TECHNICAL REPORT #51

CONTRACT NONR-N00014-76-C-0050

PROJECT NR 017-653

Supervisor: Professor Walter Kohn

DEPARTMENT OF PHYSICS
UNIVERSITY OF CALIFORNIA SAN DIEGO
LA JOLLA, CALIFORNIA

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August 1979
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**Title:** Absence of Crystalline Order in Two-Dimensions

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**Contract or Grant Number:** NONR 80014-76-C-0050

**Project, Task, or Work Unit Number:** CR-17-653

**Report Date:** August 1979

**Number of Pages:** 7

**Security Class. (of Report):** UNCLASSIFIED

**DISTRIBUTION STATEMENT:** Approved for public release; distribution unlimited.

**ABSTRACT:**
It is proved that in two dimensions a system of electrons embedded in a uniform neutralizing positive background and interacting by a potential given by $e^2/r$, cannot exhibit long range crystalline order at any finite temperature.
Absence of Crystalline Order in Two-Dimensions

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Abstract

It is proved that in two dimensions a system of electrons embedded in a uniform neutralizing positive background and interacting by a potential given by $e^2/r$ cannot exhibit long range crystalline order at any finite temperature.

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Crane and Adams$^{(1)}$ have recently presented evidence for a crystalline transition of electrons trapped on a surface of liquid helium. This quasi two-dimensional system has been canonically modeled as a system of electrons in two dimensions, interacting by a potential given by $e^2/r$, and neutralized by a uniform positive background. Here $r$ is the two dimensional distance between two points. In this context it is important to know if at any finite temperature such a system can display true long-range crystalline order in the thermodynamic limit.

We want to emphasize that the crystalline order in this system is not ruled out by the classic work of Mermin$^{(2)}$. However, from arguments due to Landau$^{(3)}$ and Peteris$^{(4)}$ it is simple to show that a harmonic solid cannot exist in the thermodynamic limit. Although the arguments by Landau and Peteris can be challenged as not being rigorous enough, recent Monte Carlo simulations by Gunn, Chakravarty and Chester$^{(5)}$ indicated, although not conclusively, the validity of the conclusion drawn from the Landau-Peteris argument. On the basis of the numerical work it was conjectured in ref. 3 that although a rigorous proof does not exist, the general conclusion of Landau and Peteris should be valid.

In this paper we shall prove rigorously that a true long-range crystalline order cannot exist in the thermodynamic limit, thus placing Landau-Peteris argument on a firmer basis for the interpretation of the interaction mentioned above. The proof makes use of the Popolitov's inequality$^{(6)}$ as discussed by Mermin$^{(2)}$. Although the proof differs in some essential aspects from that of Mermin, the general strategy is similar. We therefore follow him quite closely, making essential changes where necessary.

Because of the long-range nature of the interaction, some complications arise. Although these complications are well known, it is important to state them clearly. (a) In order to obtain physically meaningful results the interaction $e^2/r$ is replaced by $e^2/r \exp(-r)$. Since we are interested in the B.L. & J.R.

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of the neutral medium our limiting procedure will be first \( N = -\), \( A = -\).

\( W/A = n = \) constant and then \( \nu = 0 \). Here \( N \) is the total number of electrons and \( A \) the area. (b) Another related assumption \( \delta \) is that the equilibrium state of the system does not have microscopic surface charge. Although it is highly implausible that any equilibrium state violating this assumption could lower free energy, we are not aware of a proof of this.

It will become evident later that even with these assumptions stated above, the original proof of Mermin does not go through because of the presence of the long wavelength limit. Consider, then, \( N \) classical electrons enclosed in a box of area \( A \). A uniform neutralizing positive background is assumed to exist. The interaction energy \( \mathcal{U} \) is,

\[
\mathcal{U} = \frac{4}{\mathcal{A}} \sum_{i,j} \frac{e^2}{|\mathbf{r}_i - \mathbf{r}_j|} \exp(-|\mathbf{r}_i - \mathbf{r}_j|) + \frac{e^2}{2\mathcal{A}} \int \frac{d\mathbf{r}}{A} \int \frac{d\mathbf{r}'}{A} \exp(-|\mathbf{r} - \mathbf{r}'|) \tag{1}
\]

In Eq. (1) the first term is the electron-electron interaction, the second the interaction between the electrons and the positive background, and the third the self energy of the background.

We adopt the same criterion for crystallinity as Mermin, i.e.,

\[
\langle \mathcal{E}_n \rangle _{\mathcal{B}} = 0, \quad \text{if \( n \) \ is a reciprocal lattice vector.}
\]

\[
\langle \mathcal{E}_n \rangle _{\mathcal{B}} 
eq 0, \quad \text{for at least one \( n \) reciprocal lattice vector,}
\]

where,

\[
\mathcal{E}_n = \sum_{\mathbf{r}} e^2 \mathcal{U}(\mathbf{r}) e^{i\mathbf{n} \cdot \mathbf{r}}
\]

\(
\frac{1}{N} \mathcal{E}_n = \sum_{\mathbf{r}} e^2 \mathcal{U}(\mathbf{r}) e^{i\mathbf{n} \cdot \mathbf{r}}
\]

Here \( \langle \mathcal{E}_n \rangle _{\mathcal{B}} \) denotes the canonical ensemble average with respect to the interaction energy \( \mathcal{U} \) and the integrations are over the interior of a box of area \( A \). We now consider the Schwarz inequality

\[
\langle \mathcal{E}_n \rangle _{\mathcal{B}} \leq \frac{1}{N} \sum_{\mathbf{r}} e^2 \mathcal{U}(\mathbf{r}) e^{i\mathbf{n} \cdot \mathbf{r}}
\]

and choose \( C \) and \( \mathbf{B} \) to be

\[
C = \sum_{\mathbf{r}} e^2 \mathcal{U}(\mathbf{r}) e^{i\mathbf{n} \cdot \mathbf{r}}
\]

\[
\mathcal{B} = -\hbar^2 \sum_{\mathbf{r}} \left( \frac{k^2}{2m} \right) (\mathcal{U}(\mathbf{r}) e^{i\mathbf{n} \cdot \mathbf{r}})
\]

The function \( \mathcal{U}(\mathbf{r}) \) is chosen as follows \( \delta \). Consider \( \mathcal{B} \) to be a set of points within a distance \( a \) of the walls of the box, where \( a \) is independent of \( L \).

It is now defined by the following conditions: (1) \( 0 \leq \mathcal{B} \leq \frac{L}{2} \), (2) \( \mathcal{B} = 1 \) everywhere else, and (3) \( \langle \mathcal{E}_n \rangle _{\mathcal{B}} \leq \mathcal{E} \), for some \( \mathcal{E} \) independent of \( L \). It will become clearer later on that we have gone too great pains in introducing the functions \( \mathcal{U}(\mathbf{r}) \). The intention has been to make the surface term arising from the integrations by parts vanish. The contributions arising from \( \mathcal{U}(\mathbf{r}) \) could have been entirely avoided with a slightly careless use of the periodic boundary condition. The choice of \( \mathcal{B} \), Eq. (5), is one of the key points of our proof and serves to project out the longitudinal part of the force and suppresses the plasmas in the denominator of the right hand side of the inequality, Eq. (3). With these choices of \( C \) and \( \mathcal{B} \), Eqs. (4) and (5), it is straightforward to derive the inequality

\[
\frac{1}{N} \langle \mathcal{E}_n \rangle _{\mathcal{B}} \leq \frac{1}{N} \sum_{\mathbf{r}} e^2 \mathcal{U}(\mathbf{r}) e^{i\mathbf{n} \cdot \mathbf{r}}
\]

where
\[ \Delta(t) = \left( e^{2} - (k \cdot \zeta)^{2} \right)^{1/2} \left| \int_{r_{11}}^{r_{12}} p(r_{1}) \, e^{-i \sigma_{11} T} \right|^{2}, \tag{1} \]

\[ K_{k} = \frac{\hbar T}{\pi} \int_{r_{11}}^{r_{12}} p(r_{1}) \left( f(r_{1}) - f(r_{1}) \right)^{2} \]

\[ - \frac{\hbar T}{\pi} \int_{r_{11}}^{r_{12}} p(r_{1}) \left( f(r_{1}) - f(r_{1}) \right)^{2} \int_{r_{11}}^{r_{12}} e^{-i \sigma_{11} T} \]

\[ + \frac{1}{2 \hbar T} \int_{r_{11}}^{r_{12}} \int_{r_{11}}^{r_{12}} \left[ f(r_{1}) - f(r_{1}) \right] e^{i \sigma_{11} T} \int_{r_{11}}^{r_{12}} e^{-i \sigma_{11} T} \]

\[ \frac{1}{N-1} \int_{r_{11}}^{r_{12}} \int_{r_{11}}^{r_{12}} e^{i \sigma_{11} T} \]

The distribution functions are defined to be,

\[ P(r_{1}) = \frac{e^{-\beta E_{r_{1}}}}{\int_{r_{11}}^{r_{12}} e^{-\beta E_{r_{1}}} \, dr_{1}}, \tag{9} \]

\[ P(r_{1}, r_{2}) = \frac{e^{-\beta E_{r_{1}, r_{2}}}}{\int_{r_{11}}^{r_{12}} \int_{r_{11}}^{r_{12}} e^{-\beta E_{r_{1}, r_{2}}} \, dr_{1} \, dr_{2}}. \tag{10} \]

The inequality (4) is now multiplied by a positive Gaussian \( g(k \cdot \zeta) \) centered at \( k \cdot \zeta = 0 \), and summed over \( k \) to get

\[ \frac{1}{A} \sum_{k} \left( \frac{1}{A} \sum_{k} \frac{\Delta(k)}{\lambda(k)} \right) \geq \frac{\lambda_{k} T \sigma}{A} \sum_{k} \frac{\Delta(k)}{\lambda(k)}. \tag{11} \]

The strategy now will be to proceed to the thermodynamic limit, show that 

\[ \Delta(k) \sim \lambda^{2} \] for small \( k \) and then take the limit \( u \to 0 \). The proof will then be completed by showing that the left hand side of the inequality is bounded.

Proceeding to the thermodynamic limit then allows us to write

\[ \int_{0}^{\frac{\beta}{2 \hbar}} \frac{1}{2} \int_{0}^{\beta} \frac{dE}{E} \left( \frac{1}{2} \int_{0}^{\beta} \frac{dE}{E} \right) \]

The inequality (12) can be strengthened further with the help of tricks similar to those used by Fernández (2) and obtain

\[ \int_{0}^{\beta} \frac{1}{2} \frac{1}{2} \int_{0}^{\beta} \frac{dE}{E} \left( \frac{1}{2} \int_{0}^{\beta} \frac{dE}{E} \right) \]

Now,

\[ \lim_{\beta \to 0} \int_{0}^{\beta} \frac{1}{2} \frac{1}{2} \int_{0}^{\beta} \frac{dE}{E} \left( \frac{1}{2} \int_{0}^{\beta} \frac{dE}{E} \right) \]

The first term on the right hand side of Eq. (14) vanishes since we are now in the thermodynamic limit. The argument is identical to that used by Mermin (5). The second term can also be seen to vanish after converting it to a surface integral.
The next two terms also vanish but are trickier. Consider first the third term, here because of the integral at least one of the integrals get restricted to $Q_3$ and let us call that integration $T_3$. Now

$$\frac{2}{k} \int_{Q_3} \int_{Q_3} \frac{e^{-E_{12}}}{r_{12}} \left[ (u^2_{12} + u_{12}) + 1 \right] f(r_1)f(r_2) r_{12} dT_{12}$$

$$\leq \frac{2}{k} \int_{Q_3} \frac{e^{-E_{12}}}{r_{12}} \left[ (u^2_{12} + u_{12}) + 1 \right] f(r_1)f(r_2) r_{12} dT_{12}$$

$$- \left( \frac{\pi}{4} \right)^{-1} \left( \frac{\pi}{4} \right)^{-1}$$

Similarly the fourth term can be estimated to be

$$\frac{k^2}{28} \int_{Q_3} \int_{Q_3} \frac{e^{-E_{12}}}{r_{12}} \left[ (1 + f(r_1)) \left[ 1 + f(r_2) \right] - u_{12}^2 \right] f(r_1)f(r_2) r_{12} dT_{12}$$

$$\leq \frac{5k^2}{48} \int_{Q_3} \int_{Q_3} \frac{e^{-E_{12}}}{r_{12}} f(r_1)f(r_2) r_{12} dT_{12}$$

$$- \left( \frac{k}{4} \right)^{-4}$$

(15)

(16)

In both of the estimates, Eqs. (15) and (16), we have followed the prescription that the thermodynamic limit is taken before $\nu \to 0$. If this were not the case both of these estimates would have diverged. Also, since we are in the thermodynamic limit the last term can also be seen to be zero by explicit integration. It is easy to verify that if we had made use of the function $\beta$ as chosen by Kama\(h \)\(e \)\(2 \), the term analogous to the last term in Eq. (16) would have diverged as $\nu \to 0$. A closer look would reveal that the leading term in that case is proportional to $k$ and not $k^2$, thus substantiating our earlier remarks about the choice of $\beta$. We then have,

$$\lim_{\nu \to 0} \frac{k}{\nu} \int_{Q_3} \frac{e^{-E_{12}}}{r_{12}} \left[ (1 + f(r_1)) \left[ 1 + f(r_2) \right] - u_{12}^2 \right] f(r_1)f(r_2) r_{12} dT_{12}$$

$$= \frac{k}{\nu} \int_{Q_3} \frac{e^{-E_{12}}}{r_{12}} \left[ (1 + f(r_1)) \left[ 1 + f(r_2) \right] - u_{12}^2 \right] f(r_1)f(r_2) r_{12} dT_{12}$$

$$\to 0$$

(17)

The limit $\nu \to 0$ now allows us to identify the right hand side of Eq. (17), as proportional to the internal energy per electron of a system of electrons embedded in a neutralizing uniform positive background and interacting by a potential $\epsilon^2/r$. In two dimensions. Hence,

$$\lim_{\nu \to 0} \frac{k}{\nu} \int_{Q_3} \frac{e^{-E_{12}}}{r_{12}} \left[ (1 + f(r_1)) \left[ 1 + f(r_2) \right] - u_{12}^2 \right] f(r_1)f(r_2) r_{12} dT_{12}$$

$$= \lim_{\nu \to 0} \left( \frac{k}{\nu} \int_{Q_3} \frac{e^{-E_{12}}}{r_{12}} \left[ (1 + f(r_1)) \left[ 1 + f(r_2) \right] - u_{12}^2 \right] f(r_1)f(r_2) r_{12} dT_{12} \right)$$

$$= \frac{k}{\nu} \int_{Q_3} \frac{e^{-E_{12}}}{r_{12}} \left[ (1 + f(r_1)) \left[ 1 + f(r_2) \right] - u_{12}^2 \right] f(r_1)f(r_2) r_{12} dT_{12}$$

$$\leq \frac{k}{\nu} \int_{Q_3} \frac{e^{-E_{12}}}{r_{12}} \left[ (1 + f(r_1)) \left[ 1 + f(r_2) \right] - u_{12}^2 \right] f(r_1)f(r_2) r_{12} dT_{12}$$

$$- \left( \frac{\pi}{4} \right)^{-1} \left( \frac{\pi}{4} \right)^{-1}$$

(18)

For this system a rigorous lower bound has recently been obtained by Toma\(e \)\(i \)\(3 \),

$$\frac{E_{\text{int}}}{\nu} \geq \frac{1}{\nu} \int_{Q_3} \frac{e^{-E_{12}}}{r_{12}} \left[ (1 + f(r_1)) \left[ 1 + f(r_2) \right] - u_{12}^2 \right] f(r_1)f(r_2) r_{12} dT_{12}$$

$$\geq \frac{1}{\nu} \int_{Q_3} \frac{e^{-E_{12}}}{r_{12}} \left[ (1 + f(r_1)) \left[ 1 + f(r_2) \right] - u_{12}^2 \right] f(r_1)f(r_2) r_{12} dT_{12}$$

where $a$ is the average interparticle distance. Thus we have,

$$\lim_{\nu \to 0} \frac{k}{\nu} \int_{Q_3} \frac{e^{-E_{12}}}{r_{12}} \left[ (1 + f(r_1)) \left[ 1 + f(r_2) \right] - u_{12}^2 \right] f(r_1)f(r_2) r_{12} dT_{12}$$

$$\leq \frac{1}{\nu} \int_{Q_3} \frac{e^{-E_{12}}}{r_{12}} \left[ (1 + f(r_1)) \left[ 1 + f(r_2) \right] - u_{12}^2 \right] f(r_1)f(r_2) r_{12} dT_{12}$$

which can be used in the denominator to strengthen the inequity (13) further.

One additional information is required. This is

$$\left( \frac{\kappa^2}{\nu} \int_{Q_3} \frac{e^{-E_{12}}}{r_{12}} \left[ (1 + f(r_1)) \left[ 1 + f(r_2) \right] - u_{12}^2 \right] f(r_1)f(r_2) r_{12} dT_{12} \right)^2$$

(19)

The right hand side of Eq. (21) can be replaced in the thermodynamic limit by $(\sqrt{\kappa^2/\nu})^2$ since by assumption the equilibrium state does not have macroscopic surface charge (2).

To complete the proof it needs to be shown that the left hand side of the inequality, Eq. (13), is finite. This will then imply that a finite quantity is greater than or equal to infinity and hence the inequalities involved that
The next two terms also vanish but are trickier. Consider first the third term, here because of the integrand at least one of the integrations get restricted to \( G_a \) and let us call that integration \( \r_2 \). Now

\[
\frac{2}{N} \int \frac{dx_1}{x_1} \int \frac{dx_2}{x_2} \frac{e^{-r_1^2} \r_2}{r_1^2} \left( \frac{x_1^2 + \nu r_2}{r_1} + 1 \right) P(r_1, r_2) \leq \frac{2}{N} \int \frac{dx_1}{x_1} \int \frac{dx_2}{x_2} \frac{e^{-r_1^2} \r_2}{r_1^2} \left( x_1^2 + \nu r_2 + 1 \right) P(r_1, r_2) \leq \left( \frac{N}{2} \right) e^{-\beta} \tag{15} \]

Similarly the fourth term can be estimated to be

\[
\frac{k^2}{2N} \int \frac{dx_1}{x_1} \int \frac{dx_2}{x_2} \frac{e^{-r_1^2} \r_2}{r_1^2} \left[ 1 - 11(r_1^2) \right] P(r_1, r_2) \leq \frac{k^2}{2N} \int \frac{dx_1}{x_1} \int \frac{dx_2}{x_2} \frac{e^{-r_1^2} \r_2}{r_1^2} \left[ 1 + r_1^2 \right] P(r_1, r_2) \leq \left( \frac{N}{2} \right) e^{-\beta} \tag{16} \]

In both of the estimates, Eqs. (15) and (16), we have followed the prescription that the thermodynamic limit is taken before \( \nu \to 0 \). If this were not the case both of these estimates would have diverged. Also, since we are in the thermodynamic limit the last term can also be seen to be zero by explicit integration. It is easy to verify that if we had made use of the function \( B \) as chosen by Kramers \( \text{(2)} \), the term analogous to the last term in Eq. (14) would have diverged as \( \nu \to 0 \). A closer look would reveal that the leading term in that case is proportional to \( \nu \) and not \( \nu^2 \), thus substantiating our earlier remarks about the choice of \( B \). We then have,

\[
\lim_{\nu \to 0} \int_0^\infty dx_1 \frac{e^{-r_1^2} \r_2}{r_1^2} \int \frac{dx_2}{x_2} \int \frac{dx_3}{x_3} \int \frac{dx_4}{x_4} \int \frac{dx_5}{x_5} \int \frac{dx_6}{x_6} \left[ \frac{e^{-r_1^2} \r_2}{r_1^2} \left( x_1^2 + \nu r_2 + 1 \right) P(r_1, r_2) \right] \leq \left( \frac{N}{2} \right) e^{-\beta} \tag{17} \]

The limit \( \nu \to 0 \) now allows us to identify the right hand side of Eq. (17) as proportional to the internal energy per electron of a system of electrons neutralized by a neutralizing uniform positive background and interacting by a potential \( \pi \nu^2 r^2 \) in two dimensions. Here,

\[
\lim_{\nu \to 0} \int_0^\infty dx_1 \frac{e^{-r_1^2} \r_2}{r_1^2} = -\frac{1}{2} \left( \frac{\pi \nu^2}{4} \right) \tag{18} \]

For this system a rigorous lower bound has recently been obtained by Tisor \( \text{(5)} \).

\[
\frac{\pi \nu^2}{4} = \frac{4}{3} \left( \frac{\pi \nu^2}{4} \right) \tag{19} \]

where \( \nu \) is the average interparticle distance. Thus we have,

\[
\lim_{\nu \to 0} \frac{k^2}{2N} \int \frac{dx_1}{x_1} \frac{dx_2}{x_2} \frac{e^{-r_1^2} \r_2}{r_1^2} \leq \frac{1}{9} \left( \frac{\pi \nu^2}{4} \right) \tag{20} \]

which can be used in the denominator to strengthen the inequality \( \text{(13)} \) further. One additional information is required. This is

\[
\left[ \frac{k}{2N} \int \frac{dx_1}{x_1} \frac{dx_2}{x_2} \frac{e^{-r_1^2} \r_2}{r_1^2} \right]^2 \leq \left[ \frac{k}{2N} \int \frac{dx_1}{x_1} \frac{dx_2}{x_2} \frac{e^{-r_1^2} \r_2}{r_1^2} \right]^2 \tag{21} \]

The right hand side of Eq. (21) can be replaced in the thermodynamic limit by \( \frac{2}{N} \beta' \), since by assumption the equilibrium state does not have macroscopic surface charge \( \text{(7)} \).

To complete the proof it needs to be shown that the left hand side of the inequality, Eq. (13), is finite. This will then imply that a finite quantity is greater than or equal to infinity and hence the inescapable conclusion that

\[
\]
\( \langle T \rangle \) vanishes at any finite temperature. One way to proceed is to follow Mermin\(^{(2)}\). The important ingredient is of course the free energy of the system considered in this paper. This, thanks to Totsuji\(^{(9)}\), can be shown to have both an upper and a lower bound. The other alternative is to follow Sorkine\(^{(10)}\) word for word and arrive at the conclusion that the left hand side of the inequality \((13)\) is bounded.

We would like to thank M. B. Fogel and S. K. Ma for many useful discussions. This work has been supported by the National Science Foundation.

References