ASYMPTOTIC NORMALITY OF A VARIANCE ESTIMATOR OF A LINEAR COMBINATION
OF A FUNCTION OF ORDER STATISTICS

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Abstract

An estimator of the asymptotic variance of (a randomly stopped)
linear combination of a function of order statistics is considered and its
asymptotic normality is studied under appropriate regularity conditions.
A comparative study of the regularity conditions pertaining to the
asymptotic normality and strong convergence of linear combinations of
functions of order statistics and their estimated asymptotic variances is
also made.

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sure convergence; empirical distribution; linear combination of order statistics; quantile process;
stopping time; Wiener process.

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1. Introduction.

Let \( \{X_i, i \geq 1\} \) be a sequence of independent and identically distributed random variables (i.i.d.r.v.) with a continuous distribution function (d.f) \( F \), defined on the real line \( \mathbb{R} = (-\infty, \infty) \). For every \( n \geq 1 \), let \( X_{n,1}, \ldots, X_{n,n} \) be the order statistics corresponding to \( X_1, \ldots, X_n \) and consider the statistics

\[
T_{n,k} = n^{-1} \sum_{i=1}^{k} c_{n,i} h(X_{n,i}), \quad 1 \leq k \leq n, \tag{1.1}
\]

where \( \{c_{n,i}, 1 \leq i \leq n; n \geq 1\} \) is a triangular array of (known) real constants and \( h \) is a specified function. Actually, if we let \( g = h \circ F^{-1} \) and \( \xi_{n,i} = F(X_{n,i}), 1 \leq i \leq n \) (so that \( \xi_{n,1}, \ldots, \xi_{n,n} \) are the ordered r.v. of a sample of size \( n \) from the uniform \((0,1) \) d.f), we may rewrite (1.1) as

\[
T_{n,k} = n^{-1} \sum_{i=1}^{k} c_{n,i} g(\xi_{n,i}), \quad 1 \leq k \leq n. \tag{1.2}
\]

Under suitable regularity conditions (on \( g \) and the \( c_{n,i} \)), for \( k/n \rightarrow \alpha \) \((0 < \alpha \leq 1)\),

\[
n^b(T_{n,k} - \mu(\alpha))/\sigma(\alpha) \overset{\mathbb{P}}{\rightarrow} N(0,1), \tag{1.3}
\]

where for each \( \alpha \in (0,1) \), \( \mu(\alpha) \) (asymptotic mean) and \( \sigma^2(\alpha) \) (asymptotic variance) are functionals of \( g \) and the score function \( J \) (which generates the \( c_{n,i} \)). (1.3) has been proved under diverse regularity conditions by a host of research workers (viz. [1, 3, 4, 5, 6, 7, 8]). Stigler (1969) has also shown that under suitable regularity conditions,

\[
n \text{Var}(T_{n,k})/\sigma^2(\alpha) \rightarrow 1 \text{ as } n \rightarrow \infty. \tag{1.4}
\]

Let \( \{\tau_n, n \geq 1\} \) be a class of stopping times, where, for each \( n \geq 1 \),

\( \tau_n \) is defined in terms of \( X_{n,1}, \ldots, X_{n,n} \) and it assumes values in
Gardiner and Sen (1978) have shown that if \( n^{-1} \rightarrow \alpha \in (0,1) \)
and the regularity conditions pertaining to (1.3) hold, then
\[
-\frac{1}{n}\left( T_{n,n} - \mu(n^{-1} \tau_n) / \sigma(\alpha) \right) \overset{L}{\rightarrow} N(0,1),
\]
(1.5)
while if \( n^{-1}(\tau_n - \alpha) \rightarrow 0 \), then in (1.5), \( \mu(n^{-1} \tau_n) \) may also be replaced by \( \mu(\alpha) \).

In a variety of practical applications, \( \mu(\alpha) \) can be related to the basic (viz., location or scale) parameters of \( F \), and thereby, confidence intervals or tests of significance for \( \mu(\alpha) \) can be transmitted to yield parallel conclusions for these parameters. In this context, one confronts the problem of estimating \( \sigma^2(\alpha) \) and natural estimators of \( \sigma^2(\alpha) \) can be derived from the sample. The object of the present investigation is to consider such an estimator of \( \sigma^2(\alpha) \) and to study its asymptotic normality.

Along with the preliminary notions, the main theorems are presented in Section 2 and their proofs are considered in Section 3. Section 4 is devoted to some general remarks including a comparative study of the regularity conditions pertaining to the almost sure (a.s.) convergence and asymptotic normality of \( T_{n,k} \) and the estimator of \( c^2(\alpha) \). For the convenience of presentation, some of the technicalities are postponed to the Appendix.

2. Preliminary notions and the main theorems.

Define \( g \) as in after (1.1)

and assume that for every \( \theta \in (0,1) \), \( g \) is of bounded variation in

\((0, 1-\theta)\). For each \( n (> 1) \), define \( J_n \) on \([0,1]\) by letting \( J_n(t) = c_{n,i} \) for \((i-1)/n < t \leq i/n, 1 \leq i \leq n \) and \( J_n(0) = c_{n,1} \). Also, let

\[ T_n(t) = n^{-1} \sum_{i=1}^{n} I(\xi_{n,i} \leq t), \quad t \in [0,1] \]

be the empirical df. Then \( T_{n,k} \)
in (1.2) can be expressed as
We define a bounding function

\[ B(\cdot, \alpha) = \{ B(t, \alpha) = M t^{-a_1} (1-t)^{-a_2}, t \in (0,1) \} \]

where \( M(0 < M < \infty), \alpha = (a_1, a_2) \) and \( a_1, a_2 \) are real numbers. Also, for fixed \( \beta(>0) \) and \( \delta(>0) \), we define

\[ q_\beta = \{ q_\beta(t) = (t(1-t))^{\beta-\delta/2}, t \in (0,1) \}. \]

Then, we make the following assumptions:

[A1]: \( |g| \leq B(\cdot, \alpha) \) for some \( \alpha = (a_1, a_2) \).

[A2]: There exists a \( J_n \), defined on \((0,1)\), such that

\[ |J_n| \leq B(\cdot, b) \text{ and } |J| \leq B(\cdot, b), \forall n, \]

where \( b = (b_1, b_2) \) with real \( b_1, b_2 \) and except on a set of \( t \)'s of \( |g| \)-measure zero, both \( J \) is continuous at \( t \) and \( J_n \to J \) uniformly in some neighborhood of \( t \) as \( n \to \infty \).

For each \( \alpha \in (0,1] \), let us then define

\[ \mu_n(\alpha) = \int_0^\alpha J_n(t)g(t)dt, \]

\[ \sigma^2(\alpha) = \int_0^\alpha \int_0^\alpha (s \wedge t - st)J(s)J(t)dg(s)dg(t); \quad a \wedge b = \min(a, b). \]

Note that if

\[ a_1 + b_1 = a_2 + b_2 = 1/2 - \delta \]

then \( \int_0^1 B(\cdot, b)q_\beta d|g| < \infty \) and it follows from assumptions A1, A2 that both \( \mu_n(\alpha) \) and \( \sigma^2(\alpha) \) are finite and then (1.3) holds [cf. Shorack (1972)]. If, in addition \( n^{-1/2} R \alpha \in (0,1) \) and \( g \) admits a derivative.
at \( a \) or \( n^{-\frac{1}{b}}(t_n - na) = O(1) \) and \( g \) is continuous at \( a \) then (1.5) obtains [cf. Gardiner & Sen (1978)].

In the current paper, we consider the following estimator of \( \sigma^2(a) \):

\[
\hat{\sigma}^2_n(a) = \int_0^a \int_0^a \left( \tau_n(s \wedge t) - \tau_n(s)\tau_n(t) \right) J_n(\tau_n(s))J_n(\tau_n(t)) d\tau(s)d\tau(t) \tag{2.8}
\]

which can also be written as

\[
\hat{\sigma}^2_n(a) = n^{-\frac{1}{b}} \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} c_{n,i}c_{n,j} [n(i \wedge j) - ij][h(x_{n,i+1}) - h(x_{n,i})][h(x_{n,j+1}) - h(x_{n,j})] + r_n, \tag{2.9}
\]

where \( n = \min(k; \xi_{n,k} \leq a) \) and \( r_n = O(p(\xi_n^{-1})) \). Also, as in Sen (1978), \( \hat{\sigma}^2_n(a) \) can be interpreted as the conditional variance of \( nT_{n,h} \) given \( \{X_{n+k,1}, 1 \leq j \leq n+k \text{ and } k \geq 1\} \). Our main concern is to study regularity conditions pertaining to the asymptotic normality of \( n^{-\frac{1}{b}}(\sigma^2_n(a) - \sigma^2(a)) \).

For this purpose we need some additional regularity conditions:

[A3]: \( n^{b} \int_0^1 \left| J_n(\tau_n(t)) - J(\tau(t)) \right| d\tau(t) \leq 0 \), as \( n \to \infty \).

[A4]: except on a set of \( t \)'s of \( |g| \) measure zero, \( J'(t) = (d/dt)J(t) \) exists and is continuous at \( t \), and for some \( \xi = (\xi_1, \xi_2) \),

\[
|J'| \leq \beta(\xi, \xi) \text{ where } 0 \leq \xi_1 - b_1, \xi_2 - b_2 \leq 1, \tag{2.10}
\]

with \( \beta \) defined in [A2].

Let us now write \( 1 \) for the identity function on \( (0,1) \) and let

\[
J_1 = 1J, \quad J_2 = (1 - 1)J; \tag{2.11}
\]

\[
L_1(t) = 2 \int_0^1 J_2(t) d\tau, \quad L_2(t) = 2 \int_0^t J_1(t) d\tau, 0 < t < 1; \tag{2.12}
\]

\[
l_0 = l_1J_0(1) + l_2J'(2). \tag{2.13}
\]
Define
\[
\gamma^2 = \int_0^1 \int_0^1 (s \wedge t - st) l_0(s) l_0(t) dg(s) dg(t).
\] (2.14)

Then, we have the following.

Theorem 1. Suppose that A1, A2, A3 and A4 hold and
\[
\int_0^1 R(\cdot, b) q_0 d|g| < \infty.
\] (2.15)

Then, both \( \sigma^2(1) \) and \( \gamma^2 \) are finite and
\[
n^b (\sigma^2_n(1) - \sigma^2(1)) / \gamma \xrightarrow{t \to \infty} N(0,1).
\] (2.16)

The proof is considered in the next section. We may remark here that
in (2.11) through (2.14), if we let \( J(t) = 0 \) for \( t \geq a \) (when \( 0 < a < 1 \))
and denote the resulting expression in (2.14) by \( \gamma_a^2 \), then (2.16) holds for
\[
n^b (\sigma^2_n(a) - \sigma^2(a)) / \gamma_a.
\] Hence, for the sake of simplicity, we consider the case of \( a = 1 \) and, for notational convenience, write \( \sigma^2_n(1) = \sigma^2_n \), \( \sigma^2(1) = \sigma^2 \). We may also remark that whenever \( L_0 \) in (2.13) is integrable
with respect to the signed measure \( g \) on \((0,1)\), a more convenient form of
(2.14) can be obtained. Define \( G_0 \) on \((0,1)\) by
\[
G_0(t) = \int_0^t l_0(s) dg(s), 0 < t < 1.
\] (2.17)

Then, a pedestrian calculation leads us to
\[
\gamma^2 = \int_0^1 G_0^2(t) dt - \left( \int_0^1 G_0(t) dt \right)^2.
\] (2.18)

Now let us suppose \( 0 < a < 1 \) and set \( \sigma^2_n = \sigma^2_n(n^{-1} (0, a)) \). In the statement
of A4 we assume additionally that \( J \) is continuous at \( a \) and \( J_n \to J \)
uniformly in some neighborhood of \( a \) as \( n \to \infty \). For \( t \in (0, a) \) we define
\[
L_1^+(t) = 2 \int_t^a \int_0^1 J_0(t) dg \quad \text{and} \quad L_0^+ = L_1^+ \int_0^1 J_1(t) dt + L_2^+ \int_0^1 J_2(t) dt.
\] Let \( (\gamma^2) = \int_0^1 \int_0^1 (s \wedge t - st) l_0(s) l_0(t) dg(s) dg(t) \).
Theorem 2. With the remarks noted above suppose that A1 through A6 hold together with (2.15). Then both \( \sigma^2(a) \) and \( \gamma^+(a) \) are finite and if, in addition, \( n^{-1} \int_a \rangle \), then

\[
n^{-1} (\sigma^2_n - \sigma^2_n(n^{-1})^\gamma(a)) \geq B(0,1)
\]

while if \( n^{-1}(n^{-1} - m) \geq 0 \) and \( g \) admits a derivative at \( a \) then in

(2.19) \( \sigma^2(n^{-1}) \) may also be replaced by \( \sigma^2(a) \).

3. Proofs of Theorems.

Note that by (2.4) and (2.6),

\[
0 \leq \sigma^2(a) = \int_0^\infty s(t-t)J(s)J(t)dg(s)dg(t) \leq 2\int_0^1 (t(1-t))^\gamma |J(t)|d|g(t)|^2
\]

\[
\leq 2\int_0^1 (t(1-t))^\gamma |J(t)|d|g(t)|^2, \forall \alpha \in (0,1]
\]

\[
\leq 2n^2\int_{b_1}^1 Bq_y|d|g||^2, \forall \alpha \in (0,1].
\]

Now (2.15) ensures the less restrictive condition \( \int_0^1 Bq_y|d|g| < \infty \) and so \( \sigma^2(a) < \infty \) for every \( a \in (0,1] \). Similarly, on noting that under (2.15), by (2.11), (2.12) and (2.13)

\[
|L_0(t)| \leq M^* (t(1-t))^{-\gamma} B(t, b), \forall t \in (0,1), \text{ for some } M^* < \infty, (3.2)
\]

we have by (2.14) and (3.2),

\[
\gamma^2 \leq 2\int_0^1 (t(1-t))^\gamma |L_0(t)|d|g(t)|^2
\]

\[
\leq 2(M^* \int_0^1 (t(1-t))^\gamma B(t, b)|d|g(t)|^2
\]

\[
< 2M^2 (\int B(\cdot, b)q_y|d|g||^2 < \infty, \text{ by } (2.15)
\]

(3.3)
Note that by (2.6), (2.11) and (2.12),
\[ \sigma_n^2 = 2 \int_0^1 s(1-t)J(s)J(t)dg(s)dg(t) - \frac{1}{2} \int_0^1 L_1(t)dl_2(t). \]  
(3.4)

Again, if we define for each \( n > 1 \) and \( t \in [\xi_{n,1}, \xi_{n,n}] \),
\[ L_{n,1}(t) = 2 \int_{\xi_{n,1}}^{\xi_{n,n}} (1-\tau_n)J(\tau_n)dg \]  
and \( L_{n,2}(t) = 2 \int_{\xi_{n,1}}^{t} \tau_n J(\tau_n)dg \),
(3.5)
with both \( L_{n,1} \) and \( L_{n,2} \) set equal to zero otherwise, we may write
\[ \sigma_n^2 = (1/2) \int_0^1 L_{n,1}(t)dl_{n,2}(t). \]  
(3.6)

From (3.4) and (3.6), we have
\[ n^2(\sigma_n^2 - \sigma^2) = \frac{1}{2} \left[ S_{n,1} + S_{n,2} + R_n \right], \]  
(3.7)
where
\[ S_{n,1} = \int_0^1 \tau_n^2(L_{n,1}(t) - L_1(t))dl_2(t), \]  
(3.8)
\[ S_{n,2} = \int_0^1 L_1(t)d\tau_n^2(L_{n,2}(t) - L_2(t)), \]  
(3.9)
\[ R_n = \int_0^1 \tau_n^2(L_{n,1}(t) - L_1(t))d(L_{n,2}(t) - L_2(t)). \]  
(3.10)

Let \( (\omega, \mathcal{B}, P) \) be the underlying probability space and let \( U = n^2(\Gamma_n^2 - 1) \) be the uniform empirical process on \([0,1]\). Suppose \( U \) denotes a standard Brownian bridge process on \([0,1]\) defined on the same probability space. \( (\omega, \mathcal{B}, P) \) may not be rich enough to support \( U \). However, by one of the usual techniques of embedding [cf. Shorack (1972)], we may construct another probability space \( (\omega^*, \mathcal{B}^*, P^*) \) where the distributions of our original variables are preserved and which is rich enough to support \( U \). Let \( Q \) be the class of all nonnegative, continuous \( q \) on \([0,1]\) which are bounded below by functions \( \tilde{q} \) nondecreasing (nonincreasing) on \([0,\frac{1}{2}] \) and \([\frac{1}{2}, 1]\),
satisfy \( \int_0^1 \frac{-2}{q} \, dt < \infty \). Let \( \rho_q(f,g) = \sup \{ |f(t) - g(t)|/q(t) : 0 < t < 1 \} \) be the usual sup-norm metric. Then, it is known that for each \( q \in \mathbb{Q} \)

\[
\rho_q(U_n, 0) = \rho_q(U_n, 0) = \rho_q(U_n, 0). \tag{3.11}
\]

Note that by our definitions,

\[
\xi_n \frac{b}{n} (L_n - L_1) = - \int_\xi_n \frac{b}{n} J_n (\Gamma_n) \, dg + \int_\xi_n (1-1) \frac{b}{n} (J_n (\Gamma_n) - J(\Gamma_n)) \, dg
\]

\[
= \int_\xi_n (1-1)J \, dg, \quad \text{otherwise} \tag{3.12}
\]

\[
\xi_n \frac{b}{n} (L_n - L_2) = \int_\xi_n \frac{b}{n} J_n (\Gamma_n) \, dg + \int_\xi_n (1-1) \frac{b}{n} (J_n (\Gamma_n) - J(\Gamma_n)) \, dg
\]

\[
= \int_\xi_n (1-1)J \, dg, \quad \text{otherwise}. \tag{3.13}
\]

Substituting (3.12) in (3.8), we write

\[
S_n = -S_n^{(1)} + S_n^{(2)} + S_n^{(3)} - S_n^{(4)}. \tag{3.14}
\]

Define \( \xi_1 = \int_{1}^{1} \int_{t}^{1} dL_n \) and let \( \chi_{n, 1}, \chi_{n, 2} \) denote the indicators of \([\xi_{n, 1}, \xi_{n, n}]\) and \([t, \xi_{n, n}]\), \( t \in (0, 1) \) respectively. Then
\[ |s_{n,1}^{(1)} - \xi_1| \leq \int_0^1 IB(\cdot, \beta) \chi_{n,1} \, d|q| \left( \int_0^1 |\chi_{n,2} \cup J_n(\Gamma_n) - UJ| \, d|q| \right) + \int_0^1 IB(\cdot, \beta) \overline{\chi}_{n,1} \, d|q| \left( \int_0^1 |UJ| \, d|q| \right) \]
\[ = S_{n,11}^{(1)} + S_{n,12}^{(1)}, \text{ say}, \]

where \( \chi_{n,1} \) is the indicator of the complement of \( \{\xi_{n,1}, \xi_{n,2}\} \) relative to \((0,1)\). To handle \( S_{n,11}^{(1)} \) note that for \( \chi_{n,1} = 1 \) and \( \chi_{n,2} = 1 \) we have

\[ |\chi_{n,2} \cup J_n(\Gamma_n) - UJ| \leq |U_n - U|B(\cdot, \beta) + |J_n(\Gamma_n) - J||U_n|. \tag{3.16} \]

Furthermore, \( |J_n(\Gamma_n) - J| \leq 2B(\Gamma_n, \beta) \vee B(1, \beta) \), and since \( 0 < \Gamma_n < 1 \), in the range under consideration, we obtain by Theorem 2 of Wellner (1977) that there exists a set \( A \subset \Omega \) such that \( P(A) = 1 \) and for each \( \omega \in A \) there exists an integer \( n_\omega \) for which \( n \geq n_\omega \) implies

\[ |J_n(\Gamma_n) - J| \leq M^0 B(\cdot, \beta) q_1 \tilde{q}, \tag{3.17} \]

where \( M^0 < \infty \) is a constant and \( \tilde{q} = (1(1 - 1))^{b-\delta/4}. \) For such \( \omega \) and \( n \), therefore, from (3.11) and (3.17), the right hand side of (3.16) is bounded by

\[ \rho_{\dot{q}_1 q_2} (U_n, U) B(\cdot, \beta) q_1 \tilde{q} + M^0 q_1 (U_n, 0) B(\cdot, \beta) q_1 = O_p(1) B(\cdot, \beta) q_1 \tag{3.18} \]

whenever \( \chi_{n,1} = 1 \) and \( \chi_{n,2} = 0 \). When \( \chi_{n,1} = 1 \) and \( \chi_{n,2} = 0 \), however the left hand side of (3.16) is again dominated by \( O_p(1) B(\cdot, \beta) q_1 \). We note that \( \Gamma_n \to 1 \) uniformly on \([0,1]\) and thus by [A4], \( J_n(\Gamma_n) \to J \) (a.s.), pointwise a.e. \( |q| \). Since \( \xi_{n,n} \to 1 \) a.s. and from (3.11) we have for each \( t \in (0,1), \chi_{n,2} U J_n(\Gamma_n) \to UJ \) (a.s.), pointwise a.e. \( |q| \). Hence the dominated convergence theorem applies and for each \( t \in (0,1), \), we obtain
\[ x_{n,1}(t) \int_0^1 |x_{n,2} U_{n,n}(t^n) - U \int d|g| \] 

Again for each \( t \in (0,1) \), we have using the upper bound in (3.18)

\[ IB(\cdot, b) \left\{ \int_0^1 |x_{n,2} U_{n,n}(t^n) - U \int d|g| \right\} \]

\[ \leq \{1(1 - 1)\}^{\delta/2} B(\cdot, b)q_d \left\{ \int_0^1 B(\cdot, b)q_d \right\} \mathcal{O}_p(1), \] 

where the right hand side is a \(|g|\)-integrable function. It then follows from (3.19) and the dominated convergence theorem that \( S_{n,1}^{(1)} \mathcal{O}_p 0 \) as \( n \to \infty \).

To handle \( S_{n,12}^{(1)} \) we write

\[ S_{n,12}^{(1)} \leq \int_0^{\xi_{n,1}} IB(\cdot, b) d|g| \left\{ \int_t^1 u B(\cdot, b) d|g| \right\} \]

\[ + \int_{\xi_{n,1}}^1 IB(\cdot, b) d|g| \left\{ \int_t^1 u B(\cdot, b) d|g| \right\} . \] 

The first term on the right hand side may be bounded by

\[ \int_0^{\xi_{n,1}} IB(\cdot, b) d|g| \left\{ \int_t^1 u B(\cdot, b) q_d \right\} \mathcal{O}_p(0,0) \]

\[ \leq \{1(1 - 1)\}^{\delta/2} B(\cdot, b) q_d \left\{ \int_0^1 B(\cdot, b) q_d \right\} \mathcal{O}_p(1) \]

\[ = \mathcal{O}_p(1), \text{ since } \xi_{n,1} \mathcal{O}_p 0 \text{ and the integral converges}. \]

The same argument will also show that the second term on the right hand side of (3.21) is \( \mathcal{O}_p(1) \). Hence, finally it follows from (3.15) that

\[ S_{n,1}^{(1)} \mathcal{O}_p \xi_{n,1} = \int_0^1 \int_0^1 u W d|g| dL_2, \text{ as } n \to \infty. \] 

Next, we note that by [A3] and the definition of \( S_{n,1}^{(2)} \).
\[ |S^{(2)}_{n,1}| \leq \int_0^1 IB(\cdot, b) d|g| \{ \int_0^1 (1 - I) n^4 |J_n(\Gamma'_n) - J(\Gamma'_n)| d|g| \}
\]
\[ \leq \left( \int_0^1 (1 - I) \delta / 2 B(\cdot, b) q \right) \{ \int_0^1 n^4 |J_n(\Gamma'_n) - J(\Gamma'_n)| d|g| \}
\]
\[ = O(1) \cdot o(1). \]  
(3.23)

To handle \( S^{(3)}_{n,1} \) we note that it may be written in the form
\[ S^{(3)}_{n,1} = \int_0^1 dL(t) \lambda_{n,1}(t) \{ \int_t^1 \chi_{n,2} (1 - I) \Upsilon_n (J(\Gamma'_n) - J)/(\Gamma'_n - I) d|g| \} \]  
(3.24)

where the indicators \( \chi_{n,1}, \chi_{n,2} \) were defined preceding (3.15). Define
\[ \tau_2 = \int_0^1 \{ \int_t^1 (1 - I) \Upsilon d|g| \} dL(t). \]

Then
\[ |S^{(3)}_{n,1} - \tau_2| \leq S^{(1)}_{n,13} + S^{(1)}_{n,14}, \]  
(3.25)

where
\[ S^{(1)}_{n,13} = \int_0^1 IB(\cdot, b) \chi_{n,1} d|g| \{ \int_t^1 (1 - I) |\chi_{n,2} \Upsilon_n (J(\Gamma'_n) - J)/(\Gamma'_n - I) - \Upsilon'| d|g| \}, \]  
(3.26)
\[ S^{(1)}_{n,14} = \int_0^1 IB(\cdot, b) \chi_{n,1} d|g| \{ \int_t^1 (1 - I) |\Upsilon'| d|g| \}. \]  
(3.27)

The analysis of \( S^{(1)}_{n,13} \) is very similar to that of \( S^{(1)}_{n,11} \). Note that
\[ |J(\Gamma'_n) - J|/|\Gamma'_n - I| \leq B(\Gamma'_n, \xi) \vee B(1, \xi) \]  
by [A4]. Once again since \( 0 < \Gamma'_n < 1 \) in the range under consideration in (3.26) we may invoke Theorem 2 of

Wellner (1972): for some \( A^* \subseteq \Omega \) with \( P(A^*) = 1 \), there exists for each
\[ \omega \in A^*, \text{ an integer } n^*_\omega \text{ such that for } n \geq n^*_\omega \]
\[ |J(\Gamma'_n) - J|/|\Gamma'_n - I| \leq M_0 B(\cdot, \xi)/\tilde{q} \]  
(3.28)

where \( M_0 (< \omega) \) is a constant and \( \tilde{q} \) is defined as in (3.17). By steps similar to (3.16) through (3.19) for \( S^{(1)}_{n,11} \) and the continuity of \( J' \), we obtain, for each \( t \in (0,1) \)
\[
\chi_{n,1}(t) \int_0^1 (1 - 1) \left| \frac{x_{n,2}u_n (J(g_n) - J)/(g_n - 1)}{uJ' d|q|} \right|^2 0 . \quad (3.29)
\]

Furthermore in view of [A4],
\[
IB(\cdot, \mathcal{L}) = \int_0^1 \left| \left( 1 - 1 \right) \frac{x_{n,2}u_n (J(g_n) - J)/(g_n - 1)}{uJ' d|q|} \right| \mathcal{L} \]
and the right hand side is a \(|q|\)-integrable function. Hence from (3.29) and the dominated convergence theorem, we obtain \( S_{n,13}^{(1)} \to 0 \) as \( n \to \infty \).

Finally, from (3.27) and [A4]
\[
S_{n,14}^{(1)} \leq \int_0^{\xi_{n,n}} IB(\cdot, \mathcal{L}) d|q| \left( \int_0^1 (1 - 1) \left| u |B(\cdot, \mathcal{L}) d|q| \right| \right)
\]
and as in the treatment of (3.21) the first term on the right hand side of (3.31) may be bounded by
\[
\left( \int_0^{\xi_{n,n}} IB(\cdot, \mathcal{L}) d|q| \left( \int_0^1 (1 - 1) \frac{1}{2} B(\cdot, \mathcal{L}) q_4 d|q| \right) \right)_{\mathcal{L}} 0_{\mathcal{L}}
\]
and \( \xi_{n,n} \mathcal{L} - \mathcal{L} \), since the integral converges and \( \xi_{n,n} \to 0 \).

The same argument applies to the second term on the right hand side of (3.31) and so we have \( S_{n,14}^{(3)} \to 0 \) and hence finally from (3.25)
\[
S_{n,1}^{(3)} \int_0^1 \left( (1 - 1)uJ' d|q| \right) \mathcal{L}_2 . \quad (3.32)
\]

In the Appendix, Lemma 1, we show \( S_{n,1}^{(4)} \to 0 \) as \( n \to \infty \). Thus from (14), (3.22), (3.23), (3.32) and the above, it follows that
\[ S_{n,1} \xrightarrow{\mathbb{P}} \xi_2 - \xi_1 \quad \text{as} \quad n \to \infty. \]  

(3.33)

The analysis of \( S_{n,2} \) is entirely analogous, and hence, in the interest of brevity, we omit the details and present only the final result:

\[ S_{n,2} \xrightarrow{\mathbb{P}} \xi_3 + \xi_4 \quad \text{as} \quad n \to \infty, \]  

(3.34)

where

\[ \xi_3 = \int_0^1 \Phi_1 \, dg \quad \text{and} \quad \xi_4 = \int_0^1 \Phi_4 \, dg. \]  

(3.35)

Finally, a very similar analysis leads to the conclusion that

\[ R_n \xrightarrow{\mathbb{P}} 0 \quad \text{as} \quad n \to \infty. \]  

(3.36)

Hence from (3.7), (3.33), (3.34) and (3.36) we obtain that

\[ n^2 (\sigma_n^2 - \sigma^2) \xrightarrow{\mathbb{P}} \nu \{ \xi_2 - \xi_1 + \xi_3 + \xi_4 \} \quad \text{as} \quad n \to \infty. \]  

(3.37)

In Lemma 2 of the Appendix we show that

\[ \nu \{ \xi_2 - \xi_1 + \xi_3 + \xi_4 \} = \int_0^1 \Psi_0 \, dq = S, \text{ say}, \]  

(3.38)

where \( \Psi_0 \) is defined by (2.13). Therefore with \( \gamma^2 \) defined by (2.14), \( S \) has the normal distribution with mean 0 and variance \( \gamma^2 \). Q.E.D.

The proof of Theorem 2 proceeds very much along the same lines and so we omit some details here. Corresponding to (2.12) and (3.5) we define for each \( n \geq 1, \)

\[ I^*_{n,1}(t) = \sum_{i=1}^{t-1} \left[ 1 - \left( \frac{1}{n} \right)^{i} \right] \, \Phi_{n}(\gamma_n), \quad \text{for} \quad t \in [\xi_{n,1}^*, \xi_{n,1}^{n-1}] \]  

(3.39)

with \( I^*_{n,1} \) set equal to zero otherwise and

\[ I^*_{1,n}(t) = \sum_{i=1}^{t-1} \left[ 1 - \left( \frac{1}{n} \right)^{i} \right] \, \Phi_{2}(\gamma_2), \quad \text{for} \quad t \in [0, \xi_{n,1}^{n-1}], \]  

(3.40)

with \( I^*_{1,n} \) set equal to zero otherwise.
For simplicity we shall write $\mathbf{L}_1^\ast$ for $\mathbf{L}_{1,n}^\ast$ in the sequel. Now

$$a_n^2 = \frac{1}{2} \int_0^1 L_{n,1}^\ast dL_{n,2}^2 \quad \text{and} \quad \sigma^2(n^{-1} L_n) = \frac{1}{2} \int_0^1 L_{1}^\ast dL_{2}^2. \quad (3.41)$$

Therefore corresponding to (3.7) through (3.10) we have

$$n^2(a_n^2 - \sigma^2(n^{-1} L_n)) = \frac{1}{2} \left( S_{n,1}^\ast + S_{n,2}^\ast + R_n^\ast \right) \quad (3.42)$$

where

$$S_{n,1}^\ast = \int_0^{n^{-1}} n^2 L_{n,1}^\ast dL_{n,2}, \quad (3.43)$$

$$S_{n,2}^\ast = \int_0^{n^{-1}} L_1^\ast d\{n^2(L_{n,2} - L_2)\}, \quad (3.44)$$

$$R_n^\ast = \int_0^{n^{-1}} n^2 L_{n,1}^\ast d(L_{n,2} - L_2). \quad (3.45)$$

The decomposition corresponding to (3.12) reads

$$b(L_{n,1}^\ast - L_1^\ast) = \int_t^{n^{-1}} u J_n^\ast (u) du + \int_t^{n^{-1}} (1 - 1)n^2 J_n^\ast (u) du + \int_t^{n^{-1}} (1 - 1)n^2 J_n^\ast (u) du, \quad t \in [\xi_{n,1}, n^{-1}].$$

$$= \int_t^{n^{-1}} (1 - 1) J_n^\ast du, \quad t \in (0, \xi_{n,1}). \quad (3.46)$$

Since $n^{-1} \xi_{n,n} \leq 1$ by assumption and $\xi_{n,n} \geq 1$, the set on which $n^{-1} \xi_{n,n} < \xi_{n,n}$ has probability which tends to one as $n \to \infty$. The argument used to examine (3.14) now applies with only minor modifications. For instance

$$\int_{\xi_{n,1}}^{\xi_{n,n}} \int_t^{n^{-1}} u J_n^\ast (u) du - \int_t^{n^{-1}} u J_n^\ast (u) du \leq 0, \quad (3.47)$$
and the usual argument shows that provided \( n^{-1} \frac{P}{n} \alpha \)

\[
\int_0^{n^{-1} \frac{P}{n} \alpha} + \int_0^{n^{-1} \frac{P}{n} \alpha} \cdot (\text{term}) = \int_0^{n^{-1} \frac{P}{n} \alpha} \cdot (\text{term}).
\]

Likewise

\[
\int_0^{n^{-1} \frac{P}{n} \alpha} + \int_0^{n^{-1} \frac{P}{n} \alpha} \cdot (\text{term}) = \int_0^{n^{-1} \frac{P}{n} \alpha} \cdot (\text{term}).
\]

where

\[
\zeta_2^* = \int_0^{n^{-1} \frac{P}{n} \alpha} \cdot (\text{term})
\]

Finally

\[
\int_0^{n^{-1} \frac{P}{n} \alpha} + \int_0^{n^{-1} \frac{P}{n} \alpha} \cdot (\text{term}) = \int_0^{n^{-1} \frac{P}{n} \alpha} \cdot (\text{term}).
\]

From (3.46) through (3.49) we obtain

\[
S_{n,1} = R_n + \zeta_2^* - \zeta_1^* = \zeta_3^* + \zeta_4^*
\]

Finally for \( S_{n,2}^* \) and \( R_n^* \) the results are

\[
S_{n,2}^* = \zeta_3^* + \zeta_4^* \quad \text{and} \quad R_n^* = 0
\]

where

\[
\zeta_3^* = \int_0^{n^{-1} \frac{P}{n} \alpha} \cdot (\text{term}) \quad \text{and} \quad \zeta_4^* = \int_0^{n^{-1} \frac{P}{n} \alpha} \cdot (\text{term})
\]

Hence from (3.42), (3.51) and (3.52) we get

\[
\frac{1}{n}(a_n^* - \alpha^2 (n^{-1} \frac{P}{n} \alpha)) \quad \zeta_2^* - \zeta_1^* + \zeta_3^* + \zeta_4^*.
\]

A minor modification of Lemma 2 of the Appendix will show that

\[
\zeta_2^* - \zeta_1^* + \zeta_3^* + \zeta_4^* = \int_0^{n^{-1} \frac{P}{n} \alpha} \cdot (\text{term}).
\]
and thus the first part of Theorem 2 is proven. For the second part we need only recognize that for \( n^{-1} \eta_n \leq a, \)

\[
\begin{align*}
\eta_n^c (t_1^{2 (n^{-1} \eta_n)} - t_2^{2 (a)}) &= -2 \eta_n^c \int_0^{n^{-1} \eta_n} \sum_{j=1}^3 J (1) \, d\alpha_j (\nu) - \int_0^{n^{-1} \eta_n} \sum_{j=1}^3 J (2) \, d\alpha_j - 1/2 \int_0^{n^{-1} \eta_n} \eta_n^c \, dL_2,
\end{align*}
\]

with a similar expression if \( n^{-1} \eta_n > a. \) Now, for the first term we may use the argument of Lemma 1 to show that it converges to zero in probability as \( n \to \infty. \) For the second term the additional assumptions on \( q \) and \( \eta_n \) gives

\[
|\eta_n^c \int_0^{n^{-1} \eta_n} \eta_n^c \, dL_2| \leq O(1) \eta_n^c |q(n^{-1} \eta_n) - q(a)|
\]

\[
\leq O(1) \eta_n^c |n^{-1} \eta_n - a| |q'(a) + o_p (1)|
\]

\[
= o_p (1).
\]

Hence Theorem 2 is proven.

4. Some general remarks.

Consider the following example. Let \( X_1, \ldots, X_n \) be iid rv's with df \( F \) and \( E[X_1]^r < \infty \) for some \( r > 4. \) We shall consider the sample variance

\[
(4.1) \quad \bar{o}_n^2 = n^{-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2
\]

where \( \bar{X}_n = n^{-1} \sum_{i=1}^n X_i \) is the sample mean. Then in the notation of Theorem 1 we have \( c_{n,i} = 1 \) for all \( i, 1 \leq i \leq n \) and \( g = F^{-1}. \) Note that \( E[X_1]^r < \infty \) implies

\[
\lim_{t \to 0^+} t (1-t) \left| F^{-1} (t) \right|^r = 0.
\]

Thus \( |g| \leq u \) on (0, 1) with \( a_1 = a_2 = 1/r. \) Also \( j = j_n = 1 \) so that \( b_1 = b_2 = 0 \) and A3, A4 hold trivially. Therefore if we take \( \delta \) such that \( 1/r = 1/4 - \delta \) we have \( \delta > 0 \) provided \( r > 4. \) By an integration by parts (2.15) holds.
Now $I_0$ of (2.13) reduces to
\[ I_0(t) = -2F^{-1}(t) + 2 \int_0^1 F^{-1}(s) ds. \]
For simplicity let us take $EX_1 = 0$ so that $\int_0^1 F^{-1}(s) ds = 0$. Then in (2.17) we may take $C_0(t) = -(F^{-1}(t))^2$. So we obtain $\gamma^2 = \nu_4 - \sigma^4$ where $\nu_4 = EX_1^4$ and $\sigma^2 = EX_1^2 = \sigma^2(1)$. Hence Theorem 1 yields
\[ n^2(\hat{\sigma}_n^2 - \sigma^2) \xrightarrow{\text{d}} N(0, \nu_4 - \sigma^4), \quad (4.2) \]
a result which is obtainable under the assumption $r = 4$. In this context Theorem 1 "just fails" to yield the slightly stronger conclusion.

The above example presents a very interesting observation pertaining to the different sets of conditions that suffice to yield the almost sure (a.s.) convergence of the statistics $T_{n,n}$, their asymptotic normality and the asymptotic normality of the estimator $\hat{\sigma}_n^2$ of the asymptotic variance. We have noted that if $A1$, $A2$ obtain and (2.7) holds then
\[ \int_0^1 B(\cdot, b)q_2 b|g| < \infty \quad (4.3) \]
and both $\nu_n(1)$ of (2.5) and $\sigma^2(1)$ of (2.6) are finite and
\[ n^2(T_{n,n} - \nu_n(1)) \xrightarrow{\text{d}} N(0, \sigma^2(1)). \quad (4.4) \]
Under $A1$, $A2$ (2.7) ensures the finiteness of $\nu_n(1)$ and (4.3) that of $\sigma^2(1)$.

To obtain the asymptotic normality of the variance estimator of $\hat{\sigma}_n^2(1)$ of (2.8) we impose the additional assumptions $A3$, $A4$ and replace (4.3) by the stronger condition (2.15). The a.s. convergence of $T_{n,n}$ has been studied by Wellner (1977). If $A1$, $A2$ obtain and $a_1 + b_1 = a_2 + b_2 = 1 - \delta$ then $\nu_n(1)$ is finite and
\( (T_{n,n} - \nu_n) \xrightarrow{a.s.} 0. \)  \hspace{1cm} (4.5)

Sen (1978) has obtained the a.s. convergence of \( \tilde{\sigma}_n^2(1) \) following a different technique.

For the "stopped statistics" \( T_{n,1_n} \) their asymptotic normality is derived in [2]. Again if \( A_1, A_2 \) obtain and (2.7) holds

\[
n^\gamma (T_{n,1_n} - \nu_n(n^{-1}1_n)) \xrightarrow{d} N(0, \sigma^2(\alpha))
\]  \hspace{1cm} (4.6)

provided \( n^{-1}1_n \in (0,1) \) and \( g \) admits a derivative at \( \alpha \) or \( n^\gamma (n^{-1}1_n - \alpha) \equiv 0 \) (1) and \( g \) is continuous at \( \alpha \). In the latter case if we further assume the stronger condition \( n^\gamma (n^{-1}1_n - \alpha) \equiv 0 \) then \( \nu_n(n^{-1}1_n) \) in (4.6) can be also replaced by \( \nu_n(\alpha) \).

The a.s. convergence of \( T_{n,1_n} \) can be discussed along the lines of Wellner (1977) assuming \( n^{-1}1_n \xrightarrow{a.s.} \alpha \).

5. **Appendix**

**Lemma 1:** Under the hypothesis of Theorem 1 and \( S_{n,1}^{(4)} \) defined by (3.12) and (3.14), we have \( S_{n,1}^{(4)} \xrightarrow{p} 0 \) as \( n \to \infty \).

**Proof.** We first write \( S_{n,1}^{(4)} \) in the form

\[
S_{n,1}^{(4)} = -(S_{n,11}^{(4)} + S_{n,12}^{(4)} + S_{n,13}^{(4)})
\]  \hspace{1cm} (5.1)

where

\[
S_{n,11}^{(4)} = n^\gamma \int_{\xi_{n,n}}^{\xi_{n,n}} 1\{\bar{\xi}_n\} \left( \int_{\xi_{n,n}}^{1} (1 - 1) \bar{\xi}_n dq \right),
\]

\[
S_{n,12}^{(4)} = n^\gamma \int_{0}^{\xi_{n,1}} 1\{\bar{\xi}_n\} \left( \int_{\xi_{n,n}}^{1} (1 - 1) \bar{\xi}_n dq \right),
\]

\[
S_{n,13}^{(4)} = n^\gamma \int_{\xi_{n,n}}^{1} 1\{\bar{\xi}_n\} \left( \int_{\xi_{n,n}}^{1} (1 - 1) \bar{\xi}_n dq \right).
\]
Therefore,

\[ |S_{n,11}^{(4)}| \leq n \int_{\xi_{n,1}}^{\xi_{n,n}} 1_B(\cdot, \beta)d|g| \left( \int_{\xi_{n,n}}^{1} (1 - 1_B(\cdot, \beta))d|g| \right). \tag{5.2} \]

Now

\[ \int_{\xi_{n,1}}^{\xi_{n,n}} 1_B(\cdot, \beta)d|g| \leq \xi_{n,n}^{3/4} \int_{\xi_{n,1}}^{\xi_{n,n}} 1_B(\cdot, \beta)d|g|, \tag{5.3} \]

and

\[ \int_{\xi_{n,n}}^{1} (1 - 1_B(\cdot, \beta))d|g| \leq (1 - \xi_{n,n})^{1/2} (1 - 1_B(\cdot, \beta))d|g|, \tag{5.4} \]

and therefore from (5.2),

\[ |S_{n,11}^{(4)}| \leq n \int_{\xi_{n,1}}^{\xi_{n,n}} (1 - 1_B(\cdot, \beta))d|g| \left( \int_{\xi_{n,n}}^{1} (1 - 1_B(\cdot, \beta))d|g| \right) \]

\[ \leq \left( \int_{\xi_{n,1}}^{\xi_{n,n}} (1 - 1_B(\cdot, \beta))d|g| \right) \left( \int_{\xi_{n,n}}^{1} (1 - 1_B(\cdot, \beta))d|g| \right) \xi_{n,n}^{3/4} (1 - \xi_{n,n})^{1/2} \]

\[ = o_p(1) o_p(1) o_p(1) = o_p(1). \]

The argument for \( S_{n,13}^{(4)} \) is entirely analogous while for \( S_{n,12}^{(4)} \) the steps are similar except that we use \( n \xi_{n,1} = o_p(1) \). Hence \( S_{n,1i}^{(4)} = o_p(1) \) for \( i = 1, 2, 3 \) and the lemma follows from (5.1).

**Lemma 2.** With \( \xi_1, \xi_2, \xi_3, \xi_4 \) defined as in (3.22), (3.32) and (3.35) equation (3.38) holds.

**Proof.** From (2.11) and (3.35) we have

\[ \xi_3 + \xi_4 = \int_0^1 1_U1^{(1)}d_R. \tag{5.5} \]

Also from (3.22) and (3.32)

\[ \xi_2 - \xi_1 = \int_0^1 dL_2 \left( \int_1^0 1_U1^{(2)}d_R \right). \tag{5.6} \]
Integrating by parts, we obtain

\[ 
\psi_2 - \psi_1 = \int_0^1 L_2 \psi_{(2)} \, dg + L_2(0) \left( \int_0^t \psi_{(2)} \, dg \right) |_{t=0}^{t=1}. 
\]

We shall show

\[ 
\lim L_2(t) \int_t^1 \psi_{(2)} \, dg = 0 \quad (5.7)
\]

where the limit is taken in each of the two cases \( t \to 0^+ \) and \( t \to 1^- \). In what follows this is to be interpreted whenever the limit is not explicitly stated.

Now for each \( t \in (0,1) \)

\[ |L_2(t)\int_t^1 \psi \, dg| \leq 2 \rho d \int_0^t \rho \, dg \int_t^1 B(q, \rho) \, dg. \quad (5.8) \]

In view of relations similar to (5.3) and (5.4) the function on the right hand side of (5.8) is dominated by

\[ \left( \int_0^t (1 - t)^{\delta/2} B(q, \rho) \, dg \right) \left( \int_t^1 (1 - 1)^{\delta/2} B(q, \rho) \, dg \right). \quad (5.9) \]

and so by (2.15), (5.9) must vanish in the limit as \( t \to 0^+ \) and \( t \to 1^- \).

Hence

\[ 
\lim L_2(t) \int_t^1 \psi \, dg = 0. \quad (5.10)
\]

Again for each \( t \in (0,1) \)

\[ |L_2(t)\int_t^1 (1 - t) \psi \, dg| \leq 2 \rho d \int_0^t \rho \, dg \int_t^1 (1 - 1) B(q, \rho) \, dg. \quad (5.11) \]

and the function on the right hand side of (5.11) is dominated by

\[ \left( \int_0^t (1 - t)^{\delta/2} B(q, \rho) \, dg \right) \left( \int_t^1 (1 - 1) B(q, \rho) \, dg \right). \quad (5.12) \]
It follows from (2.10) and (2.15) that (5.12) vanishes in the limit as $t \to 0^+$ and $t \to 1^-$. So

$$\lim_{t \to 0^+} L_2(t) \int_t^1 (1 - 1) U_j' dg = 0. \tag{5.13}$$

But (5.10) and (5.13) imply (5.7) and therefore $\zeta_2 - \zeta_1 = \int_0^1 L_2 U_j' dq$. Hence using (5.5) and (2.13),

$$h(\zeta_2 - \zeta_1 + \zeta_3 + \zeta_4) = \int_0^1 U_0 dq,$$

which is (3.38).
References


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