w})(p). $$

But the function $\hat{w}$ satisfies the equation $(I + \tilde{K} + \sigma S)\hat{w} = 0$ and so we may write
\[(4.10) \ \partial v/\partial n^+ (p) = -\bar{v}S \partial (p) = -\bar{v}(p), \text{ for almost all } p \in \partial D.\]

Hence, the function \(v\) is a solution of the exterior homogeneous Robin problem which assumes boundary values in the sense of Theorem 3.5 and so vanishes identically according to the uniqueness theorem discussed above. We may conclude \(\bar{S}v = 0\) and this establishes the lemma.

This last result, together with the preceding lemma, shows that if \(k\) is not an eigenvalue of the interior homogeneous Dirichlet problem, then the pair of integral equations (4.1) - (4.2) has a simultaneous solution. Moreover, there can be at most one such solution as we now show.

**Lemma 4.4:** If \(u \in L^2(\partial D)\) is a solution of the homogeneous system of boundary integral equations

\[(4.11) \ (I + \bar{R}^* + S\bar{\sigma})u = 0\]

\[(4.12) \ (-\sigma + K\sigma + D_n^*)u = 0.\]

Then \(u = 0, p \in \partial D\)

Note that since the function \(u\) is assumed to satisfy the equation (4.11), we may again invoke the regularity theorem to establish the continuity of \(u\). Indeed, it may well be smoother since we assume as an hypothesis that it lies in the domain of the operator \(D_n^*\).

**Proof:** Again, define the function \(v\) on \(D_\) by the relation

\[v(P) := (D + S\sigma)u(P), \text{ } P \in D_\]
where \( u \) is a solution of the homogeneous system (4.11)-(4.12). Then, as before, \( v \) takes on boundary values in the sense of Theorem 3.5 and, since \( u \) is an \( L_2 \)-solution of equation (4.11) which has a continuous right hand member, it is also continuous. Taking limits as the point \( P \in D_- \) approaches the boundary of \( D \), and using the jump conditions for the double layer operator with continuous density, we have

\[
(4.13) \quad v(p) = u(p) + \bar{K}u(p) + (S\sigma)u(p)
\]

and so \( v(p) = 0 \) for all \( p \in \partial D \) since \( u \) is a solution of equation (4.11). Hence the function \( v \) is a solution of the homogeneous Dirichlet problem in \( D_- \). There are, then, two possibilities: either (i) \( v \) vanishes identically in \( D_- \), or (ii) \( v \) is a non-trivial eigenfunction of the interior homogeneous Dirichlet problem for the Helmholtz equation. In case (ii) the function \( v \) has regular normal derivative in the sense of Lyapunov (see Smirnov [14] p. 586 and p. 675).

In the first case, we must have that \((Du + S\sigma u)(p) = 0\) for all \( P \in D_- \). Now define the function \( v_+ \) in \( D_+ \) by

\[
(4.14) \quad v_+(p) := (Du + S\sigma u)(p), \quad P \in D_+,
\]

which, like \( v_- \), takes boundary values in the sense of Theorem 3.5. Using the jump relations for the double layer operator as the point \( P \) approaches \( \partial D \) from \( D_+ \), we may write
(4.15) \( v_+^*(p) = -u(p) + K\times u(p) + S(u)(p), \ p \in \partial D. \)

Since, for merely bounded and measurable densities, the jump relations for the normal derivative of the single layer operator hold only almost everywhere on \( \partial D, \) we may argue in a manner similar to the proof of Lemma 4.1 and conclude that, for almost all \( p \in \partial D, \)

(4.16) \( \frac{\partial v_+}{\partial n_p^+} = (S\times u)(p) + (K\times u)(p) + D_n u(p). \)

Note that since \( k \) is not an eigenvalue of the interior Dirichlet problem, Lemma 4.1 assures the existence of \( D_n u(p). \) But, from equation (4.11), we have \( K\times u + S\times u = -u \) and so, with (4.15), \( v_+^*(p) = -2u(p), \ p \in \partial D, \) while from equation (4.12) we have that, with (4.16), \( \frac{\partial v_+}{\partial n_p^+} (p) = 2(S\times u)(p) \) almost everywhere on \( \partial D. \)

Comparing these last two results, we see that the relation

(4.17) \( \frac{\partial v_+}{\partial n_p^+} (p) + (S\times u_+)(p) = 0 \)

holds almost everywhere on \( \partial D \) and we may invoke the uniqueness theorem to conclude that \( v_+ \) vanishes identically in \( D_+ \) and that \( v_+^*(p) = 0 \) almost everywhere on \( \partial D. \) It follows since \( v_+ = -2u \) that \( u(p) = 0 \) almost everywhere on \( \partial D. \) But the function \( u \) is continuous on \( \partial D \) and consequently vanishes on \( \partial D, \) which establishes the result in case (i).
If the second possibility obtains, then the function $v$ is not identically zero. But, as before,

$$
(4.18) \quad \frac{\partial v}{\partial n_p} (p) = (D_n u - \sigma u + K u) (p)
$$

for almost all $p \in \partial D$, and so, since the function $u$ satisfies equation $(4.12)$, $\frac{\partial v}{\partial n_p} (p) = 0$ almost everywhere on $\partial D$. But then the function $v$ is a solution to the interior Helmholtz problem which vanishes on the boundary and whose normal derivative vanishes almost everywhere on $\partial D$. This leads to a contradiction and completes the proof.

It remains for us to establish that, if $k$ is an eigenvalue of the interior homogeneous Dirichlet problem, then the system $(4.1)-(4.2)$ of boundary integral equations admits a simultaneous solution.

We recall the following result from [6, p. 224 and p. 277].

**Lemma 4.5:** There exists a non-trivial function $\hat{w}$ such that

$$
(4.19) \quad \hat{w} + K \hat{w} = 0
$$

if and only if $(S \hat{w})(p) = 0$, $p \in \partial D$. Moreover such a function exists if and only if $k$ is an eigenvalue of the interior Dirichlet problem. Finally, $\hat{v}$ is an eigenfunction if and only if it may be represented in the form $\hat{v}(p) = -1/2 (S \hat{w})(p)$, $p \in \partial D$ where $\hat{w} + K \hat{w} = 0$.

Clearly such a function $\hat{w}$ satisfies the boundary integral equation.
and having made this observation, we may prove the following result which is analogous to Theorem 4.3 of [6].

**Lemma 4.6:** The function $\tilde{v}$ is an eigenfunction of the interior Dirichlet problem if and only if $\tilde{v}$ can be represented in the form

$$\tilde{v} = \frac{\partial}{\partial n} w^* + \sigma w^*, \quad P \in D_-$$

where $w^*$ satisfies the homogeneous boundary integral equation

$$\frac{\partial}{\partial n} w = 0.$$

**Proof:** Assume, first, that $\tilde{w}^*$ is a non-trivial solution of the given boundary integral equation and hence with the regularity Theorem 3.1, is continuous. Define a function $\tilde{v}$ in $D_-$ by

$$\tilde{v}(P) := \frac{\partial}{\partial n} w^* + \sigma w^*(P), \quad P \in D_-,$$

which takes boundary values in the sense of Theorem 3.5. Then, as in the proof of Lemma 4.1, the function $\tilde{v}$ vanishes on $\partial D$ and hence either vanishes identically in $D_-$ or is a non-trivial eigenfunction of the interior Dirichlet problem. To see that $\tilde{v}$ is not identically zero, notice that if this were the case then its normal derivative would vanish on $\partial D$. But, then, for almost all $p \in \partial D$, we would have, using the jump conditions as in previous proofs,
(4.24) \[ \partial \hat{w} / \partial n_p^- (p) = (D_n^{-} \hat{w}^- + K^* \hat{w}^- - \sigma \hat{w}^-) (p) = 0 \]

where \( D_n^{-} \) exists since \( \frac{\partial \hat{w}^-}{\partial n_p^-} \) and \( \frac{\partial \hat{w}^-}{\partial n_p^+} \) exist and with Theorem 3.4, which applies because \( \hat{w}^- \) is continuous, \( D_n^{-} \) exists. Thus the function \( \hat{w} \) is a solution of the homogeneous boundary integral equation

(4.25) \[ (D_n + K^* \sigma) \frac{\partial}{\partial n} \hat{w}^- = 0 \]

Now define a function \( \hat{u} \) on \( D_+ \) by

(4.26) \[ \hat{u}(p) := (D \hat{w}^- + S \hat{w}^-), \quad p \in D_+, \]

which, again, takes on boundary values in the sense of Theorem 3.5. Again, using the jump conditions, we have for \( P \rightarrow P^+ \),

(4.27) \[ \hat{u}(p) := (-\hat{w}^- + K^* \hat{w}^- + S \hat{w}^-) (p), \quad p \in \partial D, \]

and

(4.28) \[ \frac{\partial \hat{u}}{\partial n_p^+} (p) = (\sigma \hat{w}^- + K \hat{w}^- + D_n \hat{w}^-) (p), \text{ a.e. on } \partial D. \]

But since \( (I + K^* S \sigma) \hat{w}^- = 0 \) we have with (4.27) that \( \hat{u}(p) = - \hat{w}^-(p) \) for all \( p \in \partial D \), while from (4.25) and (4.28) we conclude that \( \frac{\partial \hat{u}}{\partial n_p^+} (p) = 2(\sigma \hat{w}^-) (p) \) almost everywhere on \( \partial D \). The function \( \hat{u} \) is thus a solution to the exterior homogeneous Robin problem and, by the uniqueness theorem, must vanish everywhere. Thus \( \hat{w}^- (p) = 0 \) for all \( p \in \partial D \) which contradicts the choice of \( \hat{w}^- \). Hence \( \hat{v} \) must be a non-trivial eigenfunction.
Conversely, assume that \( \varphi \) is an eigenfunction of the interior Dirichlet problem and let \( v_i, i=1,2,...,n \), be a basis of eigenfunctions. Thus there exists scalars, \( \alpha_1, \ldots, \alpha_n \), such that
\[
v = \sum_{i=1}^{n} \alpha_i v_i.
\]
By lemma 4.5, to each \( v_i \) there corresponds a function \( \hat{w}_i \) such that \( \hat{w}_i \) satisfies
\[
I+K+\sigma S \hat{w}_i = 0, \quad \text{for } p \in \partial D.
\]

(4.29) \( (I+K+\sigma S)\hat{w}_i(p) = 0 \quad \text{for } p \in \partial D \)

(4.30) \( \hat{v}_i(p) = \frac{1}{2}(Sw_i)(p), \quad \text{for } p \in \partial D \).

Let \( W=\text{span}\{\hat{w}_i\} \). Note that the \( \hat{w}_i, i=1,2,...,n \), are linearly independent since the \( \hat{v}_i(p), i=1,2,...,n \), are chosen as a basis for the eigenspace. Hence \( \dim<\hat{\varphi}_{n+1}> = \dim W = n \) and \( W \subset \ker (I+K+\sigma S) \). But,

(4.31) \( \dim \ker (I+K+\sigma S) = \dim \ker (I+K+\sigma S)^* \)
\[= \dim \ker (I+K^*+\sigma S) \]
\[= \dim \ker (I+\bar{K}^*+\sigma S) \).\]

Let \( \{\vartheta_i\}, i=1,2,...,n \), be a basis for \( \ker (I+\bar{K}^*+\sigma S) \) and define for each \( i=1,...,n \)

(4.32) \( \vartheta_i(P) = (D+\sigma S)\vartheta_i(P), \quad \text{for } p \in \partial D \).

Since \( \vartheta_i \) are non-trivial solutions of \( (I+\bar{K}^*+\sigma S)\vartheta_i = 0 \), the first part of the proof shows that the functions \( \hat{u}_i, i=1,\ldots,n \) are eigenfunctions of the interior Dirichlet problem. Moreover, they are linearly independent for if \( \beta_i, i=1,\ldots,n \), are scalars such that \( \sum \beta_i \hat{u}_i = 0 \), then
which implies, as in the first part of the proof, that

$$\sum_{i=1}^{n} \beta_i \hat{w}_i^* = 0$$

and hence $\beta_i = 0$ for all $i=1,...,n$.

Thus every eigenfunction $\hat{\varphi}$ of the interior Dirichlet problem can be written in the form

$$\hat{\varphi} = \sum_{i=1}^{n} \alpha_i (D+Sc) \hat{w}_i^* = (D+Sc) \sum_{i=1}^{n} \alpha_i \hat{w}_i^* = (D+Sc) \hat{w}^*$$

for $\alpha_i, i=1,...,n$, appropriately chosen scalars, which completes the proof.

Finally, we may establish the following existence theorem for solutions of the system of boundary integral equations (4.1)-(4.2).

**Theorem 4.1**

Let $\sigma \in L_\infty(\partial D)$. Then there exists a unique solution of the system of boundary integral equations

$$\begin{align*}
(I + R^* + Sc)u &= Sg_+ \\
(-\sigma + Ks + D_n)u &= Kg_+ - g_+
\end{align*}$$

for each choice of $g_+ \in L_\infty(\partial D)$. 
Proof: If $k$ is not an eigenvalue of the interior Dirichlet problem for the Helmholtz equation, then Lemmas 4.3, 4.2, and 4.4 ensure that a unique solution of the system exists since $g_+ \in L^2(\partial D)$ implies that $g_+ \in L^2(\partial D)$.

If $k$ is an eigenvalue, then while Lemma 4.3 still guarantees the existence of at least one solution of equation (4.1), we cannot show immediately that such a solution will satisfy the second equation. Nor can we make any statement concerning uniqueness since solutions to equation (4.1) in this case are not unique.

Suppose, then, that $\hat{w}_1$ is a solution of equation (4.1) and define a function $v$ on $D_-$ by

\[
(4.36) \quad v(P) := (D\hat{w}_1 + S\hat{w}_1 - Sg_+)(P), \quad P \in D_-.
\]

Then, as in the proof of Lemma 4.1, $v$ is either identically zero or is a non-trivial solution of the interior homogeneous Dirichlet problem. In either case, according to Lemma 4.6, we may represent the function $v$ in the form $v = (D+Sg)\hat{w}^*$ for some solution $\hat{w}^*$ of the homogeneous boundary integral equation $(I+K^*+Sg)\hat{w}^* = 0$. Hence, for $P \in D_-,$

\[
(4.37) \quad (D+Sg)\hat{w}^*(P) = (D\hat{w}_1)(P) + (Sg)\hat{w}_1(P) - (Sg_+)(P).
\]

Now, define a function $w$ on $\partial D$ by

\[
(4.38) \quad w := \hat{w}_1 - \hat{w}^*.
\]
Then

\[(4.39) \quad w + \tilde{w}^* w + Sgw = \tilde{w}_1^* - \tilde{w}_1^* + \tilde{w}_1^* - \tilde{w}_1^* + Sg^* w_1^* = \tilde{w}_1^* + \tilde{w}_1^* + Sg^* w_1^* = Sg_+
\]

and so the function \(w\) satisfies the equation (4.1). Moreover, for all \(P \in D_-\), with (4.37),

\[(4.40) \quad (Dw + Sgw)(P) = (D\tilde{w}_1^* - Dw^* + Sg^* \tilde{w}_1^* - Sg^* w^*)(P) = Sg_+(P)
\]

or equivalently,

\[(4.41) \quad (Dw)(P) = S(g_+ - gw)(P), \text{ for all } P \in D_-.
\]

The function \(\tilde{w}_1^*\) and \(w^*\) are not only \(L_2\)-solutions of the non-homogeneous and homogeneous integral equations, respectively, but are also, according to the regularity theorem, continuous functions on \(\partial D\). Hence they are both bounded there and, consequently, the functions \(w\) and \(g_+ - gw\) are essentially bounded and integrable on \(\partial D\). So again, we may use Theorem 3.4 to conclude that
\begin{equation}
\lim_{\partial \Omega \to \partial \Omega} S(g_+ - \sigma w)(p) = (-g_+ + K g_+ + \sigma w - K \sigma w)(p)
\end{equation}

almost everywhere on \( \partial \Omega \). And so we have that \( D_n w(p) \) exists almost everywhere on \( \partial \Omega \) and \( (D_n w - \sigma w + K \sigma w)(p) = (-g_+ + K g_+)(p) \), almost everywhere on \( \partial \Omega \). Consequently, the function \( w \in L_2(\partial \Omega) \) satisfies the system of boundary integral equations. To finish the proof, we appeal to Lemma 4.4 for the uniqueness statement.

Next we establish the equivalence of the Robin Problem and its boundary integral equation formulation.

**Theorem 4.2:**

Let \( g_+ \) and \( \sigma \) be in \( L_\infty(\partial \Omega), \text{Im } k > 0 \), and \( \text{Im } K \sigma > 0 \). Then \( u_+ \) is a solution of the Exterior Robin Problem,

1. \( u_+ \in C_2(D_+) \); \( u_+ \), \( \frac{\partial u_+}{\partial n} + \varepsilon L_2(\partial \Omega) \) in the sense of theorem 3.5
2. \( (\Delta + k^2) u_+ = 0, \; P \in D_+ \)
3. \( \frac{\partial u_+}{\partial r} - i k u_+ = o(1/r) \)
4. \( \frac{\partial u_+}{\partial n} + \sigma u_+ = g_+ \) almost everywhere on \( \partial \Omega \),

if and only if

\begin{equation}
\frac{1}{2} S(g_+ - \sigma u) - \frac{1}{2} D u, \quad P \in D_+
\end{equation}

where

\begin{align}
(4.1) \quad (I + \overline{K}^* + S \sigma) u &= S g_+ \\
(4.2) \quad (-\sigma + K \sigma D_n) u &= K g_+ - g_+.
\end{align}
Proof: First assume \( u_+ \) is a solution of the Robin problem 1-4. Then Theorem 3.5 may be employed to obtain the Representation Theorem (2.13) which is the same as (4.43) above with \( u_+ \) replacing \( u \). Then the formal calculations by which (2.14) and (2.19) which are identical with (4.1) and (4.2) were obtained may be repeated. Note that (4.1) follows from the continuity of \( Sg_+ \) and \( Sou_+ \) (Theorem 3.2) and the jump relation for \( Du_+ \) which holds for \( u_+ \in L_2(\partial D) \) (Theorem 3.3). Then the regularity Theorem 3.1 ensures that \( u_+ \in C_0(\partial D) \). Hence with Theorem 3.4 and the fact that \( u_+ \) and \( \frac{\partial u_+}{\partial n} \) have normal derivatives in \( L_2(\partial D) \) the representation (2.13) may be differentiated and \( D_n u_+ \) must exist. Thus \( u_+ \) is a simultaneous solution of (4.1) and (4.2) hence by Theorem 4.1 is the unique solution.

Conversely assume \( u \) is the unique solution of (4.1) and (4.2) and \( u_+ \) is defined by (4.43). Clearly \( u_+ \) satisfies the Helmholtz equation and the radiation condition in \( D_+ \).

The regularity Theorem 3.1 ensures that \( u \) is continuous on \( \partial D \), hence \( Sou \) is continuous and has a normal derivative in \( L_2(\partial D) \) (Theorems 3.2, 3.3, and 3.6). Moreover \( u \) must be in the domain of \( D_n \). Hence \( u_+ \) and \( \frac{\partial u_+}{\partial n} \) are in \( L_2(\partial D) \). Then taking limits as \( P \to p^+ \) we find that on \( \partial D \)

\[
(4.44) \quad u_+ = \frac{1}{2} S(g_+ - su) + \frac{u - 1}{2} K^* u
\]

and with (4.1)

\[
(4.45) \quad u_+ = u \text{ on } \partial D.
\]

Also
\begin{equation}
\frac{\partial u_+}{\partial n} = \frac{1}{2} g_+ + \frac{1}{2} K g_+ - \frac{\sigma u}{2} - \frac{1}{2} K \sigma u - \frac{1}{2} D_n u
\end{equation}

and with (4.2)

\begin{equation}
\frac{\partial u_+}{\partial n} = g_+ - \sigma u \quad \text{on } \partial D
\end{equation}

hence

\begin{equation}
\frac{\partial u_+}{\partial n} + \sigma u_+ = g_+ \quad \text{on } \partial D.
\end{equation}

and \(u_+\) satisfies the Robin problem.

In conclusion, we remind the reader of the closing remarks of Section 2. There we derived, formally, boundary integral equations, of the same form as equations (4.1) and (4.2), but with different right hand members which were suitable for describing the total field for the scattering problem. In that derivation, the right hand sides represent, up to a constant factor, the incident field and its normal derivative respectively. As the results of this section do not depend on the particular form of the function appearing on the right hand side, we see that for sufficiently smooth incident fields the analysis of this section is valid and we may state the following theorem.

**Theorem 4.3:**

Let \(u^i\) denote an incident field and \(u = u^i + u^s\) the total field when \(u^s\) denoted the field scattered by the obstacle \(D_\_\). Then, for any \(\sigma \in \mathcal{L}_\infty (\partial D)\), there exists a unique solution of the system of boundary integral equations.
(4.49) \((I + \Sigma + K^*) u = 2u^i\)

(4.50) \((-\sigma + K\sigma + D_n) u = 2(\frac{\partial u^i}{\partial n_p})\).

The equivalence of solutions of this pair of boundary integral equations and the scattering problem follows, mutatis mutandis, from Theorem 4.2.
APPENDIX - On the generalised boundary values of single and double layers.

In Theorem 3.5 the notion of a family of parallel surfaces \( \{ S_\delta^- \} \) contained in \( D_- \) was introduced. We may define another family \( \{ S_\delta^+ \} \) contained in \( D_+ \) which converges to \( \partial D \) in the same sense as \( \{ S_\delta^- \} \). With the understanding that the layers may approach their boundary values from \( D^+ \) or \( D^- \) we state and prove Theorem 3.6.

**Theorem:**

If \( \partial D \) is a Lyapunov surface (or a finite number of disjoint Lyapunov surfaces) and \( u \in L^2(\partial D) \) then \( S_u \), \( D_u \) and \( \frac{3}{3n} S_u \) assume boundary values in the sense of Theorem 3.5, namely:

\[
A-1 \quad \int_{\partial D} |(S_u)(F_\delta^+(p)) - (S_u)(p)|^2 dS_p = o(\delta^0),
\]

\[
A-2 \quad \int_{\partial D} |(D_u)(F_\delta^+(p)) - (D_u)(p)|^2 dS_p = o(\delta^0),
\]

and

\[
A-3 \quad \int_{\partial D} \left| \frac{3}{3n} (S_u)(F_\delta^+(p)) - \frac{3}{3n} (S_u)(p) \right|^2 dS_p = o(\delta^0)
\]

**Proof:**

Considering equation A-1 first, we denote by \( P_\delta \) the image on \( S_\delta^+ \) of a point \( p \) on \( \partial D \), i.e., \( P_\delta = F_\delta^+(p) \). Recall that

\[
A-4 \quad |P_\delta - p| := R(P_\delta, p) = o(\delta) \quad \text{uniformly in } p.
\]
With (2.3) the integrand in $A-1$ becomes

\[ A-5 \quad |(Su)(P_\delta)-(Su)(p)|^2 = | \int_{\partial D} \mu(q) \left[ \gamma_1(P_\delta,q) - \gamma_1(p,q) \right] dS_q |^2 \]

\[ = \frac{1}{4\pi^2} \left| \int_{\partial D} \mu(q) \left[ \frac{ikR(P_\delta,q)}{R(P_\delta,q)} - \frac{ikR(p,q)}{R(p,q)} \right] dS_q \right|^2. \]

It may be shown that for $\text{Im} k > 0$,

\[ A-6 \quad \left| \frac{e^{ikR(P_\delta,q)}}{R(P_\delta,q)} - \frac{e^{ikR(p,q)}}{R(p,q)} \right| \leq \frac{(1+|k| D)R(P_\delta,p)}{R(P_\delta,q)R(p,q)} \]

where $D$ is the diameter of $\partial D$. From the definition of $S_\delta$ it follows that there is a constant $c$ such that,

\[ A-7 \quad R(P_\delta,p) \leq c\delta. \]

Thus there is a constant $c$, independent of $\delta$, such that $A-5$ gives rise to

\[ A-8 \quad |(Su)(P_\delta)-(Su)(p)|^2 \leq c^2 \left[ \int_{\partial D} \frac{|\mu(q)|^2 dS_q}{R(P_\delta,q)R(p,q)} \right]^2 \]

\[ \leq c^2 \int_{\partial D} \frac{dS_q}{R(P_\delta,q)R(p,q)} \int_{\partial D} \frac{|\mu(q)|^2}{R(P_\delta,q)R(p,q)} dS_q. \]
where Schwartz' inequality has been used. It also may be established that for \( R(P_\delta, p) < \frac{d}{2} \) where \( d \) is sufficiently small (depending on the Lyapunov constants of the surface)

\[
A-9 \quad \frac{1}{R(p_\delta, q)} \leq \frac{2}{R(p, q)}, \forall p, q \in \Omega D
\]

and

\[
A-10 \quad \frac{1}{R(p_\delta, q)} \leq \frac{2}{R(p_\delta, p)}, \forall p, q \in \Omega D.
\]

Employing these relations selectively we find, for any \( \alpha (0,1) \),

\[
A-11 \quad |(Su)(P_\delta) - (Su)(p)|^2 \leq c \delta^\alpha \left( \int_{\partial D} \frac{ds_q}{R(p, q)} \right) \int_{\partial D} \frac{|u(q)|^2}{R(p, q)} ds_q.
\]

The term \( \int_{\partial D} \frac{S_q}{R(p, q)} \) is an integral with a weak singularity and differentiable density \((=1)\) hence is a continuous function of \( p \) on \( \partial D \) while the term \( \int_{\partial D} \frac{|u(q)|^2}{R(p, q)} ds_q \) is a single layer with integrable density hence is integrable ([4], p. 110) and the validity of A-1 follows.

Next consider A-2 which involves somewhat different estimates.

With (2.4), (2.5), (3.18) and (3.19) we have
Introducing Gauss' integral which has the form

$$\int_{\partial D} |\nabla u(p)|^2 = \int_{\partial D} \nabla^2 u(p)$$

where

$$\int_{\partial D} \nabla^2 u(p)$$

Introducing Gauss' integral which has the form

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where

$$\int_{\partial D} \nabla^2 u(p)$$

equation A-12 may be written

$$|\nabla u(p)|^2 = \int_{\partial D} \nabla^2 u(p)$$

The triangle inequality was employed in obtaining the last inequality.
It may be shown that for $\text{Im} k > 0$

\[ A-16 \quad \left| \frac{\partial}{\partial n_q} \left[ \gamma (P_\delta, q) - \gamma (P_\delta, q) + \gamma (p, q) - \gamma (p, q) \right] \right| = \]

\[ = \frac{1}{2\pi} \left| \frac{\partial}{\partial n_q} \left[ \frac{ikR(P_\delta, q)}{R(P_\delta, q)} - \frac{ikR(p, q)}{R(p, q)} \right] \right| \leq \frac{c \delta}{R(P_\delta, q) R(p, q)} \]

where $c$ is a constant which depends on $k$ and the diameter of $\partial D$ but is independent of $p$ and $\delta$. Hence the first term on the right hand side of $A-15$ may be estimated as

\[ A-17 \quad \left| \int_{\partial D} \frac{\partial u(q)}{\partial n_q} \left[ \gamma (P_\delta, q) - \gamma (P_\delta, q) + \gamma (p, q) - \gamma (p, q) \right] dS_q \right|^2 \]

\[ \leq c_1 \delta^2 \left( \int_{\partial D} \frac{|u(q)|}{R(P_\delta, q) R(p, q)} dS_q \right)^2 \]

which is exactly the same as the estimate $A-7$ hence the subsequent analysis applies here as well. The second term on the right of $A-15$ is analysed with the help of the following estimate which may be shown to hold on a Lyapunov of order $\alpha$:

\[ A-18 \quad \left| \frac{\partial}{\partial n_q} \left[ \gamma (P_\delta, q) - \gamma (P_\delta, q) + \gamma (p, q) - \gamma (p, q) \right] \right| \leq \frac{c_1 R(p, q)}{R^3(p, q)} + \frac{c_2 R(P_\delta, P)}{R(p, q) R^2(p, q)} \]

where $c_1$ and $c_2$ are constants. Then
for modified constants $a_1$ and $a_2$. The first integral on the right of A-19, with Schwartz' inequality, becomes

\[
A-20 \quad \left( \int_{\partial D} \frac{|u(q) - u(p)|}{R^3(\delta, q)} \, ds_q \right)^2 \leq \int_{\partial D} \frac{ds_q}{R^3(\delta, q)} \int_{\partial D} \frac{|u(q) - u(p)|^2}{R^3(\delta, q)} \, ds_q.
\]

Defining $\Sigma_d(p)$, a patch on $\partial D$ as,

\[
A-21 \quad \Sigma_d(p) = \{ q \in \partial D \mid R(p, q) < d \}
\]

where $d$ is sufficiently smaller than the Lyapunov radius associated with $\partial D$ so that A-9 and A-10 may be used, it follows that

\[
A-22 \quad \int_{\partial D} \frac{ds_q}{R^3(\delta, q)} = \int_{\partial D \setminus \Sigma_d(p)} \frac{ds_q}{R^3(\delta, q)} + \int_{\Sigma_d(p)} \frac{ds_q}{R^3(\delta, q)} \leq \frac{8A}{d^2} + c \int_0^d \int_0^{2\pi} \frac{d\rho}{(R^2(\delta, p) + \rho^2)^{3/2}}.
\]
The constants $A$ and $c$ are of no particular importance for our present purpose but we note that $A$ is the surface area of $3D$.

In the second integral in A-22 we have introduced polar coordinates $(p,\phi)$ on the disk tangent to $3D$ at $p$ and used the inequalities (e.q. Günter[4])

\[ A-23 \quad dS_q \leq 2\rho d\rho d\phi, q \in \Sigma_d(p) \]

and

\[ A-24 \quad R(p_\delta, q) \geq \frac{1}{2} \sqrt{R^2(p_\delta, p) + R^2(p, q)} \geq \frac{1}{2} \sqrt{R^2(p_\delta, p) + \rho^2}. \]

The integral in A-22 is easily evaluated and we have

\[ A-25 \quad \int \frac{dS_q}{R^3(p_\delta, q)} \leq \frac{8A}{d^3} + 2\pi c \left[ \frac{1}{R(p_\delta, p)} - \frac{1}{(R^2(p_\delta, p) + d^2)^{1/2}} \right] \leq \frac{c_1}{d} \]

for some constant $c_1$. Similarly the second integral in (A-20) may be written

\[ A-26 \quad \int \frac{|u(q) - u(p)|^2}{R^3(p_\delta, q)} dS_q \leq 16 \int_{3D \setminus \Sigma_d(p)} \frac{|u(q)|^2 + |u(p)|^2}{d^3} dS_q \]

\[ + 8 \int_{\Sigma_d(p)} \frac{|u(q) - u(p)|^2}{(d^2 + R^2(p, q))^{3/2}} dS_q \]

\[ \leq c_1 ||u||^2_{L^2(3D)} + c_2 ||u(p)||^2 + c_3 \int_0^d \int_0^{2\pi} \frac{d\theta}{(d^2 + R^2(p, q))^{3/2}} ||u(p+q) - u(p)||^2 \]
where again we use polar coordinates on the plane tangent to \( p \) to parameterize \( \Sigma_d(p) \) and employ (A-23), (A-24), and (3.21). The vector \( q_1 \) in local rectangular coordinates centered at \( p \) is

\[
A-27 \quad q_1 = (\xi, \eta, \zeta(\xi, \eta)) \quad , \quad \xi = \rho \cos \phi \quad , \quad \eta = \rho \sin \phi
\]

where \( \zeta \) is a local representation of the surface, hence depends on \( p \), but also satisfies, since \( d \) is smaller than the Lyapunov radius

\[
A-28 \quad |\zeta| < \alpha \rho^{1+\alpha} \quad , \quad \alpha = \text{Lyapunov index}.
\]

Hence for \( d \) sufficiently small

\[
A-29 \quad |q_1| < 2d.
\]

Substituting these results in A-20 we have

\[
A-30 \quad \left( \int_{\partial D} \frac{|u(q) - u(p)|}{\partial_3 (p_\delta, q)} \, dS_q \right)^2 \leq \frac{1}{8} \left( c_1 \|u\|_{L_2(\partial D)}^2 + c_2 |u(p)|^2 \right)
\]

\[
+ c_3 \int_0^{2\pi} \int_0^1 \frac{d}{d\rho} \frac{d}{d\phi} \frac{\rho |u(p+q_1) - u(p)|^2}{(\rho^2 + \rho_\delta^2)^{3/2}}
\]

The second integral on the right in A-19 is treated similarly yielding
A-31 \[ \left\{ \int_{\mathcal{D}} \frac{|\mu(q) - \mu(p)|}{R(P, q) R^{-\alpha}(p, q)} \, ds_q \right\}^2 \leq \frac{1}{\delta^{1+a}} \left( b_1 |u|_{L^2(\mathcal{D})}^2 + b_2 |u(p)|^2 \right) + b_3 \int_0^1 \int_0^{2\pi} \frac{|\mu(p+q_1) - \mu(p)|^2}{\rho^{1-a} \sqrt{\theta^2 \delta^2 + \rho^2}} \, d\rho \, d\phi \]

Incorporating A-30 and A-31 into A-19 we have, for appropriately redefined constants,

\[ A-32 \left\{ \int_{\mathcal{D}} |\mu(q) - \mu(p)|^{\beta/\gamma} \, ds_q \right\}^{\frac{\beta}{\gamma}} \leq c_1 \delta |u|_{L^2(\mathcal{D})}^2 \]

Integrating over \( \mathcal{D} \) and employing Fubini's theorem, which applies since \( u \in L^2(\mathcal{D}) \) and both \( \frac{\rho}{(\theta^2 \delta^2 + \rho^2)^{3/2}} \) and \( \frac{1}{\rho^{1-a} \sqrt{\theta^2 \delta^2 + \rho^2}} \) are integrable in the \( \rho, \phi \) variables for \( \delta > 0 \), we obtain

\[ A-33 \left\{ \int_{\mathcal{D}} |\mu(q) - \mu(p)| \frac{\beta}{\gamma} \, ds_q \right\}^{\frac{\beta}{\gamma}} \leq (c_1 + c_2) \delta |u|_{L^2(\mathcal{D})}^2 \]

\[ + c_3 \delta \int_0^1 \int_0^{2\pi} \frac{\rho}{(\theta^2 \delta^2 + \rho^2)^{3/2}} \left( \int_{\mathcal{D}} |\mu(p+q_1) - \mu(p)|^2 \, ds_p \right) \]
The integral \[ \int_{\partial D} |u(p+q_1) - u(p)|^2 dS_p \] is the $L_2$ modulus of continuity of $u$ which is $O(|q_1|^2)$ (Stein [16]) although for present purposes it is sufficient to have $O(|q_1|\beta)$, $\beta > 0$. Then with (A-29)

\[ A-34 \int_{\partial D} |u(p+q_1) - u(p)|^2 dS_p < c \rho^\beta \]

and (A-33) becomes

\[ A-35 \int_{\partial D} \int_{\partial D} |u(q) - u(p)|^2 dS_q [\gamma(q, p) - \gamma(q, p)] dS_p ]^2 dS_p \]

\[ \leq c_1 \delta \|u\|^2_{L_2(\partial D)} + c_2 \delta \int_0^d \frac{\rho^{1+\beta}}{(\theta^2 - \rho^2 + \rho^2)^{3/2}} d\rho + c_3 \delta^{1+\alpha} \int_0^d \frac{d\rho}{\rho^{1-\beta}} \frac{1}{\sqrt{\theta^2 + \rho^2}} \]

\[ \leq c_4 \delta \|u\|^2_{L_2(\partial D)} + c_2 \delta^8 \int_0^\infty \frac{s^{1+\beta}}{(\theta^2 + s^2)^{3/2}} ds + c_3 \delta^{2+\beta} \int_0^\infty \frac{ds}{s^{1-\beta} \sqrt{\theta^2 + s^2}}, \beta < 1 \]

When $\beta < 1$, both integrals are bounded and the right hand side is $O(\delta^\beta)$. If $\beta > 1$ even more rapid decay of (A-34) with $\delta$ is easily shown. In all cases the validity of (A-2) is thus ensured.
The proof of A-3 involves an argument similar to that used in establishing A-2 which we merely suggest. With (2.3) (3.16), (3.17) and (A-13) we obtain

\[ A - 38 \quad \left| \frac{3}{2n_p} (S_u P_\delta) \frac{3}{2n_q} (S_u) (p) \right|^2 \leq 4 \left( \int_{\partial D} |u(q)| \left| \frac{3}{2n_p} [\gamma (P_\delta, q) - \gamma_o (P_\delta, q)] \right. \right. \\
\left. \left. + \gamma_o (p, q) - \gamma(p, q) \right| dS_q \right)^2 \\
+ 4 \left[ \int_{\partial D} |u(q)| \left| \frac{3}{2n_p} [\gamma_o (P_\delta, q) - \gamma_o (P_\delta, q)] \right. \right. \\
\left. \left. + \frac{3}{2n_q} [\gamma_o (P_\delta, q) - \gamma_o (P_\delta, q)] \right| dS_q \right)^2 \\
+ 4 \left[ \int_{\partial D} |u(p) - u(q)| \left| \frac{3}{2n_q} [\gamma_o (P_\delta, q) - \gamma_o (P_\delta, q)] \right. \right. \\
\left. \left. \left| dS_q \right|^2 \right) \right.

The first integral on the right may be estimated in the same way as (A-17) since (A-16) remains valid when \( \frac{3}{2n_p} \) is replaced by \( \frac{3}{2n_q} \). The third integral is precisely the same as (A-19) and is treated identically. The second integral is estimated using the following inequality

\[ A - 37 \quad \frac{3}{2n_p} [\gamma_o (P_\delta, q) - \gamma_o (p, q)] + \frac{3}{2n_q} [\gamma_o (P_\delta, q) - \gamma_o (p, q)] \leq \frac{c R(P_\delta, p)}{R(P_\delta, q) R^{2-\alpha} (p, q)} \]
which enables us to write

\[ A-38 \int_{\partial D} |u(q)| \left| \frac{\partial}{\partial p} \gamma_0(p, q) - \gamma_0(p, q) \right| + \frac{\partial}{\partial q} \left[ \gamma_0(p, q) - \gamma_0(p, q) \right] |dS_q| \]

\[ \leq c \delta^2 \left( \int_{\partial D} \frac{|u(q)|}{R(P_\delta, q) R^{2-\alpha}(p, q)} dS_q \right)^2 \]

\[ \leq 2c \delta^2 \left( \int_{\partial D} \frac{|u(q) - u(p)|}{R(P_\delta, q) R^{2-\alpha}(p, q)} dS_q \right)^2 + \]

\[ 2c \delta^2 |u(p)|^2 \left( \int_{\partial D} \frac{dS_q}{R(P_\delta, q) R^{2-\alpha}(p, q)} \right)^2. \]

The first term in the right also appears in A-19 and is treated as it was there. The second term is \( o(\delta^0) \) since the integral may be shown to be \( O\left(\frac{1}{\delta^{1-\alpha}}\right) \). This completes the proof of Theorem 3.6.

We remark that while we have proven the theorem for Lyapunov surfaces of index \( \alpha \), the proof is contingent upon the existence of families of parallel surfaces, which we have only asserted for index 1. This is sufficient for our use of the theorem in Section IV, however should the families exist under the weaker hypothesis (see Burago, Maz'ja and Sapeznikova [1]) then Theorem 3.6 will hold in that case as well.
References


