Subharmonic Solutions of a Forced Wave Equation

Paul H. Rabinowitz
Mathematics Department
University of Wisconsin
Madison, Wisconsin 53706

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Introduction

In a recent paper [1], we established the existence of subharmonic solutions of forced Hamiltonian systems of ordinary differential equations. The goal of this note is to show that subharmonics also occur for a class of semilinear wave equations.

To be more precise, let $z(t) = (z_1(t), \ldots, z_{2n}(t))$, $H: \mathbb{R}^{2n} \to \mathbb{R}$, and consider the Hamiltonian system of ordinary differential equations:

$$\frac{dz}{dt} = JH_z(t, z), \quad J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

where $I$ denotes the identity matrix in $\mathbb{R}^n$. Suppose $H(t, 0) = 0$, $H(t, z) > 0$, and $H$ is $T$ periodic in $t$. It was shown in [1] that if $H$ satisfies appropriate additional conditions near $z = 0$ and $z = -$, then (0.1) possesses an infinite number of distinct subharmonic solutions, i.e. for each $k \in \mathbb{N}$, (0.1) has a solution $z_k(t)$ of period $kT$ and infinitely many of the functions $z_k$ are distinct. For single second order equations of the form

$$v'' + g(t, v) = 0$$

with $g$ $T$-periodic in $t$, more delicate such results were obtained earlier under related hypotheses by Jacobowitz [2].
Further work on this question was carried out by Hartman [3] who weakened the hypotheses of [2] and improved the conclusions.

We will show how analogues of some of the results of [1] can be obtained for a family of forced semilinear wave equations. Thus consider

\[
\begin{cases}
  u_{tt} - u_{xx} + f(x,t,u) = 0 & \quad 0 < x < \ell \\
u(0,t) = 0 = u(\ell,t)
\end{cases}
\]

(0.3)

where \( f \) is \( T \) periodic in \( t \). It was shown in [4] that (0.3) possesses a nontrivial classical \( T \) periodic solution provided that \( T \in \mathbb{Q} \), i.e. \( T \) is a rational multiple of \( t \), and \( f \) satisfies appropriate conditions. Recently a slightly stronger result has been obtained by Brezis, Coron, and Nirenberg [5]. In the following section we will prove that the hypotheses required in [4] for the above existence theorem imply that (0.3) also has subharmonic solutions: for all \( k \in \mathbb{N} \), (0.3) possesses a \( kT \) periodic solution \( u_k \) and infinitely many of these functions are distinct. The proof relies on an amalgam of ideas from [1] and [4].

\[1. \textbf{The existence theorem}\]

Suppose \( f : [0,\ell] \times \mathbb{R}^2 \to \mathbb{R} \) and satisfies

\( (f_1) \quad f(x,t,0) = 0, \ f_x(x,t,r) > 0 \) for \( 0 \neq r \) near 0, and \( f(x,t,r) \) is strictly monotonically increasing in \( r \) for all \( r \in \mathbb{R} \).

\( (f_2) \quad f(x,t,r) = o(|r|) \) at \( 4 = 0 \)

\( (f_3) \quad \text{There are constants } \quad u > 2 \quad \text{and} \quad \bar{r} > 0 \quad \text{such that} \)

\[\]
\[ 0 < \mu F(x,t,r) = \int_{0}^{r} f(x,t,s) ds \leq r f(x,t,r) \]

for \(|r| \geq \bar{r}\)

(f_4) There is a constant \(T > 0\) such that \(f(x,t + T,r) = f(x,t,r)\) for all \(x,t,r\).

Note that \((f_3)\) implies that

\[(1.1) \quad F(x,t,r) \geq a_1 |r|^\mu - a_2 \]

for some constants \(a_1 > 0, a_2 > 0\) and for all \(r \in \mathbb{R}\), i.e. \(F\) grows at a more rapid rate than quadratic at \(r = 0\).

We will prove the following theorem:

Theorem 1.2: Let \(f \in C^2([0,\ell] \times \mathbb{R}^2, \mathbb{R})\) and satisfy \((f_1) - (f_4)\). If \(T \in \mathbb{Q}\), then for all \(k \in \mathbb{N}\), the problem

\[(1.3) \quad \begin{cases} u_{tt} - u_{xx} + f(x,t,u) = 0 , & 0 < x < \ell \\ u(0,t) = 0 = u(\ell,t) \end{cases} \]

possesses a nonconstant \(kT\) periodic solution \(u_k \in C^2\). Moreover infinitely many of the functions \(u_k\) are distinct.

Before giving the proof of Theorem 1.2, several remarks are in order. Since \(T \in \mathbb{Q}\) implies that \(kT \in \mathbb{Q}\) for all \(k \in \mathbb{N}\), the first assertion of the theorem is a special case of Theorem 4.1 and Corollary 4.14 of [4]. However, since we do not know \(kT\) is an minimal period of \(u_k\), the functions \(u_k\) may all represent the same \(T\) periodic
function or possibly a finite number of distinct periodic functions. Thus what is new and of interest here is that in fact infinitely many of the functions $u_k$ must be distinct.

To establish this result we will show that on the one hand, if only finitely many of the functions $u_k$ were distinct, a corresponding variational formulation of (1.3) would have an unbounded subsequence of critical values, $c_k$, with corresponding critical points representing reparametrizations of the same function. The growth of the $c_k$'s will be like $k^2$. On the other hand it turns out that $c_k$ grows at most linearly in $k$, a contradiction.

To make this statement, which contains variants of ideas in [1], more precise, a closer inspection must be made of the existence mechanism of [4]. For convenience we take $t = \pi$ and $T = 2\pi$. Fixing $k \in \mathbb{N}$, we seek a solution of (1.3) which is $2\pi k$ periodic in $t$. It is convenient to rescale time $t = kr$ so that the period becomes $2\pi$ and (1.3) transforms to

$$
\begin{align*}
&u_{tt} - k^2(u_{xx} - f(x, kr, u)) = 0 \\
&u(0, t) = 0 = u(\pi, t); u(x, t + 2\pi) = u(x, t)
\end{align*}
$$

The solution of (1.4) is obtained via an approximation argument. Three approximations are made. First observe that the wave operator part of (1.4), $u_{tt} - k^2 u_{xx}$ has an infinite dimensional null space, $N$, in the class of functions satisfying the periodicity and boundary conditions, namely...
To provide some compactness for the problem in $N$, we perturb the wave operator by adding a term $-\beta v_{TT}$ to it where $\beta > 0$ and $v$ denotes the $L^2$ orthogonal projection of $u$ into $N$. Secondly the unrestricted rate of growth of $f(x,t,r)$ at $|r| = \infty$ creates technical problems which we bypass by suitably truncating $f$, i.e., we replace $f$ by $f_K(x,t,r)$ where $f_K$ coincides with $f$ for $|r| \leq K$, satisfies $(f_1) - (f_4)$ with $\mu$ replaced by a new constant $\bar{\mu} = \min(4, \mu)$ in $(f_3)$. Moreover $f_K$ grows like $r^3$ at $\infty$. (See Eq (5.22) of [4]). Thus we replace (1.4) by

$$
\begin{align*}
\begin{cases}
  u_{TT} - \beta v_{TT} - k^2(u_{xx} - f_K(x,k,T,u)) = 0, & 0 < x < \pi \\
  u(0,\tau) = 0 = u(\pi,\tau); u(x,\tau + 2\pi) = u(x,\tau)
\end{cases}
\end{align*}
$$

Formally (1.5) can be cast as a variational problem, namely that of finding critical points of

$$
I(u;k,\beta, K) = \int_0^{2\pi} \int_0^\pi \left[ \frac{1}{2} u_k^2 \beta v_k^2 - k^2 \left( \frac{1}{2} u^2 + F_K(x,k,T,u) \right) \right] dx \, d\tau
$$

where $F_K$ is the primitive of $f_K$. Our final approximation is to pose this variational problem in a finite dimensional space

$$
E_m = \text{span}\{\sin jx \sin n\tau, \sin jx \cos n\tau | 0 \leq j, n \leq m \}.
$$

A critical point of $I|_{E_m}$ will be a solution of the $L^2$ orthogonal projection of (1.5) onto $E_m$. 

A series of lemmas in [4] use \((f_1) - (f_4)\) and the form of \(I\) to establish the existence of a nontrivial critical point \(u_{mk}\) of \(I|_{E^m}\) as well as an estimate on the corresponding critical value \(c_{mk}\) of the form

\[
0 < c_{mk} = I(u_{mk}, k, \beta, K) \leq M_k
\]

where \(M_k\) is a constant independent of \(\beta, K,\) and \(m\). Further arguments in [4] allow successively letting \(m \to \infty\) and \(\beta \to 0\) to get a solution \(u_k\) of

\[
\begin{aligned}
u_{\tau\tau} - k^2 (u_{xx} - f_K(x, k\tau, u)) &= 0 & 0 < x < \pi \\
u(0, \tau) = n(\pi, \tau); u(x, \tau + 2\pi) &= u(x, \tau)
\end{aligned}
\]

with \(c_k = I(u_k, k, 0, K) \leq M_k\). Moreover for \(K = K(k)\) sufficiently large, \(||u_k||_{L^\infty} \leq K\) so \(f_K(x, k\tau, u_k) = f(x, k\tau, u_k)\) and \(u_k\) satisfies (1.4). Lastly a separate argument shows \(c_k > 0\) so \(u_k \neq 0\) via \((f_1)\) and the form of \(I\).

Returning to the question of how many of the functions \(u_k\) are distinct, we will first study the dependence of \(M_k\) on \(k\). To do so requires a closer look at how the bound \(M_k\) is determined. Lemma 1.13 of [4] provides a minimax characterization of \(I(u_{mk}, k, \beta, K)\) which in turn yields the bound \(M_k\).

Let

\[
W_{mk} = \text{span}\{\sin jx \sin n\tau, \sin jx \cos n\tau | 0 \leq j, n \leq m \}
\]

and \(n^2 \leq j^2 k^2\), \(v_k = a_k \sin x \sin (k + 1)\tau\).
and $a_k$ is chosen so that $||\varphi_k||_{L^2} = 1$.

Set $V_{mk} = W_{mk} \oplus \text{span } \{\varphi_k\}$. It was shown in [4] that

\begin{equation}
0 < c_{mk} \leq \max_{u \in V_{mk}} I(u; k, \beta, K)
\end{equation}

(Note that $I \to -\infty$ as $||u||_{L^2} \to \infty$ via (f$_3$) so we have a max rather than a sup in (1.9)). Let $z = z_{mk}$ denote the point in $V_{mk}$ at which the max is attained. We can write

\begin{equation}
z = ||z||_{L^2}(\gamma \xi + \delta \varphi_k)
\end{equation}

where $\xi \in W_{mk}$ with $||\xi||_{L^2} = 1$ and $\gamma^2 + \delta^2 = 1$.

Substituting (1.10) into (1.9) and using the form of $I$ yields

\begin{equation}
k^2 \int_0^{2\pi} \int_0^{\pi} F_K(x, k\tau, z) \, dx \, d\tau \leq \frac{1}{2} \int_0^{2\pi} \int_0^{\pi} (z^2 - k^2 \varphi_k^2) \, dx \, d\tau
\leq \frac{\delta^2}{2} ||z||_{L^2}^2 \int_0^{2\pi} \int_0^{\pi} (\varphi_k^2 - k^2 \varphi_k^2) \, dx \, d\tau
\leq \overline{M} ||z||_{L^2}^2 k
\end{equation}

where $\overline{M}$ is independent of $k$ and $m$ (as well as $\beta$ and $K$). Since $F_K$ satisfies (1.1) with a constant $\overline{\mu}$ independent of $K$, (1.11) shows that

\begin{equation}
k(a_1 ||z||_{L^2}^{\overline{\mu}} - a_3) \leq \overline{M} ||z||_{L^2}^2
\end{equation}
By the Hölder inequality we find that

\[(1.13) \quad k(a_4 |z| \|\vec u\|_{L^2} - a_3) \leq \bar M |z| \|_{L^2}^2 \]

which implies that

\[(1.14) \quad \|z\|_{L^2} \leq \bar M_1 \]

with \(\bar M_1\) independent of \(m, k, \beta, K\). Returning to \((1.9)\) and using \((1.14)\) yields

\[(1.15) \quad c_{mk} = I(u_{mk}; k, \beta, K) \leq \bar M_2 k \]

with \(\bar M_2\) independent of \(m, k, \beta, K\). It follows that \(c_k\) satisfies the same estimate:

\[(1.16) \quad c_k = I(u_k; k, 0, K) \leq \bar M_2 k \]

To complete the proof of Theorem 1.2, we will show that
\[(1.16)\) is violated if more than finitely many solutions \(u_k\) correspond to the same function in the original \(t\) variables. To present the idea in its simplest setting, suppose first that all of the functions \(u_k(x, \tau)\) are reparameterizations of \(u_1(x,t)\). Then \(u_k(x, \tau) = u_1(x, k\tau) = u_1(x,t) \equiv u(x,t)\). For \(K = K(k)\) sufficiently large we have
(1.17) \[ c_k = \int_0^{2\pi} \int_0^\pi \left[ \frac{1}{2} u_{kt}^2 - k^2 \left( \frac{u_{kx}}{2} + F(x, kt, u_k) \right) \right] dx \, dt \]

\[ = k \int_0^{2\pi k} \int_0^\pi \left[ \frac{1}{2} u_t^2 - u_x^2 \right] dx \, dt \]

\[ = k^2 \int_0^{2\pi} \int_0^\pi \left[ \frac{1}{2} u_t^2 - u_x^2 \right] dx \, dt \]

\[ = k^2 c_1 \]

since \( u \) is \( 2\pi \) periodic in \( t \). The positivity of \( c_1 \) and (1.17) show that \( c_k \) tends to infinity like \( k^2 \) contrary to the bound (1.16). This argument shows (1.3) has at least one \( 2\pi k \) periodic solution distinct from \( u_1(x, t) \).

For the general case we argue similarly. Suppose two solutions \( u_j(x, \tau) \) and \( u_k(x, \tau) \) correspond to the same function of \( (x, t) \), i.e. \( u_j(x, \tau) = u_j(x, \frac{t}{j}) \equiv v(x, t) \equiv u_k(x, \frac{t}{k}) \). Thus \( u_j(x, \tau) = v(x, j\tau) \) and \( u_k(x, \tau) = v(x, k\tau) \). Since \( v(x, t) \) is both \( 2\pi j \) and \( 2\pi k \) periodic in \( t \), there are \( j_1, k_1, \sigma \in \mathbb{N} \) such that \( j = \sigma j_1, k = \sigma k_1 \) and \( v \) is \( 2\pi \sigma \) periodic in \( t \). (We can take \( \sigma \) to be the greatest common divisor of \( j \) and \( k \).) Arguing as in (1.17) yields

(1.18) \[ c_k = k \int_0^{2\pi \sigma} \int_0^\pi \left[ \frac{1}{2} (v_t^2 - v_x^2) - F(x, t, v) \right] dx \, dt \]

\[ = k^2 \frac{2\pi \sigma}{\sigma} \int_0^{2\pi \sigma} \int_0^\pi \left[ \frac{1}{2} (v_t^2 - v_x^2) - F(x, t, v) \right] dx \, dt \]

\[ = \frac{k^2}{\sigma} A \]

and
Thus if there is a sequence $u_{k_i}$ of solutions of (1.4) corresponding to the same function $v$, by (1.18) - (1.19) we have

\begin{equation}
(1.20)
\frac{k_i^2}{\sigma^A}
\end{equation}

where $\sigma \in \mathbb{N}$ is the greatest common divisor of $\{k_i\}$. Hence $c_{k_i} \to \infty$ like $k_i^2$ contrary to (1.16) and the proof of Theorem 1.2 is complete.

Remark 1.21: Note that if $F(x,t,r)$ and $F_K$ satisfy

$$F, F_K \geq a_1 |r|^\nu$$

for some $\nu > 2$, it follows from (1.11) that

$$\|z\|_2 \leq a_5^\frac{1}{\nu-2}$$

and therefore

$$c_k \leq a_6^\frac{1-2}{\nu-2} = a_6^\frac{\nu-4}{\nu-2}$$

Thus if $\nu < 4$, $c_k \to 0$ as $k \to \infty$. Further restrictions on $F$ (as in [1]) imply $u_k \to 0$ as $k \to \infty$. 

Remark 1.22: Existence of infinitely many distinct subharmonic solutions was also established in [1] for a family of subquadratic Hamiltonian systems, i.e. Hamiltonian systems where $H$ grows less rapidly than quadratically as $|z| \to \infty$. There are several existence theorems for periodic solutions of semi-linear wave equations in which the primitive of the forcing term is subquadratic [6-10]. We believe the conclusions of this paper carry over to the subquadratic case via the arguments used here and in [1].
References


