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CONVEX PROGRAMS AND THEIR CLOSURES

by

C. E. Blair¹

J. Borwein²

and

R. G. Jeroslow³

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¹ University of Illinois, Urbana

² Dalhousie University

³ Carnegie-Mellon University and Georgia Institute of Technology

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Management Sciences Research Group
Graduate School of Industrial Administration
Carnegie-Mellon University
Pittsburgh, Pennsylvania 15213
Abstract

We extend the limiting Lagrangean equation to

\[ \lim_{\theta \to 0^+} \sup_{\lambda \in \mathbb{R}} \inf_{x \in \text{affine support}} \left( f_0(x) + \theta (w x + w_1) + \sum_{h \in H} \lambda^h \sum_{h \in H} \lambda^h \right) = v(P), \]

and the results on affine supports from which it was deduced, to a very general setting that subsumes the previous constraint qualifications.

A simple example shows the need for some constraint qualification.

Key Words:

(1) Nonlinear programming
(2) Lagrangean
(3) Convexity
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C. E. Blair, 1 J. Borwein, 2 and R. G. Jeroslow 3

For a convex function \( f : D \to \mathbb{R} \) \((D \subseteq \mathbb{R}^n, D \text{ convex})\) the closure \( \text{cl}(f) : \text{cl}(D) \to \mathbb{R} \cup \{+\infty\} \) is defined by

\[
\text{cl}(f)(y) = \sup \{h(y) | h \text{ linear affine, } h(x) \leq f(x) \text{ for all } x \in D\}
\]

where \( \text{cl}(D) \) is the closure of the convex set \( D \).

It is well known that: (i) \( \text{cl}(f)(x) \leq f(x) \) for all \( x \in D \); (ii) \( \text{cl}(f) \) is convex; (iii) \( \text{cl}(f)(x) = f(x) \) for all \( x \in \text{relint}(D) \), where \( \text{relint}(D) \) denotes the relative interior of the convex set \( D \).

For a convex optimization problem (with possibly infinitely many constraints)

\[
\begin{align*}
\inf & \quad f_0(x) \\
\text{subject to} & \quad f_h(x) \leq 0 \text{ for } h \in H \\
& \quad x \in K
\end{align*}
\]

with optimal value denoted \( v(P) \), the closure is

\[
\begin{align*}
\inf & \quad \text{cl}(f_0)(x) \\
\text{subject to} & \quad \text{cl}(f_h)(x) \leq 0 \text{ for } h \in H \\
& \quad x \in \text{cl}(K)
\end{align*}
\]

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1University of Illinois, Urbana.

2Dalhousie University.

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with optimal value denoted $v(P)$. We assume throughout that $(P)$ is consistent.

Duffin [1] and Jeroslav [2] show that when $(P)$ and $(P')$ have the same optimal value, a "limiting Lagrangean" exists, in the sense that (using the homotopy form of the limiting Lagrangean of [2, equation (50)])

$$\lim_{\theta \to 0^+} \sup_{\lambda \in \Lambda} \inf_{x \in X} \left\{ f(x) + \theta (wx + w_1) + \sum_{h \in H} \lambda_h f_h(x) \right\} = v(P)$$

for $w \in \mathbb{R}^n$ and $w_1 \in \mathbb{R}$ suitably chosen, where $\Lambda$ denotes that space of vectors $(\lambda_h | h \in H)$ which are nonnegative and only finitely non-zero. Moreover, from the value equality $v(P) = v(P')$ also follows "fine detail" from which (2) is deduced, as e.g., [2, Theorem 3] and [2, Corollary 3].

It is also established in [2] that $v(P) = v(P')$ holds in many instances in which the usual constraint qualifications, such as the existence of Slater points, may fail to hold even for $|H|$ finite. This is because the limiting Lagrangean (2) is not related to issues of linear affine, or even rather more general, supports to the perturbation function of $(P)$. This aspect of the limiting Lagrangean was already present in the first limiting Lagrangean, due to R. J. Duffin [1].

The purpose of this note, is to extend the validity of the limiting Lagrangean (and Theorem 3 and Corollary 3 of [2]) to a rather broad setting that is associated with the ordinary Lagrangean in the case of $|H|$ finite. We show, in this setting, that the limiting Lagrangean holds again under weaker hypotheses than the ordinary Lagrangean, even for $|H|$ finite; and our result also treats $|H|$ infinite.
Let $D_h$ be the domain of definition of $f_h$. [2] showed that $v(P) = v(P')$ if $\text{relint}(K) \subseteq \text{relint}(D_h)$ for all $h \in \{0\} \cup H$ and there was an $x_0 \in \text{relint}(K)$ such that $f_h(x_0) \leq 0$ for all $h \in H$. These latter hypotheses were called (CQ) in [2].

In this note we show:

**Theorem:** Let $H'$ denote those indices $h \in \{0\} \cup H$ such that $f_h$ is not closed.

$$v(P) = v(P')$$ if there is an $x_0$ satisfying this constraint qualification:

$$x_0 \in \text{relint}(K) \cap \bigcap_{h \in H'} \text{relint}(D_h) \quad \text{and} \quad f_h(x_0) \leq 0 \quad \text{for} \quad h \in H'.$$

(The intersection over an empty set is defined to be $\mathbb{R}^n$.

**Proof:** If $x$ is feasible for $P$ it is also feasible for $P'$ because $\text{cl}(f_h)(x) \leq f_h(x)$. Since $\text{cl}(f_0)(x) \leq f_0(x), v(P') \leq v(P)$.

To show that $v(P') = v(P)$, let $x$ be any feasible point of $P'$. For $0 < \lambda < 1$ if $y = \lambda x + (1 - \lambda)x_0$, $y \in K$ and $y \in \text{relint}(D_h)$ for all $h \in H'$, by the Accessibility Lemma [5, 3.2.1.1]. By (iii), $\text{cl}(f_h)(y) = f_h(y)$ for all $h \in H'$; therefore $\text{cl}(f_h)(y) = f_h(y)$ for all $h \in H \cup \{0\}$.

Since $\text{cl}(f_h)(x_0) \leq f_h(x_0) \leq 0$ and $\text{cl}(f_h)(x) \leq 0$ for $h \in H$, one easily shows (by considering $f_h$ and $\text{cl}(f_h)$ on $[x, x_0]$) that $f_h(y) = \text{cl}(f_h)(y) \leq 0$ for $h \in H$. So $y$ is a feasible point for $P$.

By semi-continuity $\lim_{\lambda \to 1} \text{cl}(f_0)(y) = \text{cl}(f_0)(x)$. But since $\text{cl}(f_0)(y) = \lambda^{-1} f_0(y)$ for all $\lambda < 1$ this implies $v(P) \leq \text{cl}(f_0)(x)$.

Since $x$ was arbitrary, this shows $v(P) \leq v(P')$. Hence $v(P) = v(P')$, as desired.

Q.E.D.
REMARK: The same proof shows that, if $K$ is closed, one obtains $v(P) = v(P')$ from:

$$x_0 \in K \cap \bigcap_{h \in H} \text{relint}(D_h)$$

and $f_h(x_0) \leq 0$ for $h \in H$.

In one of the constraint qualifications of [2], it is assumed that $H' = \emptyset$ and $K$ is closed, in which case (4) becomes that constraint qualification (CQ)'. Trivially, (4) implies also the constraint qualification (CQ) of [2].

COROLLARY: Suppose that (P) has at least two different feasible points, and that none of the sets $K$ or $D_h$ for $h \in H'$ contains any line segments in $K \setminus \text{relint}(K)$ or $D_h \setminus \text{relint}(D_h)$ (where $H'$ is as defined in the theorem).

Then $v(P) = v(P')$.

PROOF: Let $x_a \neq x_b$ both be feasible in (P). Since $x_a, x_b \in K$ and $K$ contains no line segment in $K \setminus \text{relint}(K)$, $x_0 = (x_a + x_b)/2 \in \text{relint}(K)$.

Similarly, $x_0 \in \text{relint}(D_h)$ for $h \in H'$. Trivially, $f_h(x_0) \leq 0$ for $h \in H$.

The result now follows from the theorem.

Q.E.D.

Some constraint qualification is needed to insure that $v(P) = v(P')$, even for $|H|$ finite. For example, consider this instance of a convex program:
\[
\begin{align*}
\inf \ x_1 \\
\text{subject to} \quad x_2 &\leq 0 \\
f_2(x_1, x_2) &\leq 0 \\
(x_1, x_2) &\in K
\end{align*}
\]

where \( K = \{(x_1, x_2) | 0 \leq x_1 \leq 1 \text{ and } x_2 \geq 0\} \) and

\[
f_2(x_1, x_2) = \begin{cases} 
0, & 0 \leq x_1 \leq 1 \text{ and } x_2 > 0; \\
1 - x_1, & 0 \leq x_1 \leq 1 \text{ and } x_2 = 0; \\
+\infty, & \text{otherwise}.
\end{cases}
\]

Here \( v(P) = 1 \) and \( v(P^c) = 0 \), since \( \text{cl}(f_2)(x_1, x_2) \equiv 0 \) if \( 0 \leq x_1 \leq 1 \) and \( x_2 \geq 0 \). In this example, \( H^- = \{2\} \), as \( f_2 \) is not continuous on line segments that begin in the interior of \( K \) and end in the boundary segment \( \{(x_1, x_2) | x_2 = 0 \text{ and } 0 \leq x_1 < 1\} \). Here also \( x_2 \leq 0 \) and \( (x_1, x_2) \in K \) implies \( x_2 = 0 \), so \( (x_1, x_2) \notin \text{relint}(D_2) \); hence (3) fails.

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REFERENCES


We extend the limiting Lagrangean equation
\[
\lim_{\delta \to 0^+} \sup \inf \{ f(x) + \theta(wx + w_1) + \sum_{h} \lambda_h f_h(x) \} = v(P),
\]
and the results on affine supports from which it was deduced, to a very general setting that subsumes the previous constraint qualifications.

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