CONVERGENCE PROOF OF ECONOMIC REPRESENTATION
OF TRANSCENDENTAL FUNCTIONS

TECHNICAL REPORT
July 15, 1960

M.R.I. Project No. 2229-P

Contract No. Nonr-2638(00)(X)

For

Applied Mathematics Laboratory
David W. Taylor Model Basin
Washington, D. C.

DISTRIBUTION STATEMENT A
Approved for public release
Distribution Unlimited

COLUMBIA UNIVERSITY
HUDSON LABORATORIES
CONTRACT NO-ONR-27135

LEVEL

DDC FILE COPY

78 03 07 094

79 08 03 117
<table>
<thead>
<tr>
<th>REPORT NUMBER</th>
<th>2. GOVT ACCESSION NO.</th>
<th>3. RECIPIENT'S CATALOG NUMBER</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

4. TITLE (and Subtitle)
CONVERGENCE PROOF OF ECONOMIC REPRESENTATION OF TRANSCENDENTAL FUNCTIONS

5. TYPE OF REPORT & PERIOD COVERED
Technical Report

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s)
Fields, Jerry L. and Luke, Yudell L.

8. CONTRACT OR GRANT NUMBER(S)
N0m-2638(00)(X)

9. PERFORMING ORGANIZATION NAME AND ADDRESS
Midwest Research Institute

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
2229-P

11. CONTROLLING OFFICE NAME AND ADDRESS
Office of Naval Research, Code 220
800 North Quincy Street
Arlington, VA 22217

12. REPORT DATE
15 JUL 1960

13. NUMBER OF PAGES

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)

15. SECURITY CLASS. (of this report)
UNCLASSIFIED

16. DISTRIBUTION STATEMENT (of this Report)
Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)
CONVERGENCE PROOF OF ECONOMIC REPRESENTATION OF TRANSCENDENTAL FUNCTIONS

by

Jerry L. Fields
Yudell L. Luke

M.R.I. Project No. 2229-P

Contract No. Nonr-2638(00)(X)

TECHNICAL REPORT
July 15, 1960

For

Applied Mathematics Laboratory
David W. Taylor Model Basin
Washington, D. C.
PREFACE

This report covers research initiated by the Applied Mathematics Laboratory, David W. Taylor Model Basin, Washington, D. C. The research work upon which this report is based was accomplished by Midwest Research Institute under Contract No. Nonr-2638(00)(X).

The numerics in this report are related to work previously done in earlier reports, and the authors wish to take this opportunity to again thank those members of the Midwest Research Institute staff who materially contributed. They are Geraldine Coombs, Betty Kahn, Anna Lee Samuels and Wanda Shelp.

J.L.F. and Y.L.L.

Approved for:

MIDWEST RESEARCH INSTITUTE

S. L. Levy, Manager
Mathematics and Physics Division

July 15, 1960
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page No.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Summary and Introduction</td>
<td>1</td>
</tr>
<tr>
<td>1. Convergence of the Rational Approximations</td>
<td>2</td>
</tr>
<tr>
<td>2. Numerics</td>
<td>13</td>
</tr>
<tr>
<td>Bibliography</td>
<td>16</td>
</tr>
</tbody>
</table>
CONVERGENCE PROOF OF ECONOMIC REPRESENTATION OF TRANSCENDENTAL FUNCTIONS

by

Jerry L. Fields
and
Yudell L. Luke

Summary and Introduction

In a previous report [1], it was shown that a large class of transcendental functions could be represented by a rational function, the ratio of two polynomials, together with a remainder term. More specifically, if \( E(z) \) is defined by a Laplace integral, then

\[
E(z) = \frac{\phi_n}{f_n + R_n}, \quad R_n = F_n/f_n
\]

where \( \phi_n \) and \( F_n \) are polynomials of degree \( n \) and \( R_n \) is the remainder. In Section I it is shown that this can be considered a regular summation technique.

By proper choice of some parameters, \( f_n \) is a polynomial of hypergeometric form. If, except for a multiplicative factor, \( E(z) \) has a generalized hypergeometric series representation, then \( F_n \) is a sum of \( n \) generalized hypergeometric series.

Only in certain cases could it be shown that the rational representation converges, i.e., for a fixed, \( \lim_{n \to \infty} R_n(z) = 0 \). One of the principal difficulties lies in the fact that asymptotic expansions of a certain class of hypergeometric polynomials were known only for a few special cases, the classical Jacobi polynomials, for example. Asymptotic expansions of this class of hypergeometric polynomials now exist, see [2] and the convergence question can be further explored.

Numbers in square brackets pertain to references at end of report.
The structure of the numerator of the error term, i.e., \( F_n \) is complicated and only in a few special cases has it been possible to represent it in a useful form. This drawback is partly alleviated by the fact that one of the above-mentioned special cases includes the Whittaker functions and their natural generalizations. For those cases where \( F_n \) can be put into a useful form, we prove convergence of the rational representation.

In the case of particular interest, i.e., the Whittaker functions, we represent \( F_n \) as an integral and then find a suitable bound for it. This coupled with an asymptotic estimate of \( f_n \) enables us to prove that \( \lim_{n \to \infty} R_n = 0 \) and also to obtain an asymptotic bound for the error. This is the essence of Section I. In Section II, numerics are presented to display the effectiveness of this bound.

1. Convergence of the Rational Approximations

In [1], it was shown how to formally generate a rather general sequence of rational approximations for the generalized hypergeometric function [3]

\[
E(z) = _{p+1}F_q \left( \sigma, \alpha_1, \ldots, \alpha_p \left| \begin{array}{c} \lambda \vspace{1mm} \\ z \end{array} \right. \right), \tag{1.1}
\]

where

\[
mF_n(x) = mF_n \left( \alpha_1, \ldots, \alpha_m \left| \begin{array}{c} \beta_1, \ldots, \beta_n \vspace{1mm} \end{array} \right| x \right) = \sum_{k=0}^{m} \frac{\prod_{t=1}^{m} \frac{\sigma_{t}^{(k)}}{k!}}{\prod_{t=1}^{n} \frac{\beta_{t}^{(k)}}{k!}} (x)^k, \tag{1.2}
\]

\[
(\omega)_k = \frac{\Gamma(\omega+k)}{\Gamma(\omega)}
\]

Throughout this paper we employ a contracted notation and write
Thus \( \alpha_m \) is interpreted as \( \prod_{k=1}^{m} \alpha_k \), and a similar remark holds for \( \rho_k \). We assume that none of the \( \alpha_k \)'s, or \( \rho_k \)'s are zero or a negative integer. Also, it is assumed that the difference of any numerator parameter \( \alpha_j \), and any denominator parameter \( \rho_t \), is never equal to zero. Empty terms in expression such as (1.3) are replaced by unity. Two distinct sequences of rational approximations were developed in the above paper [1]. They are as follows.

**Type I or Homogeneous Case**

\[
E(z) = \frac{\Phi_n(z, \vartheta)}{f_n(\vartheta)} + R_n(z, \vartheta) , \quad R_n(z, \vartheta) = \frac{F_n(a_k, z, \vartheta)}{f(\vartheta)} \tag{1.4}
\]

\[
\Phi_n(z, \vartheta) = \sum_{r=0}^{n} \frac{(-1)^r (\alpha_{\vartheta})^r (\frac{z}{\vartheta})^r}{r! \left( \rho_{\vartheta} \right)_r} f_n[r](\vartheta) \tag{1.5}
\]

(1.5)

\[
f_n[r](\vartheta) = \sum_{k=r}^{n} a_{n-k} \vartheta^k , \quad f_n[0] = f_n(\vartheta) \tag{1.6}
\]
\[ F_n(a_k, z, \gamma) = \frac{\gamma^r}{\Gamma(\sigma)} \int_0^\infty e^{-zt} t^{\sigma-1} \sum_{k=0}^{\infty} \frac{a_{n-k}(\lambda, \gamma, t)^k}{k!} \frac{\sum_{r=0}^{\infty} \left( \frac{\alpha_p r+k+1}{\lambda t} \right)^{r+1}} \left( \frac{\rho_q r+k+1}{r+k+1} \right)^{r+1} dt \]  

\[ = \frac{\gamma^r}{\Gamma(\sigma)} \int_0^\infty e^{-zt} t^{\sigma-1} \sum_{k=0}^{\infty} \frac{a_{n-k}(\lambda, \gamma, t)^k}{k!} \frac{\sum_{r=0}^{\infty} \left( \frac{\alpha_p r+k+1}{\lambda t} \right)^{r+1}} \left( \frac{\rho_q r+k+1}{r+k+1} \right)^{r+1} dt \]  

\[ = \frac{\gamma^r}{\Gamma(\sigma)} \int_0^\infty e^{-zt} t^{\sigma-1} \sum_{k=0}^{\infty} \frac{a_{n-k}(\lambda, \gamma, t)^k}{k!} \frac{\sum_{r=0}^{\infty} \left( \frac{\alpha_p r+k+1}{\lambda t} \right)^{r+1}} \left( \frac{\rho_q r+k+1}{r+k+1} \right)^{r+1} dt \]  

where \( \alpha_p = \prod_{t=1}^{p} \alpha_t \), and \( \rho_q = \prod_{t=1}^{q} \rho_t \). Here the coefficients \( a_k \) are as yet arbitrary, and \( \gamma \) is unrestricted except that \( |\gamma/z| \leq 1 \).

**Type II or Non-Homogeneous**

\[ B(z) = \frac{\Phi_n(z, \gamma)}{f_n(\gamma)} + R_n(z, \gamma) \quad R_n(z, \gamma) = \frac{F_n(a_k, z, \gamma)}{f_n(\gamma)} \]  

\[ \Phi_n(z, \gamma) = \sum_{r=0}^{\infty} \frac{(\sigma)_r (\alpha_p)_r}{(1)_r (\rho_q)_r} \left( \frac{\lambda}{z} \right)^r f_n^{[r+1]}(\gamma) \]  

\[ f_n^{[r]}(\gamma) = \sum_{k=0}^{\infty} a_{n-k} \gamma^k \quad f_n^{[0]}(\gamma) = f(\gamma) \]
\[ F_n(a, z, \gamma) = \frac{z^\gamma}{\Gamma(\gamma)} \int_0^\infty e^{-zt} \sigma t^{-1} \sum_{k=0}^n a_{n-k}(\lambda \gamma)^k \sum_{r=0}^\infty \frac{(\sigma \rho_q)^r_k}{(\rho_q)^r_k} \frac{(\lambda t)^r}{(\rho_q)^r_k} \frac{dt}{(z^r_k)^k} \]  

(1.13)

\[ = \frac{z^\gamma}{\Gamma(\gamma)} \int_0^\infty e^{-zt} \sigma t^{-1} \sum_{k=0}^n a_{n-k}(\lambda \gamma)^k \sum_{r=0}^\infty \frac{(\sigma \rho_q)^r_k}{(\rho_q)^r_k} \frac{(\lambda t)^r}{(\rho_q)^r_k} \frac{dt}{(z^r_k)^k} \]  

(1.14)

\[ = \sum_{r=0}^\infty \frac{(\sigma \rho_q)^r_k}{(\rho_q)^r_k} \frac{(\lambda \gamma)^k}{(z^r_k)^k} \sum_{k=0}^n a_{n-k}(\sigma \rho_q)^r_k \frac{(\alpha \rho_q)^r_k}{(\rho_q)^r_k} \frac{(\lambda \gamma)^k}{(z^r_k)^k} \]  

(1.15)

where, as before, the \( a_k \)'s are unspecified and \( \gamma \) is subject only to the restriction \( |\gamma/z| \leq 1 \).

The difference between the two sequences is that when \( z = \gamma \), in the Type I representation, \( \varphi_n(0) \neq 0 \), whereas for the Type II representation under the same condition, \( \varphi_n'(0) = 0 \). For some particular case one of the representations may be more desirable than the other. Also, it sometimes happens that one sequence approaches the desired limit monotonically from above, and the other monotonically from below, thus yielding rational inequalities for the functions in question. The above formal sequences were derived, strictly speaking, with the restrictions \( p \leq q \) and \( \text{Re}(z) > \text{Re}(\lambda) \) if \( p = q \), while \( \text{Re}(z) > \text{Re}(\lambda) \) if \( p < q \). However, these restrictions may be weakened whenever the resulting expressions are defined and make sense.

The merit of \([1]\) lies in the fact that it motivates how the arbitrary coefficients \( a_k \) should be chosen and gives a closed form expression for the error. The \( a_k \)'s were chosen in \([1]\) to agree under certain circumstances, at least, with the rational approximations given by application of Lanczos' \( \tau \)-method (see \([4]\)) to the linear differential equation satisfied by \( E(z) \) (see \([3]\)). The choices made for the \( a_k \) were

**Type I - Homogeneous**

\[ a_{n-k} = \frac{c_{nk}(\rho_q)^k}{(\alpha_{p+1})^k} \]  

if \( \sigma = 1 \)  

(1.16)
\[ a_{n-k} = \frac{C_{nk}(q)_k}{(\alpha p+1)_k (\sigma +1)_k \lambda^k \lambda^k} \quad \text{if } \tau \neq 1 \tag{1.17} \]

where

\[ C_{nk} = \frac{(-n)_k (n+\alpha +\beta +1)_k}{(\beta +1)_k k!} \]

**Type II - Non-Homogeneous**

\[ a_{n-k} = \frac{C_{nk}(q-1)_k}{(\alpha p)_k (\sigma)_k \lambda^k} \tag{1.18} \]

and \( C_{nk} \) is defined as above. With this selection of the coefficients \( a_k \), we show that the rational sequences defined by (1.4) - (1.9) and (1.10) - (1.15) converge for two general classes of functions defined as follows:

Class I is composed of those hypergeometric functions such that \( p \leq q \) in (1.1) and Class II is made up of the hypergeometric functions such that \( p = q+1 \) in (1.1). Class II includes the Whittaker functions, modified Bessel functions, the Weber Parabolic Cylinder functions, and various special cases thereof. For completeness, we define a Class III composed of all hypergeometric functions such that \( p \geq q+1 \) in (1.1).

Considering \( B(z) \) as a formal infinite series, we define the partial sum of this series by

\[ P_n(z) = \sum_{r=0}^{n-1} \frac{(\sigma)_r (\alpha p)_r (\lambda/z)^r}{(\beta q)_r r!} \tag{1.19} \]

Rearrangement of (1.5) and (1.11) gives

\[ \varphi_n(z, \gamma) = \sum_{k=n}^{n} a_{n-k} \gamma^k P_k(z) \tag{1.20} \]
where \( a = 0 \) for the Type I or Homogeneous Case, 
\[ a = 1 \] for the Type II or Non-Homogeneous Case.

Thus our rational approximations may be considered as a summation technique. A method of summability is said to be regular if it sums a convergent series to its ordinary sum. We now show that our economization process is a regular summability process.

**Theorem.** If

\[
\begin{align*}
(1) \quad & \phi_n(z, \gamma) = \sum_{k=a}^{n} a_{n-k} \gamma^k p_k(z), \quad f_n(\gamma) = \sum_{k=0}^{n} a_{n-k} \gamma^k, \\
(2) \quad & \lim_{k \to \infty} p_k(z) = E(z), \quad \text{for fixed } z, \\
(3) \quad & a_{n-k} > 0, \quad \gamma > 0, \quad \text{and} \\
(4) \quad & \lim_{n \to \infty} f_n(\gamma) = \infty, \quad f_{n+1}(\gamma) > f_n(\gamma) \quad \text{for fixed } \gamma, \\
\end{align*}
\]

then 
\[
\lim_{n \to \infty} \left| \frac{\phi_n(z, \gamma)}{f_n(\gamma)} - E(z) \right| = 0 \quad \text{for fixed } \gamma \text{ and } z.
\]

**Proof.**

Given \( \epsilon > 0 \), there exists a positive integer \( N \) (usually dependent on \( z \)) such that \( n > N \) implies \( |p_k(z) - E(z)| < \epsilon/3 \). Then

\[
\phi_n(z, \gamma)/f_n(\gamma) = \sum_{k=a}^{N} \left\{ a_{n-k} \gamma^k \left[ p_k(z) - E(z) - \epsilon/3 \right] \right\} + \epsilon/3 \sum_{k=0}^{N} a_{n-k} \gamma^k
\]

\[
+ \frac{\sum_{k=N}^{n} a_{n-k} \gamma^k \left[ p_k(z) - E(z) \right]}{f_n(\gamma)} - a_n \left[ \epsilon/3 + E(z) \right] \]

- 7 -
Thus, for $n > N$, 

\[
\left| \Phi_n(z, \gamma) - \Phi_n(z) - \Phi(z) \right| \leq \sum_{k=0}^{N} \left\{ a_{n-k} \frac{\left| \Phi_k(z) - \Phi(z) - c/3 \right|}{\Phi_n(z)} \right\} + a_n \left( \frac{c}{3} \right) 
\]

\begin{align*}
+ \frac{c}{3} \frac{\Phi_n(z)}{\Phi_n(z)}
\end{align*}

Since $N$ is fixed, $n$ can be chosen large enough such that the right hand side of the above is less than $c$. This proves the theorem. From the results of [2], and (1.16) - (1.18), we see that convergence for Class I of the hypergeometric functions is proved, when $\lambda > 0$.

We now consider Class II, i.e., the situation where $p = q + 1$. Treating the non-homogeneous or Type II case only, for $p = 1$ and $q = 0$, from (1.15) and (1.18) we get

\[
F_n(z, \gamma) = \frac{\gamma}{\Gamma(\gamma)} \int_0^\infty e^{-zt} t^{\gamma-1} \sum_{k=0}^{n} C_{nk}(\gamma/z)^k \binom{\gamma}{k} \text{F}_2 \left( \frac{\alpha_1+k, \gamma+k, \lambda}{k+1, \gamma} \right) dt.
\]  

By use of the relationships, see [3]

\[
p+1 \text{F}_q \left( \begin{array}{c} \alpha_p, \sigma_1, \sigma_2 \\ \rho_1, \rho_2 \end{array} | z \right) = \frac{\Gamma(\sigma_1+\sigma_2)}{\Gamma(\sigma_1)\Gamma(\sigma_2)} \int_0^1 u^{\sigma_1-1}(1-u)^{\sigma_2-1} p \text{F}_q \left( \begin{array}{c} \alpha_p, \rho_1, \rho_2 \end{array} | zu \right) du,
\]

\[\text{Re } \sigma_1 > 0, \text{ Re } \sigma_2 > 0\]

and

\[
2 \text{F}_1 \left( \begin{array}{c} \alpha, \beta \\ z \end{array} | z \right) = (1-z)^{-\beta} 2 \text{F}_1 \left( \begin{array}{c} \gamma-\alpha, \beta \\ \gamma \end{array} | \frac{z}{z-1} \right),
\]

we can write
\[ F_n(z, y) = \frac{z \sigma}{\Gamma(\sigma-1)\Gamma(1-\alpha_1)\Gamma(\alpha_1)} \int_0^1 \int_0^1 e^{-zt\sigma-l_u\alpha_1} (1-l_u)^{-\alpha_1-1} (l-t \lambda u)^{\alpha_1-1} \, dt \, du \]

\[ \times \, _3F_2 \left( \begin{array}{c} -n, n+\alpha+\beta+1, 1 \\ \alpha_1, 1+ \beta \end{array} \right) \left( \frac{y}{z}, \frac{1-t \lambda u}{1-t \lambda} \right) \, dv \, du \, dt \]

\[ \text{Re} \lambda < 0, \text{Re}(z) > 0, \, 0 < \text{Re} \alpha_1 < 1, \text{Re} \sigma > 1 \quad . \quad (1.24) \]

Defining

\[ M^{II}_1 = \max_{0 \leq n \leq 1} \left| _3F_2 \left( \begin{array}{c} -n, n+\alpha+\beta+1, 1 \\ \alpha_1, 1+ \beta \end{array} \right) \left( \frac{y}{z}, \frac{1-t \lambda u}{1-t \lambda} \right) \right| \quad , \quad (1.25) \]

and noticing that

\[ \left| 1-t \lambda u \right| \geq 1, \, 0 \leq t \leq \infty, \, 0 \leq u \leq 1, \]

\[ 0 \leq \left| u \left( \frac{y}{z} \right) \frac{1-t \lambda uv}{1-t \lambda u} \right| \leq 1, \quad \left| y/z \right| \leq 1, \, 0 \leq v \leq 1 \quad \}

\[ (1.26) \]

it is easily seen by direct computation that

\[ F_n(z, y) \leq M^{II}_1 \quad . \quad (1.27) \]

By repeated use of (1.22) and the same method of proof as above, it can be shown that

\[ F_n(z, y) \leq M^{II}_{q+1} \quad . \quad (1.28) \]

\[ M^{II}_{q+1} = \max_{0 \leq n \leq 1} \left| _qF_{q+2} \left( \begin{array}{c} -n, n+\alpha+\beta+1, \rho_q-1, 1 \\ \beta+1, \alpha_{q+1} \end{array} \right) \left( \frac{y}{z} \right) \right| \quad . \quad (1.28) \]
From [2], if \((2\omega + \alpha + \beta + 3/2) > 0\), where

\[
2\omega = \left\{ 1/2 + \sum_{i=1}^{q} \alpha_i - \left( q + \beta + \sum_{i=1}^{q+1} \alpha_i \right) \right\}, \tag{1.29}
\]

\[
M_{q+1}^{II} = \left| \frac{\Gamma(\beta+1)\Gamma(\alpha_{q+1}) N^{4\omega+\alpha+\beta+1} \left\{ 1+\alpha(1/N) \right\}}{\Gamma(\rho_{q+1})\Gamma(2\omega+\alpha+\beta+3/2)} \right| \right\} \quad \text{.} \tag{1.30}
\]

\[
N^2 = n(n+\alpha+\beta+1) \]

Thus \(F_n(z, \lambda)\) is bounded by a term of algebraic order in \(n\).

On the other hand, by (1.18), (1.12) and a result given in [2],

\[
F_n(z) = q^{n+3}F_{q+3} \left( \begin{array}{c}-n, n+\alpha+\beta+1, \rho_{q+1}^{-1}, 1+\beta \end{array} \bigg| \frac{\lambda}{\lambda} \right) \]

\[
= \frac{\Gamma(\alpha_{q+1})\Gamma(\beta)\Gamma(1+\beta)}{\Gamma(\rho_{q+1})2\pi^{3/2}} \left( \frac{N^2 \lambda}{\lambda} \right)^{3/2} \exp \left\{ 3 \left( \frac{N^2 \lambda}{\lambda} \right)^{1/3} - \left( \frac{N^2 \lambda}{\lambda} \right) \right\} \right\} \] \tag{1.31}

\[
+ \frac{q \left( \frac{-2}{\lambda} \right)^{1/3} + 0 \left( \frac{\rho_{q+1}^{7/3}}{N^{4/3}} \right)} {\left( \frac{N^2 \lambda}{\lambda} \right)^{1/3}} \text{ terms of algebraic order in } N \right\} \]
where

\[ N^2 = n(n+\alpha+\beta+1) \]

\[ \tilde{\omega} = \frac{2\omega+1/2-\sigma}{3} \]

\[ \lambda = -\lambda , \text{Re}(\lambda) \geq 0 \]

\[ \varphi(\gamma/\lambda) = + \frac{1}{15} (\gamma/\lambda)^2 - (\alpha+\beta+2/3+2\tilde{\omega})(\gamma/\lambda) \]

\[ + \frac{(D_1-D_2)}{3} (2D_1+D_2=1) + E_2 = E_1 = 2/9 \]

\[ D_1 = 1 + \sum_{t=1}^{q} (\rho_{t}^{-1}) , \quad D_2 = \sum_{t=1}^{q+3} \alpha_t , \quad \alpha_{q+2} = \sigma , \quad \alpha_{q+3} = 1+\beta \]

\[ E_1 = D_1 - 1 + \sum_{n=2}^{q} \sum_{j=1}^{n-1} (\rho_{n}^{-1})(\rho_{j}^{-1}) \]

\[ E_2 = \sum_{n=2}^{q+3} \sum_{j=1}^{n-1} (\alpha_n)(\alpha_j) \]

Since \( f_n(\gamma) \) is of exponential order in \( n \),

\[ \lim_{n \to \infty} R_n(z, \gamma) = \lim_{n \to \infty} \frac{F_n(z, \gamma)}{f_n(z, \gamma)} = 0 . \]  \hfill (1.33)

Similarly in the homogeneous or Type I case, one can show

\[ F_n(z, \gamma) \leq \frac{\lambda\sigma}{q^{z+1}} \frac{I}{M_q} \]  \hfill (1.34)
where

\[
M_q^{(1)} = \max_{0 \leq v \leq 1} \left| q^3 F_{q+2} \left( \begin{array}{c} -n, n+\alpha + \beta + 1 \\ \beta +1, \alpha_q + 1 \end{array} \bigg| v \right) \right| \quad (\sigma \neq 1)
\]

\[
= \max_{0 \leq v \leq 1} \left| q^3 F_{q+2} \left( \begin{array}{c} -n, n+\alpha + \beta + 1 \\ \beta +1, \alpha_q + 1 \end{array} \bigg| v \right) \right| \quad (\sigma = 1)
\]

Again, by [2], if \((2n+\alpha+\beta+3/2) > 0\), where

\[
2n = \frac{1}{2} - q - \beta + \sum_{i=1}^{q} \rho_i - \sum_{i=1}^{q+1} \alpha_q
\]

\[
= \frac{1}{2} - q - \beta + \sum_{i=1}^{q} \rho_i - \sum_{i=1}^{q+1} \alpha_q
\]

\[
M_q^{(1)} = \left| \frac{\Gamma(\beta+1) \Gamma(\alpha_q+1)}{\Gamma(1+\rho_q) \Gamma(2n+\alpha+\beta+3/2)} \left\{ 1 + o\left(\frac{1}{N}\right) \right\} \right| \quad (\sigma \neq 1)
\]

\[
= \left| \frac{\Gamma(\beta+1) \Gamma(1+\alpha_q+1)}{\Gamma(\rho_q) \Gamma(2n+\alpha+\beta+3/2)} \left\{ 1 + o\left(\frac{1}{N}\right) \right\} \right| \quad (\sigma = 1)
\]

\[N^2 = n(n+\alpha+\beta+1)\]
Thus, by (1.34) and (1.37), $F_n(z, \gamma)$ is again bounded by a term of algebraic order in $n$. Combining this with (1.51), we again have

$$\lim_{n \to \infty} F_n(z, \gamma) = \lim_{n \to \infty} \frac{F_n(z, \gamma)}{f_n(\gamma)} = 0 .$$

(1.38)

It calls for remark that the bound for the integral in (1.24) obtained by ignoring the oscillatory nature of the hypergeometric function in its integrand over the entire region of integration. Thus, the asymptotic bound for the remainder is conservative as the rational approximations converge much more rapidly than indicated by (1.28). Realistic estimates of $F_n(z, \gamma)$ and so of $R_n(z, \gamma)$ seem to be much more difficult. However, though (1.28) is quite rough, it is easy to use and not excessively misleading.

2. Numerics

To give a qualitative idea of the effectiveness of the bound $M_{\text{II}}$ for $F_n(z, \gamma)$, we consider an example given in [1] (Table 8.2)). The example is connected with the modified Bessel function $K_\nu(z)$.

$$K_\nu(z) = e^{-\pi(\nu/2z)^{1/2}}E(z)$$

$$E(z) = 2F_0(1/2 - \nu, 1/2 + \nu) {1/2z}$$

(2.1)

Using the non-homogeneous or Type II approach,
\[ f_n(y) = \, _3F_3\left(\begin{array}{c}
-\frac{n+\alpha}{2}, \frac{n+\alpha}{2} + 1, 1 \\
\frac{\beta}{2}, \frac{1}{2} - \nu, \frac{1}{2} - \nu
\end{array}\mid -2y\right) \]

\[ \phi_n(z, y) = (1/4 - \nu^2) \sum_{r=0}^{n} \frac{c_{n,r}(\nu/z)^r}{(1/2 + \nu + r)(1/2 - \nu + r)} \]

\[ \times \, _3F_3\left(\begin{array}{c}
-\frac{n+r}{2}, \frac{n+\alpha}{2} + \frac{1}{2} + r, \frac{1}{2} + r \\
\frac{\beta}{2} + \frac{1}{2} + \nu + r, \frac{3}{2} - \nu + r
\end{array}\mid -2y\right) \]  

(2.2)

\[ E(z, y) = \frac{\phi_n(z, y)}{f_n(z, y)} + R_n(z, y) \]

\[ = E_n(z, y) + R_n(z, y) \]

If \( \alpha = \beta = -1/2, \nu = 0, y = z, \) and denoting asymptotic values as computed from (1.30) and (1.31) by \( \sim \), we have to four significant figures,

<table>
<thead>
<tr>
<th>( z )</th>
<th>( f_3(z) )</th>
<th>( \sim f_3(z) )</th>
<th>( R_3(z, y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>298.3</td>
<td>295.5</td>
<td>5.598 x 10^{-4}</td>
</tr>
<tr>
<td>1.0</td>
<td>1 265.</td>
<td>1 276.</td>
<td>3.145 x 10^{-4}</td>
</tr>
<tr>
<td>2.0</td>
<td>6 515.</td>
<td>7 007.</td>
<td>5.84 x 10^{-5}</td>
</tr>
<tr>
<td>5.0</td>
<td>72 400.</td>
<td>121 000.</td>
<td>0.0</td>
</tr>
<tr>
<td>7.0</td>
<td>184 300.</td>
<td>507 800.</td>
<td>-1.0 x 10^{-6}</td>
</tr>
<tr>
<td>10.0</td>
<td>506 600.</td>
<td>3 994 000.</td>
<td>-8.0 x 10^{-7}</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( z )</th>
<th>( F_3(z, y) )</th>
<th>( \sim M_1^{\text{II}} )</th>
<th>( \sim M_2^{\text{II}} )</th>
<th>( \sim M_1^{\text{II}} / \sim f_3(z) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.1670</td>
<td>-9.4</td>
<td>9.42</td>
<td>3.188 x 10^{-2}</td>
</tr>
<tr>
<td>1.0</td>
<td>0.3977</td>
<td></td>
<td></td>
<td>7.382 x 10^{-3}</td>
</tr>
<tr>
<td>2.0</td>
<td>0.5805</td>
<td></td>
<td></td>
<td>1.344 x 10^{-3}</td>
</tr>
<tr>
<td>5.0</td>
<td>0.0</td>
<td></td>
<td></td>
<td>7.785 x 10^{-5}</td>
</tr>
<tr>
<td>7.0</td>
<td>-0.1843</td>
<td></td>
<td></td>
<td>1.855 x 10^{-5}</td>
</tr>
<tr>
<td>10.0</td>
<td>-0.4053</td>
<td></td>
<td></td>
<td>2.359 x 10^{-6}</td>
</tr>
</tbody>
</table>

(2.3)
The above example shows that our error analysis is quite conservative, as expected. For the present, a pragmatic view should be taken concerning the error. That is, if more precise information is required, in those cases where convergence is assured, one should compute for successive values of \( n \) and accept the common digits as correct. To illustrate, in the above example for \( K_0(z) \), if

\[
E_3(0.5) = 0.85932
\]

\[
E_4(0.5) = 0.85976
\]

and it is quite reasonable to say

\[
E(0.5) = 0.860
\]

with a possible error of one unit in the third decimal place.
BIBLIOGRAPHY


## DISTRIBUTION LIST

**Contract No. Nonr-2638(00)(X)**

<table>
<thead>
<tr>
<th>No. of Copies</th>
<th>Recipient</th>
</tr>
</thead>
<tbody>
<tr>
<td>14</td>
<td>CHBUSHIPS</td>
</tr>
<tr>
<td>1</td>
<td>10 Tech. Library (Code 312)</td>
</tr>
<tr>
<td>1</td>
<td>1 Tech. Asst. to Chief (Code 106)</td>
</tr>
<tr>
<td>1</td>
<td>1 Electronic Computer Div. (Code 260)</td>
</tr>
<tr>
<td>1</td>
<td>1 Asst. Chief for Field Activities (Code 700)</td>
</tr>
<tr>
<td></td>
<td>1 Asst. Chief for Nuclear Propulsion (Code 1500)</td>
</tr>
<tr>
<td>1</td>
<td>CHBUAER</td>
</tr>
<tr>
<td>1</td>
<td>CHBUORD</td>
</tr>
<tr>
<td>1</td>
<td>CHBUSANDA</td>
</tr>
<tr>
<td>1</td>
<td>CHBUCENSUS</td>
</tr>
<tr>
<td>1</td>
<td>CHNBR</td>
</tr>
<tr>
<td>1</td>
<td>NAVSHIPYD BSN</td>
</tr>
<tr>
<td>1</td>
<td>NAVSHIPYD CHASN</td>
</tr>
<tr>
<td>1</td>
<td>NAVSHIPYD LBEACH</td>
</tr>
<tr>
<td>1</td>
<td>NAVSHIPYD NARE</td>
</tr>
<tr>
<td>1</td>
<td>NAVSHIPYD NYK</td>
</tr>
<tr>
<td>1</td>
<td>NAVSHIPYD NORVA</td>
</tr>
<tr>
<td>1</td>
<td>NAVSHIPYD SPRAN</td>
</tr>
<tr>
<td>1</td>
<td>NAVSHIPYD PHILA</td>
</tr>
<tr>
<td>1</td>
<td>NAVSHIPYD PTSMH</td>
</tr>
<tr>
<td>1</td>
<td>NAVSHIPYD PUG</td>
</tr>
<tr>
<td>1</td>
<td>NAVSHIPYD PEARL</td>
</tr>
</tbody>
</table>
### DISTRIBUTION LIST (Continued)

<table>
<thead>
<tr>
<th>No. of Copies</th>
<th>Recipient</th>
</tr>
</thead>
</table>
| 1             | CO & DIR, USNML  
Philadelphia, Pennsylvania |
| 1             | CO & DIR, USNEL  
San Diego, California |
| 1             | CO & DIR, USNRDL  
San Francisco, California |
| 1             | CO & DIR, USN Trg. Device Ctr.  
Computer Branch  
Fort Washington, New York |
| 1             | CO, USNCML  
St. Paul, Minnesota |
| 1             | CDR, USNFG  
Dahlgren, Virginia |
| 3             | CDR, USNOTS  
China Lake, California  
1 Pasadena Annex  
1 Tech. Library  
1 Michelson Lab (Code 5038) |
| 1             | CDR, USNOL  
White Oak, Silver Spring, Maryland |
| 1             | DIR, USNEES  
Annapolis, Maryland |
| 1             | DIR, USNRL  
Washington 25, D. C. |
| 1             | SUPT, USN Postgrad School  
Monterey, California  
Attn: Library, Technical Reports Section |
| 1             | SUPT, US Naval Academy  
Department of Math  
Annapolis, Maryland |
<table>
<thead>
<tr>
<th>No. of Copies</th>
<th>Recipient</th>
</tr>
</thead>
</table>
| 1            | CG, Aberdeen Proving Ground  
               Aberdeen, Maryland     |
| 1            | DIR, National Bureau of Standards  
               Washington 25, D. C.  |
| 1            | Chief, AFSCM  
               Washington 25, D. C.  |
| 1            | DIR, Langley Aero Lab  
               Langley Field, Virginia |
| 1            | DIR, Lewis Flight Propulsion Lab  
               NACA  
               Cleveland 11, Ohio |
| 1            | CDR, Wright Air Development Center  
               Attn: WCRRN-4  
               Wright-Patterson Air Force Base, Ohio |
| 1            | CG, Frankford Arsenal  
               Head, Math Section  
               Pitman-Dunn Laboratories  
               Philadelphia 37, Pennsylvania |
| 1            | CG, White Sands Proving Ground  
               Flight Determination Lab  
               Las Cruces, New Mexico |
| 1            | USAEC, Technical Information Service  
               Oak Ridge, Tennessee |
| 2            | U. S. Atomic Energy Commission  
               Washington 25, D. C.  
               Attn: Tech. Library |
| 1            | CO, Diamond Ordnance Fuse Laboratory  
               Washington 25, D. C.  
               Attn: Library |
| 1            | George Washington University  
               Logistics Research  
               Washington, D. C. |
<table>
<thead>
<tr>
<th>No. of Copies</th>
<th>Recipient</th>
</tr>
</thead>
</table>
| 1            | Johns Hopkins University  
|              | Applied Physics Laboratory  
|              | Silver Spring, Maryland |
| 1            | University of California  
|              | Librarian, Numerical Analysis  
|              | Los Angeles 24, California |
| 1            | Carnegie Institute of Technology  
|              | Pittsburgh, Pennsylvania |
| 1            | Hudson Laboratory  
|              | Columbia University  
|              | Dobbs Ferry, New York |
| 2            | Harvard University  
|              | Cambridge, Massachusetts  
|              | 1 Department of Math  
|              | Attn: Prof. J. L. Walsh  
|              | 1 Computation Laboratory |
| 1            | Institute for Advanced Study  
|              | Princeton, New Jersey |
| 2            | Massachusetts Institute of Technology  
|              | Cambridge, Massachusetts  
|              | Attn: Computation Center |
| 2            | New York University  
|              | New York, New York  
|              | 1 Inst. of Math Sciences  
|              | 1 AEC Computing Facility |
| 1            | DIR, Research Center  
|              | Ohio State University  
|              | Columbus, Ohio |
| 1            | Pennsylvania State University  
|              | Department of Math  
<p>|              | University Park, Pennsylvania |</p>
<table>
<thead>
<tr>
<th>No. of Copies</th>
<th>Recipient</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Princeton University</td>
</tr>
<tr>
<td></td>
<td>Princeton, New Jersey</td>
</tr>
<tr>
<td></td>
<td>Attn: Library</td>
</tr>
<tr>
<td>1</td>
<td>University of California</td>
</tr>
<tr>
<td></td>
<td>Berkeley, California</td>
</tr>
<tr>
<td>2</td>
<td>University of Illinois</td>
</tr>
<tr>
<td></td>
<td>Urbana, Illinois</td>
</tr>
<tr>
<td></td>
<td>1 Department of Math</td>
</tr>
<tr>
<td></td>
<td>1 Electronic Digital Computer Project</td>
</tr>
<tr>
<td>2</td>
<td>University of Maryland</td>
</tr>
<tr>
<td></td>
<td>College Park, Maryland</td>
</tr>
<tr>
<td></td>
<td>1 Department of Math</td>
</tr>
<tr>
<td></td>
<td>1 Inst. for Fluid Dynamics and Applied Math</td>
</tr>
<tr>
<td>1</td>
<td>Willow Run Research Center</td>
</tr>
<tr>
<td></td>
<td>University of Michigan</td>
</tr>
<tr>
<td></td>
<td>Ypsilanti, Michigan</td>
</tr>
<tr>
<td>1</td>
<td>University of Washington</td>
</tr>
<tr>
<td></td>
<td>Department of Math</td>
</tr>
<tr>
<td></td>
<td>Seattle, Washington</td>
</tr>
<tr>
<td>1</td>
<td>Yale University</td>
</tr>
<tr>
<td></td>
<td>New Haven, Connecticut</td>
</tr>
<tr>
<td>1</td>
<td>State College of Washington</td>
</tr>
<tr>
<td></td>
<td>Department of Math</td>
</tr>
<tr>
<td></td>
<td>Pullman, Washington</td>
</tr>
<tr>
<td>1</td>
<td>Stanford University</td>
</tr>
<tr>
<td></td>
<td>Palo Alto, California</td>
</tr>
<tr>
<td></td>
<td>Attn: Dr. G. E. Forsythe</td>
</tr>
<tr>
<td>1</td>
<td>Johns Hopkins University</td>
</tr>
<tr>
<td></td>
<td>Charles and 34th Streets</td>
</tr>
<tr>
<td></td>
<td>Baltimore 18, Maryland</td>
</tr>
<tr>
<td>1</td>
<td>Rutgers University</td>
</tr>
<tr>
<td></td>
<td>New Brunswick, New Jersey</td>
</tr>
<tr>
<td></td>
<td>Attn: Prof. E. P. Starke</td>
</tr>
<tr>
<td>No. of Copies</td>
<td>Recipient</td>
</tr>
<tr>
<td>--------------</td>
<td>-----------</td>
</tr>
</tbody>
</table>
| 1            | Brown University  
Divison of Engineering  
Providence, Rhode Island  
Attn: Dr. R. D. Kodis |
| 1            | Princeton University  
Princeton, New Jersey  
Attn: Prof. H. J. Maehly  
Chief of Computer Project |
| 1            | University of Rochester  
Department of Math  
Rochester, New York |
| 1            | The Scripps Institute of Oceanography  
University of California  
LaJolla, California  
Attn: Dr. Walter Munk |
| 1            | Dr. Werner C. Rheinboldt, Director  
Computing Center, 112 Hinds Hall  
Syracuse University  
Syracuse 10, New York |
| 1            | DIR, Combustion Engineering, Inc.  
Windsor, Connecticut |
| 1            | DIR, Westinghouse Electric Corporation  
Bettis Atomic Power Division  
P.O. Box 1468  
Pittsburgh 30, Pennsylvania |
| 2            | Argonne National Laboratory  
P.O. Box 299  
Lemont, Illinois |
| 1            | Armour Research Foundation  
35 W. 33rd Street  
Chicago 16, Illinois |
| 1            | Battelle Memorial Institute  
505 King Avenue  
Columbus, Ohio |
<table>
<thead>
<tr>
<th>No. of Copies</th>
<th>Recipient</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Brookhaven National Laboratory</td>
</tr>
<tr>
<td></td>
<td>Upton, Long Island, New York</td>
</tr>
<tr>
<td>1</td>
<td>Cornell Aero Laboratory, Inc.</td>
</tr>
<tr>
<td></td>
<td>Buffalo, New York</td>
</tr>
<tr>
<td>1</td>
<td>Curtiss-Wright Corporation</td>
</tr>
<tr>
<td></td>
<td>Research Division</td>
</tr>
<tr>
<td></td>
<td>Clifton, New Jersey</td>
</tr>
<tr>
<td>1</td>
<td>Douglas Aircraft Company, Inc.</td>
</tr>
<tr>
<td></td>
<td>Santa Monica Division</td>
</tr>
<tr>
<td></td>
<td>3000 Ocean Park Boulevard</td>
</tr>
<tr>
<td></td>
<td>Santa Monica, California</td>
</tr>
<tr>
<td>1</td>
<td>Engineering Research Associates</td>
</tr>
<tr>
<td></td>
<td>St. Paul 4, Minnesota</td>
</tr>
<tr>
<td>1</td>
<td>IBM Corporation</td>
</tr>
<tr>
<td></td>
<td>590 Madison Avenue</td>
</tr>
<tr>
<td></td>
<td>New York 22, New York</td>
</tr>
<tr>
<td>1</td>
<td>Knolls Atomic Power Laboratory</td>
</tr>
<tr>
<td></td>
<td>General Electric Company</td>
</tr>
<tr>
<td></td>
<td>Math Analysis Unit</td>
</tr>
<tr>
<td></td>
<td>Schenectady, New York</td>
</tr>
<tr>
<td>1</td>
<td>Lincoln Lab</td>
</tr>
<tr>
<td></td>
<td>B-125</td>
</tr>
<tr>
<td></td>
<td>Lexington, Massachusetts</td>
</tr>
<tr>
<td>1</td>
<td>Lockheed Aircraft Corporation</td>
</tr>
<tr>
<td></td>
<td>Missile Systems Division</td>
</tr>
<tr>
<td></td>
<td>Sunnyvale, California</td>
</tr>
<tr>
<td>1</td>
<td>Lockheed Aircraft Corporation</td>
</tr>
<tr>
<td></td>
<td>Van Nuys, California</td>
</tr>
<tr>
<td>1</td>
<td>Los Alamos Scientific Lab</td>
</tr>
<tr>
<td></td>
<td>Los Alamos, New Mexico</td>
</tr>
<tr>
<td>No. of Copies</td>
<td>Recipient</td>
</tr>
<tr>
<td>--------------</td>
<td>-----------</td>
</tr>
<tr>
<td>1</td>
<td>Remington Rand Division of Sperry Rand Electronic Computer Department 315 Fourth Avenue New York 10, New York</td>
</tr>
<tr>
<td>1</td>
<td>Ramo-Wooldridge Corporation 8820 Bellanca Avenue Los Angeles 45, California</td>
</tr>
<tr>
<td>1</td>
<td>Rand Corporation Santa Monica, California</td>
</tr>
<tr>
<td>1</td>
<td>Sandia Corporation Albuquerque, New Mexico Attn: Library</td>
</tr>
<tr>
<td>1</td>
<td>United Aircraft Corporation 400 Main Street East Hartford 8, Connecticut</td>
</tr>
<tr>
<td>1</td>
<td>Vitro Corporation of America 261 Madison Avenue New York 16, New York</td>
</tr>
<tr>
<td>1</td>
<td>Oregon State College Corvallis, Oregon Attn: Professor W. Milne</td>
</tr>
<tr>
<td>1</td>
<td>Professor A. C. Aitken University of Edinburgh Edinburgh, Scotland</td>
</tr>
<tr>
<td>1</td>
<td>Professor D. E. Rutherford University of St. Andrews St. Andrews, Scotland</td>
</tr>
<tr>
<td>1</td>
<td>Professor J. L. Synge Dublin Institute for Advanced Studies Dublin, Eire</td>
</tr>
<tr>
<td>No. of Copies</td>
<td>Recipient</td>
</tr>
<tr>
<td>--------------</td>
<td>-----------</td>
</tr>
</tbody>
</table>
| 1            | Professor T. S. Broderick  
Trinity College  
Dublin, Eire |
| 1            | Professor R. E. Langer  
Mathematics Research Center  
University of Wisconsin  
Madison 6, Wisconsin |
| 1            | Professor Wallace Givens  
Department of Mathematics  
Wayne State University  
Detroit 2, Michigan |
| 1            | Dr. A. S. Householder  
Oak Ridge National Laboratories  
Oak Ridge, Tennessee |
| 1            | Professor John Todd  
Department of Math  
California Institute of Technology  
Pasadena, California |
| 1            | Dr. Franz Alt  
Mathematical Computation Laboratory  
National Bureau of Standards  
Washington 25, D. C. |
| 1            | Dr. E. W. Cannon  
Applied Mathematics Division  
National Bureau of Standards  
Washington 25, D. C. |
| 1            | Miss Irene Stegun  
Mathematical Computation Laboratory  
National Bureau of Standards  
Washington 25, D. C. |
| 1            | Mr. Cecil Hastings, Jr.  
136 Kualoa Street  
Kailua, Hawaii |
<table>
<thead>
<tr>
<th>No. of Copies</th>
<th>Recipient</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Professor E. J. McShane</td>
</tr>
<tr>
<td></td>
<td>University of Virginia</td>
</tr>
<tr>
<td></td>
<td>Charlottesville, Virginia</td>
</tr>
<tr>
<td>1</td>
<td>Professor J. B. Rosser</td>
</tr>
<tr>
<td></td>
<td>Department of Math</td>
</tr>
<tr>
<td></td>
<td>Cornell University</td>
</tr>
<tr>
<td></td>
<td>Ithaca, New York</td>
</tr>
<tr>
<td>1</td>
<td>Professor D. H. Lehmer</td>
</tr>
<tr>
<td></td>
<td>Department of Math</td>
</tr>
<tr>
<td></td>
<td>University of California</td>
</tr>
<tr>
<td></td>
<td>Berkeley 4, California</td>
</tr>
<tr>
<td>1</td>
<td>Professor Arthur Erdélyi</td>
</tr>
<tr>
<td></td>
<td>Department of Math</td>
</tr>
<tr>
<td></td>
<td>California Institute of Technology</td>
</tr>
<tr>
<td></td>
<td>Pasadena, California</td>
</tr>
<tr>
<td>5</td>
<td>Navy Department</td>
</tr>
<tr>
<td></td>
<td>David W. Taylor Model Basin</td>
</tr>
<tr>
<td></td>
<td>Washington, D. C.</td>
</tr>
<tr>
<td></td>
<td>Attn: Dr. John W. Wrench, Jr.</td>
</tr>
<tr>
<td>1</td>
<td>Professor Bertram Bussell</td>
</tr>
<tr>
<td></td>
<td>Department of Engineering</td>
</tr>
<tr>
<td></td>
<td>University of California</td>
</tr>
<tr>
<td></td>
<td>Los Angeles 24, California</td>
</tr>
<tr>
<td>75</td>
<td>Midwest Research Institute</td>
</tr>
<tr>
<td></td>
<td>Attn: Yudell L. Luke</td>
</tr>
<tr>
<td>1</td>
<td>Midwest Research Institute</td>
</tr>
<tr>
<td></td>
<td>Attn: Mathematics and Physics Division</td>
</tr>
<tr>
<td>1</td>
<td>Midwest Research Institute</td>
</tr>
<tr>
<td></td>
<td>Attn: Library</td>
</tr>
</tbody>
</table>