METRICALLY UPPER SEMICONTINUOUS MULTIFUNCTIONS AND THEIR INTERSECTIONS

Szymon Dolecki

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

January 1980

Received November 9, 1979

Approved for public release
Distribution unlimited

Sponsored by
U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

Air Force Office
of Scientific Research
Washington, D.C. 20332

Office of Naval Research
Arlington, Virginia 22217
We present a variety of methods for establishing metric upper semicontinuity. We give conditions for the intersection of metric upper semicontinuous multifunctions to be metric upper semicontinuous and discuss their applicability.

AMS(MOS) Subject Classification: 90C30, 52A99, 54C60

Key Words: semicontinuity, stability, Lusternik theorem, decisive separation.

Work Unit No. 5 - Operations Research

Mathematics Research Center and Mathematics Department, University of Wisconsin, Madison, and Institute of Mathematics, Polish Academy of Sciences, Śniadeckich 8, Warsaw, Poland.

The study of metric upper semicontinuity (stability) is of importance in optimization theory. The report discusses classical and recent techniques of establishing metric upper semicontinuity and provides their extensions. The metric upper semicontinuity of intersections, the importance of which has been recognized only recently, and several applications to optimization problems are discussed.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.
METRICALLY UPPER SEMICONTINUOUS MULTIFUNCTIONS
AND THEIR INTERSECTIONS

Szymon Dolecki

A multifunction \( \Gamma \) from a topological space \( Y \) into a metric space \((X, \rho)\) is called metrically upper semicontinuous at \( y_0 \) if for each \( r' > 0 \) there is a neighborhood \( W \) of \( y_0 \) such that

\[
(0.1) \quad \Gamma W \subseteq B(\Gamma y_0, r)
\]

where for \( A \subseteq X, B(A, r) = \{ x : B(x, r) \cap A \neq \emptyset \}, B(x, r) = \{ v : \rho(x, v) < r \} \) and \( \Gamma W = \bigcup_{y \in W} \Gamma y \). Such multifunctions are frequently called upper Hausdorff semicontinuous (u.H.s.c.).

Classical methods of establishing metric upper semicontinuity and their recent extensions (Sections 2 and 3) apply to certain classes of multifunctions. But, what is really needed in applications (e.g., duality in optimization theory, exact penalty methods, sensitivity) is metric upper semicontinuity of the intersections of some multifunctions from the above mentioned classes.

A dramatic suspense is caused by the fact that the intersection of two metrically upper semicontinuous multifunctions need not be itself metrically upper semicontinuous. This fact was recognized in [2] and a way of getting around this difficulty in some applications was proposed in [7]. In [15] Rolewicz gave a geometrical sufficient condition (for the intersection of u.H.s.c. multifunctions to be u.H.s.c) and applied it to some open problems concerning continuously differentiable maps. His sufficient condition however does not apply to some other important problems.

\[\text{Mathematics Research Center and Mathematics Department, University of Wisconsin, Madison, and Institute of Mathematics, Polish Academy of Sciences, Sniadeckich 8, Warsaw, Poland}\]

\[\text{Sponsored by the United States Army under Contract No. DAAG29-75-C-0024, the Air Force Office of Scientific Research Grant AFOSR-79-0018, and the Office of Naval Research Contract 041-404.}\]
In Section 4 we provide a general necessary and sufficient condition for metric upper semicontinuity of intersections. As a special case it yields the Rolewicz condition and enables us to refine and to extend the results of [15] (Section 5).

Section 6 provides an example of applications of our general condition to problems of different nature than those at Section 5.

1. Some basic facts about metric upper semicontinuity.

Let \((Y, \delta)\) be a metric space. A function \(q: \mathbb{R}_+ \to \mathbb{R}_+ \cup \{\infty\}\) is called a rate of semicontinuity at \(y_0\), if

\[
\Gamma B(y_0, q(r)) \subseteq B(\Gamma y_0, r) \quad r > 0
\]

A function \(\beta: \mathbb{R}_+ \to \mathbb{R}_+ \cup \{\infty\}\) is called a modulus of semicontinuity at \(y_0\), if

\[
\Gamma B(y_0, r) \subseteq B(\Gamma y_0, \beta(r)) \quad r > 0.
\]

\(\Gamma\) is u.H.s.c at \(y_0\), if and only if there is a rate \(q\) strictly positive (there is a modulus \(\beta\) such that \(\lim_{r \to 0} \beta(r) = 0\)).

We say that \(\Gamma\) is graph-closed at \(y_0\), if

\[
\Gamma y_0 = \bigcap_{W \in \mathcal{B}(y_0)} \text{cl} \, \Gamma W
\]

where \(\mathcal{B}(y_0)\) is a neighborhood basis at \(y_0\) and \(\text{cl}\) stands for the closure. Note that \(\Gamma\) is graph-closed at \(y_0\) for each \(y_0\) in \(Y\), if and only if the graph of \(\Gamma\) \(\mathcal{Q}(\Gamma) = \{(y, x): x \in \Gamma y\}\) is closed. It is a simple observation that

...
1.1 Proposition

If \( \Gamma y_0 \) is a closed set and \( \Gamma \) is u.H.s.c at \( y_0 \), then \( \Gamma \) is graph-closed at \( y_0 \).

(Topological) upper semicontinuity implies metric upper semicontinuity, the converse statement being true under some auxiliary assumptions (see [6]).

A multifunction \( \Gamma \) is said to be metrically continuous (Hausdorff continuous) at \( y_0 \), if for each \( r > 0 \) there is a neighborhood \( W \) of \( y_0 \) such that (0.1) holds and for \( y \) in \( W \)

\[
\Gamma y_0 \subseteq B(\Gamma y, r)
\]

Formulas (0.1) and (1.4) may be rephrased: \( h(\Gamma y_0, \Gamma y) < r \), where \( h \) stands for the Hausdorff distance: \( h(A_1, A_2) = \inf \{ r : A_1 \subseteq B(A_2, r), A_2 \subseteq B(A_1, r) \} \).

If there are a positive function \( \beta \) tending to 0 with its argument and a \( \delta > 0 \) such that \( \beta \) is a modulus of semicontinuity of \( \Gamma \) at each \( y \) in \( B(y_0, \delta) \), then \( \Gamma \) is metrically continuous about \( y_0 \). In other words,

\[
h(\Gamma y_1, \Gamma y_2) < \beta(\rho(y_1, y_2)), \quad y_1, y_2 \in B(y_0, \delta)
\]

If \( \beta \) is linear about 0 and (1.5) holds we say that \( \Gamma \) is (locally) Lipschitz at \( y_0 \). If (0.1) holds and, (1.4) holds for all \( y \) in \( W \cap \Gamma^{-1}X \), then we say that \( \Gamma \) is domain continuous at \( y_0 \).

We say that \( \Gamma \) is lower semicontinuous at \( (y, x) \) at a rate \( q \), if

\[
\Gamma^{-1}B(x, r) \supseteq B(y, q(r))
\]
is uniformly lower semicontinuous at \((y_0', x_0')\) if there is a neighborhood \(Q\) of \((y_0', x_0')\) and a (strictly) positive function \(q\) such that for each \((y, x)\) in \(Q \cap Q(\Gamma)\), \(\Gamma\) is lower semicontinuous at \((y, x)\) at the rate \(q\).

1.2 Proposition [2]

\(\Gamma\) is uniformly (at a rate \(q\)) lower semicontinuous at \((y_0', x_0')\), if and only if there are neighborhoods \(V\) of \(x_0\) and \(W\) of \(y_0\) such that for each \(y\) in \(W\) the multifunction \(\Gamma(y)\) defined by

\[
\Gamma(y) = \Gamma (y) \quad \Gamma (y) \neq V \cap rz, \quad z \neq y
\]

is u.h.s.c at \(y\) at a rate \(\tilde{q}\), where for some \(r_0 > 0\) \(\tilde{q}(r) = q(r)\) as \(r \geq r_0\).

A simple but important generalization of this proposition we obtain by restricting the multifunction \(\Gamma\) to its effective domain \(\Gamma^{-1}X = \{y: \Gamma y \neq \emptyset\}\). Then, all the notions may be related to the metric space \(\Gamma^{-1}X\): for example, \(\Gamma\) is domain lower semicontinuous at \((y, x)\) at a rate \(q\), if

\[
\Gamma^{-1}B(x, r) \supset B(y, q(r)) \cap \Gamma^{-1}X.
\]

Observe, however, that metric upper semicontinuity when related to its domain remains unchanged. The resulting theorem is obtained from Proposition 1.2 by replacing lower semicontinuity by domain lower semicontinuity and "for each \(y\) in \(W\)" by "for each \(y\) in \(W \cap \Gamma^{-1}X\)."

The importance of domain semicontinuities was recognized by Levine and Pomerol [11].
1.3 Example

Let \( X \) be a Banach space and let \( f \) be a nonzero continuous linear form on \( X \). Let \( \Gamma : \mathbb{R}^2 \to 2^X \) be given by:

\[
\Gamma(r_1, r_2) = \{ x : f(x) = r_1, f(.): r_2 \}.
\]

\( \Gamma \) is not lower semicontinuous, but it is uniformly u.H.s.c at each point of its domain \( \{ (r_1, r_2) : r_1 = r_2 \} \).

1.4 Corollary

\( \Gamma \) is uniformly (domain) lower semicontinuous at \( (y_0, x_0) \) at a rate \( q \), if and only if there are neighborhoods \( U \) of \( x_0 \) and \( W \) of \( y_0 \) and a number \( r_0 > 0 \) such that for \( x \) in \( V \) and for \( y \) in \( W \) (in \( W \cap \Gamma^{-1}X \))

\[
\text{dist}(y, \Gamma^{-1}x) < q(r) \Rightarrow \text{dist}(x, \Gamma y) < r, r < r_0.
\]

Proof

It is enough to prove "non domain" part of the corollary. We rephrase the conclusion of Proposition 1.2 for each \( y \) in \( W \) for \( r < r_0 \).

\[
\Gamma B(y, q(r)) \cap V \subset B(\Gamma y, r).
\]

Let \( x \) be in \( \Gamma B(y, q(r)) \cap V \). Equivalently, \( x \) is in \( V \) and \( \Gamma^{-1}x \) intersects \( B(y, q(r)) \). In other words \( \text{dist}(y, \Gamma^{-1}x) < q(r) \). On the other hand \( x \) is in \( B(\Gamma y, r) \), whenever \( B(x, r) \) meets \( \Gamma y \), or \( \text{dist}(x, \Gamma y) < r \).
2. Where does one encounter metric upper semicontinuity

The Banach open mapping theorem states that if a continuous linear operator $F$ maps a Banach space $X$ onto a Banach space $Y$, then the multifunction $F^{-1} : Y \rightarrow 2^X$ is lower semicontinuous at $(0, 0)$ at a linear rate. Linearity implies uniform lower semicontinuity everywhere, hence $F^{-1}$ is metrically continuous, in fact, Lipschitz. More generally, if $\Gamma : Y \rightarrow 2^X$ is a multifunction with closed convex graph such that $y_0 \in \text{Int} \Gamma^{-1}X$, then for every $x_0$ in $\Gamma y_0$, $\Gamma$ is lower semicontinuous at $(y_0, x_0)$, thus, by convexity, uniformly lower semicontinuous at $(y_0, x_0)$ ([16]).

Consequently, in view of Proposition 1.2, the multifunctions $\Gamma(y)$ defined in (1.7) are u.H.s.c (this property we call sometimes $\delta$-semicontinuity). Moreover, if $y_0 \in \text{Int} \Gamma^{-1}X$, then all graph closed convex multifunctions which are "close" to $\Gamma$ are lower semicontinuous at $(y_0, x_0)$ at a uniform universal rate (a perturbation theorem [2], see [10] for a special case). This nice property of families of graph-convex multifunctions was used in proving uniform lower semicontinuity of non-convex multifunctions (that can be approximated by families of graph-convex multifunctions) by Lusternik [12]: let $F$ be a continuously differentiable (about $x_0$) mapping into a Banach space $Y$ such that $F'(x_0)X = Y$; then $F^{-1}$ is uniformly lower semicontinuous (at a piecewise linear rate) at $(F(x_0), x_0)$. Extensions of the Lusternik theorem were proposed by Ioffe-Tikhomirov [9], Robinson [14], Ioffe [8] and the present author [2][3].

In this type of results, an approximation theorem [2] (an extension of the Pták theorem [13]) is very useful. We shall need only its special form:

(*) Actually Lusternik proved a special consequence of uniform lower semicontinuity.
2.1 Theorem [2]

Let \( \Gamma \) be a multifunction from a metric space into a complete metric space \( X \). Suppose that there are a neighborhood \( Q \) of \((y_0, x_0)\), numbers \( r_j, 0 < r_j \leq b \) such that for \((y, x)\) in \( Q \cap Q(\Gamma) \)

\[
B(\Gamma^{-1}B(x, r), \delta r) \supseteq B(y, br), 0 < r \leq r_0.
\]

Then \( \Gamma \) is uniformly lower semicontinuous at \((y_0, x_0)\) at a rate \( q \) such that \( q(r) = (b - \delta r, r \leq r_1) \).

Theorem 2.1 may be related to the domain of \( \Gamma \). Then (2.1) becomes

\[
B(\Gamma^{-1}B(x, r), \delta r) \supseteq B(y, br) \cap \Gamma^{-1}X, 0 < r \leq r_0
\]

and the thesis is that of uniform domain lower semicontinuity. The applicability of Theorem 2.1 appears through the following scheme: Let \( X \) be a complete metric space, \( Y, Z \) Banach spaces, \( \Gamma: Y \rightrightarrows X \) a multifunction, \( \{\Gamma'(y, x)\} \) a family of (graph-) convex, closed multifunctions from \( Y \) into \( Z \) (such that \( 0 \in \Gamma'(y, x)0 \) for each \((y, x))\).

Consider the following assumptions: there are numbers \( 0 < a < 1 \) and \( c > 0 \) such that for each \( \delta > 0 \) there is an \( r_0 > 0 \) such that

\[
B(\Gamma^{-1}B(x, r), \delta r) \supseteq y + \Gamma'(y, x)^{-1}B(0, ar) \cap B(0, cr)
\]
for \(|x - x_0| < r_0, |y - y_0| < r_0\) \(y \in \Gamma^{-1}x, r \leq r_0\) and for each \(\varepsilon > 0\) there are a neighborhood \(V\) of \((y_0, x_0)\) and \(v_1\) such that

\[(2.3) \; B(\Gamma' (y, x)^{-1}B(0, ar) \cap B(0, cr), \varepsilon r) \supset \Gamma' (y_0, x_0)^{-1}B(0, ar) \cap B(0, cr) \quad r \leq r_0\]

Such a family \(\{\Gamma' (y, x)\}\) is called an image nearly inner approximation of \(\Gamma\) at \((y_0, x_0)\) (see [2] [7] [3] for special cases).

**2.2 Proposition**

Let \(\{\Gamma' (y, x)\}\) be an inia of \(\Gamma\) at \((y_0, x_0)\). If

\[(2.4) \; 0 \in \text{Int} \; \Gamma' (y_0, x_0)^{-1}X,\]

then \(\Gamma\) is uniformly lower semicontinuous at \((y_0, x_0)\) at a piecewise linear rate.

**Proof**

It follows from the Baire category theorem and from the convexity of \(\Gamma' (y_0, x_0)\) that there are \(b_0\) and \(r_1\) such that

\[\Gamma' (y_0, x_0)^{-1}B(0, ar) \supset B(0, b_0 r), \quad r \leq r_1\]

Let \(b_1 > \varepsilon > 0\). On setting \(b_1 = \min(b_0, c)\) \(r_2 = \min(r_0, r_1)\) we have, in view of (2.3), that

\[B(\Gamma' (y, x)^{-1}B(0, ar) \cap B(0, cr), \varepsilon r) \supset B(0, b_1 r), r \leq r_2\]

for \((y, r)\) in \(V\).
Now, we choose $\vartheta$ such that $\vartheta + \varepsilon < b_1$ and in view of (2.2) we obtain

$$B(\Gamma^{-1}B(x, r), (\vartheta + \varepsilon)r) \supset B(y, b_1r)$$

for $|x - x_0| < r_2, |y - y_0| < r_2, y \in \Gamma^{-1}x, r \in r_2$. In virtue of Theorem 2.1 the proof is complete.

The above scheme embraces various convex approximations of multifunctions (continuous differentiability and more generally strict differentiability, Levitin-Milyutin-Osmolovskii approximation (see [2]), their combinations and extensions [3]). We are going to present two concrete results, not most general, but having some important implications.

2.3 Example

Let $G: X \to Y$ be a continuously differentiable mapping about a point $x_0$ of a Banach space $X$ and let $D$ be a closed convex cone in a Banach space $Y$. Consequently for each $\vartheta > 0$ there is $r_0 > 0$ such that

$$|G(x + h) - G(x) - G(x)h| < \vartheta |h|,$$

for $|x - x_0| < r_0, |h| < r_0$. Define

$$\Gamma y = \{x : y \in G(x) + D\}$$

and note that
(2.7) \[ \Gamma'(y, x)z = \{ h : z \in G'(x)h + D \} \quad y \in \Gamma^{-1}x \]

collects an image nearly inner approximation of \( \Gamma \) at \( (G(x_0), x_0) \)

As a corollary of Proposition 2.2 one has

2.4 Proposition ([14])

If

(2.8) \[ G'(x_0)x + D = Y, \]

then \( \Gamma \) of (2.6) is uniformly lower semicontinuous at \( (G(x_0), x_0) \) at a piecewise linear rate.

Let \( G \) be a family of subsets of a Banach space \( X \). A family \( r' \) of closed convex cones is said to be uniformly tangent to \( \{ G \} \) at \( x_0 \), if for each \( \varepsilon > 0 \) there is an \( r_0 > 0 \) such that for \( x \in \partial A \) (boundary of \( A \), \( ||x - x_0|| < r_0 \), \( A \in G \) there is an element \( A'(x) \) of \( G' \) such that for \( ||v - x|| < r_0 \), one has

(a) if \( v \) is in \( A'(x) \), then
\[ \text{dist} (v, A) < \varepsilon ||v - x|| \] \[ (2.9) \]
(b) if \( v \) is in \( A \), then
\[ \text{dist} (v, A'(x)) < \varepsilon ||v - x||. \]

2.5 Proposition (see [3])

Let \( \Gamma \) be given by (2.6), \( \Gamma'(x, y) \) by (2.7) and suppose that (2.5) holds.
The family \( \{ x + \Gamma(y, x)(0) \} \) is uniformly tangent to \( \{ \Gamma(y) \} \) at \( x_0 \).

Proof

It follows from Proposition 2.4 and Corollary 1.3 that there are numbers \( k > 0, r_0 > 0 \) such that for \( |v - x_0| < r_0', |z - y_0| < r_0 \)

\[
(2.11) \quad \text{dist}(v, \Gamma z) \leq k \text{dist}(z, \Gamma^{-1}v).
\]

Let \( \epsilon = k\delta \) and let \( v \) be in \( x + \Gamma'(y, x)(0), (y \in G(x) + D) \) equivalently \( 0 = \Gamma'(y, x)^{-1}(v - x) = G'(x)(v - x) + D \) thus there is \( d \) in \( D \) such that

\[
0 = G'(x)(v - x) + d \quad \text{and in view of (2.5) and (2.6), G(x) is in B} (\Gamma^{-1}v, \epsilon, |v - x|), \quad \text{hence recalling (2.11)}
\]

\[
\text{dist} (v, \Gamma(G(x))) \leq k\delta \|v - x\|
\]

Let \( v \) be in \( \Gamma G(x) \). Thus there is \( d \) in \( D \) such that \( G(v) + d = G(x) \) and by (2.5)

\[
\|G'(x)(v - x) + d\| < \delta \|v - x\|
\]

By (2.8) and the continuity of \( G'(x) \) there is an element \( w \) of \( X \) such that \( G(x)(v - w) = 0 \) and \( |w - x| \leq k \|G'(x)(v - x) + d\| < k\delta \|v - x\| \).

(In fact \( k \) may be taken the same as in \( (2.11) \)).

The following result is that of uniform lower semicontinuity of a multifunction \( \Gamma : Y \to 2^X \), where \( Y \) is a Banach space and \( S \) is a closed subset of a Banach space admitting a uniform tangent family. This
result generalizes Proposition 2.4 and is very close to that of Ioffe [8].

Theorem 2).

In our case (in comparison to [8]) the set $S$ will be more specific, the multifunction $\Gamma$ more general, the sufficient condition (controllability) much easier to verify and the conclusion stronger (uniformity). Let $G$ be a continuously differentiable mapping from $X$ into a Banach space $Z$, $D$ a closed convex cone in $Z$ such that

\[(2.12) \quad G'(x_0)X + D = Z\]

Set

\[(2.13) \quad S = \{x : 0 \in G(x) + D\}\]

Denote by

\[(2.14) \quad T(x) = \{v : 0 \in G(x)v + D\}\]

Let $F$ be a continuously differentiable mapping of $X$ into a Banach space $Y$, $C$ a closed convex cone in $Y$. Consider the multifunction $\Gamma : Y \to 2^S$ defined by

\[(2.15) \quad \Gamma y = \{x \in S : y \in F(x) + C\}\]

2.6 Theorem

If (2.12) holds and
(2.16) \[ F'(x_0)(T(x_0)) + C = Y, \]

then the multifunction \( \varGamma \) of (2.15) is uniformly lower semicontinuous at \((F(x_0), x_0)\) at a piecewise linear rate.

Proof

We shall show that the multifunction \( \varGamma'(y, x) : Y \to 2^X \) defined by

(2.17) \[ \varGamma'(y, x)z = \{ h \in T(x) : z \in F'(x)h + C \} \quad \text{for } y \in \varGamma^{-1}x. \]

is an image nearly inner approximation of \( \varGamma \) at \((F(x_0), x_0)\). It will be then enough to apply Proposition 2.2, since our assumption in view of (2.17) implies (2.4). It follows from Proposition 2.5 that \( \{x + T(x)\}, x \in S, \) is a family uniformly tangent to \( S \) about \( x_0 \). Consequently, there is a family of functions \( \{\xi_x\} \)

(2.18) \[ \xi_x : x + T(x) \to S \]

such that for each \( 1 \leq \delta > 0 \) there is an \( r_0 > 0 \) such that if \( \|x - x_0\| < r_0 \) and \( \|v - x\| < r_0 \), then

\[ \|\xi_x(v) - v\| \leq \delta \|v - x\|. \]

Therefore, for \( r \leq r_0 \)

\[ \xi_x(B(x, r) \cap x + T(x)) \subseteq B(x, (1 + \delta)r) \cap S \]

There is an \( r_1 \leq r_0 \) such that in \( B(x_0, 3r_1) \) the mapping \( F \) is Lipschitz.
continuous with constant 1. Hence for each \( v \) in \( B(x, r) \cap x + T(x) \) there is an element \( \xi_x(v) \) in \( S \cap B(x, (1 + \delta)r) \) such that
\[
\Gamma(v) - F(\xi_x(v)) = v - x
\]
or
\[
(2.19) \quad B(\Gamma(3(x, (1 + \delta)r) \cap S), \delta r) = F(B(x, r) \cap x + T(x)).
\]

On the other hand, there is an \( r_2 \leq r_1 \) such that for
\[
\|x - x_0\| \leq r_2, \quad |v - x| \leq r_2
\]
\[
\|\Gamma(v) - F(x) - F'(x)(v - x)| < \delta |v - x|
\]
in particular this is true for \( v - x \) in \( T(x) \), thus
\[
(2.20) \quad B(F(B(x, r) \cap x + T(x)), \delta r) = F(x) + F'(x)B(0, r) \cap T(x).
\]

On recalling (2.15) and (2.17) we conclude that
\[
(2.21) \quad B(\Gamma^{-1}B(x, r), 2\delta r) \supset y + \Gamma'(y, x)^{-1}B(0, \frac{r_2}{2})
\]
for \( r \leq \frac{r_2}{2} \), \( \|x - x_0\| < \frac{r_2}{2}, y \in \Gamma^{-1}x \), hence (2.2) is satisfied.

We shall show now that the family \( \{T(x)\} \) is continuous in the following sense. For each \( \epsilon > 0 \) there is \( r > 0 \) such that for \( x_1 - x_0 < r, |x_2 - x_0| < r \) one has that if \( h \in T(x_1) \), then \( \text{dist}(h, T(x_2)) \leq \eta |h| \). In fact, for each \( \epsilon > 0 \) there is \( r_0 > 0 \) such that, for \( x_1, x_2 \) in \( B(x_0, r_0) \)
\[
\|G'(x_1)h - G'(x_2)h\| \leq \epsilon |h|
\]
Let \( h \in T(x_1) \). There is \( d \) in \( D \) such that \( G'(x_1)h + d = 0 \). On the other hand,

\[
dist\left(0, \Gamma'(G(x_2), x_2)^{-1}h\right) \leq \|G'(x_1)h + d\| = \epsilon.
\]

(2.22)

By the uniform lower semicontinuity of the family \( \Gamma'(y, x) \) about \((y_0, x_0)\) there are \( m \geq 0 \) and \( r_0 \geq r_1 > 0 \) such that

\[
dist(h, \Gamma'(G(x_2), x_2)0) \leq m \cdot \|G'(x_2)h - d\| \leq m \epsilon \|h\|.
\]

(the last estimate following from (2.22)). Set \( m \epsilon \leq \eta \). Consequently, \( x - B(0, r) \cap T(x) \) and \( x - \Gamma'(x)(B(0, r) \cap T(x)) + C \) are continuous multi-functions and (2.3) is satisfied.

We shall now provide another condition equivalent to (2.16) combined with (2.12). Let \( \mathcal{J}: X \to Y \times Z \) be given by

\[
\mathcal{J}(x) = (F(x), G(x))
\]

and let \( C = G \times D \).

2.7 Lemma

Combined, conditions (2.12) and (2.16) are equivalent to

\[
(2.23) \quad \mathcal{J}(x_0)X + C = Y \times Z
\]
Proof.

It follows from (2.23) that for each $y$ in $Y$ there is an $x$ such that $(y, 0) \in (F'(x_0)x + C, G'(x_0)x + D)$ what amounts to (2.16). Similarly, (2.12) follows immediately from (2.23).

Suppose (2.12) and (2.16) and take $(y, z) \in Y \times Z$. By (2.12) there is an $x$ such that $z \in G'(x_0)x + D$. By (2.16) there is an $\widetilde{x}$ such that $0 \in G'(x_0)\widetilde{x} + D$ and $y - F'(x_0)x \in F'(x_0)\widetilde{x} + C$. Consequently $z \in G'(x_0)(x + \widetilde{x}) + D$ and $y \in F'(x_0)(x + \widetilde{x}) + C$.

3. On farther extensions

Proposition 2.4 may be generalized by replacing the cone $D$ by a cone-valued multifunction.

Let $G$ be a continuously differentiable (about $x_0$) mapping from a Banach space $X$ valued in a Banach space $Y$. Consider, as well, a multifunction $C : X \to 2^Y$, such that for each $x$, $Cx$ is a closed convex cone in $Y$ with the vertex at $0$. We assume that for each $\epsilon > 0$ there is a neighborhood $Q$ at $x_0$ such that for every $x$ from $Q$ and for each $h$ from $Cx_0$ there exists an $h(x)$, $||h(x)|| = ||h||$ in $Cx$ such that

$$||h(x) - h|| \leq \epsilon ||h||.$$
Rephrasing, for each $\varepsilon > 0$ there is a neighborhood $Q$ of $x_0$ such that for each $x$ in $Q$

\begin{equation}
B(Cx \cap B(0, r), \varepsilon r) \supset Cx_0 \cap B(0, r),
\end{equation}

Define

\begin{equation}
\Gamma y = \{x : y \in G(x) + Cx\}
\end{equation}

The multifunction

\begin{equation}
\Gamma'(x, y)z = \{h : z \in G'(x)h + Cx\}, \quad y \in \Gamma'^{-1}x
\end{equation}

is an image nearly inner approximation of $\Gamma$ at $(x_0, G(x_0))$.

Indeed, (2.2) (with $c = +\infty$) follows directly from the definitions.

In order to check (2.3) let $v \in \Gamma'(y_0, x_0)^{-1}B(0, r) \cap B(0, r)$; there are $g$ in $B(0, r)$ and $h$ in $Cx_0$ such that $v = G'(x_0)g + h$. Certainly

\[ ||h|| \leq ||v|| + ||G'(x_0)|| ||g||, \]

thus $h$ is in $B(0, (1 + k)r)$ where $k > ||G'(x_0)||$. There exists $r_0 > 0$ such that if $||x_0 - x|| < r_0$ then there is an $h(x)$ in $Cx$ such that (3.1) holds.

On the other hand, by the continuity of $G'(\cdot)$, there is an $r_1 < r_0$ such that

\[ ||G'(x)g - G'(x_0)g|| \leq \varepsilon ||g|| \quad \text{as} \quad ||x - x_0|| < r_1. \]

Therefore the element $w = G'(x)g + h(x)$ of $\Gamma'(y, x)^{-1}B(0, r)$ satisfies

\[ ||v - w|| \leq \varepsilon (2 + k)r \quad \text{and by convexity there is} \quad z \text{ in} \quad \Gamma'(y, x)B(0, r) \cap B(0, r) \]
such that \( v - x = 2^{r} (2 + k) r \). Therefore for the discussed multifunctions (2.8) holds where \( a = 1, c = 1, \varepsilon \) is replaced by \( 2^{r} (2 + k) \).

As an immediate conclusion we formulate

**Theorem 3.1**

If

(3.5) \[ G'(x_0)x + Cx_0 = Y, \]

then the multifunction (3.3) is uniformly lower semicontinuous at \((F(x_0), x_0)\) at a piecewise linear rate.

Another way of generalizing results of type of Proposition 2.4 or Theorem 2.6 may become a temptation, when one observes that if for a continuous linear operator \( F \) (from a Banach space \( X \) into a Banach space \( Y \)) we have that \( FX \) is closed, then the multifunction \( F^{-1} \) is u.H.s.c (domain Lipschitz continuous). This is due to the fact that \( F^{-1} \) is uniformly domain lower semicontinuous (its domain \( FX \) is itself a Banach space, hence we may apply the Banach open mapping theorem).

However, an attempt to replace, say (2.8) in Proposition 2.4 by an assumption that \( G'(x_0)x + D \) is closed, may be discarded quickly by noting, that even for \( D = \{0\} \) the property "\( G'(x)x \) is closed" is unstable (see e.g. [10] p. 57).

There remains however a possibility of generalization, when \( G'(x_0)x \) is of finite codimension (the property which is stable) we eliminate this possibility too.
Example 3.2

Let $X$ be a Hilbert space which orthonormal basis is denoted by $\{e_i\}_{i=1}^\infty$. Let $G : X \rightarrow \mathbb{R}^2$ be given by

$$G(x) = (g_1(x), g_2(x)),$$

where $g_1(x) = x_1$ (where $x = \sum_{i=1}^\infty x_i e_i$) and $g_2(x) = x_1 + \sum_{i=2}^\infty \frac{x_i^2}{i}$. Of course, $G$ is continuously differentiable and $G'(0)X$ has finite co-dimension. But the multifunction $G^{-1}$ is not domain lower semicontinuous. To see this observe that $G^{-1}(0, 0) = \{0\}$ and for $r < 0$

$$G^{-1}(r, 0) = \{x : x_1^2 = -\sum_{i=2}^\infty \frac{x_i^2}{i} \}$$

Thus for $|r|$ arbitrarily small there are $x$ in $G^{-1}(r, 0)$ that do not belong to $B(0, 1)$.

3.3 Example

Modifying Example 3.2 set $g_2(x) = x_1 + \sum_{i=2}^\infty x_i^2$. In this case the dimension of $G'(0)X$ is one, but $G^{-1}$ is uniformly domain lower semicontinuous (at 0) (for details see Example 4.6).
4. General conditions on metric upper semicontinuity of intersections

Let $A_1, A_2$ be subsets of a metric space $(X, p)$. We say that $A_1$ and $A_2$ separate decisively, if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that

\[(4.1) \quad B(A_1 \cap A_2, \varepsilon) \supseteq B(A_1, \delta) \cap B(A_2, \delta)\]

When $A_1$ and $A_2$ do not intersect, we understand that $B(A_1 \cap A_2, \varepsilon)$ is empty for each $\varepsilon$; then condition (4.1) becomes that there is a $\delta > 0$ such that $B(A_1, \delta) \cap B(A_2, \delta) = \emptyset$.

When $A_1$ is a subset of $A_2$, then the sets separate decisively and $\delta$ may be taken equal to $\varepsilon$.

4.1 Lemma

Let $X$ and $Z$ be two metric spaces, $p$ a mapping of $X$ onto $Z$ such that $p$ and $p^{-1}$ are uniformly continuous. If $A_1, A_2$ separate decisively, then $p(A_1), p(A_2)$ separate decisively.

Proof

Let $\delta > 0$. There is $\varepsilon > 0$ such that for each $x$, $B(p(x), \delta) \supseteq p(B(x, \varepsilon))$. Consequently

\[B(p(A_1) \cap p(A_2), \delta) = B(p(A_1 \cap A_2), \delta) \supseteq p(B(A_1 \cap A_2, \varepsilon)).\]

In view of (4.1) and the above inclusion, there is $\delta$ such that

\[B(p(A_1) \cap p(A_2), \delta) \supseteq p(B(A_1, \delta) \cap B(A_2, \delta)).\]
By the uniform continuity of $p^{-1}$ there is a $\xi$ such that

$$B(A_1, \delta) \supseteq p^{-1}B(p(A_1), \xi), \quad p(A_2, \delta) \supseteq p^{-1}B(p(A_2), \xi)$$

thus

$$B(p(A_1) \cap p(A_2), \delta) \supseteq B(p(A_1), \xi) \cap B(p(A_2, \xi)) .$$

4.2 Lemma

The sets $A_1$ and $A_2$ separate decisively, if and only if there exists a function $h : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\}$ continuous at $(0, 0)$ and $h(0, 0) = 0$ such that

$$\text{(4.2)} \quad \text{dist}(x, A_1 \cap A_2) \leq h(\text{dist}(x, A_1), \text{dist}(x, A_2))$$

Such a function $h$ is called a modulus of separation.

Proof

Assume that such a function exists and set $\varepsilon > 0$. By the continuity of $h$ at $(0, 0)$, there is a $\delta > 0$ such that $h(r_1, r_2) < \varepsilon$, if $r_1 < \delta$ and $r_2 < \delta$. If an element $x$ is in $B(A_1, \delta) \cap B(A_2, \delta)$ (or equivalently if $\text{dist}(x, A_1) < \delta$ and $\text{dist}(x, A_2) < \delta$), then by (4.2)

$$\text{dist}(x, A_1 \cap A_2) < \varepsilon,$$

hence it belongs to $B(A_1 \cap A_2, \varepsilon)$.

On the other hand, if $A_1$ and $A_2$ separate decisively, then for each $n \geq 1$ there is a $\delta_n > 0$ such that

$$\text{(4.3)} \quad B(A_1 \cap A_2, \frac{1}{n}) \supseteq B(A_1, \delta_n) \cap B(A_2, \delta_n) .$$
We define the functions $h_n : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\}$ by setting

$$h_n(r_1, r_2) = \begin{cases} \frac{1}{n}, & \text{if } r_1, r_2 < \delta_n \\ +\infty, & \text{otherwise} \end{cases} \quad (4.4)$$

and define

$$h(r_1, r_2) = \inf_n h_n(r_1, r_2) \quad (4.5)$$

The function $h$ of (4.5) satisfies the listed properties. (It is obviously continuous and 0 at $(0, 0)$.)

Take an arbitrary $x$. If $\max(\text{dist}(x, A_1), \text{dist}(x, A_2)) > 1$, then by (4.4)

$$h(\text{dist}(x, A_1), \text{dist}(x, A_2)) = +\infty \quad (4.5)$$

and (4.2) is fulfilled. If

$$\delta_{n+1} < \max(\text{dist}(x, A_1), \text{dist}(x, A_2)) < \delta_n,$$

then by (4.3)

$$\text{dist}(x, A_1 \cap A_2) < \frac{1}{n},$$

thus $h$ satisfies (4.2).

Let us give some attention to those pairs of sets $A_1, A_2$ that do not separate decisively. This means, by definition, that there exists an $\varepsilon > 0$ and a sequence $\{x_n\}$ such that
\[ \text{dist}(x_n', A_1 \cap A_2) \geq \varepsilon_0 \]

and

\[ \lim_{n\to\infty} \text{dist}(x_n, A_1) = \lim_{n\to\infty} \text{dist}(x_n, A_2) = 0 \]

4.3 Example

Let \( X = \mathbb{R}^2 \), \( A_1 = \{(x, y) : y = 0\} \), \( A_2 = \{(x, y) : y = e^{-x}\} \). The intersection of these sets is empty but for each \( \delta \) the sets \( B(A_1, \delta) \) and \( B(A_2, \delta) \) meet. The sequence \((n, 0)\) satisfies (4.6), (4.7).

4.4 Example (compare [7])

Let \( X \) be a Hilbert space which orthonormal basis is denoted by \( \{e_n\}_{n=1}^{\infty} \). Let

\[ A_1 = \left\{ \sum_{n=1}^{\infty} t_n e_n \in X : \sum_{n=1}^{\infty} \frac{t_n^2}{n^2} \leq 1 \right\} \]

and

\[ A_2 = \left\{ \sum_{n=1}^{\infty} t_n e_n \in X : t_1 = 1 \right\} . \]

The only common element of these sets is \( \{e_1\} \). The sequence \( \{e_1 + e_n\}_{n=1}^{\infty} \) is distant from \( e_1 \) by one. On the other hand, it is a subset of \( A_2 \) and for each \( n \), \( e_1 + e_n \) is distant from the element \( 1 - \frac{1}{n} e_1 + e_n \) of \( A_1 \) by \( 1 - \sqrt{1 - \frac{1}{n}} \).

In first of the two examples the sequence satisfying (4.6) (4.7) is unbounded; in the latter example it is bounded. Nevertheless in both cases,
it is not compact. Compactness implies decisive separation, but is not necessary.

4.5 Lemma

If one of the sets $A_1, A_2$ is compact and the other closed, then the sets separate decisively.

Proof

Since $X$ is metric it is enough to consider sequential compactness. We shall prove that no sequence can satisfy (4.6) and (4.7). Indeed, assume that $A_1$ is compact and there are an $\varepsilon_0$ and a sequence $\{x_n\}$ satisfying (4.6) (4.7). Consequently there are sequences $\{y_n\}$ in $A_1$, $\{z_n\}$ in $A_2$ and $\{\delta_n\}$, $\delta_n \downarrow 0$ such that $\rho(x_n, y_n) < \delta_n$ and $\rho(x_n, z_n) < \delta_n$. There is a subsequence $\{y_{n_k}\}$ convergent to an element, say $y_\infty$, of $A_1$, thus $\{x_{n_k}\} \{z_{n_k}\}$ converge to $y_\infty$, too. On the other hand, $y_\infty$ is in $A_2$ (in view of the closedness of $A_2$). This contradicts (4.6).

4.6 Example

$X$ stands for a Hilbert space, like in Example 4.4. Put

$$A_1 = \{ \sum_{n=1}^{\infty} t_n e_n \in X : \sum_{n=1}^{\infty} t_n^2 \leq 1 \}$$

and

$$A_2 = \{ \sum_{n=1}^{\infty} t_n e_n \in X : t_1 = 1 \}$$

The only common point is $\{e_1\}$. These sets separate decisively. To show
that, denote by \( r^2 = \sum_{n=2}^{\infty} t_n^2 \). Then \( A_1 \) is included in

\[
\{ \sum_{n=1}^{\infty} t_n e_n \in X : r \leq 1, t_1 \leq \sqrt{1-r^2} \}
\]

Fig. 1

We infer, that if \( \text{dist} (x, A_1) + \text{dist} (x, A_2) < \ell \), then \( \text{dist} (x, A_1 \cap A_2) < s \).

In order to relate \( s \) to \( \ell \) we observe that

\[
\frac{s}{1+\ell} = \frac{s-r}{\ell}
\]

\[
\frac{1-\sqrt{1-r^2}}{\ell} = \frac{1}{1+\ell}
\]

thus

\[
(4.8) \quad s = \sqrt{(2+\ell)} \ell \cdot
\]

Therefore for \( \varepsilon > 0 \) we may pick \( \delta = (\frac{\varepsilon}{2})^2 \) if it is less than 1.

The following theorem shows how decisive separation characterizes the metrically upper semicontinuous multifunctions, the intersection of which is also metrically upper semicontinuous.
4.7 Theorem

a) Let $\Gamma_1, \Gamma_2$ be multifunctions from a topological space $Y$ into subsets of $X$, u.H.s.c at $y_0$. If $\Gamma_1y_0$ and $\Gamma_2y_0$ separate decisively, then the intersection $\Gamma_1 \cap \Gamma_2$ is u.H.s.c at $y_0$.

Moreover, there is a modulus of semicontinuity $\beta$ of $\Gamma_1 \cap \Gamma_2$ such that

$$\beta(r) = h(\beta_1(r), \beta_2(r)),$$

where $h$ is a modulus of separation of $\Gamma_1y_0$ and $\Gamma_2y_0$, $\beta_i$ is a modulus of semicontinuity of $\Gamma_i$ (at $y_0$), $i = 1, 2$.

b) If the sets $A_1$ and $A_2$ do not separate decisively, then there are a (metric) space $Y$ and u.H.s.c (at $y_0$) multifunctions $\Gamma_1, \Gamma_2$:

$$Y \rightarrow 2^X,$$

such that

$$\Gamma_1y_0 = A_1, \quad \Gamma_2y_0 = A_2,$$

the intersection of which is not u.H.s.c at $y_0$.

Proof

a) Let $\epsilon > 0$. There is a $\delta > 0$ such that $B(\Gamma_1y_0 \cap \Gamma_2y_0, \epsilon) 
\supset B(\Gamma_1y_0, \delta) \cap B(\Gamma_2y_0, \delta)$. Since $\Gamma_1, \Gamma_2$ are u.H.s.c. at $y_0$, there are neighborhoods $W_1, W_2$ of $y_0$ such that $\Gamma_1W_1 \subset B(\Gamma_1y_0, \delta)$ and $\Gamma_2W_2 \subset B(\Gamma_2y_0, \delta)$. Consequently, $(\Gamma_1 \cap \Gamma_2)(W_1 \cap W_2) \subset B(\Gamma_1y_0 \cap \Gamma_2y_0, \epsilon)$.

To show (4.9) assume that $x$ is in $(\Gamma_1 \cap \Gamma_2)B(y_0, r)$. Thus, by semicontinuity $\text{dist } (x, \Gamma_1y_0) < \beta_1(r)$ and $\text{dist } (x, \Gamma_2y_0) < \beta_2(r)$. In view of (4.2) we may set $\beta(r) = h(\beta_1(r), \beta_2(r))$. 
b) Suppose that $A_1$ and $A_2$ do not separate decisively and let 
\( \{x_n\} \) be a sequence satisfying (4.6) (4.7). Define $Y = [0, -\infty)$ and 
\[
\Gamma_1^n = \Gamma_2^n = \{x_n\}, \quad n = 1, 2, \ldots
\]
\[
\Gamma_1^0 = A_1, \quad \Gamma_2^0 = A_2
\]
\[
\Gamma_1^y = \Gamma_2^y = \emptyset, \quad \text{otherwise.}
\]
So defined multifunctions are u.H.s.c. at 0, but their intersection is not.

4.8 Remark

Let $A_1, A_2$ be given. If $A_1$ is a subset of a ball, say $B(x_0, r_0)$ and for $r_1 > r_0$ the sets $A_1$ and $A_2 \cap B(x_0, r_1)$ separate decisively then $A_1$ and $A_2$ separate decisively. Moreover there is a modulus of separation of the latter equal to that of the former for small $r_1, r_2$.

4.9 Remark

Let $C_1 \subset A_1$, $C_2 \subset A_2$ be such that $C_1 \cap C_2 = A_1 \cap A_2$. If $A_1$ and $A_2$ separate decisively, then $C_1$ and $C_2$ too (with the same modulus).

4.10 Remark

A decisive separation property has itself a semicontinuity character. For two given subsets $A_1, A_2$ of $X$ define the multifunction $\Delta : \mathbb{R}_+ \to 2^X$ by
\[
\Delta 0 = A_1 \cap A_2
\]
\[
\Delta r = B(A_1, r) \cap B(A_2, r), \quad r > 0
\]
$A_1, A_2$ separate decisively, if and only if the multifunction $\Delta$ is metrically upper semicontinuous at 0.
5. Rolewicz theorem and localization of metric upper semicontinuity.

In [15] Rolewicz introduced, what we call, c-stars and d-convex sets, and proved that for two u.H.s.c multifunctions \( \Gamma_1, \Gamma_2 \) for which \( \Gamma_1 y_0 \) is a c-star and \( \Gamma_2 y_0 \) is d-convex \((c > d)\), \( \Gamma_1 \cap \Gamma_2 \) is u.H.s.c at \( y_0 \). His proof amounts, in practice, to demonstrating that \( \Gamma_1 y_0 \) and \( \Gamma_2 y_0 \) separate decisively.

Let \( X \) be a normed space. A subset \( A \) of \( X \) is called c-convex at \( x_0 \) (cA), if for every \( x \) in \( A \) and for each \( 0 < a < 1 \) there exists \( x_\alpha \) in \( A \) such that

\[
(5.1) \quad \| x_\alpha - (ax + (1 - a)x_0) \| \leq (1 - \alpha)c \| x - x_0 \|
\]

Of course, every convex set is c-convex for each \( c > 0 \) at each point.

The union of c-convex (at \( x_0 \)) sets is c-convex at \( x_0 \).

5.1 Lemma ([15])

Let \( F : X \to Y \) be continuously differentiable about \( x_0 \) and such that \( F'(x_0)X = Y \). Then for each \( c > 0 \), there exists a ball \( Q \) (centered at \( x_0 \)) such that the set

\[
\{ x : F(x) = F(x_0) \} \cap Q
\]

is c-convex at \( x_0 \).

A subset \( A \) of \( X \) is called a c-star at \( x_0 \), if for each \( x \) in \( A \) the convex hull of
(5.2) \[ B(x_0, c|x-x_0|) \cup \{x\} \]

is included in A.

It is a simple observation that A is a c-star at \( x_0 \), if and only if for each \( x \) in A

(5.3) \[ \bigcup_{0 < a < 1} B(ax_0 + (1-a)x, a c|x-x_0|) \subseteq A \]

Certainly, each c-star at \( x_0 \) is c-convex at \( x_0 \).

5.2 Proposition

Every c-star at \( x_0 \) not equal to the whole space is bounded.

Proof

Suppose that A is a c-star at \( x_0 \) and there is a sequence \( \{x_n\} \) such that \( ||x_n - x_0|| \geq n \).

Then by (5.3) for each \( n = 1, 2, \ldots \)

\[ B(x_0, cn) \subseteq B(x_0, c||x_n - x_0||) \subseteq A, \]

hence \( A = X \).

We shall give a simpler proof of

5.3 Proposition [15]

Every bounded convex set A for which \( x_0 \) is an interior point, is a c-star at \( x_0 \).
Proof

By assumptions there are numbers \( 0 < \epsilon < M \) such that
\[
B(x_0, \epsilon) \subseteq B(x_0, M).
\]
We set \( c = \frac{\epsilon}{M} \) and observe that for each \( x \in A \) \( B(x_0, c \cdot |x - x_0|) \subseteq B(x_0, cM) \subseteq A \). By convexity of \( A \) the convex hull (5.2) is a subset of \( A \).

We note that for a family \( \{A_i\} \) of \( c \)-stars at \( x_0 \), \( U \), \( A_i \) and \( \bigcup_{i \in I} A_i \) are \( c \)-stars at \( x_0 \). Consequently, if \( \{A_i\} \) is a family of convex sets such that \( B(x_0, r) \subseteq A_i \subseteq B(x_0, M) \) for each \( i \), then \( \bigcup_{i \in I} A_i \) is a \( \frac{r}{M} \)-star at \( x_0 \).

5.4 Theorem (Rolewicz [15])

Let \( 0 < d < c \leq 1 \). If \( A_1 \) is a \( d \)-convex set at \( x_0 \) and \( A_2 \) is a \( c \)-star at \( x_0 \), then \( A_1 \) and \( A_2 \) separate decisively and there is a modulus of separation of the form

\[
h(r_1, r_2) = m \cdot (r_1 + r_2),
\]

where \( m \) depends only on \( d \) and \( c \).

The Rolewicz theorem is especially useful in localization of metric upper semicontinuity. It is known [7] that if \( \Gamma \) is u.H.s.c at \( y_0 \) (with modulus \( \beta \)) then for each \( \epsilon > 0 \) and each neighborhood \( Q \) of \( x_0(\epsilon \Gamma y_0) \), there is a neighborhood \( Q_0 \subset Q \) of \( x_0 \) such that \( Q_0 \cap \Gamma \) is u.H.s.c at \( y_0 \) with modulus \( (1 + \epsilon)\beta \). The following example shows that \( Q_0 \) cannot in general be replaced by a ball about \( x_0 \).
5.5 Example

Let $E$ be a (nonseparable) Hilbert space, $\{r_\epsilon\}_{\epsilon \in \mathbb{R}}$ a family of orthonormal vectors in $E$, $\chi$ a characteristic function of $\{r_\epsilon\}_{\epsilon \in \mathbb{R}}$.

$$A = \{(e, r) \in E \times \mathbb{R} : 0 \leq r \leq \chi(e)\}.$$ 

Equip $E \times \mathbb{R}$ with the norm

$$\| (e, r) \| = \sup(\| e \|, |r|)$$

and define $\Gamma : E \to 2^{E \times \mathbb{R}}$ by

$$\Gamma y = A + (y, 0).$$

Of course $\Gamma$ is closed-valued continuous multifunction, but no multifunction $\Delta$ of form

$$\text{cl} \ B(0, R) \cap \Gamma \quad \quad \quad \quad \quad B(0, R) \cap \Gamma$$

is u.H.s.c at 0. We shall show this fact for $R \leq 1$. Let $0 < \epsilon < \frac{R}{2}$ and pick an $r$ such that $R < r < R + \frac{\epsilon}{2}$. The element $x = ((r - \epsilon)e, R)$ is in $\Delta(-\epsilon e, r)$ and $x \notin B(\Delta 0, \frac{R}{2})$. As $\| -\epsilon e \| = \epsilon$ and $\epsilon$ was chosen arbitrarily $\Delta$ is not u.H.s.c at 0.

It follows immediately from the definition that for each $r > 0$ the ball $B(x_0, r)(or \text{cl} \ B(x_0, r))$ is a 1-star at $x_0$. Therefore, if $\Gamma$ is an u.H.s.c multifunction (at $y_0$) and $\Gamma y_0$ is $d$-convex at $x_0$ with $d \leq 1$, then $B(x_0, r) \cap \Gamma$ (and $\text{cl} \ B(x_0, r) \cap \Gamma$) is u.H.s.c at $y_0$ in virtue of Theorems 5.4 and 4.7.
5.6 Lemma

Let \( \{A(x)\}_{x \in X} \) be uniformly tangent to a family \( G \) at \( x_0 \).

For each \( d > 0 \) and every neighborhood \( Q \) of \( x_0 \) there are neighborhoods \( Q_0 \subset Q_1 \subset Q \) of \( x_0 \) such that for each \( A \) of \( G \) such that \( A \cap Q_0 \neq \emptyset \) the set \( A \cap Q_1 \) is \( d \)-convex at each \( v \) in \( A \cap Q_0 \).

Proof

Fix \( d > 0 \) and choose a neighborhood \( Q \) of \( x_0 \). Let \( \varepsilon < \frac{d}{3} \) and let \( r_0 \) correspond to \( \varepsilon \) in (2.9) (2.10) and be such that \( B(x_0, r_0) \subset Q \). Set \( Q_0 = B(x_0, r_0) \), \( Q_1 = B(x_0, \frac{2r_0}{3}) \).

Let \( v \) be in \( Q_0 \cap A \) for \( A \) in \( G \) and let \( x \) be in \( A \cap Q_1 \).

By (2.10) there is \( v_1 \) in \( A'(x) \) such that

\[
(5.5) \quad \| v_1 - v \| \leq \varepsilon \| v - x \|
\]

The element \( (1 - \alpha)v_1 + \alpha x \) lies in \( A'(x) \), thus there is an \( x_\alpha \) in \( A \) such that

\[
\| x_\alpha - (\alpha x + (1 - \alpha)v_1) \| \leq \varepsilon (1 - \alpha) \| v_1 - x \|
\]

what combined with (5.5) implies

\[
(5.6) \quad \| x_\alpha - (\alpha x + (1 - \alpha)v_1) \| \leq (1 - \alpha) \varepsilon (1 + \varepsilon) \| v - x \|
\]

We estimate, taking into account (5.5) and (5.6)

\[
\| x_\alpha - (\alpha x + (1 - \alpha)v) \| \leq (1 - \alpha) \varepsilon (2 + \varepsilon) \| v - x \|
\]
which is less than \((1 - \alpha) d \| v - x \|\), if we assume that \(d \leq 3\). The proof is complete.

The above lemma enables us to prove the following result concerning the Lipschitz continuity of the multifunction \(\Gamma\) (2.15), \(\Gamma : Y \to 2^S\) where \(S\) is defined by (2.13) under condition (2.12).

### 5.7 Theorem

If (2.16) holds, then there are a neighborhood \(W\) of \(y_0\), a neighborhood \(Q\) of \(x_0\) and numbers \(c, r_0 > 0\) such that for \(x\) in \(Q\), for \(y\) in \(\Gamma^{-1}(x) \cap W\) and \(r \leq r_0\) the multifunction

\[ B(x, r) \cap \Gamma \]

is Lipschitz continuous about \(y\) with constant \(c\).

**Proof**

Let \(d < 1\). In view of Lemma 2.7 and Proposition 2.5 applied to the multifunction

\[ \Delta(y, z) = \{ x : (y, z) \in \mathcal{F}(x) + C \} \]

the family \(\{\Delta(y, z)\}_{(y, z)}\) possesses a uniformly tangent family at \(x_0\). A fortiori, the family \(\Gamma y = \Delta(y, 0)\) has a uniformly tangent family at \(x_0\).

In virtue of Lemma 5.6 in every neighborhood \(V\) of \(x_0\) there are neighborhoods \(Q_0 \subseteq Q_1 \subseteq V\) of \(x_0\) such that for each \(y\) and \(x\) in \(\Gamma y \cap Q_0\), \(\Gamma y \cap Q_1\) is \(d\)-convex at \(x\).

Choose positive numbers \(r_0\) and \(s_0\) such that \(B(x_0, 2s_0) \subseteq Q_0\).
Let $x$ be in $Q = B(x_0, s_0)$ and let $r \leq r_0$. The ball $B(x, r)$ is a $\frac{r - s}{r}$-star at each $v$ in $B(x, s)$. We choose $s$ so that

$$s = \frac{s_0}{r_0}$$

to guarantee that $B(x, r)$ is a $c$-star at such $v$.

On the other hand, by Theorem 2.6 there is a neighborhood $W$ of $y_0$ such that for every $y$ in $\Gamma^{-1}x \cap W$ there is a neighborhood $W_y$ of $y$ with $\Gamma^{-1}G(x, s) \supset W_y$ or equivalently such that for each $z$ in $W_y$, $\Gamma z \cap B(x, s) \neq \emptyset$. From the first part of the proof we know that $\Gamma z \cap Q_1$ is $d$-convex at each $v$ in $\Gamma z \cap B(x, s)$.

Now each ball $B(x, s)$ in consideration has the property that $B(x, s) + B(0, r_0 - s_0) \subset Q_1$, thus we conclude on recalling Theorem 5.4 and Remark 4.8 that there is a function $h$ of the form (5.4) for $r < r_0 - s_0$ which is a modulus of separation for each $B(x, s)$ and $\Gamma z$ described above.

We conclude that the same is true about the sets $B(x, s) \cap S$ and $\Gamma z$ (Remark 4.9).

From the assumptions of our theorem in view of Theorem 2.6 and Proposition 1.2 there is a neighborhood $V$ of $x_0$ and a neighborhood $W$ of $y_0$ such that for $y$ in $W$ the multifunctions (1.7) are metrically upper semicontinuous (at a universal piecewise linear rate). We may assume that $V$ that appears at the beginning of the proof is equal to this just introduced.

On the other hand, $y - B(x, s) \cap S$ is a constant, thus metrically upper semicontinuous multifunction (with the rate $q(r) = \infty$).
Therefore, by Theorem 4.7 (a), the multifunction $\Gamma \cap B(x, r)$ is u.H.s.c. at each element of $W_y$.

The uniform linear rate of semicontinuity (of all these multifunctions) at all points implies Lipschitz continuity.

It is of great importance in optimization to establish the Lipschitz continuity of so-called primal functionals [2] [15] [7]. Let $f$ be a real-valued function on $S$ (2.13) locally Lipschitz continuous about $x_0$. The primal functional of $(f, \Gamma)$ restricted to $Q$ is the real-valued function on $Y$:

$$f_{\Gamma Q}(y) = \inf_{x \in \Gamma y \cap Q} (f(x))$$

5.8 Corollary

Under the assumptions of Theorem 2.6, there are neighborhoods $W$ of $y_0$, $Q$ of $x_0$ and a number $r_0$ such that for each $x$ in $Q$ for every $y$ in $\Gamma^{-1}x \cap W$ and for $r = r_0$, the primal functional of $(f, \Gamma)$ restricted to $B(x, r)$ is locally Lipschitz at $y$. There is a universal Lipschitz constant for all such primal functionals.

Proof.

Apply Theorem 5.7 together with [5].
6. Decisive separation of weakly separated sets

The sets $A_1, A_2$, the decisive separation of which we discuss in this section, have the property

\[(6.1) \quad x \in A_1 \cap A_2 \Rightarrow x \in \partial A_1 \cap \partial A_2\]

where $\partial$ stands for the topological boundary. We shall consider sets of the form

\[(6.2) \quad A_1 = \{x : f_1(x) \leq 0\}, \quad A_2 = \{x : f_2(x) \leq 0\},\]

where $f_1, f_2$ are real-valued functions on a Banach space $X$.

Property (6.1) links the study with optimization theory; decisive separation of sets satisfying (6.1) is crucial in sensitivity theory, a branch of optimization ([4]).

6.1 Proposition

Suppose that the functions $f_1, f_2$ are continuous and the sets (6.2) satisfy (6.1). Then if the set

\[A_1 \cap A_2\]

is nonempty it is the set of all the global solutions of the problem

\[(6.3) \quad f_1(x) = \inf, \quad f_2(x) \leq 0\]
Proof

If \( x \) is in \( A_2 \), then \( f_1(x) \geq 0 \), because otherwise by the continuity of \( f_1 \) there would be a neighborhood \( Q \) of \( x \) such that \( Q \subseteq \{ v : f_1(v) < 0 \} \subseteq A_1 \), contradicting (6.1). Since \( A_1 \cap A_2 \) is nonempty, there is an element \( \hat{x} \) of \( A_2 \) such that \( f_1(\hat{x}) \leq 0 \); consequently \( \inf f_1(x) = 0 \) and every element of \( A_1 \cap A_2 \) is a solution of (6.3). On the other hand every solution \( x \) of (6.3) satisfies \( f_1(x) \leq 0 \) and \( f_2(x) \leq 0 \).

6.2 Corollary

If the functions \( f_1, f_2 \) are differentiable and the sets (6.2) satisfy (6.3), then for each \( x \) in \( A_1 \cap A_2 \) there are positive numbers \( \lambda_1, \lambda_2 \) not both zero such that

\[
(6.4) \quad \lambda_1 f_1'(x) + \lambda_2 f_2'(x) = 0.
\]

If besides \( f_1, f_2 \) are twice differentiable, then

\[
(6.5) \quad \lambda_1 f_1''(x) + \lambda_2 f_2''(x) \geq 0.
\]

Formulas (6.4) and (6.5) follow from well-known necessary conditions for an \( x \) to be a local minimum at (6.3). Under additional assumptions the sets of type (6.2) associated with (6.3) fulfill (6.1) ([4]).

In the sequel, we shall assume that for an \( x_0 \) in \( A_1 \cap A_2 \),
Then there will be a strictly positive $r$ such that
\[(6.7) \quad f_1'(x_0) + r f_2'(x_0) = 0\]

6.3 Theorem

Let $A_1, A_2$ satisfy (6.1) and be of the form (6.2) where $f_1, f_2$ are continuously differentiable.

Assume that an $x_0$ in $A_1 \cap A_2$ satisfies (6.6).

Let $i$ be an isomorphism of $\ker f_1'(x_0) \times \mathbb{R}$ onto $X$ such that $i(0, 0) = x_0$. Then there are neighborhoods $W_1$ of 0 in $\ker f_1'(x_0)$, $W_2$ of 0 in $\mathbb{R}$ and $Q$ of $x_0$ in $X$ and real-valued functions $\varphi_1, \varphi_2$ on $\ker f_1'(x_0)$ such that $Q = \bar{i}(W_1 \times W_2)$ and

$$\{x : f_1(x) = 0\} \cap Q = \bar{i}\{y, r : r = \varphi_1(y), y \in W_1\}$$

The sets $A_1 \cap Q, A_2 \cap Q$ separate decisively, if and only if the multifunction $\Delta$:

\[(6.8) \quad \Delta r = W_1 \cap \{y : \varphi_1(y) - \varphi_2(y) = r\}\]

is metrically upper semicontinuous at 0.

Proof

It follows from Proposition 2.5 and from (6.6) that

$$Y = \ker f_1'(x_0) = \ker f_2'(x_0)$$
tangent (at \( x_0 \)) to \( \{ x : f_1(x) = 0 \} \) and to \( \{ x : f_2(x) = 0 \} \). We shall represent these two sets locally about \( x_0 \) as functions on \( Y \).

Let \( i \) be an isomorphism of \( Y \times \mathbb{R} \) and \( X \) such that \( i(0, 0) = x_0 \). For a function \( f \) on \( X \) define \( \tilde{f} : Y \times \mathbb{R} \rightarrow \mathbb{R} \) by

\[
\tilde{f}(y, r) = f \circ i(y, r).
\]

The partial derivatives of \( f \) are

\[
(6.10) \quad \tilde{f}_y(y, r)h = f'(i(y, r)) \circ i(h, 0)
\]

\[
\tilde{f}_r(y, r)s = f'(i(y, r)) \circ i(0, s)
\]

If \( \ker f(x_0) = Y \), then \( \tilde{f}'(0, 0) = 0 \) and \( \tilde{f}'(0, 0)\mathbb{R} = \mathbb{R} \). Consequently there are neighborhoods \( W_1 \) of \( 0 \) in \( Y \) and \( W_2 \) of \( 0 \) in \( \mathbb{R} \) and a function \( \phi : W_1 \rightarrow W_2 \) such that

\[
(6.11) \quad \{(y, r) : r = \phi(y), \ y \in W_1\} = \{(y, r) : \tilde{f}(y, r) = 0\} \cap (W_1 \times W_2)
\]

The set \( (6.11) \) is the preimage by \( i \) of

\[
\{x : f(x) = 0\} \cap Q
\]

where \( Q \) is a neighborhood of \( x_0 \). If now \( \phi_1, \phi_2 \) correspond to \( f_1 \) and \( f_2 \), then we may pick \( W_1, W_2 \) and \( Q \quad \text{good for both the functions.} \)

By Lemma 4.1 the decisive separation of the sets \( A_1 \cap Q \) and
and $A_2 \cap Q$ is equivalent to the same property of $\phi_1^{-1}(A_1 \cap Q)$.

We may assume without loss of generality that $\phi_1(y) \geq \phi_2(y)$ for $y \in W_1$

and consider the decisive separation of the sets

$$C_1 = \{(y, r) : r \geq \phi_1(y)\} \cap W_1 \times W_2, \quad G_2 = \{(y, r) : r \leq \phi_2(y)\} \cap W_1 \times W_2.$$

The neighborhoods $Q, W_1, W_2$ might be chosen so that for $(y, r) \in W_1 \times W_2$

$$\phi_1(y) - r \leq 2 \text{dist}((y, r), C_1)$$

$$r - \phi_2(y) \leq 2 \text{dist}((y, r), C_2).$$

Indeed, for $(y, r)$ in $W_1 \times W_2$ we have that

$$\text{dist}((y, r), C_1) \leq \phi_1(y) - r$$

$$\text{dist}((y, r), C_2) \leq r - \phi_2(y)$$

Thus

$$\text{dist}((y, r), C_1) + \text{dist}((y, r), C_2) \leq \phi_1(y) - \phi_2(y).$$

The derivative of the function $\phi_1$ is

$$\phi'_1(y) = -\frac{\partial^T(y, \phi_1(y))}{\partial y} \cdot \frac{\partial^T(y, \phi_2(y))}{\partial y}$$

thus is continuous and vanishes at 0.

It follows that for all $y, \tilde{y}$ in a neighborhood of 0 (say $W_1$)
\[ \varphi_1(y) \leq \varphi_1(\tilde{y}) + ||\tilde{y} - y||. \]

Consequently, for \( y, \tilde{y} \) in \( W_1 \)

\[ \varphi_1(y) - r \leq |\varphi_1(\tilde{y}) - r| + ||\tilde{y} - y|| \leq 2 \sqrt{(\varphi_1(\tilde{y}) - r)^2 + ||\tilde{y} - y||^2}. \]

Therefore, if \( y \) is in \( W_1 \) and \( \varphi_2(y) \leq r \leq \varphi_1(y) \), then

(6.15) \[ \varphi_1(y) - \varphi_2(y) \leq 2 \left( \text{dist}((y, r), C_1) + \text{dist}((y, r), C_2) \right). \]

On the other hand, \( (y, r) \) is in \( C_1 \cap C_2 \) whenever \( y \) is in \( A = \{ z : \varphi_1(z) - \varphi_2(z) = 0 \} \) and \( r = \varphi_1(z) \). We have the estimates

\[ \text{dist}(y, A) \leq \text{dist}((y, r), C_1 \cap C_2) \leq \sqrt{\text{dist}^2(y, A) + (\varphi_1(y) - \varphi_2(y))^2}, \]

which together with (6.13) and (6.15) complete the proof.

6.4 Example

Assume now that, in addition to hypotheses of Theorem 6.3, the functions \( f_1, f_2 \) are twice continuously differentiable. In view of Corollary 6.2 and Formula (6.6) there is \( \lambda > 0 \) such that

\[ f_1'(x_0) = -\lambda f_2'(x_0) \]

\[ f_1''(x_0) + \lambda f_2''(x_0) \geq 0. \]

We shall assume that there is a \( k > 0 \) such that
If formulae (6.1), (6.6) and (6.16) hold, then there is a neighborhood $Q$ of $x_0$ such that the sets $A_1 \cap Q$ and $A_2 \cap Q$ separate decisively.

Proof

In view of Theorem 6.3 we should prove that the multifunction (6.8) is u.H.s.c at $0$. Denote: $\psi(y) = \varphi_1(y) - \varphi_2(y)$, where $\varphi_1, \varphi_2$ are those introduced in the previous proof. Using formula (6.14) we may compute the second derivative of $\varphi(= \varphi_1, \varphi_2)$

$$
\varphi''(y)(h, h) = \frac{1}{\tilde{\xi}_r(y, \varphi(y))^2} \left( \tilde{\xi}_r''(y, \varphi(y))h + \tilde{\xi}_r''(y, \varphi(y)) \cdot \varphi'(y)h \right) + \tilde{\xi}_r''(y, \varphi(y))h
$$

Since $Y = \ker f'(x_0)$ we have

$$
\varphi''(0)(h, h) = -\frac{1}{f'(x_0)(0, 0)} \frac{f''(x_0)(0, h, 0)}{f'(x_0)(0, 0, 0, 0)}
$$

Therefore, by (6.7)
We identify \( \langle h, 0 \rangle \) with \( h \). In view of our choice \( c_1 \not= c_2 \), \( f_1(x_0)(0, 1) < 0 \), hence by (6.16)

\[
\psi''(0)(h, h) \geq k_1 |h|^2
\]

Since \( \varrho(0) \cdot \psi(0) = 0 \), there is a neighborhood \( V \) of \( 0 \) in which

\[
\psi(h) \geq \frac{k_1}{2} |h|^2.
\]

The proof is complete.
7. Conclusion

Proposition 6.5 constitutes a simple example of higher order sufficient condition for uniform lower semicontinuity. It applies to the multifunction \( \psi^{-1} \), when the first derivative vanishes (critical point). In similar circumstances first order conditions (by which we understand the Lusternik-type conditions presented in Sections 2 and 3) cannot be used and higher derivatives should be taken into account in establishment of semicontinuity properties of multifunctions. It was pointed out in Section 6 that higher order sufficient conditions are crucial in sensitivity theory, where the nature of problems excludes applicability of first order conditions.

Another failure of first order conditions is illustrated in Example 3.3. Note that the discussed multifunction \( G^{-1} \) may be represented as the intersection of two u.H.s.c multifunctions defined on \( \mathbb{R}^2 \) and valued in \( X \), namely

\[
G^{-1}(r_1, r_2) = \{ x : g_1(x) = r_1 \} \cap \{ x : g_2(x) = r_2 \}.
\]

Again we face a problem of the metric upper semicontinuity of an intersection. In a similar context a use of higher order conditions may turn out of great value, when the usual constraint qualifications fail. This conclusion may sound like an introduction to a study of higher order conditions for semicontinuity. I hope to carry out such a study one day.
References


forschung und Statistik, to appear

J. 7 (25) (1975), 438 - 441.
We present a variety of methods for establishing metric upper semicontinuity. We give conditions for the intersection of metric upper semicontinuous multifunctions to be metric upper semicontinuous and discuss their applicability.