CONFIDENCE INTERVALS ON A RATIO OF VARIANCES IN THE TWO-FACTOR -- ETC(U)

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CONFIDENCE INTERVALS ON A RATIO OF VARIANCES
IN THE TWO-FACTOR NESTED COMPONENTS OF VARIANCE MODEL

by

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Technical Report Number 3
May 1, 1980

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PREPARED UNDER CONTRACT
N00014-78-C-0463 (NR 042-402)
FOR THE OFFICE OF NAVAL RESEARCH
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ABSTRACT

Consider the two-factor nested components of variance model

\[ Y_{ijk} = \mu + A_i + B_{ij} + C_{ijk}, \]

where \( \text{Var}[A_i] = \sigma_A^2, \text{Var}[B_{ij}] = \sigma_B^2, \)

\( \text{Var}[C_{ijk}] = \sigma_C^2. \)

Confidence intervals are derived for \( \frac{\sigma_A^2}{\sigma_C^2}, \frac{\sigma_B^2}{(\sigma_A^2 + \sigma_C^2)} \) and \( \frac{\sigma_C^2}{(\sigma_A^2 + \sigma_C^2)}. \)

KEY WORDS: Confidence intervals on ratios of variances.
1. Introduction

Consider the two-factor nested components-of-variance model given by

\[ Y_{ijk} = \mu + A_i + B_{ij} + C_{ijk} \]

for

\[ i = 1, 2, \ldots, I > 1; j = 1, 2, \ldots, J > 1; \text{ and } k = 1, 2, \ldots, K > 1; \]

where \( E[A_i] = 0; \) \( \text{Var}[A_i] = \sigma_A^2; \) \( E[B_{ij}] = 0; \) \( \text{Var}[B_{ij}] = \sigma_B^2; \) \( E[C_{ijk}] = 0; \)

and \( \text{Var}[C_{ijk}] = \sigma_C^2. \) The random variables \( Y_{ijk} \) are observable; the random variables \( A_1, \ldots, A_I; B_{11}, \ldots, B_{IJ}; C_{11}, \ldots, C_{IJK} \) are pairwise uncorrelated and unobservable and are jointly normally distributed; \( \mu, \sigma_A^2, \sigma_B^2, \) and \( \sigma_C^2 \) are unobservable parameters. The parameter space \( \Omega \) is defined by

\[ \Omega = \{ (\mu, \sigma_A^2, \sigma_B^2, \sigma_C^2): -\infty < \mu < \infty, \sigma_A^2 \geq 0, \sigma_B^2 \geq 0, \sigma_C^2 \geq 0 \}. \]

These specifications define a two-factor nested components-of-variance model with equal numbers in the subclasses and the ANOVA table is displayed in Table 1.

Table 1.

ANOVA table for two-factor nested components-of-variance model with equal numbers in the subclasses

<table>
<thead>
<tr>
<th>Source</th>
<th>d.f.</th>
<th>S.S.</th>
<th>M.S.</th>
<th>E.M.S.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total</td>
<td>IJK</td>
<td>( \sum Y^2_{ijk} )</td>
<td></td>
<td>( \sum Y^2_{ijk} )</td>
</tr>
<tr>
<td>Mean</td>
<td>1</td>
<td>( IJK \bar{Y}^2 )</td>
<td></td>
<td>( IJK \bar{Y}^2 )</td>
</tr>
<tr>
<td>Factor A</td>
<td>( n_1 = I-1 )</td>
<td>( \sum (\bar{Y}_{i..} - \bar{Y})^2 )</td>
<td>( S^2_1 )</td>
<td>( \theta_1 = \sigma_A^2 + \sigma_B^2 + \sigma_C^2 )</td>
</tr>
<tr>
<td>B within A</td>
<td>( n_2 = I(J-1) )</td>
<td>( \sum (\bar{Y}<em>{ij..} - \bar{Y}</em>{i..})^2 )</td>
<td>( S^2_2 )</td>
<td>( \theta_2 = \sigma_B^2 + \sigma_C^2 )</td>
</tr>
<tr>
<td>Error</td>
<td>( n_3 = IJ(K-1) )</td>
<td>( \sum (Y_{ijk} - \bar{Y}_{ij..})^2 )</td>
<td>( S^2_3 )</td>
<td>( \theta_3 = \sigma_C^2 )</td>
</tr>
</tbody>
</table>
In this model there are several functions of the variance components that may be of interest in applied problems. These include
\[ \sigma^2_A, \sigma^2_B, \sigma^2_C, \sigma^2_C/(\sigma^2_C + \sigma^2_B), \sigma^2/(\sigma^2_C + \sigma^2_A), \sigma^2/(\sigma^2_A + \sigma^2_B + \sigma^2_C), \sigma^2_B/(\sigma^2_A + \sigma^2_B + \sigma^2_C), \]
and \[ \sigma^2_C/(\sigma^2_A + \sigma^2_B + \sigma^2_C). \]
The only functions of \( \sigma^2_A, \sigma^2_B, \sigma^2_C \) given above for which an exact size confidence interval exists is \( \sigma^2_C \) and \( \sigma^2_C/(\sigma^2_C + \sigma^2_B) \).

Approximate size confidence intervals for \( \sigma^2_A \) and \( \sigma^2_B \) have been given by Moriguti (1954), Bulmer (1956) and Howe (1974). Approximate size confidence intervals for \( \sigma^2_A/(\sigma^2_A + \sigma^2_B + \sigma^2_C), \sigma^2_B/(\sigma^2_A + \sigma^2_B + \sigma^2_C) \) and \( \sigma^2_C/(\sigma^2_A + \sigma^2_B + \sigma^2_C) \) have been given by Graybill and Wang (1979). In this paper we give approximate size confidence intervals for \( \sigma^2_C/(\sigma^2_A + \sigma^2_B), \sigma^2_A/(\sigma^2_A + \sigma^2_B), \sigma^2_A/\sigma^2_C \) and \( \sigma^2_C/\sigma^2_A \).

Actually we obtain approximate size confidence intervals for \( \sigma^2_A/\sigma^2_C \) only since \( \sigma^2_A/\sigma^2_C, \sigma^2_A/(\sigma^2_A + \sigma^2_C), \) and \( \sigma^2_C/(\sigma^2_A + \sigma^2_C) \) can be obtained from these.

In Section 2 the lower limit of the upper confidence interval is derived, in Section 3 the upper limit of the lower confidence interval is given, and in Section 4 is a short discussion of other methods that could possibly be used for confidence intervals on \( \sigma^2_A/\sigma^2_C \).
2. **Lower Limit of the Upper Confidence Interval on \( \sigma_A^2 / \sigma_C^2 \)**

Since \( Y \ldots, S_1^2, S_2^2, \) and \( S_3^2 \) are complete sufficient statistics for this problem, we will require the upper confidence interval to be a function of them. Write

\[
g(\bar{Y} \ldots, S_1^2, S_2^2, S_3^2) < \frac{\theta_1 - \theta_2}{\theta_3} < \infty
\]

for the \( 1 - \alpha \) upper confidence interval where the function \( g(\bar{Y} \ldots, S_1^2, S_2^2, S_3^2) \), the lower confidence point, is to be determined.

Using the notation in Table 1 observe that \( \frac{\theta_1 - \theta_2}{\theta_3} = \frac{JK\sigma_B^2}{\sigma_C^2} \), so an upper confidence interval on \( \frac{\theta_1 - \theta_2}{\theta_3} \) is equivalent to an upper confidence interval on \( \frac{\sigma_A^2}{\sigma_C^2} \).

Since \( \sigma_A^2 / (\sigma_A^2 + \sigma_C^2) \) is a function of \( \theta_1, \theta_2, \theta_3 \) only, this is unchanged if any constant \( c \) is added to \( Y_{ijk} \) in the model given in Section 1. Thus the lower confidence point \( g(\bar{Y} \ldots, S_1^2, S_2^2, S_3^2) \) should also be unchanged if \( c \) is added to \( Y_{ijk} \). Let \( c = -\bar{Y} \ldots \); thus \( \bar{Y} \ldots + c \) is zero and \( S_1^2, S_2^2, S_3^2 \) are unchanged when \( Y_{ijk} \) is replaced by \( Y_{ijk} + c \) (or specifically by \( Y_{ijk} - \bar{Y} \ldots \)). Hence \( g(\bar{Y} \ldots, S_1^2, S_2^2, S_3^2) \) becomes \( g(0, S_1^2, S_2^2, S_3^2) \) and the lower confidence point is a function of \( S_1^2, S_2^2, \) and \( S_3^2 \) only. So the objective is to find a function of \( S_1^2, S_2^2, S_3^2 \), say \( f(S_1^2, S_2^2, S_3^2) \) such that

\[
P[f(S_1^2, S_2^2, S_3^2) < (\theta_1 - \theta_2)/\theta_3] \approx 1 - \alpha.
\]

If \( Y_{ijk} \) is replaced by \( cY_{ijk} \) for \( c \neq 0 \), then \( (\theta_1 - \theta_2)/\theta_3 \) is unchanged. Thus we require \( f(c^2S_1^2, c^2S_2^2, c^2S_3^2) = f(S_1^2, S_2^2, S_3^2) \). Let \( c^2 = 1/S_2^2 \), then \( f(S_1^2, S_2^2, S_3^2) = f(S_1^2/S_2^2, 1, S_3^2/S_2^2) = h(S_1^2/S_2^2, S_3^2/S_2^2) \), so the lower confidence point of \( (\theta_1 - \theta_2)/\theta_3 \) is a function of \( S_1^2/S_2^2 \) and \( S_3^2/S_2^2 \) only.
Since the maximum likelihood estimator of \( \frac{\theta_1 - \theta_2}{\theta_3} = \frac{\theta_1/\theta_2 - 1}{\theta_3/\theta_2} \) is of the form \( \frac{s_1^2 - s_2^2}{s_3^2} = \frac{s_1^2/s_2^2 - 1}{s_3^2/s_2^2} \), we require \( h(s_1^2/s_2^2, s_3^2/s_2^2) \) to be

(a) monotonic increasing in \( s_1^2/s_2^2 \); (b) monotonic decreasing in \( s_3^2/s_2^2 \).

Let \( \theta = \frac{s_2^2 - s_1^2}{s_3^2} \), then from Mood et al. (1974, p. 180).

\[
\text{Var}(\theta) = \text{Var}(\frac{s_1^2 - s_2^2}{s_3^2}) = \frac{2n_2^2}{n_1(n_3 - 2)^2} \frac{\theta_1^2}{\theta_3} + \frac{2n_2^3}{n_2(n_3 - 2)^2} \frac{\theta_2^2}{\theta_3^2} + \frac{2n_2^2}{(n_3 - 4)(n_3 - 2)^2} \left( \frac{\theta_1}{\theta_3} - \frac{\theta_2}{\theta_3} \right)^2 + \frac{4n_3^2}{n_1(n_3 - 4)(n_3 - 2)^2} \frac{\theta_2^2}{\theta_3^2} + \frac{4n_3^2}{n_2(n_3 - 4)(n_3 - 2)^2} \frac{\theta_2^2}{\theta_3^2}
\]

If we replace the \( \theta_1 \) by UMVU estimators and denote the resulting \( \text{Var}(\hat{\theta}) \) by \( \hat{\text{Var}}(\hat{\theta}) \), then \( \hat{\text{Var}}(\hat{\theta}) = c_1 s_1^2/s_3^2 + c_2 s_2^2/s_3^2 + (c_3 s_1^2/s_3^2 - c_4 s_2^2/s_3^2)^2 \) where \( c_1, c_2, c_3 \) and \( c_4 \) are appropriate constants which are functions of \( n_1, n_2, \) and \( n_3 \).
So a large sample lower confidence point for \( \frac{\hat{\theta}_1 - \hat{\theta}_2}{\hat{\theta}_3} \) is

\[
\hat{\theta} = N_a \sqrt{\text{Var}(\hat{\theta})} = \frac{s_1^2}{s_3^2} - N_a \left\{ c_1 \frac{s_1^4}{s_3^4} + c_2 \frac{s_2^4}{s_3^4} + (c_3 \frac{s_1^2}{s_3^2} - c_4 \frac{s_2^2}{s_3^2})^2 \right\}^{1/2} = \frac{s_2^2}{s_3^2} \left[ \frac{s_1^2}{s_2^2} - 1 - N_a \left\{ c_1 \left( \frac{s_1^2}{s_2^2} \right)^2 + c_2 + (c_3 \frac{s_1^2}{s_2^2} - c_4^2) \right\}^{1/2} \right] = \frac{s_2^2}{s_3^2} q \left( \frac{s_1^2}{s_2^2} \right)
\]

where \( N_a \) is the upper \( a \) probability point of a standard normal p.d.f.

Therefore, in general we require the lower confidence point, \( h(S_1^2/S_2^2, S_3^2/S_2^2) \), of \( \frac{\hat{\theta}_1 - \hat{\theta}_2}{\hat{\theta}_3} \) to be of the form \( \frac{s_2^2}{s_2^2} q \left( \frac{s_1^2}{s_2^2} \right) \), and we determine the function \( q \left( \frac{s_1^2}{s_2^2} \right) \) such that

\[
P\left[ -\frac{S_1^2}{S_3^2} q \left( \frac{s_1^2}{s_2^2} \right) \leq \frac{\hat{\theta}_1 - \hat{\theta}_2}{\hat{\theta}_3} \right] = 1 - a \tag{2.1}
\]

is close to \( 1 - a \). We require \( q \left( \frac{s_1^2}{s_2^2} \right) \) to satisfy (1), (2), (3) below.

1. When the hypothesis \( H_0: \sigma_A^2 = 0 \) vs. \( H_a: \sigma_A^2 > 0 \) is accepted for a size \( a \) test the confidence interval should include zero, and when \( H_0 \) is rejected, \( h(S_1^2/S_2^2, S_3^2/S_2^2) \) should be an increasing function of \( S_1^2/S_2^2 \).

To test \( H_0: \sigma_A^2 = 0 \) vs. \( H_a: \sigma_A^2 > 0 \) the hypothesis \( H_0 \) is accepted if and only if \( S_1^2/S_2^2 < F_a: n_1, n_2 \) (This test is uniformly most powerful unbiased). Thus

\[
h(S_1^2/S_2^2, S_3^2/S_2^2) = 0 \quad \text{when} \quad S_1^2/S_2^2 \leq F_a: n_1, n_2
\]

\[
h(S_1^2/S_2^2, S_3^2/S_2^2) > 0 \quad \text{and increasing in} \quad S_1^2/S_2^2 \quad \text{when} \quad S_1^2/S_2^2 > F_a: n_1, n_2
\]
Since \( h(S_1^2/S_2^2, S_3^2/S_2^2) = \frac{s_2^2}{s_3^2} q(s_1^2/s_2^2) \) we obtain

\[
q(s_1^2/s_2^2) = 0 \quad \text{when } S_1^2/S_2^2 \leq F\alpha: n_1, n_2
\]

\[
q(s_1^2/s_2^2) > 0 \text{ and increasing in } S_1^2/S_2^2 \quad \text{when } S_1^2/S_2^2 > F\alpha: n_1, n_2
\]

(2) When \( J \rightarrow \infty \) (hence \( n_2 \rightarrow \infty \) and \( n_3 \rightarrow \infty \) ) the confidence interval will be required to have an "exact" confidence coefficient \( 1 - \alpha \). When \( J \rightarrow \infty \) it follows that \( n_2 \rightarrow \infty \) and \( n_3 \rightarrow \infty \) and from this it follows that \( S_2^2 + \theta_2 \) in probability and \( S_3^2 + \theta_3 \) in probability. Start with

\[
P\left[ F_{\alpha: n_1, \infty} \right] = 1 - \alpha
\]

and use the result of \( J \rightarrow \infty \), i.e. replace \( S_2^2 \) and \( S_3^2 \) by their "equivalent" values \( \theta_2 \) and \( \theta_3 \) respectively, to obtain

\[
P\left[ \frac{S_2^2}{S_3^2} \left( \frac{S_1^2}{S_2^2} - 1 \right) \leq \frac{\theta_1 - \theta_2}{\theta_3} \right] = 1 - \alpha
\]

Hence when \( J \rightarrow \infty \)

\[
q(s_1^2/s_2^2) = 0 \quad \text{when } S_1^2/S_2^2 \leq F\alpha: n_1, \infty
\]

\[
q(s_1^2/s_2^2) = \frac{s_1^2}{s_2^2} - 1 \quad \text{when } S_1^2/S_2^2 > F\alpha: n_1, \infty
\]

(3) If \( \sigma_A^2 \rightarrow \infty \), the quantity \( \frac{\theta_1 - \theta_2}{\theta_3} \) is dominated by \( \theta_1/\theta_3 \), and we want

\[
P\left[ \frac{s_2^2}{s_3^2} \left( \frac{s_1^2}{s_2^2} \right) \leq \frac{\theta_1}{\theta_3} \right] = 1 - \alpha
\]

to be equal to \( 1 - \alpha \). This requires \( q(s_1^2/s_2^2) \) to behave like

\[
q(s_1^2/s_2^2) = \frac{s_1^2}{s_2^2} \left( 1 + c(s_1^2/s_2^2) \right) \quad \text{where}
\]

\[
q(s_1^2/s_2^2) = \frac{s_1^2}{s_2^2} \left( 1 + c(s_1^2/s_2^2) \right) \quad \text{where}
\]
\[ l(S_1^2/S_2^2) \to 0 \text{ as } S_1^2/S_2^2 \to \infty \]

Any function \( q(S_1^2/S_2^2) \) satisfying conditions (1), (2), (3) will give an exact confidence coefficient in the three limiting cases \( \theta_1/\theta_2 = 1, \theta_1/\theta_2 = \infty \) and \( J = \infty \).

The simplest function satisfying those conditions is the linear function

\[ q(S_1^2/S_2^2) = a_1 S_1^2/S_2^2 + b_1 \]

where \( a_1 \) and \( b_1 \) are functions of \( n_1, n_2, n_3 \), and \( \alpha \) and are determined by the conditions (1), (2), and (3). However, this did not give results as good as desired so a more general function was used, namely

\[ q(S_1^2/S_2^2) = [a_1 S_1^2/S_2^2 + b_1 + c_1(S_1^2/S_2^2)^{-1}] / F_\alpha: n_1, n_3 \]

(2.2)

From condition (3) \( a_1 = 1 \).

From condition (2) \( b_1(n_1, =, \infty) = -F_\alpha: n_1, =: c_1(n_1, =, \infty) = 0 \)

From condition (1) \( F_\alpha: n_1, n_2 + b_1 + c_1/F_\alpha: n_1, n_2 = 0 \) or

\[ c_1 = -F_\alpha: n_1, n_2 (F_\alpha: n_1, n_2 + b_1). \]

Let \( b_1(n_1, n_2, n_3) = -F_\alpha: n_1, = \) for all \( n_2 \) and \( n_3 \), then

\[ c_1 = F_\alpha: n_1, n_2 (F_\alpha: n_1, = -F_\alpha: n_1, n_2), \]

and

\[ q(S_1^2/S_2^2) = [S_1^2/S_2^2 - F_\alpha: n_1, = + F_\alpha: n_1, n_2 (F_\alpha: n_1, = - F_\alpha: n_1, n_2) S_2^2/S_1^2] / F_\alpha: n_1, n_3 \]

Thus a \( 1 - \alpha \) upper confidence interval on \((\theta_1 - \theta_2)/\theta_3\) is \( L_2 \leq (\theta_1 - \theta_2)/\theta_3 < \infty \)

where \( L_2 \) is defined by

\[ L_2 = 0 \quad \text{if} \quad S_1^2/S_2^2 \leq F_\alpha: n_1, n_2 \]

(2.3)

\[ L_2 = \frac{S_2^2}{S_3^2 F_\alpha: n_1, n_3} \left[ S_1^2/S_2^2 - F_\alpha: n_1, = + F_\alpha: n_1, n_2 (F_\alpha: n_1, = - F_\alpha: n_1, n_2) S_2^2/S_1^2 \right] \]

\[ \text{if} \quad S_1^2/S_2^2 > F_\alpha: n_1, n_2 \]
Note that $L_2 = 0$ if and only if the $\alpha$ level test of $H_0: \sigma_A^2 = 0$
is accepted, so $P[L_2 = 0] = P[S_1^2/S_2^2 < F_{n_1, n_2}] < 1 - \alpha$ and $P[L_2 = 0] = 1 - \alpha$
if and only if $\sigma_A^2 = 0$. The probability associated with Equation (2.3) is a function
of the unknown parameter $\rho = \theta_1/\theta_2$ and is exactly equal to $1 - \alpha$ when $\rho$ is one
or infinity or when $J$ is infinite.

The excellence of this approximation is indicated by Table 2, calculated by
simulation. Columns 7, 8, and 9 of Table 2 contain the range of probabilities of
$L_2 \leq (\theta_1 - \theta_2)/\theta_3$ as the unknown parameter $\theta_1/\theta_2$ varies from 1 to $\infty$. The
approximation appears to be quite satisfactory even for small sample sizes.

The remainder of this section is devoted to the study of the behavior of
$P = \frac{\sigma_A^2}{\sigma^2} q(S_1^2/S_2^2) \leq \frac{\theta_1 - \theta_2}{\theta_3}$ for all values of $n_1, n_2,$ and $n_3$. From Table 2

$P$ appears to get closer to $1 - \alpha$ as the value of $K$ (hence $n_3$) increases.
In fact as $K \to \infty$ (hence $n_3 \to \infty$) the problem is reduced to the interval
estimation of $\sigma_A^2$ in the one-factor model and the method discussed in this
section is equivalent to Moriguti's method (1954). From this one knows that the
error in $P$ is of the order $n_2^{-2}$, i.e. $P = 1 - \alpha + O(n_2^{-2})$. Another way to
examine the behavior of $P$ is to expand $P$ in powers of $n_2^{-1}$ and $n_3^{-1}$.
The algebraic details of this work are heavy (see Bulmer (1957)). The resulting
expansion is

$$P = 1 - \alpha + a_0 + a_{12}/n_2 + a_{13}/n_3 + a_{22}/n_2^2 + a_{33}/n_3^2 + a_{23}/n_2n_3 + O(n_2^{-2}, n_3^{-3}).$$

This assures that as the values of $J$ and $K$ (hence $n_2$ and $n_3$) increase
the accuracy of the approximation gets better.

In Table 2, $P$ is between 0.9500 and 0.9597 when $I = 3$, $J = 3$, $K = 3$, and
$1 - \alpha = 0.95$ and when $I = 7$, $J = 3$, $K = 3$, $P$ is between 0.9500 and 0.9581.
A study of the values of $P$ when $I$ is large ($n_1$, $n_2$, and $n_3$ are large, but
$R_1 = n_1/n_2$, $R_2 = n_1/n_3$ remain constant) is in Wang (1979).
3. Upper Limit of the Lower Confidence Interval on $\sigma^2_A/\sigma^2_C$

Since $P[\frac{\theta_1 - \theta_2}{\theta_3} < f(S^2_1, S^2_2, S^2_3)] = 1 - P[f(S^2_1, S^2_2, S^2_3) \leq \frac{\theta_1 - \theta_2}{\theta_3}]$, we use the confidence coefficient $\alpha$ in the lower limit of the upper confidence interval in Equation (2.3) to obtain a lower $1 - \alpha$ confidence interval on $(\theta_1 - \theta_2)/\theta_3$ given by $0 < (\theta_1 - \theta_2)/\theta_3 < U$ where

$$U = \frac{S^2_2}{S^2_2 F_{1-\alpha:n_1,n_3}} \left[ S^2_1/S^2_2 - F_{1-\alpha:n_1,n_2} - F_{1-\alpha:n_1,n_2} (F_{1-\alpha:n_1,n_2} - F_{1-\alpha:n_1,n_2}) S^2_2/S^2_1 \right]$$

if $S^2_1/S^2_2 > F_{1-\alpha:n_1,n_2}$  \hspace{1cm} (3.1)

$$U = 0$$

if $S^2_1/S^2_2 < F_{1-\alpha:n_1,n_2}$

We could determine how close the confidence coefficient of this confidence interval is to the nominal $1 - \alpha$ by simulation. However, due to the expense of computer simulation we chose a different route. We used $q_1(S^2_1/S^2_2) = a_{11} S^2_1/S^2_2 + b_1$ and conditions similar to (1), (2), (3) of Section 2 to obtain the confidence interval $0 < (\theta_1 - \theta_2)/\theta_3 < U_1$, where $U_1$ is given by

$$U_1 = 0$$

if $S^2_1/S^2_2 < F_{1-\alpha:n_1,n_2}$  \hspace{1cm} (3.2)

$$U_1 = \frac{S^2}{S^2_3} (S^2_1/S^2_2 F_{1-\alpha:n_1,n_3} - F_{1-\alpha:n_1,n_2}/F_{1-\alpha:n_1,n_2})$$

if $S^2_1/S^2_2 > F_{1-\alpha:n_1,n_2}$

Note that $U_1 = 0$ and $U = 0$ if and only if the $1 - \alpha$ level test of $H_0: \sigma^2_A = 0$ is accepted. Also note that conditions (2) and (3) of Section 2 are satisfied by the confidence intervals given in Equations (3.1) and (3.2).
The probability associated with Equation (3.2) depends on the value of \( \rho = \frac{\theta_1}{\theta_2} \), \( n_1, n_2, n_3 \) and can be easily calculated if \( n_1 \) is even; we get

\[
P[\frac{\theta_1 - \theta_2}{\theta_3} \leq \frac{S_2^2}{S_3^2} \left( \frac{S_1^2}{S_2^2 F_{1-\alpha, n_1, n_2}} - F_{1-\alpha, n_1, n_3} \right)]
\]

\[
= \frac{1}{c+1} \frac{n_2/2}{d+1} \frac{n_3/2}{n_1/2-1} \sum_{y=0}^{1} \frac{1}{y!2^y} \left( \frac{c}{c+1} + \frac{d}{d+1} \right)^y
\]

where \( c = \frac{R_1 F_{1-\alpha, n_1, n_2}}{\rho} \), \( d = (\rho-1) R_2 F_{1-\alpha, n_1, n_3}/\rho \) (see Wang (1979)).

The results of the probabilities of \( (\theta_1 - \theta_2)/\theta_3 \leq U \) are given in Table 3 for various values of \( I, J, K \) and for \( 1-\alpha = 0.09, 0.95, 0.99 \). The actual probabilities are quite close to the specified probabilities even for small sample sizes. We expect the results to be even better if the more general confidence interval \( 0 < (\theta_1 - \theta_2)/\theta_3 \leq U \) is used where \( U \) is given in Equation (3.1).

4. Comparison with Other Methods.

The literature does not contain any references that have been evaluated and directly relate to confidence intervals on \( \sigma_A^2/\sigma_C^2 \). Perhaps Satterthwaite's (1946) method could be used but this procedure is extremely poor when used to place confidence intervals on the difference of expected mean squares (i.e. on \( (\theta_1 - \theta_2)/\theta_3 = J K \sigma_A^2/\sigma_C^2 \)). Broemeling (1969) presents a method for placing simultaneous confidence intervals on \( \sigma_A^2/\sigma_C^2 \) and \( \sigma_B^2/\sigma_C^2 \). This method can be used to place confidence intervals on \( \sigma_A^2/\sigma_C^2 \).

We use Equation (15) in Broemeling (1969) to obtain

\[
P[0 < K J \sigma_A^2/\sigma_C^2 \leq \frac{S_2^2}{S_1^2} F_{1-\alpha, n_1, n_3}] \geq (1 - \alpha)^2
\]  
(4.1)
which can be used for a lower confidence interval on $KJq^2_A/\sigma^2_c$ with confidence coefficient greater than or equal to $(1 - \alpha)^2$. Clearly the $1 - \alpha$ lower confidence interval in Equation (3.2) above is shorter than the $(1 - \alpha)^2$ confidence interval in Equation (4.1). Thus the confidence interval on $\sigma^2_A/\sigma^2_c$ derived from the procedure by Broemeling is not as good as the method presented in this paper.

REFERENCES


Table 2
Confidence Coefficients for an Upper Confidence Interval on $\frac{1-\theta_2}{\theta_3}$ (on $\sigma^2_1/\sigma^2_0$) Using Equation (2.3)

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<th>n_2</th>
<th>n_3</th>
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<th>1-(\alpha = .95)</th>
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Table 3
Confidence Coefficients for a Lower Confidence Interval on $\frac{1 - \theta}{\theta} \left( \text{on } \frac{\sigma_{A}}{\sigma_{C}} \right)$ Using Equation (3.2)

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<th>n₂</th>
<th>n₃</th>
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**CONFIDENCE INTERVALS ON A RATIO OF VARIANCES IN THE TWO-FACTOR NESTED COMPONENTS OF VARIANCE MODEL**

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**CONTRACT OR GRANT NUMBER(S)**
N00014-78-C-0463

**REPORT DATE**
May 1980

**DISTRIBUTION STATEMENT (of this report)**
APPROVED FOR PUBLIC RELEASE: DISTRIBUTION UNLIMITED

**ABSTRACT**
Confidence intervals on ratios of variances
ABSTRACT

Consider the two-factor nested components of variance model \( Y_{ij} = \mu + A_i + B_{ij} + C_{ij}, \) where \( \text{Var}(A_i) = \sigma_A^2, \)
\( \text{Var}(B_{ij}) = \sigma_B^2, \) \( \text{Var}(C_{ij}) = \sigma_C^2. \)

Confidence intervals are derived for \( \hat{\sigma}_A^2, \sigma_A^2/(\sigma_B^2 + \sigma_C^2) \)
and \( \sigma_C^2/\sigma_C^2. \)