A LOWER BOUND FOR ON-LINE ONE-DIMENSIONAL BIN PACKING ALGORITHM--ETC(U)

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## A Lower Bound for On-Line One-Dimensional Bin Packing Algorithms

Let $L = (p_1, p_2, \ldots, p_n)$ be a list of real numbers in the interval $(0, 1]$. The one-dimensional bin packing problem is to place the $p_i$'s into a minimum number of unit-capacity bins. For any algorithm $A$, let $A(L)$ denote the number of bins used by $A$ in packing $L$ and let $OPT(L)$ denote the minimum number of bins needed to pack $L$. It is shown that, for any on-line algorithm $A$,

$$\lim_{n \to \infty} \left( \frac{A(L)}{OPT(L)} \right) > 1.536.$$
A LOWER BOUND FOR ON-LINE ONE-DIMENSIONAL BIN PACKING ALGORITHMS

by

Donna J. Brown

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A LOWER BOUND FOR ON-LINE ONE-DIMENSIONAL BIN PACKING ALGORITHMS

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Abstract

Let \( L = (p_1, p_2, \ldots, p_n) \) be a list of real numbers in the interval \((0, 1]\). The one-dimensional bin packing problem is to place the \( p_i \)'s into a minimum number of unit-capacity bins. For any algorithm \( A \), let \( A(L) \) denote the number of bins used by \( A \) in packing \( L \) and let \( \text{OPT}(L) \) denote the minimum number of bins needed to pack \( L \). It is shown that, for any on-line algorithm \( A \),

\[
\lim_{n \to \infty} \frac{\max A(L)}{\text{OPT}(L)} = 1.536.
\]

This work was supported by the Joint Services Electronics Program (U.S. Army, U.S. Navy, and U.S. Air Force) under contract N00014-79-C-0424.

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I. Introduction

Let \( L = (p_1, p_2, \ldots, p_n) \) be a list of real numbers in the interval \((0, 1]\). The one-dimensional bin packing problem is to place the \( p_i \)'s into a minimum number of unit-capacity bins; i.e., the sum of the numbers in each bin can be at most 1. Because this problem is known to be NP-hard [8], much work has been done in the study of heuristic algorithms with guaranteed performance bounds [12, 13, 14, 16].

In this paper we are concerned with algorithms for which the pieces (numbers) in list \( L \) are available one at a time, and each piece must be placed in some bin before the next piece is available; such an algorithm is referred to as on-line [12, 13, 16]. The performance measure used is the ratio of the number of bins used by an algorithm \( A \) in packing list \( L \), \( A(L) \), to the optimum (minimum) number of bins required to pack the list, \( \text{OPT}(L) \).

Example 1. Consider the list \( L_1 = (3/4, 1/6, 1/6, 2/3, 1/4) \). One possible packing algorithm is the well known First-Fit (FF) Algorithm [12, 13, 14], which places each piece in the first bin which has enough available space. As shown in Figure 1a, this algorithm leads to a packing which uses three bins. An optimal packing requires only two bins (see Figure 1b). Notice that \( \text{FF}(L_1) = \frac{3}{2} \text{OPT}(L_1) \).

We are interested, however, in the ratio \( \frac{A(L)}{\text{OPT}(L)} \) for lists \( L \) with many pieces. In particular, we wish to determine a lower bound on the performance ratio

\[
\lim_{n \to \infty} \max_{\text{OPT}(L) = n} \frac{A(L)}{\text{OPT}(L)}.
\]
a) Packing $L_1$ by the First-Fit Algorithm: $FF(L_1) = 3$.

b) An optimal packing of $L_1$: $OPT(L_1) = 2$.

Figure 1. Packings of $L_1$ from Example 1.
Example 2. For $n$ even, let the list $L_2$ consist of $n$ pieces of size $3/8$ and $n$ pieces of size $5/8$. The First-Fit Algorithm uses $\frac{3n}{2}$ bins, compared to an optimal packing of $n$ bins (see figures 2a and 2b). Thus, we know that, for the First-Fit Algorithm,

$$\text{FF}(L_2) \geq \frac{3}{2} \text{OPT}(L_2).$$

(In fact, it is known [12,13], that there is a list $L$ for which $\text{FF}(L) = \frac{17}{10} \text{OPT}(L)$.)

We shall show that there is no algorithm which can always use fewer than $1.536 \text{OPT}(L)$ number of bins. Thus, for any packing algorithm $A$,

$$\lim_{n \to \infty} \left[ \max_{\text{OPT}(L) = n} \frac{A(L)}{\text{OPT}(L)} \right] > 1.536$$

This lower bound is an improvement over the bound of 1.5 proved by Yao [16].

On the upper bound side, Yao in [16] gave an algorithm with a performance ratio of $5/3$, an improvement over the $17/10$ of the First-Fit Algorithm. Brown [41] has an algorithm with a slightly better performance ratio of about 1.65.

Much work has recently been done with two-dimensional bin packing. Various algorithms [1, 2, 3, 7, 9] have been proposed, many using ideas from one-dimensional packing algorithms [12,13,14]. Some work on two-dimensional lower bounds has also been done [5, 6, 15]. In particular, the 1.536 lower bound presented in this paper extends immediately to two dimensions and gives a 1.536 lower bound for any on-line two-dimensional algorithm which packs pieces in order of decreasing or increasing height or increasing width [6].
a) Packing $L_2$ by the First-Fit Algorithm: \( \text{FF}(L_2) = \frac{3n}{2} \).

b) An optimal packing of $L_2$: \( \text{OPT}(L_2) = n \).

Figure 2. Packings of $L_2$ from Example 2.
II. An Example

Yao [16] used a list consisting of pieces of sizes $\frac{1}{6} - 2\varepsilon$, $\frac{1}{3} + \varepsilon$, $\frac{1}{2} + \varepsilon$ in order to obtain his $\frac{3}{2}$ lower bound for any on-line bin packing algorithm. In this section we show that the result can be improved to $\frac{109}{71} > 1.535$ by considering a list with pieces sized $\frac{1}{42} - 3\varepsilon$, $\frac{1}{7} + \varepsilon$, $\frac{1}{3} + \varepsilon$, $\frac{1}{2} + \varepsilon$. In Section III the method is generalized to a list with pieces of $t$ different sizes. The work in this section is therefore only a special case of what will be shown, but it is presented here to illustrate the method and therefore make the proof of the main theorem easier to understand. (Also, $\frac{109}{71}$ is not much smaller than 1.536.)

Let $\varepsilon$ be a small positive number, $0 < \varepsilon < \frac{1}{43 \cdot 42 \cdot 3}$. For $n$ a multiple of 42, consider the list $L = L_1 L_2 L_3 L_4$, where

$L_1$ consists of $n$ pieces of size $\frac{1}{42} - 3\varepsilon$,
$L_2$ consists of $n$ pieces of size $\frac{1}{7} + \varepsilon$,
$L_3$ consists of $n$ pieces of size $\frac{1}{3} + \varepsilon$,
$L_4$ consists of $n$ pieces of size $\frac{1}{2} + \varepsilon$.

Noting that

$$\text{OPT}(L_1) = \frac{n}{42},$$
$$\text{OPT}(L_1 L_2) = \frac{n}{6},$$
$$\text{OPT}(L_1 L_2 L_3) = \frac{n}{2},$$
$$\text{OPT}(L) = n,$$

we can define the ratios

$$r_1(n) = \frac{A(L_1)}{\text{OPT}(L_1)} = \frac{42}{n} A(L_1),$$
$$r_2(n) = \frac{A(L_1 L_2)}{\text{OPT}(L_1 L_2)} = \frac{6}{n} A(L_1 L_2),$$

(2.1)
\[ r_3(n) = \frac{A(L_1L_2L_3)}{\text{OPT}(L_1L_2L_3)} = \frac{2}{n} A(L_1L_2L_3), \]
\[ r_4(n) = \frac{A(L)}{\text{OPT}(L)} = \frac{1}{n} A(L). \]

We shall prove that
\[ \max\{r_1(n), r_2(n), r_3(n), r_4(n)\} \geq \frac{109}{71}. \]

Let \( B \) denote the set of bins packed by an algorithm \( A \), after the pieces in \( L_1L_2L_3 \) have been packed. Each bin \( b_w \in B \) (\( 1 \leq w \leq |B| \)) contains \( m_{1,w} \) pieces of size \( \frac{1}{42} - 3\epsilon \), \( m_{2,w} \) pieces of size \( \frac{1}{7} + \epsilon \), and \( m_{3,w} \) pieces of size \( \frac{1}{3} + \epsilon \). (Note that \( m_{1,w}, m_{2,w}, \) and \( m_{3,w} \) are nonnegative integers, \( 0 \leq m_{1,w} \leq 42, 0 \leq m_{2,w} < 7, 0 \leq m_{3,w} < 3. \)) For notational convenience, we shall omit the double subscript and simply write \( m_j \) when we mean \( m_{j,w} \). We define the set of bins \( \alpha_i \) (\( 1 \leq i \leq 3 \)) as follows:
\[ \alpha_i = \{ b_w \in B | b_w \text{ is at least half full, } m_i \neq 0, \text{ and } m_j = 0 \text{ for } 1 \leq j < i \}. \]

In other words, a bin \( b_w \) is in
\[ \alpha_1 \text{ if } \frac{1}{42} m_1 + \frac{1}{7} m_2 + \frac{1}{3} m_3 > \frac{1}{2} \text{ and } m_1 \neq 0, \]
\[ \alpha_2 \text{ if } \frac{1}{7} m_2 + \frac{1}{3} m_3 > \frac{1}{2} \text{ and } m_2 \neq 0, m_1 = 0, \]
\[ \alpha_3 \text{ if } \frac{1}{3} m_3 > \frac{1}{2} \text{ and } m_3 \neq 0, m_1 = m_2 = 0. \]

Similar, we define \( \beta_i \) (\( 1 \leq i \leq 3 \)) to be:
\[ \beta_i = \{ b_w \in B | b_w \text{ is less than half full, } m_i \neq 0, \text{ and } m_j = 0 \text{ for } 1 \leq j < i \}. \]

Thus, a bin \( b_w \) is in
\[ \beta_1 \text{ if } \frac{1}{42}m_1 + \frac{1}{7}m_2 + \frac{1}{3}m_3 < \frac{1}{2} \text{ and } m_1 \neq 0 \]
\[ \beta_2 \text{ if } \frac{1}{7}m_2 + \frac{1}{3}m_3 < \frac{1}{2} \text{ and } m_2 \neq 0, m_1 = 0 \]
\[ \beta_3 \text{ if } \frac{1}{3}m_3 < \frac{1}{2} \text{ and } m_3 \neq 0, m_1 = m_2 = 0. \]

Letting \(|\alpha_1| (|\beta_1|)\) represent the number of bins in \(\alpha_1 (\beta_1)\), we have

\[ A(L_1) = |\alpha_1| + |\beta_1| \]
\[ A(L_1L_2) = |\alpha_1| + |\beta_1| + |\alpha_2| + |\beta_2| \] (2.2)
\[ A(L_1L_2L_3) = |\alpha_1| + |\beta_1| + |\alpha_2| + |\beta_2| + |\alpha_3| + |\beta_3| \]

Notice that no two pieces of size \(\frac{1}{2} + \epsilon\) will fit in the same bin, nor
will any of the \(n\) pieces of size \(\frac{1}{2} + \epsilon\) fit in an \(\alpha_1, \alpha_2, \) or \(\alpha_3\) bin, so

\[ A(L) \geq n + |\alpha_1| + |\alpha_2| + |\alpha_3|. \] (2.3)

Let us assume that

\[ \max[r_1(n), r_2(n), r_3(n), r_4(n)] < \frac{109}{71}. \] (2.4)

Combining equations (2.1), (2.2), and (2.3), this tells us

\[ \frac{n}{42} \cdot \frac{109}{71} > |\alpha_1| + |\beta_1| \]
\[ \frac{n}{6} \cdot \frac{109}{71} > |\alpha_1| + |\beta_1| + |\alpha_2| + |\beta_2| \]
\[ \frac{n}{2} \cdot \frac{109}{71} > |\alpha_1| + |\beta_1| + |\alpha_2| + |\beta_2| + |\alpha_3| + |\beta_3| \] (2.5)
\[ n \cdot \frac{109}{71} > |\alpha_1| + |\alpha_2| + |\alpha_3| + n \]

Because there are \(n\) pieces of size \(\frac{1}{42} - 3\epsilon\), \(n\) of size \(\frac{1}{7} + \epsilon\), and \(n\) of
size \(\frac{1}{3} + \epsilon\),
\[ n = \sum_{b_w \in B} m_1 \]
\[ n = \sum_{b_w \in B} m_2 \]
\[ n = \sum_{b_w \in B} m_3 \]

From (2.6), we immediately have

\[ -\frac{4}{42} n = -\frac{4}{42} \sum_{b_w \in B} m_1 \]
\[ -\frac{1}{2} n = -\frac{1}{2} \sum_{b_w \in B} m_2 \]
\[ -n = -\sum_{b_w \in B} m_3 \]

Summing equations (2.5) and (2.7),

\[ \frac{109}{71} n\left(\frac{1}{42} + \frac{1}{6} + \frac{1}{2} + 1\right) - n\left(\frac{4}{42} + \frac{1}{2} + 1\right) \]

\[ > 4|\alpha_1| + 3|\beta_1| + 3|\alpha_2| + 2|\beta_2| + 2|\alpha_3| + |\beta_3| + n \]

\[ -\frac{4}{42} \sum_{m_1} - \frac{1}{2} \sum_{m_2} - \sum_{m_3} \]

Simplifying inequality (2.8) and rearranging terms:
\[
\sum_{\beta \in B} \left( \frac{4}{42} m_1 + \frac{1}{2} m_2 + m_3 \right) > 4|\alpha_1| + 3|\beta_1| + 3|\alpha_2| + 2|\beta_2| + 2|\alpha_3| + |\beta_3|
\]

\[
\sum_{\beta \in \alpha_1} \left( \frac{4}{42} m_1 + \frac{1}{2} m_2 + m_3 \right) + \sum_{\beta \in \beta_1} \left( \frac{4}{42} m_1 + \frac{1}{2} m_2 + m_3 \right) \]

\[
+ \sum_{\beta \in \alpha_2} \left( \frac{1}{2} m_2 + m_3 \right) + \sum_{\beta \in \beta_2} \left( \frac{1}{2} m_2 + m_3 \right) + \sum_{\beta \in \alpha_3} m_3 + \sum_{\beta \in \beta_3} m_3
\]

\[
> 4|\alpha_1| + 3|\beta_1| + 3|\alpha_2| + 2|\beta_2| + 2|\alpha_3| + |\beta_3|.
\]  

(2.9)

By considering separately each of the summations on the left hand side, we show that inequality (2.9) gives a contradiction.

(a) For \( b_w \in \alpha_1 \):

\[
\frac{1}{42} m_1 + \frac{1}{7} m_2 + \frac{1}{3} m_3 \leq 1
\]

\[
\frac{4}{42} m_1 + \frac{1}{2} m_2 + m_3 < 4
\]

(b) For \( b_w \in \beta_1 \):

\[
\frac{1}{42} m_1 + \frac{1}{7} m_2 + \frac{1}{3} m_3 < \frac{1}{2}
\]

\[
\frac{4}{42} m_1 + \frac{1}{2} m_2 + m_3 < 2
\]

(c) For \( b_w \in \alpha_2 \):

\[
\frac{1}{7} m_2 + \frac{1}{3} m_3 \leq 1
\]

\[
m_2 + 2 m_3 \leq 6 + \frac{1}{7} m_2
\]

Since the left hand side is an integer, \( m_2 + 2m_3 \leq 6 \)

\[
\frac{1}{2} m_2 + m_3 \leq 3
\]
(d) For \( b \in \beta_2 \):
\[
\frac{1}{7} m_2 + \frac{1}{3} m_3 < \frac{1}{2}
\]
\[
\frac{1}{2} m_2 + m_3 < 2
\]

(e) For \( b \in \alpha_3 \):
\[
\frac{1}{3} m_3 < 1
\]
\[
m_3 \leq 2
\]

(f) For \( b \in \beta_3 \):
\[
\frac{1}{3} m_3 < \frac{1}{2}
\]
\[
m_3 \leq 1
\]

Combining (a) - (f),

\[
\sum_{b \in \alpha_1} \left( \frac{4}{42} m_1 + \frac{1}{2} m_2 + m_3 \right) + \sum_{b \in \beta_1} \left( \frac{4}{42} m_1 + \frac{1}{2} m_2 + m_3 \right)
\]
\[
+ \sum_{b \in \alpha_2} \left( \frac{1}{2} m_2 + m_3 \right) + \sum_{b \in \beta_2} \left( \frac{1}{2} m_2 + m_3 \right) + \sum_{b \in \alpha_3} m_3 + \sum_{b \in \beta_3} m_3
\]

\[
< 4|\alpha_1| + 3|\beta_1| + 3|\alpha_2| + 2|\beta_2| + 2|\alpha_3| + |\beta_3|
\]

This contradicts inequality (2.9). The assumption in (2.4) must be incorrect, from which we conclude that

\[
\max \left\{ \frac{A(L_1)}{\text{OPT}(L_1)}, \frac{A(L_1 L_2)}{\text{OPT}(L_1 L_2)}, \frac{A(L_1 L_2 L_3)}{\text{OPT}(L_1 L_2 L_3)}, \frac{A(L)}{\text{OPT}(L)} \right\} \geq \frac{109}{71}
\]
III. The Main Result

Define the sequence of integers \( \{a_n\} \), for \( n \geq 1 \), by

\[
a_1 = 2
\]

\[
a_{n+1} = 1 + \frac{1}{a_1 a_2 \ldots a_n}
\]

Thus, \( \{a_n\} = \{2, 3, 7, 43, 1807, 3263443, \ldots\} \),

and notice that

\[
\sum_{i=1}^{n} \frac{1}{a_i} = \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{43} + \frac{1}{1807} + \ldots = 1.
\]

This sequence has been studied by Golomb [10,11] and it is conjectured that the closest approximation to 1 from below, which is a sum of \( k \) reciprocal integers, is given by

\[
\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_k} = 1 - \frac{1}{a_{k+1} a_{k+2} \ldots a_{k+1}}
\]

for every positive integer \( k \).

In the proof of our lower bound result, we shall make use of the following simple lemma.
Lemma. Let \( \{a_k\} \) be the sequence of integers defined above in (1). Then, for \( 1 \leq k \leq j \),

\[
\frac{j + 1}{a_k} \geq \frac{k}{a_k - 1}
\]

Proof:

We first observe that

\[
a_k \geq k + 1
\]

Then

\[
(k+1)a_k - (k+1) \geq k a_k
\]

\[
\frac{k+1}{a_k} \geq \frac{k}{a_k - 1}
\]

and so, for \( j \geq k \),

\[
\frac{j + 1}{a_k} \geq \frac{k}{a_k - 1}.
\]

Motivated by the work in Section II, we now state and prove our main result.

**Theorem.** For any on-line one-dimensional packing algorithm \( A \),

\[
\lim_{n \to \infty} \max_{OPT(L) = n} \frac{A(L)}{OPT(L)} \geq \frac{1}{\sum_{i=1}^{t} \frac{1}{a_i - 1}} > 1.5363
\]

Proof:

For any positive integer \( t \geq 3 \), let \( \epsilon \) be a small fixed number,

\[
0 < \epsilon < \frac{1}{a_\epsilon (a_\epsilon - 1)(t-1)}.
\]
We define pieces $p_1, \ldots, p_t$ to be of sizes

\[ p_1 = \frac{1}{a_t-1} - (t-1)\epsilon \]

and

\[ p_j = \frac{1}{a_{t+1-j}} + \epsilon, \]

for $2 \leq j \leq t$. Consider the list $L = L_1 L_2 \ldots L_t$, where each $L_i$ consists of $n$ pieces of size $p_i$, for $n$ some multiple of $a_t - 1$. Then, for

\[ 1 \leq k \leq t, \]

\[ \text{OPT}(L_1 L_2 \ldots L_k) = \frac{n}{a_{t+1-k}} \]

(3.2)

and we can define the ratios

\[ r_k(n) = \frac{A(L_1 L_2 \ldots L_k)}{\text{OPT}(L_1 L_2 \ldots L_k)}. \]

(3.3)

We shall prove that

\[ \max_{1 \leq k \leq t} \{ r_k(n) \} \geq R_t, \]

(3.4)

where

\[ R_t = \frac{\sum_{i=1}^{t-1} \frac{i}{a_i-1}}{\sum_{i=1}^{t-1} \frac{1}{a_i-1}}. \]

(3.5)

Let $B$ denote the set of bins packed by an algorithm $A$, after the $(t-1)n$ pieces in list $L_1 L_2 \ldots L_{t-1}$ have been packed. Each bin $b_w \in B$ ($1 \leq w \leq |B|$) contains $m_{i,w}$ pieces of size $p_i$, for all $1 \leq i \leq t-1$. For
notational convenience, we shall omit the double subscript and simply write
\( m_i \) when we mean \( m_{i,j} \). Note that \( 0 \leq m_j < a_{t+1-j} \), for \( 1 \leq j \leq t-1 \). For
\( l \leq k \leq t-1 \), the set \( \alpha_k \) is defined to consist of those bins \( b_w \in B \) which are
at least half full and in which the smallest piece has size \( p_k \). Similarly,
we define \( \beta_k \) to be the set of bins \( b_w \in B \) which are less than half full and in
which the smallest piece has size \( p_k \). So \( |\alpha_k|(|\beta_k|) \) represents the number of
bins in \( \alpha_k (\beta_k) \), and, for \( 1 \leq k \leq t-1 \)

\[
A(L_1 L_2 \ldots L_k) = \sum_{i=1}^{k} (|\alpha_i| + |\beta_i|).
\]  

(3.6)

Having packed \( L_1 L_2 \ldots L_{t-1} \), we note that it will not be possible to
place any of the remaining \( n \) pieces of size \( p_\ell \) in any \( \alpha_k \) bin. So we
also have

\[
A(L_1 L_2 \ldots L_\ell) \geq n + \sum_{i=1}^{t-1} |\alpha_i|.
\]  

(3.7)

Let us assume that

\[
\max_{1 \leq i \leq t} \{r_i(n)\} < R_\ell.
\]  

(3.8)

Making use of equations (3.2), (3.3), (3.6), and (3.7), this assumption
leads to the following inequalities, for \( 1 \leq k \leq t-1 \):

\[
\frac{n}{a_{t+1-k-1}} \cdot R_\ell > \sum_{i=1}^{k} (|\alpha_i| + |\beta_i|)
\]  

\[
n \cdot R_\ell > n + \sum_{i=1}^{t-1} |\alpha_i|
\]  

(3.9)
Because there are $n$ pieces of each size $p_i$, we note that

$$n = \sum_{b \in B} \frac{m}{t-k+1}$$

for all $k$ in the range $2 \leq k \leq t$. Thus,

$$- \frac{k}{\alpha_k} \cdot n = - \frac{k}{\alpha_k} \sum_{b \in B} m_{t-k+1} \quad (3.10)$$

Summing equations (3.9) and (3.10) over $k$ gives

$$nR_t \sum_{k=1}^{t-1} \frac{1}{a_{t+1-k}} + nR_t - n \sum_{k=2}^{t} \frac{k}{\alpha_k}$$

$$> \sum_{k=1}^{t-1} \sum_{i=1}^{k} (|\alpha_i| + |\beta_i|) + n + \sum_{i=1}^{t-1} |\alpha_i| - \sum_{k=2}^{t} \frac{k}{\alpha_k} \sum_{b \in B} m_{t-k+1}$$

From (3.5), we observe that

$$R_t = \frac{1 + \sum_{k=2}^{t} \frac{k}{\alpha_k}}{1 + \sum_{k=1}^{t-1} \frac{1}{a_{t+1-k}}}$$

and so inequality (3.11) can be simplified to give

$$\sum_{k=2}^{t-1} \frac{k}{\alpha_k} \sum_{b \in B} m_{t-k+1} > \sum_{k=1}^{t-1} \sum_{i=1}^{k} (|\alpha_i| + |\beta_i|) + \sum_{i=1}^{t-1} |\alpha_i| \quad (3.12)$$

Inequality (3.12) further simplifies to give
The remainder of this proof consists of showing that (3.13) gives a contradiction. In particular, we shall show that

$$\sum_{k=2}^{t} \frac{k}{a_{k-1}^{m_{t-k+1}}} \geq \sum_{j=1}^{t-1} ((j+1)|\alpha_{t-j}| + j|\beta_{t-j}|)$$

(3.13)

for any bin $b \in \alpha_{t-j}$ ($1 \leq j \leq t-1$) and that

$$\sum_{k=2}^{t} \frac{k}{a_{k-1}^{m_{t-k+1}}} \leq j + 1$$

(3.14)

for any bin $b \in \beta_{t-j}$ ($1 \leq j \leq t-1$). From this we deduce that the assumption in (3.8) is incorrect, thereby proving the assertion of (3.4). The theorem follows immediately.

We first prove assertion (3.14). For $b \in \alpha_{t-j}$, then

$$p_1 m_1 + p_2 m_2 + \ldots + p_{t-1} m_{t-1} \leq 1$$

(3.16)

and $p_{t-j} m_{t-j}$ is the first nonzero term. There are two cases.

1. Assume that $j \leq t-2$. Then

$$\sum_{i=1}^{j+1} \frac{1}{a_{i}^{m_{t-i+1}}} \leq 1$$

and

$$\sum_{i=2}^{j} \frac{1}{a_{i}^{m_{t-i+1}}} \leq 1 + \frac{1}{a_{j+1}^{m_{t-j}}}$$

(3.17)
Recalling that $m_j < a_{t+1-j}$, then we know

$$m_{t-j} < a_{j+1} \quad (3.18)$$

Also, as a consequence of (3.1),

$$a_{j+2} - 1 = a_{j+1}(a_{j+1} - 1) \quad (3.19)$$

Using (3.18) and (3.19), inequality (3.17) gives

$$\frac{1}{a_{j+1}-1} m_{t-j} + \sum_{i=2}^{j+1} \frac{1}{a_i} m_{t-i+1} < 1 + \frac{1}{a_{j+1}-1} \quad (3.20)$$

From (3.1), we note that $a_{j+1} - 1$ is divisible by $a_i$, for all $i \leq j$. Thus, the left hand side of (3.20) is a multiple of $\frac{1}{a_{j+1}-1}$, and we have

$$\frac{1}{a_{j+1}-1} m_{t-j} + \sum_{i=2}^{j+1} \frac{1}{a_i} m_{t-i+1} \leq 1.$$ 

Thus,

$$\frac{j+1}{a_{j+1}-1} m_{t-j} + \sum_{i=2}^{j+1} \frac{j+1}{a_i} m_{t-i+1} \leq j+1.$$ 

Applying the Lemma,

$$\frac{j+1}{a_{j+1}-1} m_{t-j} + \sum_{i=2}^{j+1} \frac{j}{a_i-1} m_{t-i+1} \leq j+1$$

and we have proved inequality (3.14) for $j \leq t-2.$
(ii) Assume that \( j = t - 1 \); i.e., \( b_w \in \alpha_1 \). Since \( p_i > \frac{1}{a_{t+1-i}} \) for \( 2 \leq i \leq t - 1 \), we conclude from (3.16) that

\[
\frac{1}{a_{t-1}} - (t-1)\epsilon m_1 + \sum_{i=2}^{t-1} \frac{1}{a_i} m_{t-i+1} < 1.
\]

Recalling how we chose \( \epsilon \),

\[
\frac{1}{a_{t-1}} m_1 + \sum_{i=2}^{t-1} \frac{1}{a_i} m_{t-i+1} < 1 + \frac{m_1}{a_t(a_t-1)}
\]  

(3.21)

Because \( m_1 \leq a_t - 1 \), the right hand side of (3.21) is less than \( 1 + \frac{1}{a_t} \).

As in case (i), we also note that the left hand side of (3.21) is a multiple of \( \frac{1}{a_{t-1}} \) and that \( \frac{1}{a_{t-1}} > \frac{1}{a_t} \). Thus,

\[
\frac{1}{a_{t-1}} m_1 + \sum_{i=2}^{t-1} \frac{1}{a_i} m_{t-i+1} \leq 1
\]  

(3.22)

Similar to case (i), we multiply both sides of (3.22) by \( t \) and apply the Lemma in order to obtain the desired result:

\[
\sum_{i=2}^{t} \frac{i}{a_{i-1}} m_{t-i+1} \leq t.
\]

We now prove assertion (3.15). For \( b_w \in \beta_{t-j} \), then

\[
p_1 m_1 + p_2 m_2 + \ldots + p_{t-1} m_{t-1} < \frac{1}{2}
\]

and \( m_{t-j} \) is the first nonzero term. There are two cases.
(i) Assume that \( j \leq t - 2 \). Then

\[
\sum_{i=2}^{i+1} \frac{1}{a_i} m_{t-i+1} < \frac{1}{2} \tag{3.23}
\]

Multiplying both sides of (3.23) by \( j + 2 \) and then applying the Lemma,

\[
\sum_{i=2}^{j+1} \frac{i}{a_i} m_{t-i+1} < \frac{1+2}{2}. \tag{3.24}
\]

For \( j = 2 \), \( \frac{1+2}{2} \leq j \) and the result is proved. For \( j = 1 \), (3.24) reduces to \( m_{t-1} < \frac{3}{2} \). Since \( m_{t-1} \) is an integer, this says \( m_{t-1} \leq 1 \) and once again the desired result holds.

(ii) Assume that \( j = t - 1 \); i.e., \( b_w \in \beta_1 \). Similar to inequality (3.21), we have

\[
\frac{1}{a_{t-1} m_1} + \sum_{i=2}^{t-1} \frac{1}{a_i} m_{t-i+1} < \frac{1}{2} + \frac{1}{a_t} \tag{3.25}
\]

Multiplying both sides of (3.25) by \( t \) and applying the Lemma,

\[
\sum_{i=2}^{t} \frac{i}{a_i} m_{t-i+1} < \frac{t}{2} + \frac{t}{a_t}
\]

For \( t \geq 3 \),

\[
\frac{t}{a_t} < \frac{t-2}{2}
\]

and so

\[
\sum_{i=2}^{t} \frac{i}{a_i} m_{t-i+1} < t - 1
\]

and the theorem is proved. \( \blacksquare \)
References


