Cross Field Jetting of Energetic Ions
Produced by Rayleigh-Taylor Instability

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October 3, 1979

This research was sponsored by the Defense Nuclear Agency under Subtask Code S99QAXHC047, work unit 15 and work unit title Debris Injection.
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Performing Organization Name and Address:
Naval Research Laboratory
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Controlling Office Name and Address:
Defense Nuclear Agency
Washington, DC 20305

Monitoring Agency Name and Address:

Monitor Date:
October 3, 1979

Report Date:
October 3, 1979

Report Date:
October 3, 1979

Number of Pages:
34

Security Classification (of this report):
UNCLASSIFIED

Distribution Statement (of this report):
Approved for public release; distribution unlimited.

Distribution Statement (of the abstract entered in Block 20, if different from Report):

Supplementary Notes:
This research was sponsored by the Defense Nuclear Agency under Subtask Code S99QAXHC047, work unit 15 and work unit title Debris Injection.

Key Words (Continue on reverse side if necessary and identify by block number):
Rayleigh-Taylor instability
Debris transport
Cross-field transport
Trapped betas

Abstract (Continue on reverse side if necessary and identify by block number):

The phenomena of cross field jetting of debris following a nuclear detonation is explained by the Rayleigh-Taylor instability. The growth rate for the mode is calculated as well as thresholds for onset and wavelength. A model for this phenomena is developed and the amount of debris transported across the field lines is estimated.

NRL Memorandum Report 4068

PERIOD COVERED:
Interim report on a continuing NRL problem.

PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS:
NRL Problem 67H02-53
DNA Subtask S99QAXHC047

PERIOD COVERED:
Interim report on a continuing NRL problem.

PERIOD COVERED:
Interim report on a continuing NRL problem.

Approved for public release; distribution unlimited.
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CROSS FIELD JETTING OF ENERGETIC IONS PRODUCED BY RAYLEIGH–TAYLOR INSTABILITY

I. Introduction

A high altitude nuclear explosion (HANE) can lead to cross field transport of debris which in turn produces energetic trapped betas on L shells considerably higher than the one on which the detonation took place. The jetting model currently used in the SPECTER code is an estimate from ad hoc arguments. In this work the cross field transport of charged debris, during expansion of the debris sphere, is modeled by considering the Rayleigh-Taylor instability in a linear calculation. The model considers three phenomena. Two provide energy sources for the instability and the third provides a damping mechanism. The damping mechanism allows estimates of minimum wavelengths for the modes as well as critical thickness and temperatures for onset of the instability.

The first phenomenon which can initiate the Rayleigh-Taylor instability is the laminar like accelerations of a percentage of the ions within the shock. Essentially this leads to a three species description of the plasma. Particle simulations of an advancing shock have demonstrated the existence of these accelerated ions which have densities of approximately 10% of the ion density in the shock. The third species causes a space charge with the resultant electric field. The result of this is the Rayleigh-Taylor instability. The second source of energy for the instability is the curvature of the magnetic field lines caused by the bubble expansion. This force is more apparent during early phases of the bubble evolution. Finally, there is damping of the instability provided by finite Larmor radius corrections to the momentum equations (i.e. magnetic viscosity). The damping is a

Note: Manuscript submitted June 25, 1979.
function of the instability wavelength and is directly coupled to the thickness of the debris shell as well as the perpendicular pressure of the debris.

In this report a simplified model for prediction of these effects will be presented. Details of the calculation of all three phenomena are discussed in Appendix A and B and some sample estimate of growth rates, wavelengths and mass transport are presented in Appendix C.

II. Model

In this model we make use of the growth and damping rates represented by Eq. A-2 and Eqs. B-9, D, 11. Use is also made of some scaling relations provided by Dr. Robert Clark. These relations have been previously provided under this contract. As in Appendix B, we work in cylindrical coordinates where the B field is in the \( \hat{\theta} \) direction and displacements are in the \( \hat{r} \) and \( \hat{z} \) direction, Fig. 1.

The growth rate from the laminar acceleration is represented by the following equations

\[
\gamma_L = k V_A \left( \frac{f}{2L_n L_m (k^2 - 1/L_n^2)} \right)^{1/2} \tag{II-1}
\]

where \( L_n \equiv \frac{\rho_n'}{\rho_0}, L_m \equiv \frac{B_n'}{B_0} \), \( V_A \) is the Alfven speed and \( f \) is the fraction of ions accelerated in the shock, typically 10%. In this report \( L_n \) is taken equal to \( L_m \). Equation II-1 provides one of the restrictions on \( \lambda_z \):

\[
\lambda_z < 2\pi L_n \tag{II-2}
\]

It should also be noted that as \( \lambda_z \to 0, \gamma_L \to \text{constant} \). The damping rate
produced by finite larmor radius effects is calculated in Appendix B.

and is found to be

\[ \gamma_u = -\frac{3}{2} \frac{k_x k_z}{\rho_o c_i}. \]  

The growth rate due to the centrifugal acceleration, calculated in Appendix B, is represented by one of two relations:

\[ \gamma_c \equiv \left( \frac{k_r T_R}{\rho_o c_i} \right)^{1/3} \]  

or if \((B')^2 > \frac{\frac{4\pi}{L_n}}{L_n} \) or \(1 > \frac{\frac{4\pi}{L_n}}{B_o L_n} \)

\[ \gamma_{cII} = \frac{B_o}{L_m} \left( \frac{1}{\rho_0 \omega} \right)^{1/3} \]

Equations II-1, 3, 5 lead to estimate of the minimum wavelength in the z direction is: the condition for growth is

\[ \gamma_L + \gamma_c > \gamma_0. \]  

Where Eq. II-5 is the limit most likely to occur in this problem,

using equations II-1, II-3, and II-5, one obtains the following relation

\[ \frac{V_A}{L_n} \left[ \left( \frac{\rho_A}{\rho_o} \right)^{1/2} + \frac{f}{2} \right] > \frac{3}{2} \frac{k_x k_z}{\rho_o c_i}, \]  

or

\[ \lambda_z > \frac{12\pi^2 L_n k_r k_z V_A \rho_A}{B_o^2 \omega \left( \frac{\rho_A}{\rho_o} \right)^{1/2} + \left( \frac{f}{2} \right)^{1/2}}. \]
Equation II-2 and II-8 provide the upper and lower bound on $\lambda_z$. One additional condition is required to complete the description that is $\lambda_z < \pi R$. This implies that the wavelength must be less than the upper half of the circumference of the bubble. These three conditions determine the time of onset for the instability and provide additional information as to possible orientation of region of maximum instability.

The next estimate required is the amount of mass involved in the instability. If we consider a typical sinusoidal oscillation as the perturbation, an estimate of amount of debris can be achieved. See figure 2.

If $\zeta_r < \lambda/4$ and $\lambda/4 < R$ we can assume angles $\theta$ and $\phi$ small and arrive at the following relation

$$
\zeta_r \sim \frac{\lambda_z^2}{16R},
$$

where $\zeta_r$ must be less than $\Delta$, the coupling shell thickness. In order for the instability to remain in the linear regime it has been shown that $\zeta_r < .2 \lambda_z$. We will take $\zeta_r \sim .1 \lambda_z$. As the instability progresses into the nonlinear regime most of the mass involved can be reasonably estimated by the area within the linear regime. The area subtended by a triangle which covers the ejected mass regions of a single oscillation is approximately

$$
A \sim \frac{1}{4} \lambda_z \zeta_r \sim \frac{\lambda_z^2}{40},
$$

The total mass ejected per lobe is then estimated by $\rho_0 A \Delta \theta$ and the total
number of debris particles is estimated by

\[ D_T \sim n_D A_{\theta} \].

The scale length, \( l_\theta \), is considered to be the wavelength in the \( \hat{\theta} \) direction. Since the Rayleigh-Taylor instability is a flute type mode typically \( k \cdot \mathbf{B} \approx 0 \), however there can be finite \( k_\parallel \) or in this case \( k_\theta \) as long as \( k_\parallel /k_\perp < (m_e/m_\perp)^{1/2} \). This establishes an upper bound on \( l_\theta \). A lower bound is approximately half the hemispheric distance in the \( \hat{\theta} \) direction so the minimum \( l_\theta = \frac{\pi R}{2} \). Here we will take the conservative estimate \( l_\theta = \lambda_\theta = \frac{\pi R}{2} \). The volume of debris involved in each outward jetting lobe can now be calculated from Eq. II-11.

The scaling used in this model was obtained from numerical calculations by Dr. Robert Clark. We begin with the usual kinetic energy relation

\[ Y_k = \frac{1}{2} M_0 u_0^2 \]  

where \( Y_k \) is the kinetic yield in ergs.

Assuming complete coupling and \( \rho_A(R) \sim \text{const} \)

\[ M_0 u_0 = \left( M_0 + \frac{4}{3} \pi R^3 \rho_A \right) u \]

more exactly \( M_0 u_0 = \left( M_0 + \lambda \pi \int_0^R R' e \rho_A(R')dR' \right) u \)

If we define \( R^* = M_0/(4/3 \pi \rho_A) \) as the radius at which a weapon mass of air is swept out then the following scaling appears:

\[ u/u_0 = 1/(1 + R^*/R^*) \]
Making use of the marginal stability criterion for the ion-ion instability
and assuming $B_e$ is less than unity, a scaling relation can be derived for the
B field in the shock as a function of radius

$$\frac{B_o}{B_0} \sim \frac{u(4\pi \rho A)^{1/2}}{2.5} \alpha$$

Using Eq. II-13 we obtain the relation

$$B_o \sim \alpha u(4\pi \rho A)^{1/2} / 2.5(1 + R^2/R^*^3)$$

where $B_o \geq B_A$ must hold. This relation is an underestimate of the
B field as the ion-ion instability has already begun. Comparisons with
simulations show that a factor of 3 is necessary to bring this relation
into agreement with the simulations. Therefore, the factor $\alpha$ is set
equal to 3 in recognition of this fact.

The scaling of temperature is obtained by noting that $T_i < T_{\text{kinetic}}$
must hold because Landau damping for the ion-ion instability occurs
where $v - v_A < \sqrt{\frac{kT_i}{m_i}}$. We then use the following scaling:

$$m_iu^2 = kT_{\text{kinetic}}$$

$$(kT_i/m_i u^2) \sim 0.2$$

therefore

$$kT_i \sim 0.4 m_i u^2 \sim 0.4 kT_{\text{kinetic}}$$

Substituting Eqs. II-16, II-17, II-13 into Eq. II-8 we obtain
This minimum wavelength can now be estimated if \( k_r \) and \( L_n \) are known. We estimate the maximum \( \lambda_r \) to be the coupling shell thickness, \( \Delta \), and estimate the density and magnetic scale lengths to be the e-folding distance of half the coupling shell thickness, \( L_n \sim 0.37 \Delta/2 \). From simulations of Dr. Clark it is found that the coupling shell thickness with respect to the magnetic bubble radius varies with time. For reasons to be discussed in the following paragraph the instability will have its onset when approximately one weapon mass has been swept out. For Starfish the time at which \( R=R^* \) is approximately 120 msec, at this time \( \Delta \sim 0.025 R \).

The estimate that the instability onset begins at approximately \( R=R^* \) is determined by Eq. II-2. At this point in the dynamics a variety of effects appear but the dominant effects are the drop in temperature and continued increase in the coupling shell thickness allowing the condition on Eq. II-1 to be met. Applying this scaling for \( k_r \) and \( L_n \), Eq. II-2 and Eq. II-18 can be written as

\[
\lambda_z < 0.37 \Delta/2
\]

II-19
\[ \lambda_z > \frac{6\pi^2(.37)(.04)(2.5)^2}{\sigma^2(4\pi\rho_A)^{3/2} \left[ (\rho_A/\rho_0)^{3/2} + (f/2)^{3/2} \right]} \]

An additional piece of information is also available from Eq. II-19. The coupling shell thickness is a function of angle with respect to the magnetic field approximated by

\[ \Delta(R,\theta) = \Delta(R)/\sin\theta, \quad \theta > \theta_0 \]

where \( \theta_0 \) is the angle at which coupling fails, \( \theta_0 \sim \sqrt{\Delta/R} \).

Equation II-19 provides the major restriction to the onset time of the instability. As can be seen, increasing \( \Delta \) leads to earlier onset of the mode, therefore there will be a preferred direction to the largest modes. Since the field lines are frozen into the plasma and the instability is a flute instability \( k \cdot B = 0 \), the region of max. \( \Delta \), where the expansion is perpendicular to the frozen field line, will be some angle between field alignment, \( \theta = 0 \) and perpendicular to the ambient field line \( \theta = \pi/2 \), but will easily satisfy the condition \( \theta > \theta_0 \). For symmetrical expansion into a B field in a constant density atmosphere, one expects two regions of earliest onset, see Fig. 3. However, the atmosphere is exponential and therefore one has a preferred direction along the ambient density gradient where \( \Delta \) will actually be the largest, Fig. 4. This argument suggests that the region of maximum growth will be in a direction slightly southward and essentially upward for Starfish, Fig. 4. This model predicts onset of the Rayleigh-Taylor instability at a time when \( R = R^* \). At this time the wavelength is estimated to be

\[ \lambda_z > \frac{m_1 c}{\varepsilon} \frac{5.48 \theta}{\sigma^2(4\pi\rho_A)^{3/2} \left[ (\rho_A/\rho_0)^{3/2} + (f/2)^{3/2} \right]} \]
and the number of debris atoms injected is represented by

\[ N_D = \frac{\pi^2 n_D \lambda Z R}{20} f_0 \]  

Here \( n_D \) is taken to be \( 0.5 \, (n_s) \) with \( n_s \approx \frac{4}{3} n_A \) where \( n_A \) is the ambient ion density, \( R \) is as usual the radius of the bubble and \( n_A \) is the ambient mass density of the ions. Generally all quantities labeled \( f_0 \) are defined in the coupling shell and the ambient is defined to be the air located at the shock.

III. Summary

The Rayleigh-Taylor instability is found to be unstable with growth rates determined by Eqs. II-1,3,5 and wavelengths determined by II-2 and II-21. The wavelength and the total number of debris particles transported across the field lines, \( N_D \), are found to be functions of altitude. The instability appears to onset when a weapon mass of air has been swept out by the expanding shock front. From considerations of the relative thickness, \( \Delta \), of the coupling shell a directional dependence can be surmised. The direction most optimal for the instability appears to be oriented in the direction of the ambient density gradient.

For the purpose of modeling the following relations appear to produce the appropriate results. The wavelength is

\[ \lambda_z = \frac{5.48 \, m_1 c}{\alpha \omega^2 (4 \pi \rho_A) \left[ \left( \frac{\rho_A}{\rho_0} \right)^{\frac{1}{2}} + (f/2)^{\frac{1}{2}} \right]^{\frac{1}{2}}} \]

where \( \alpha = 3 \), \( \rho_0 \) is the mass density in the coupling shell, \( f = 0.1 \), \( \rho_A \) is the ambient density at the coupling shell and \( m_1 = 27 \, m_\rho \). The total number of debris
particles transported across the field lines is computed by the relation

\[
N_D = \frac{\pi^2 n_D \lambda Z^R}{20} I_{\theta},
\]

with \( n_D = 0.5 n_s \) where \( n_s \) is the density in the coupling shell and \( n_s \sim 4 n_A \) where \( n_A \) is the ambient mass density at the coupling shell altitude. The scale length is taken to be \( I_{\theta} = \pi R/2 \) where \( R \approx R^* \) and \( R^* \) is the radius at which one weapon mass of air has been swept out.
Appendix A. Linear Calculation of Rayleigh-Taylor Instability

Driven by Shock Acceleration

In this calculation it is assumed that the shock is moving perpendicular to the magnetic field lines. The shock, spatial gradients and force are taken in the $\hat{z}$ direction with the magnetic field in the $\hat{\phi}$ direction. Three species are considered. The electrons and ions moving in the shock frame of reference and a third species comprising approx. 10% of the density, accelerated within the shock with a force $F = \frac{1}{2} \frac{m_v V_A^2}{L_m}$, where $V_A$ is the Alfven speed and $L_m$ is the magnetic field scale length. Further, it is assumed that the magnetic field is frozen into the plasma.

Five variables are unknown, the perturbed velocities for the two ion species and the three perturbed densities. Therefore, five equations are necessary. Consider the general momentum equation:

$$n_j \frac{\partial v_j}{\partial t} + n_j m_j (v_j \cdot \nabla) v_j = q_j \frac{v_j}{c} \times B + n_j q_j E - \nabla p_j + n_j F_j$$  \hspace{1cm} A-1

and apply the following scaling $\omega_{ci} >> \frac{3}{\partial t} V \cdot \nabla$ and $\omega_{ce} >> \omega_{ci}$. This implies that inertial terms can be neglected and that $v_{ei}/v_{ce} << v_{ie}/v_{ci}$. $F_j$ represents collisional drag or acceleration depending on the species and is in the $\hat{z}$ direction. With this ordering the electron momentum Eq. becomes:

$$n_e q_e (E + \frac{\mathbf{v}_e}{c} \times \mathbf{B}) \cdot \nabla p_e - n_e F_e = 0$$  \hspace{1cm} A-2
The ion momentum equation to lowest order in the cyclotron frequency becomes:

\[ q_i \frac{n_i}{c} (v_i \times B) + n_i q_i E - \nabla P_i - n_i F_i \quad \text{with } i = 1, 2 \tag{A-3} \]

so that the lowest order in velocity is:

\[ v_i(0) = \frac{c}{q_i n_i B^2} B \times \nabla P_i - \frac{c}{q_i B^2} B \times F_i \tag{A-4} \]

To the next order one obtains

\[ n_i m_i \left[ \frac{\partial v_i(0)}{\partial t} + (v_i(0) \cdot \nabla) v_i(0) \right] = n_i q_i \left[ \frac{E \cdot \nabla}{c} + \frac{v_i \times B}{c} \right] - \nabla P_i + n_i F_i \tag{A-5} \]

In addition to Eqs. (A2 and 5) we use the continuity equation

\[ \frac{\partial n_i}{\partial t} + \nabla \cdot (n_j v_j) = 0 \tag{A-6} \]

and the ideal gas assumption

\[ P_j = n_j T_j \tag{A-7} \]

We shall assume charge neutrality, \( \nabla \cdot J = 0 \), and incompressibility, \( \nabla \cdot v_j = 0 \). Further, we substitute \( v_i(0) \) for \( v_i \) in the equations and then write \( v_i \) to represent \( v_i(0) \). The assumption that \( v_i(0) \) is equal to the \( v_i \) is valid if \( \frac{cB \times E}{c} \gg \frac{cB \times \nabla E - cB \times F_e}{n_e q_e B^2} \) which is generally true. Therefore, Equation (A-6) becomes

\[ \frac{\partial n_i}{\partial t} + v_i(0) \cdot \nabla n_j = 0 \tag{A-8} \]
Adding Eqs. A-2 and 5 we obtain

\[ n_1 m_1 \frac{dv_1}{dt} + n_2 m_2 \frac{dv_2}{dt} = \left( n_e q_e + n_1 q_1 + n_2 q_2 \right) E + \left( n_e q_e v_e + n_1 q_1 v_1 + n_2 q_2 v_2 \right) \frac{B}{c} \]

\[- \nabla (p_e + p_1 + p_2) + n_e F_e + n_1 F_1 + n_2 F_2 \]

\[ \text{A-9} \]

Recall that \( \nabla \cdot J = 0 \) implies that one can write

\[ J = -\hat{e} \times \nabla \psi \]

where \( \psi \) is a scaler potential function such that \( J \times B = -\nabla \psi \).

Using Eq. (A-10) in Eq. (A-9) and making use of quasineutrality, we obtain

\[ n_1 m_1 \frac{dv_1}{dt} + n_2 m_2 \frac{dv_2}{dt} = -\nabla \hat{P} + \sum_j n_j F_j \]

\[ \text{A-11} \]

where \( \hat{P} = p_e + p_1 + p_2 + B \psi \) and we have assumed this problem is electrostatic. At this point two ways to approach the problem are available:

either linearize Eq. A-11 as it stands using \( f_1 = f_1 e^{i(k \cdot x - \omega t)} \) and keep Fourier components in both directions, \( \hat{x} \) and \( \hat{z} \), or move to a center of mass of motion for the ion species and then linearize. If the first option is applied one must use energy conservation to close the set of equations. The result, to first order in \( k \), is a complex cubic equation with complicated coefficients. The second method which will be followed here requires the following definitions:

\[ V = \frac{E n_j m_j v_j}{\rho} \quad \text{where} \quad \rho = E n_j m_j \]

\[ \hat{F} = \sum_j n_j F_j \] and the perturbation takes the form \( f(x, z) = \hat{F}(z)e^{i(k \cdot x - \omega t)} \).
Incompressibility is still required for the center of mass velocity as well as the individual ion velocities. Equation (A-11) becomes

$$\rho \frac{dV}{dt} = - V_p + F$$  \quad (A-12)$$

when \( V \cdot V = V \cdot V_1 - V \cdot V_2 = 0 \) is used. Taking the curl of Eq. (A-12), and noting that \( V_0 \) is in only the \( \hat{x} \) direction since the calculation is done in the reference frame of shock moving in \( \hat{z} \), and linearizing the resulting equation leads to

$$\rho_o \left\{ -i k \left( \frac{dV_{1z}}{dt} + \frac{\partial}{\partial z} \frac{dV_{1x}}{dt} \right) + \rho_o \frac{dV_{1x}}{dt} \right\} = -i k F_{x-z}$$  \quad (A-13)$$

Applying \( V \cdot V_1 = 0 \) to equation (A-13) leads to a second order equation

$$\text{or} \quad \frac{\partial}{\partial z} \rho_o \frac{\partial V_{1z}}{\partial z} - k^2 \rho \frac{V_{1z}}{\partial z} = -i k \frac{F_{x-z}}{x_0}$$  \quad (A-14)$$

Since we are in the shock frame all accelerations on the shock are zero with the exception of those generated within the shock itself. The force on the second ion species is due to a type of laminar acceleration within the shock.\(^1\) Therefore, the force term reduces to \( F_{z1} = n_{21} F_2 \). In order to proceed the force term is multiplied and divided by a factor of \( \rho \), where the ratio of the perturbed density to the perturbed center of mass density is taken as zeroth order so that

$$\bar{F}_{z1} = n_{21} F_2 + \rho \frac{n_{21} F_2}{\rho_1} = \rho \frac{F_2}{\rho_1}.$$  

Linearizing the center of mass continuity equation

$$\frac{\partial \rho}{\partial t} + V \cdot \nabla \rho = 0$$  \quad (A-15)$$
one obtains $-i(\omega - k_v x) \rho_1 + V_1 x \rho_0 = 0$. Solving for $\rho_1$ and substituting into the force term one can rewrite Eq. (A-14)

$$\frac{\partial}{\partial z} \rho_0 \frac{\partial}{\partial z} V_1 x - k_x V_1 x + \frac{k_x^2 \rho_0^2 V_1 x}{(\omega + k_x V_0 x)^2} = 0. \quad (A-16)$$

The solution to this equation is easily obtain and has the form,

$$V_1 x = e^{-\frac{\rho'}{2\rho_0}} e^{iKz}$$

where

$$K^2 = \left(\frac{\rho'}{\rho_0}\right)^2 + \frac{\rho_0 k_x}{(\omega - k_v x \rho_0)} - k_x^2.$$

The boundary conditions applied to the oscillatory portion of the solution, $q = e^{iKz}$, leads to the quantization condition.

$$K^2 d^2 = n^2 \pi^2 \quad (A-17)$$

where the boundary conditions are $q = 0$ at $z = 0$ and $q = 0$ and $z = d$. With $z$ measured from the peak of the shock profile and $d$ is the distance to the foot of the shock from $z = 0$. Using Eq. (A-17) leads directly to the equation

$$(\omega - k_v x \rho_0) = k_x \left(\frac{\rho_0 k_x}{(\rho_0 \rho_0)^2} - k_x^2\right)^{1/2} \quad (A-18)$$

Now $\rho_0^2/\rho_0 = -1/\lambda$ on the front side of the shock and there will be a growth rate from this mechanism if

$$k_x^2 + \left(\frac{n \pi}{d}\right)^2 \geq \left(\frac{\rho_0^2}{\rho_0}\right)^2 = \lambda^2 \quad (A-19)$$
If \((\frac{n\pi}{d})^2 > \frac{1}{k^2}\) the mode will become convective. One can then estimate the order of \(n\) that will allow the mode to grow. Since \(k = d e^{-1}\), then \(n\pi < \frac{1}{.37}\) must be met. This means only the \(n = 0\) terms will allow growing modes. Therefore, the wavelength of the instability must obey

\[
\lambda_x < 2\pi k .
\]  

Providing this condition is met the growth rate becomes

\[
\gamma = k V_A \left( \frac{\left| \frac{\partial \phi}{\partial \phi} \right| f}{2L_m \left( k^2 \left( \frac{\partial \phi}{\partial \phi} \right) \right)} \right)^{1/2},
\]

where \(f\) is the fraction of accelerated ions and \(L_m\) the scale length of the magnetic field. In the cylindrical coordinates employed in the main body of this work \(k_x \rightarrow k_z\).
Appendix B. Rayleigh-Taylor Instability Driven by Centrifugal Forces

To calculate the effects of centrifugal forces and the Larmor radius effect, it is necessary to work in cylindrical or spherical coordinates. For this problem cylindrical coordinates allows the necessary physics to be described; therefore, we will work in this system. Here we will work in Lagrangian with the various quantities described as follows:

\[ B_0 = (0, B_\theta (R), 0) \]
\[ R(t) = R(t_0) = \zeta (t) \]
\[ U(t_0) = \frac{R}{\partial t} = 0 \quad u_1(t) = \frac{\partial \zeta (t)}{\partial t} = i \omega \zeta \]

where \( \zeta = (\zeta_\rho, 0, \zeta_z) \) and \( f_1 = f_1 e^{i(k \cdot x - (t_0)} \)

Four equations are necessary, we use: Faraday's law

\[ -\frac{1}{c} \frac{\partial B}{\partial t} = \nabla \times u \times B, \text{ where Ohms Law} \]
\[ E + u \times B/c = 0 \]

has been used, the momentum equation

\[ \rho \frac{\partial u}{\partial t} = -\nabla \cdot \left( \rho \mathbf{u} \right) + \frac{1}{\mu_0} \left( \mathbf{B} \cdot \nabla \mathbf{B} \right) + \frac{\mathbf{u} \times \mathbf{B}}{c} \]

, the continuity equation

\[ \frac{\partial n}{\partial t} + \nabla \cdot (n \mathbf{u}) = 0 \quad \text{where we assume } \nabla \cdot \mathbf{u} = 0 \]

and finally we assume for the pressure equation, the ideal gas law

\[ P = nT. \]

Equations B-1,4 are now linearized. The off-diagonal pressure elements provide the finite Larmor radius effects. The off-diagonal pressure are represented in the following fashion,

\[ \pi_{rr} - \pi_{zz} \sim \frac{P}{c_i} \left[ \frac{\partial u_z}{\partial z} + \frac{\partial u_r}{\partial z} \right], \]

\[ \pi_{rz} \sim P \frac{1}{2c_i} \left[ \frac{\partial u_z}{\partial z} - \frac{\partial u_r}{\partial z} \right], \]
where \( \pi_{zr} = \pi_{rz} \). The set of linearized equations consisting of two components of the momentum equation, Faraday's law and the continuity equation are the placed in matrix form and the determinate set to zero to determine the frequency. After considerable algebra a dispersion relation accurate to the second order in \( k \) is obtained

\[
\omega = -\frac{G \pm (G^2 - 4A)^{1/2}}{2}
\]

where \( G = -\left( \frac{\rho}{\rho_0} - \frac{\delta \epsilon}{\rho_0 (\rho_0 - \epsilon)} \right), \ A = \alpha n_0 / \rho_0 + \frac{1}{4 \mu_0} \left( \frac{\partial B_0}{\partial t} \right)^2 - \left( \frac{B_0}{r} \right)^2 \)

with

\[
\delta = \frac{p \ w_c}{\omega_c^2} \left( \frac{3}{r} + \frac{1}{n_0} \frac{\partial n_0}{\partial r} \right)
\]

\[
\alpha = \frac{p \ w_c}{\omega_c^2} \frac{1}{r} \left( \frac{3}{r} + \frac{1}{n_0} \frac{\partial n_0}{\partial r} \right)
\]

\[
\epsilon = -\frac{p \ w_c}{2\omega_c^2} \left( \frac{1}{r} + \frac{1}{n_0} \frac{\partial n_0}{\partial r} \right) \ k_r - i k_r^2
\]

\[
\kappa = \frac{p \ w_c}{2\omega_c} \left( k_z \left( \frac{1}{r} + \frac{1}{n_0} \frac{\partial n_0}{\partial r} \right) - k_r k_z \right).
\]

Neglecting terms of order \( 1/r^2 \) and \( k^3 \)

\[
G \sim \frac{p \ k_z}{\rho_0 \ w_c} \left( \frac{3}{r} + 1/r \frac{\partial n_0}{\partial r} - 1/2 \ k_r \right)
\]

\[
B-7
\]
Equation B-6 is in a form to be handled numerically since all of the components are either zeroth order terms or must obey scaling relations which have to be postulated to proceed any further.

The functional dependence of the growth rate for the wave can be obtained by approximating or expanding some of the components of Eq. B-16. For example, if one assumes the gyro-radius terms are smaller than the curvature term then

\[ 4A > G^2 \text{ or } (G^2 - 4A)^{3/2} \sim 12A^{3/2} (1 - G^2/8A) \]

and

\[ \omega \sim - \frac{G}{2} \pm iA^{3/2} (1 - G^2/8A) \quad \text{B-8} \]

The variable, \( A \), can be approximated in three separate forms:

**case 1**

\[ A^{3/2} = \frac{ik \Gamma_{T_1} k_{n_0}^*}{\rho_0} \left[ 1 - \frac{i}{\partial_{nk} \Gamma_{T_1} k_{n_0}^*} \left( \left( \frac{\partial B_0}{\partial r} \right)^2 + \left( \frac{B_0}{r} \right)^2 \right) \right] \]

**case 2**

\[ A^{3/2} = \pm \frac{iB_0}{2r(\rho_0 n)^{3/2}} \left( 1 - \frac{(\partial B_0/\partial r)^2}{(B_0/r)^2} \right) \]

**case 3**

\[ A^{3/2} = \pm \frac{\partial B_0}{\partial r} \frac{1}{(\rho_0 n)^{3/2}} \left( 1 - \frac{B_0^2}{r^2 \left( \frac{\partial B_0}{\partial r} \right)^2} \right) \]

Since \( G \) is complex there will be both a real frequency and a growth rate.

The three cases lead to the following solutions:

**case 1** \( \text{Im} A > \text{Re} A \)

\[ \gamma \sim - \frac{3}{2} \frac{p^1}{\rho_0 \omega c_1 k^2 k^2} \left( \frac{k \Gamma_{T_1} k_{n_0}^*}{2m_1} \left( \frac{n_0'}{n_0} \right) \right)^{3/2} \left( 1 + \frac{1}{\partial_{nk} \Gamma_{T_1} k_{n_0}^*} \left[ B_0^2 + \left( \frac{B_0}{r} \right)^2 \right] \right) \quad \text{B-9} \]

\[ \omega \sim \frac{p^1 k^2}{\rho_0 \omega c_1} \left( 5/r + n_0'/n_0 \right) \mp \left( \frac{k \Gamma_{T_1} k_{n_0}^*}{2m_1} \left( \frac{n_0'}{n_0} \right) \right)^{3/2} \left( 1 + \frac{1}{\partial_{nk} \Gamma_{T_1} k_{n_0}^*} \left[ B_0^2 + \left( \frac{B_0}{r} \right)^2 \right] \right) \]
As can be seen from these relations, the finite larmor radius terms provide a damping term and the curvature affects provide a growing term. For the work of current interest only case 1 or 3 are of interest. Typically case 3 appears to be the dominate limit.
Appendix C

Here we make an estimate of the various quantities derived in Sec. II of the report. We take the following parameters from calculations by Dr. Clark.

\[ \begin{align*}
  n_i &= 27 \text{mp} \\
  R &= 220 \text{km} \\
  f &= 0.1 \\
  n_A &= 2.5 \times 10^5 \\
  \rho_A &= 6.68 \times 10^{-18}
\end{align*} \]

\[ \begin{align*}
  \Delta &\approx 0.025 R \\
  n_i &= 4n_A \\
  n_i &\approx 1 \times 10^6 \\
  n_o &\approx 0.5n_i \\
  \rho_o &\approx 4.5 \times 10^{-17}
\end{align*} \]

With these parameters we find the following sizes for the parameter. From Eq. II-21 we obtain

\[ \lambda_z \approx 2.8 \times 10^6 \text{ cm for } \alpha = 1 \]  

\[ \lambda_z \approx 3.1 \times 10^5 \text{ cm for } \alpha = 3 \]  

From Eq. II-22 the total number of debris atoms can be calculated using C-1.

\[ N_D = \frac{\pi R^2 \lambda_z R \theta z}{20} \approx 1.52 \times 10^{18} \lambda_\theta , \alpha = 3 \]  

\[ N_D \approx 9.15 \times 10^{18} \lambda_\theta, \alpha = 1 \]  

Now C-2 can be completed if we know \( \lambda_\theta \). If we use

\[ \lambda_\theta = \lambda_{\parallel} = \lambda_{\parallel} \left( \frac{m_1}{m_e} \right)^{1/2} \]
then $\lambda_0 \approx 1.09 \times 10^8$. If we use $\lambda_0 \approx \pi R$ then $\lambda_0 \approx 6.97 \times 10^7$. Actually because of the effects of the exponential atmosphere in $\Delta$ as discussed in section II. We estimate $\lambda_0 \approx \pi R/2$ and use this as a lower bound on $\lambda_0$. Then $\lambda_0 \approx 3.5 \times 10^7$ and

$$N_{D_{\text{min}}} \approx 5.3 \times 10^{25} \quad \alpha = 3$$

$$N_{D_{\text{max}}} \approx 1.7 \times 10^{26} \quad \alpha = 3$$

using the upper bound as $\lambda_0$.

Using the result from C-1 the total number of wavelengths that will fit on the bubble is

$$N_w = \frac{\pi R}{\lambda_z} \approx 2.23 \times 10^2, \quad \alpha = 3$$

$$N_w \approx 25, \quad \alpha = 1$$

The growth and damping rate have the following magnitudes. First we calculate some useful parameters

$$V_A \approx \frac{3}{(4\pi \rho)^{1/2}} \approx 3.2 \times 10^7$$

$$R \approx R^*$$

$$u \approx \frac{u_0}{2}$$

$$u_0 = \left(\frac{2Y_K}{M_0}\right)^{1/2} \approx 1.7 \times 10^8$$

$$B_0 \approx au_0 \left(4\pi \rho_A\right)^{1/2}/5$$
From Eq. II-1 and 5

\[ \gamma_L + \gamma_{CH} \sim \frac{V_A}{L_n} \left[ \left( \frac{\rho_A}{\rho_o} \right)^{1/2} + \left( \frac{\rho_o}{\rho_A} \right)^{1/2} \right] \sim 1.91 \times 10^2 \]

From Eq. II-3

\[ \gamma_D \sim 1.5 (.04) u^2 \frac{2\pi}{\Delta} \frac{2\pi}{\lambda} \frac{m_c}{\eta} \]

\[ \gamma_D \sim 1.21 \times 10^2 \]

\[ \gamma = \gamma_D + \gamma_L + \gamma_C \sim 7.1 \times 10^1 \]

This mode will e-fold quite rapidly after onset of the instability.

References


Fig. 1 - Coordinate system for the general calculation. In the bubble \( B = B(\theta) \).

Fig. 2 - Linear deformation of the plasma surface during growth of Rayleigh-Taylor instability. \( \xi_r \) is the linear displacement, typically less than 20% of \( \lambda \).
Fig. 3 - Expansion of bubble into constant density atmosphere. The development of the coupling shell is symmetric about $\Delta_{\text{min}}$. For coupling to occur $\theta > \theta_0 \sim \sqrt{\Delta/R}$. Regions of earliest onset are demarked $\Delta_{\text{max}}$.

Fig. 4 - Expansion of bubble into atmosphere with density gradient directed upward and field lines directed obliquely to the gradient. Coupling shell has lost symmetry and $\Delta_{\text{max}}$ is directed at an angle to both $\Delta n_o$ and the field line. Region for earliest onset of the instability is demarked $\Delta_{\text{max}}$. 
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