CONVEX SPECTRAL FUNCTIONS

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In this paper we characterize all convex functionals defined on certain convex sets of hermitian matrices and which depend only on the eigenvalues of matrices. We extend these results to certain classes of non-negative matrices. This is done by formulating some new characterizations for the spectral radius of non-negative matrices, which are of independent interest.

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The following result is useful in connection with matrix applications: If 

\[ \lambda_1(A) \] denotes the largest eigenvalue of a hermitian matrix \( A \), then

\[ \lambda_1(A+B) \leq \lambda_1(A) + \lambda_1(B) \]

i.e., if \( A \) and \( B \) are hermitian matrices, the largest eigenvalue of \( A+B \) is 
at most the sum of the largest eigenvalue of \( A \) and \( B \). The quantity \( \lambda_1(A) \) is 
a functional, i.e. a scalar depending on the matrix \( A \). The above example suggests 
the following problem which is solved in this paper: Determine all functionals 
\( \phi(A) \) depending only on the eigenvalues \( \lambda_1,...,\lambda_n \) of \( A \) such that \( \phi(A) \) is 
convex, i.e.

\[ \phi(aA + (1-a)B) \leq a \phi(A) + (1-a)\phi(B), \quad 0 \leq a \leq 1 \]

when \( A, B \) are hermitian. In economics and biology one very often deals with 
non-negative matrices. Denote by \( \lambda_1(A) \) the spectral radius of a non-negative 
matrix \( A \geq 0 \), i.e. the largest non-negative eigenvalue of \( A \). The fact that 
\( \lambda_1(A) > 1 \) or \( \lambda_1(A) < 1 \) plays a crucial role in the stability behaviour of the 
system. So any convexity results on \( \lambda_1(A) \) are helpful to estimate \( \lambda_1(A) \). Un-
fortunately (1) does not hold in general for \( A, B \) non-negative. In this paper 
we prove the validity of (1) for \( A, B \) non-negative if \( B-A \) is a diagonal matrix. 
We extend this result for more special type of non-negative matrices. To derive 
these results we bring new characterizations of the spectral radius of non-negative 
matrices.

The responsibility for the wording and views expressed in this descriptive summary 
lies with MRC, and not with the author of this report.
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1. Introduction

Let $A$ be an $n \times n$ matrix with complex entries. We arrange the eigenvalues of $A$ in the following order
\begin{equation}
\text{Re} \lambda_1(A) \geq \text{Re} \lambda_2(A) \geq \ldots \geq \text{Re} \lambda_n(A).
\end{equation}

By $H_n$ we denote the set of all $n \times n$ hermitian matrices. For $A \in H_n$ the classical maximal characterization states
\begin{equation}
\lambda_1(A) = \max_{(x,x)=1} (Ax,x).
\end{equation}

Thus $\lambda_1(A)$ is a convex functional on $H_n$. Ky Fan extended (1.2) [3]
\begin{equation}
\sum_{i=1}^k \lambda_i(A) = \max \sum_{i=1}^k (Ax_i, x_i).
\end{equation}

In particular $\sum_{i=1}^k \lambda_i(A)$ is a convex functional on $H_n$. A function
\begin{equation}
\phi: A \rightarrow \mathbb{R} (A \in H_n)
\end{equation}
is called a spectral function if
\begin{equation}
\phi(A) = F(\lambda_1(A), \ldots, \lambda_n(A)), F : \mathbb{R} \rightarrow \mathbb{R}, X \subseteq \mathbb{R}_n.
\end{equation}

Here $\mathbb{R}_n$ consists of all vectors $(x_1, \ldots, x_n)$, $x_1 \geq x_2 \geq \ldots \geq x_n$. In Section 2 of this paper we characterize all $F$ for which $\phi$ is a convex functional on $A$. It turns out that $F$ must be convex on $X$ and $F$ Schur's order preserving [11].

\begin{equation}
F(a) \leq F(b) \text{ if } a = (a_1, \ldots, a_n) < b = (b_1, \ldots, b_n),
\end{equation}
\begin{equation}
\sum_{i=1}^k a_i \leq \sum_{i=1}^k b_i, \quad i = 1, \ldots, n-1.
\end{equation}

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We also characterize all $F$ such that $\phi$ is strictly convex. Let $A$ be an $n \times n$ non-negative matrix. As usual denote by $r(A)$ the spectral radius of $A$. So $\lambda_1(A) = r(A)$. $r(A)$ is not a convex functional on non-negative matrices. For example consider

$$A = \begin{pmatrix} 0 & \varepsilon \\ 1 & 0 \end{pmatrix}, \quad r(A) = \sqrt{\varepsilon}.$$ 

Recently [1] Cohen proved that $r(A)$ is a convex function in $i$-th diagonal entry of $A$ for any $1 \leq i \leq n$. We extend Cohen's result namely, we show that $r(A+D)$ is convex on $D_n$ – the set of all $n \times n$ real diagonal matrices. In fact this result is a consequence of the Donsker-Varadhan characterization of $r(A)$ [2]. In Section 3 we bring more general characterizations of $r(A)$ by using a certain fundamental inequality for non-negative matrices established in [5]. This enables us to show that $\log r(e^D A)$ is also convex on $D_n$ for a non-negative $A$. If $A^{-1}$ happens to be an M-matrix then we have a stronger result. Namely, $r(DA)$ is convex on $D_n^+$ – the subset of non-negative matrices in $D_n$. This is done in Section 4.

In Section 5 we show how the results of Section 2 can be extended to the non-symmetric case by assuming that $A$ is a totally positive matrix of order $j(TP_j)$. We shall state our results in case that $A$ is a TP ($=TP_1$) matrix. That is all minors of $A$ (of all orders) are non-negative. In that case we have

$$\lambda_1(e^D A) \geq \lambda_2(e^D A) \geq \ldots \geq \lambda_n(e^D A) \geq 0, \quad D \in D_n.$$ 

If $A$ is non-singular then the last inequality is strict. Let

$$\phi(D) = F(\log \lambda_1(e^D A), \ldots, \log \lambda_n(e^D A)).$$

Then $\phi$ is convex on $A \in D_n$ if and only if $F$ is convex on $X$ and Schur's order preserving.

We remark that the results in Section 2 hold for symmetric compact operators in Hilbert space. The results of Section 3-5 can be extended to appropriate integral operators, for example, as it was pointed out in [5].
2. Convex functions on the spectrum of hermitian matrices

Let $A$ be an $n \times n$ hermitian matrix. We can view $A$ as a self adjoint operator on $\mathbb{C}^n$ endowed with the standard inner product

$$ (x, y) = y^* x , x, y \in \mathbb{C}^n . $$

Since the eigenvalues of $A$ are real we arrange them in the decreasing order

$$ \lambda_1(A) \geq \cdots \geq \lambda_n(A) . $$

Denote by $\xi_1, \ldots, \xi_n$ the corresponding set of orthonormal eigen-vectors of $A$

$$ A\xi_i = \lambda_i(A) \xi_i , \quad (\xi_i, \xi_j) = \delta_{ij}, \quad i, j = 1, \ldots, n . $$

Let $\mathcal{H}_n$ denote the set of all $n \times n$ hermitian matrices. Since $\lambda_1(A)$ has the maximal characterization

$$ \lambda_1(A) = \max_{\langle x, x \rangle = 1} \langle Ax, x \rangle , $$

$\lambda_1(A)$ is a convex function on $\mathcal{H}_n$. More generally we have [4]

Theorem 2.1. Let $(a_j)^n$ be a decreasing sequence of real numbers

$$ a_1 \geq a_2 \geq \cdots \geq a_n . $$

Then for any $A$ belonging to $\mathcal{H}_n$

$$ a_1 \geq \cdots \geq a_{l+1} \geq \cdots \geq a_{r-1} + 1 \geq \cdots \geq a_r = a_n , (i_0 = 0) . $$

Assume that the equality sign holds for some $x_1, \ldots, x_n$. Let

$$ a_1 = \cdots = a_{l_1} > a_{l_1 + 1} = \cdots = a_{l_2} > \cdots > a_{r-1} + 1 = \cdots = a_r = a_n , (i_0 = 0) . $$

Then there exists an orthonormal eigensystem of $A$ such that the following subspaces coincide

$$ [\xi_{l_1+1}, \ldots, \xi_{l_2+1}] = [x_{l_1+1}, \ldots, x_{l_2+1}] , \quad j = 0, \ldots, r-1 . $$

The characterization (2.7) in the case that $a_1 = \cdots = a_1 = 1, a_{i+1} = \cdots = a_n = 0$ was established by Fan [3].
In particular
\[(2.8) \quad \phi(A) = \sum_{i=1}^{n} a_i \lambda_i(A)\]
is a convex functional on \(H_n\) if (2.4) is satisfied. That is
\[(2.9) \quad \phi(cA + (1-c)B) \leq c\phi(A) + (1-c)\phi(B), \quad A, B \in H_n, \quad 0 \leq c \leq 1\]

We now are ready to state the problem which we solve in this section. A function
\[(2.10) \quad \phi : A \rightarrow \mathbb{R}, \quad A \in H_n\]
is called a spectral function if
\[(2.11) \quad \phi(A) = F(\lambda_1(A), \ldots, \lambda_n(A))\]
That is \(\phi\) is defined on the spectrum of \(A\). Our problem is to characterize all convex spectral functions on \(H_n\). To answer this problem we introduce some notation and definitions.

Let \(a = (a_1, \ldots, a_n)\) and \(b = (b_1, \ldots, b_n)\) be two vectors satisfying (2.4). According to [7, Sec. 2.18] \(a\) is majorized by \(b\), which is denoted by \(a \prec b\), if
\[(2.12) \quad \sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} b_i, \quad k = 1, \ldots, n-1\]
\[(2.13) \quad \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i\]
Denote
\[(2.14) \quad \lambda(A) = (\lambda_1(A), \ldots, \lambda_n(A))\]

From Theorem 2.1 we obtain

**Lemma 2.1.** Let \(A, B \in H_n\). Then
\[(2.15) \quad \lambda(A+B) < \lambda(A) + \lambda(B)\]

Moreover,
\[(2.16) \quad \lambda(A+B) = \lambda(A) + \lambda(B)\]

if and only if \(A\) and \(B\) have a common eigenvector system.
(2.17) \[ A\xi_i = \lambda_i(A)\xi_i, \quad B\xi_i = \lambda_i(B)\xi_i, \quad (\xi_i, \xi_j) = \delta_{ij}, \quad i, j = 1, \ldots, n. \]

**Proof.** Let

(2.18) \[ (A + B)\xi_i = \lambda_i(A + B)\xi_i, \quad (\xi_i, \xi_j) = \delta_{ij}, \quad i, j = 1, \ldots, n. \]

So for any \( a = (a_1, \ldots, a_n) \) which satisfies (2.4) we get

(2.19) \[
\sum_{i=1}^{n} a_i \lambda_i(A + B) = \sum_{i=1}^{n} a_i (A + B)\xi_i, \quad (\xi_i, \xi_j) = \delta_{ij}, \quad i, j = 1, \ldots, n.
\]

This establishes (2.15). Suppose that (2.16) holds. Then we must have

(2.20) \[
\sum_{i=1}^{n} a_i \lambda_i(A) = \sum_{i=1}^{n} a_i (A\xi_i, \xi_i), \quad \lambda_i(A) = \sum_{i=1}^{n} a_i \lambda_i(B).
\]

for any \( a_1 \geq a_2 \geq \cdots \geq a_n \). Choose \( a_i = n-i \). Then the equalities (2.7) imply (2.17). This conclusion is in fact is stated in Theorem 3.1 in [4].

By \( \mathbb{R}^n_+ \) denote the following subset of \( \mathbb{R}^n \)

(2.21) \[
\mathbb{R}^n_+ = \{ x \mid x = (x_1, \ldots, x_n), \quad x_1 \geq x_2 \geq \cdots \geq x_n \}.
\]

Clearly

(2.22) \[
\lambda : H_n \rightarrow \mathbb{R}^n_+ (\lambda(A) = (\lambda_1(A), \ldots, \lambda_n(A))).
\]

Let

(2.23) \[
\lambda(A) = X.
\]

Thus the function \( F \) in terms of which \( \phi \) is constructed satisfies \( F : X \rightarrow \mathbb{R} \).

Let \( D_n \) be the set of all \( n \times n \) real diagonal matrices and \( D_n^+ \) the set of all diagonal matrices

(2.24) \[
D_n = \text{diag} \{ a_1, \ldots, a_n \}, \quad a_1 \geq a_2 \geq \cdots \geq a_n.
\]

Given \( X \in \mathbb{R}^n_+ \) we require that \( A \) should be of the form

(2.25) \[
A = \lambda^{-1}(X).
\]

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Suppose \( B \in X. \) Then \( D(B) \in A. \) Thus the assumption that \( \phi \) is convex on \( A \) implies in particular that \( \phi \) is convex on \( D(B) n A. \) So we must have that \( F \) is convex on \( X \) which means also that \( X \) must be convex. Let \( D(B) \in A \) and \( P \) be a permutation matrix \( (\delta_{ij+1})_1^n \) \( (n+1 \equiv 1). \) Then

\[
(2.26) \quad \frac{1}{n} \sum_{i=1}^{n} P D(B)(P^T)^P = (\frac{1}{n} \beta_i/n)I.
\]

Here by \( P^T \) we denote the transpose of \( P. \) Therefore if \( B \in X \) then \( \vec{B} = (b, \ldots, b) \in X \)
\( (b = \frac{1}{n} \beta_i/n). \) This in particular implies that

\[
(2.27) \quad \text{if } B \in X, \ a < B, \text{ then } a \in X.
\]

**Definition 2.1.** Let \( X \subseteq \mathbb{R}^n. \) The set \( X \) is called strongly convex if \( X \) is convex and the condition (2.27) is satisfied.

**Theorem 2.2.** Let \( X \) be a strongly convex set in \( \mathbb{R}^n \) which contains at least one point \( a, \)
\( (2.28) \quad a_1 > a_2 > \ldots > a_n \).

Let \( F : X \rightarrow \mathbb{R}. \) Assume that \( F \in C^{(1)}(X). \) Consider a spectral function \( \phi : A \rightarrow \mathbb{R} (A \subseteq H) \)
where \( \phi \) and \( A \) are given by (2.11) and (2.25) accordingly. Then \( \phi \) is convex on \( A \) if and only if \( F \) is convex on \( X \) and

\[
(2.29) \quad \frac{\partial F}{\partial x_1}(a) \geq \frac{\partial F}{\partial x_2}(a) \geq \ldots \geq \frac{\partial F}{\partial x_n}(a)
\]

for any \( a \in X. \) Moreover, \( \phi \) is strictly convex on \( A, \) i.e.

\[
(2.30) \quad \phi(cA + (1-c)B) < c\phi(A) + (1-c)\phi(B), \ A \neq B, \ 0 < c < 1,
\]

if and only if \( F \) is strictly convex on \( X \) and

\[
(2.31) \quad \frac{\partial F}{\partial x_i}(a) > \frac{\partial F}{\partial x_{i+1}}(a) \text{ if } a_i > a_{i+1}.
\]

To prove the theorem we need the following theorem of Ostrowski [11] (Theorems VII and VIII).
Theorem 2.3. Let $X$ and $F$ satisfy the assumptions of Theorem 2.2. Then $F$ satisfies (2.29) if and only if

(2.32) \[ F(a) \leq F(\beta) \text{ if } a < \beta. \]

Moreover

(2.33) \[ F(a) < F(\beta) \text{ if } a < \beta \text{ and } a \neq \beta \]

if and only if the condition (2.31) holds.

Proof. Assume first that $F$ is convex on $X$. So if $\lambda(A), \lambda(B) \in X$ then

(2.34) \[ F(\lambda(A) + \lambda(B)) \leq \frac{1}{2} (F(\lambda(A)) + F(\lambda(B))). \]

According to Theorem 2.3, the assumption (2.29) implies

(2.35) \[ F(\lambda(A) + \lambda(B)) \leq \frac{1}{2} (F(\lambda(A)) + F(\lambda(B))). \]

by the virtue of (2.15). This shows that $\phi$ is convex on $A$. Assume furthermore that $F$ is strictly convex on $X$. So if $\lambda(A) \neq \lambda(B)$ the inequality sign holds in (2.34). This implies

(2.30). Suppose that $\lambda(A) = \lambda(B)$ but $A \neq B$. According to Lemma 2.1 $\lambda(A+B) \neq (\lambda(A) + \lambda(B))$.

So the additional assumption (2.31) yields the inequality sign in (2.35) according to Theorem 2.3. This manifests that $\phi$ is strictly convex on $A$. Assume now that $\phi$ is convex on $A$. In particular $\phi$ is convex on $D_n \cap A$. This immediately implies that $F$ is convex on $X$.

Furthermore if $\phi$ is strictly convex then $F$ is strictly convex. Let $\beta \in X$. So $D(\beta) \in A$. Assume that $a < \beta$. Then $D(a) \in A$. The classical result of [7, sec. 2.19] states that

(2.36) \[ G\beta = a, \]

where $G$ is some doubly stochastic matrix. The Birkhoff theorem implies

(2.37) \[ G = \sum_{i=1}^{k} a_i P_i, \quad a_i > 0, \quad \sum_{i=1}^{k} a_i = 1 \]

and $P_i$ is a permutation matrix. So

(2.38) \[ D(a) = \sum_{i=1}^{k} a_i P_i D(\beta) P_i^T. \]

So the convexity of $\phi$ implies
(2.39) \[ \psi(D(a)) \leq \sum_{i=1}^{k} a_i \psi(D(B)P_i^T) = \psi(D(B)) \]

which is equivalent to (2.32). Now (2.29) follows from Theorem 2.3. Assume furthermore that
\( \psi \) is strictly convex. Then we must have (2.33) which implies (2.31) according to Theorem 2.3.

The proof of the theorem is concluded.

Suppose
(2.40) \[ A \in H_m, m > n . \]

When we can define \( \psi : A \rightarrow \mathbb{R} \) by (2.11). That is \( \psi \) does not depend on \( \lambda_{n+1}(A), \ldots, \lambda_m(A) \),

i.e. \( \frac{3F}{3x_i} = 0 \) for \( i > n \). In that case Theorem 2.2 reads:

Corollary 2.1. Let the assumptions of Theorem 2.2 hold except that we have (2.40). Then \( \psi \) is

convex on \( A \) if and only if \( F \) is convex on \( X \), the inequalities (2.29) hold and in addition

(2.41) \[ \frac{3F}{3x_i} (a) \geq 0, a \in X . \]
3. Some characterization of the spectral radius

Let $A$ be an $n \times n$ non-negative matrix such that there exists two positive vectors $u, v$ satisfying

$$ Au = r(A)u, \quad Av^T = r(A)v, \quad u^T = (u_1, \ldots, u_n) > 0, \quad v^T = (v_1, \ldots, v_n) > 0. $$

Assume the normalization

$$ \sum_{i=1}^{n} u_i v_i = 1. $$

Let $P_n$ be the set of probability vectors

$$ P_n = \{ a | a = (a_1, \ldots, a_n), \quad a_i > 0, \quad \sum_{i=1}^{n} a_i = 1 \}. $$

In [5, Sec. 3] it was manifested

**Theorem 3.1.** Let $A$ be an $n \times n$ non-negative irreducible matrix having positive entries on the diagonal (or fully indecomposable, see Remark 3.3 in [5]). Then for any $a \in P_n$, with positive entries $(a_i > 0)$. The function $f(a) = \sum_{i=1}^{n} a_i \log \frac{(Ax)_i}{x_i}$ has a unique critical point $\xi = (\xi_1, \ldots, \xi_n)$ in the interior point of $P_n (\xi_i > 0)$ which must satisfy

$$ \min_{x > 0} \sum_{i=1}^{n} a_i \log \frac{(Ax)_i}{x_i} = \sum_{i=1}^{n} a_i \log \frac{(A\xi)_i}{\xi_i}. $$

Thus, if $a$ is chosen to be

$$ a = (u_1 v_1, \ldots, u_n v_n), $$

where $u$ and $v$ satisfy (3.1) - (3.2) then

$$ \sum_{i=1}^{n} u_i v_i \log \frac{(Ax)_i}{x_i} \geq \log r(A), $$

since $x = u$ is a critical point of $f(x)$.

From Theorem 3.1 we get

**Theorem 3.2.** Let $A$ be an $n \times n$ non-negative matrix such that $r(A) > 0$. Then

$$ \sup_{a \in P_n} \inf_{x > 0} \sum_{i=1}^{n} a_i \log \frac{(Ax)_i}{x_i} = \log r(A). $$
Suppose that there exists a positive vector \( u \) satisfying (3.1). Assume that

\[
\inf_{x > 0} \frac{1}{n} \sum_{i=1}^{n} a_i \log \frac{(Ax)_i}{x_i} = \log r(A).
\]

Then the vector \( v \)

\[
v = \left( \frac{a_1}{u_1}, \ldots, \frac{a_n}{u_n} \right)
\]

fulfills (3.1). In particular if \( A \) is irreducible then \( \alpha \) is unique and given by (3.5).

**Proof:** As the left-hand side of (3.7) is a continuous function of \( A \) it is enough to prove (3.7) for \( A \) positive. Let \( u > 0 \) be the corresponding eigenvector of \( A \). So

\[
\inf_{x > 0} \frac{1}{n} \sum_{i=1}^{n} a_i \log \frac{(Ax)_i}{x_i} = \log r(A)
\]

for any \( \alpha \) such that \( \sum_{i=1}^{n} a_i = 1 \). Thus

\[
\sup_{\alpha \in \mathbb{P}_n} \inf_{x > 0} \frac{1}{n} \sum_{i=1}^{n} a_i \log \frac{(Ax)_i}{x_i} \leq \log r(A).
\]

The above inequality together with (3.6) yields (3.7). Suppose that (3.8) holds. If \( u > 0 \) satisfies (3.1) then \( x = u \) is a minimal point for \( f(x) = \frac{1}{n} \sum_{i=1}^{n} a_i \log \frac{(Ax)_i}{x_i} \). So

\[
0 = \frac{\partial f}{\partial x_j} \bigg|_{x=u} = \frac{1}{n} \sum_{i=1}^{n} a_i \frac{a_i j}{(Ax)_i} \bigg|_{x=u} = r(A)^{-1} \frac{1}{n} \sum_{i=1}^{n} a_i u_i^{-1} a_i j - a_j u_j^{-1}.
\]

This shows that \( v \) given by (3.9) is a left eigenvector of \( A \) corresponding to \( r(A) \). If \( A \) is irreducible, then \( u \) and \( v \) are unique up to a multiple of a positive scalar. Thus \( \alpha \) is of the form (3.5) and since \( \alpha \in \mathbb{P}_n \), \( \alpha \) unique. The proof of the theorem is completed.

We now bring an extended version of Theorem 3.2 which includes (3.7) and the Donsker-Varadhan characterization [2] as its special cases.

**Theorem 3.3.** Let \( \mathbb{V} : \mathbb{R} \to \mathbb{R} \) be a continuous convex function on \( \mathbb{R} \). Define \( \phi : \mathbb{R}_+ \to \mathbb{R} \)

\[
\phi(x) = \mathbb{V}(\log x).
\]

Let \( A \) be an \( n \times n \) non-negative matrix such that \( r(A) > 0 \). Assume

\[
\mathbb{V}'(\log r(A)) \geq 0.
\]
Then

\[ \sup_{\alpha \in \mathbb{P}_n} \inf_{x > 0} \sum_{i=1}^{n} a_i \phi \left( \frac{(Ax)_i}{x_i} \right) = \phi(r(A)) . \]  

Assume that the inequality sign holds in (3.11) and suppose that there exists a positive vector \( u \) satisfying (3.1). If

\[ \inf_{x > 0} \sum_{i=1}^{n} a_i \phi \left( \frac{(Ax)_i}{x_i} \right) = \phi(r(A)) , \]

then the vector \( v \) (3.9) satisfies (3.1). In particular if \( A \) is irreducible then \( \alpha \) is unique and given by (3.5).

Proof. Let \( t_0 = \log r(A) \), \( \mathcal{W}'(t_0) = e \). Then the convexity of \( \mathcal{W} \) implies

\[ \mathcal{W}(t) \geq \mathcal{W}(t_0) + (t-t_0)\mathcal{W}'(t_0) . \]

So

\[ \sum_{i=1}^{n} a_i \phi \left( \frac{(Ax)_i}{x_i} \right) \geq \phi(r(A)) - e \log r(A) \]

\[ + e \sum_{i=1}^{n} a_i \log \frac{(Ax)_i}{x_i} , \quad \alpha \in \mathbb{P}_n . \]

As \( e > 0 \) from Theorem 3.2 and the above inequality we get

\[ \sup_{\alpha \in \mathbb{P}_n} \inf_{x > 0} \sum_{i=1}^{n} a_i \phi \left( \frac{(Ax)_i}{x_i} \right) \geq \phi(r(A)) . \]

Since \( \phi \) is continuous we may assume that \( A \) is positive. By choosing \( x = u \) the left-hand side of (3.15) we deduce an opposite inequality of (3.15). This establishes (3.12). In case the \( e > 0 \) we use the arguments of Theorem 3.2 to analyze the equality (3.12). End of proof.

Letting \( \phi(x) = e^x \) in Theorem 3.3 we obtain the Donsker-Varadhan characterization [2].

Corollary 3.1. Let the assumptions of Theorem 3.2 hold. Then

\[ \sup_{\alpha \in \mathbb{P}_n} \inf_{x > 0} \sum_{i=1}^{n} a_i \frac{(Ax)_i}{x_i} = r(A) . \]

Suppose that
If $A$ has a positive eigenvector $u$ then the conclusions of Theorem 3.2 apply.

Recall the classical characterization due to Wielandt [12]

$$\inf_{x>0} \max_{1 \leq i \leq n} \frac{(Ax)_i}{x_i} = r(A)$$

for any non-negative $A$. Assume that $\phi$ is an increasing function of $x$ on $\mathbb{R}_+$. So

$$\inf_{x>0} \sup_{0 \leq \alpha \leq P} \inf_{i=1}^n \alpha_i \phi \left( \frac{(Ax)_i}{x_i} \right) = \inf_{x>0} \phi \left( \max_{1 \leq i \leq n} \frac{(Ax)_i}{x_i} \right)$$

$$= \phi \left( \inf_{x>0} \max_{1 \leq i \leq n} \frac{(Ax)_i}{x_i} \right) = \phi \left( r(A) \right).$$

Thus if $\phi$ is increasing and satisfies the assumptions of Theorem 3.3 then we can interchange $\sup$ with $\inf$ in (3.12). The characterization (3.19) is completely equivalent to the Wielandt characterization (3.18) while (3.12) seems to be a deeper characterization.

Let $A$ be a non-negative and non-singular. Assume furthermore that $A^{-1}$ is an $M$-matrix, i.e. the off-diagonal elements of $A^{-1}$ are non-positive. Following [5] we bring another characterization of $r(A)$.

**Theorem 3.4.** Let $A$ be a non-negative and non-singular matrix such that $A^{-1}$ is an $M$-matrix.

Then

$$\inf_{x>0} \sup_{0 \leq \alpha \leq P} \inf_{i=1}^n \alpha_i \frac{x_i}{(Ax)_i} = \frac{1}{r(A)}.$$  

Assume that there exists a positive vector $u$ satisfying (3.1) and suppose

$$\sup_{x>0} \inf_{0 \leq \alpha \leq P} \alpha_i \frac{x_i}{(Ax)_i} = \frac{1}{r(A)}.$$  

Then $v$ given by (3.9) satisfies (3.1). In particular if $A$ is irreducible then $\alpha$ is unique and given by (3.5).

**Proof.** We have available the representation

$$A^{-1} = rI - B, \ B \geq 0, \ r > r(B).$$
and $B$ is reducible if and only if $A$ is reducible (e.g. [8, chap. 8]). Again, as in the proof of Theorem 3.2 one may assume that $B$ is positive. By letting $x$ to be equal to the positive eigenvector $u$ of $A$ we immediately deduce

$$\inf \frac{\sum_{i=1}^{n} a_{i} x_{i}}{\alpha \sum_{i=1}^{n} (Ax)_{i}} \geq \frac{1}{r(A)} .$$

(3.23)

Let $\alpha$ be given by (3.5). Obviously for any $x > 0$ and $y = Ax$

$$\frac{\sum_{i=1}^{n} u_{i} v_{i} (Ax)_{i}}{\sum_{i=1}^{n} u_{i} v_{i} y_{i}} = \frac{\sum_{i=1}^{n} (A^{-1} y)_{i}}{\sum_{i=1}^{n} v_{i} y_{i}} = \frac{1}{r(B)} .$$

(3.24)

From Corollary 3.1 it follows

$$\frac{\sum_{i=1}^{n} u_{i} v_{i} (By)_{i}}{\sum_{i=1}^{n} v_{i} y_{i}} \geq r(B) .$$

(3.25)

So

$$\frac{\sum_{i=1}^{n} u_{i} v_{i} (Ax)_{i}}{\sum_{i=1}^{n} u_{i} v_{i} y_{i}} \leq r(B) .$$

(3.26)

and the equality sign holds if $x = u$. This establishes (3.20). The equality (3.21) is analyzed in the same way as in Theorem 3.2.

Remark 3.1. Theorem 3.4 does not hold for arbitrary non-negative matrices, take for example $A$ to be a permutation matrix $P \neq I$. Therefore Theorem 3.4 is not a special case of Theorem 3.3.
4. Convexity properties of the spectral radius

Let $A$ be an $n \times n$ non-negative matrix. Consider the matrix $A + D$, $D \in D_n$. Assume that the eigenvalues of $A + D$ arranged in the order

$$(4.1) \quad \text{Re } \lambda_1(A) \geq \text{Re } \lambda_2(A) \geq \ldots \geq \text{Re } \lambda_n(A).$$

Let

$$(4.2) \quad \rho(D) = \lambda_1(A+D).$$

We claim that $\rho(D)$ is real. If $D$ is non-negative this fact is a consequence of the Perron-Frobenius theorem. For an arbitrary $D$ consider $A + D + aI$

$$(4.3) \quad \lambda_k(A+D+aI) = \lambda_k(A+D) + a, \ k = 1, \ldots, n$$

for a big enough $A + D + aI \geq 0$ and (4.3) implies that $\rho(D)$ is real. Moreover, by considering the matrix $B = A + D + aI$ and using the Donsker-Varadhan characterization for $B$ we get the following characterization for $\rho(D)$

$$(4.4) \quad \rho(D) = \sup_{a \in \mathbb{R}} \lambda_1(D,a).$$

Here $\lambda_1(D,a)$ is a linear functional on $D_n$

$$(4.5) \quad \lambda_1(D,a) = \frac{1}{n} \sum_{i=1}^{n} a_i d_1 + \inf_{x > 0} \frac{1}{n} \sum_{i=1}^{n} a_i \frac{(Ax)_i}{x_i},$$

$$a = (a_1, \ldots, a_n), \quad D = \text{diag}(d_1, \ldots, d_n).$$

It is a standard fact (4.4) and (4.5) imply the convexity of $\rho(D)$ on the set $D_n$. More precisely we have:

**Theorem 4.1.** Let $A$ be a fixed $n \times n$ non-negative matrix. Assume that $\rho(D)$, $D \in D_n$, is given by (4.2). Then $\rho(D)$ is a real valued convex functional on $D_n$.

$$(4.6) \quad \rho((D_1+D_2)/2) \leq (\rho(D_1) + \rho(D_2))/2.$$

Moreover, if $A$ is irreducible then the equality sign holds in (4.6) if and only if

$$(4.7) \quad D_2 - D_1 = aI$$

for some $a$. 

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Proof. As we pointed out (4.6) is a consequence of (4.4). So it is enough to analyze the equality case. Let

\[ A_1 = A + (D_1 + D_2)/2, \ A_1 u = r_1 u, \ A_1^\top v = r_1 v, \ r_1 = \rho((D_1 + D_2)/2). \]

As \( A \) is irreducible we may assume that \( u, v > 0 \) and the normalization (3.2) holds. Let \( a \) be given by (3.5). So

\[ L_1((D_1 + D_2)/2, a) = \inf_{x > 0} \sum_{i=1}^{n} \frac{(A_1 x)_i}{x_i} = \sum_{i=1}^{n} \frac{(A_1 u)_i}{u_i} = r_1. \]

If we apply the results of Section 3 in [5]

\[ f(x, B) = \sum_{i=1}^{n} a_i \frac{(Bx)_i}{x_i} \]

where \( B + bI \) is irreducible matrix, for some positive \( b \), then \( f(x, B) \) has a unique critical point in the interior of \( P_n \) which must be the minimum point \( (f(x) = \infty \) on the boundary of \( P_n \). The equality sign in (4.6) implies

\[ L_1(D_1, a) = \rho(D_1), \ L_1(D_2, a) = \rho(D_2). \]

That is

\[ f(x, A + D_1) \geq f(u, A + D_1) = \rho(D_1), \ f(x, A + D_2) \geq f(u, A + D_2) = \rho(D_2). \]

The uniqueness of the minimal point of \( f(X, B) \) implies

\[ (A + D_1) u = \rho(D_1) u, \ (A + D_2) u = \rho(D_2) u. \]

As \( u > 0 \) (4.7) follows the above equality. The proof of the theorem is completed.

The inequality (4.6) extends Cohen's result [1]. Let \( A \) be a non-negative matrix such that \( r(A) > 0 \). Clearly, for any \( D \in D_n \), \( r(e^D A) \) is also positive. Define

\[ R(D) = \log r(e^D A). \]

According to Theorem 3.2,

\[ R(D) = \sup_{a \in P_n} L_2(D, a), \]

where

\[ -15- \]
Combining (4.15) and (4.16) and using the uniqueness result stated in Theorem 3.1 as in the proof of Theorem 4.1 we deduce.

**Theorem 4.2.** Let $A$ be a fixed $n \times n$ non-negative matrix having a positive spectral radius. Assume that $R(D)$ is given by (4.14). Then $R(D)$ is a convex functional on $\mathbb{R}^n$.

(4.17)

\[ R(D_1 + D_2) \leq \frac{1}{2} (R(D_1) + R(D_2)) \]

Moreover if $A$ is irreducible and the diagonal entries of $A$ are positive (or $A$ is fully indecomposable) then the equality sign holds in (4.17) if and only if (4.7) holds for some $a$.

Assume that $A, B \in \mathbb{R}^n$ and furthermore $A$ is positive definite ($\langle Ax, x \rangle > 0$ for $x \neq 0$).

$BA$ is similar to $A^{1/2}BA^{1/2}$. This shows that $\lambda_1(BA)$ is a convex functional on $\mathbb{R}^n$ for a fixed positive definite $A$. If in addition $A$ has non-negative entries then $\lambda_1(DA)$ is convex on $\mathbb{R}^n$. This result does not apply in general for non-negative matrices. For example, take $A$ to be a permutation matrix $P \neq I$. However, $\lambda_1(DA)$ is convex on $\mathbb{R}^n$ - the set of $n \times n$ non-negative diagonal matrices if $A^{-1}$ is an M-matrix.

**Theorem 4.3.** Let $A^{-1}$ be an M-matrix. Then $r(DA)$ is a convex functional on $\mathbb{R}^n$.

(4.18)

\[ r \left( \frac{D_1 + D_2}{2} \right) A \leq \frac{1}{2} (r(D_1 A) + r(D_2 A)) \]

Moreover if $A$ is irreducible then the equality sign in (4.18) holds if and only if

(4.19)

\[ D_2 = aD_1 \]

for some positive $a$ provided that $D_1$ or $D_2$ have positive diagonal elements.

**Proof.** Using the continuity argument we may assume that in the decomposition (3.22) $B$ is positive (irreducible), i.e. $A$ is positive (irreducible). Thus if all diagonal elements of $D_0 = \text{diag} \{ d_1^0, ..., d_n^0 \}$ are positive then $D_0 A$ is positive (irreducible). According to the Perron-Frobenius theorem $r(D_0 A)$ is a simple root of the $\det(\lambda I - D_0 A) = 0$. By the implicit function theorem $r(DA)$ is an analytic function of $D$ in the neighborhood of $D_0$. Then the convexity of $r(DA)$ would follow if we show that

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(4.20) \[ r(DA) > r(D_0A) + \sum_{i=1}^{n} \left( d_{1i} - d_{1i}^{(0)} \right) \frac{dr(DA)}{3d_{1i}} D_0^{(0)} \]

for any \( D_0 \) with positive diagonal elements. Let \( \xi, \eta \) be the eigenvectors corresponding to \( D_0A \) and \( A^T D_0 \)

(4.21) \[ D_0A\xi = r(D_0A)\xi, \quad A^T D_0\eta = r(D_0A)\eta \]

\[ 0 < \xi = (\xi_1, \ldots, \xi_n), \quad 0 < \eta = (\eta_1, \ldots, \eta_n), \quad \sum_{i=1}^{n} \xi_i \eta_i = 1. \]

It can be shown that

(4.22) \[ \frac{dr(DA)}{3d_{1i}} D_0^{(0)} = n \frac{3d_{1i}}{3d_{1i}} A\xi = r(D_0A) \sum_{i=1}^{n} \frac{\xi_i}{d_{1i}^{(0)}} \xi_i \eta_i \]

This can be done by bringing \( D_0A \) to the Jordan form and using the simplicity of \( r(D_0A) \). See for example (10, II, §5.4). Thus (4.20) is equivalent to

(4.23) \[ r(DA) > r(D_0A) \sum_{i=1}^{n} \frac{d_{1i}}{d_{1i}^{(0)}} \xi_i \eta_i \]

This inequality was established in [5]. It follows directly from (3.26). Indeed suppose that \( D \) has positive diagonal elements and let

(4.24) \[ DAW = r(DA)w, \quad w = (w_1, \ldots, w_n) > 0. \]

Then according to (3.26)

\[ \frac{1}{r(D_0A)} \sum_{i=1}^{n} \frac{\xi_i}{d_{1i}^{(0)}} \frac{d_{1i}}{d_{1i}^{(0)}} (DAW)_i = \frac{n}{\sum_{i=1}^{n} \frac{\xi_i}{d_{1i}^{(0)}} \frac{d_{1i}}{d_{1i}^{(0)}} (DAW)_i} \]

which establishes (4.23) for \( D \) with positive diagonal. So (4.18) holds in the interior of \( D_n^+ \). The continuity argument implies the validity of (4.18) on \( D_n^+ \). Suppose that \( A \) is also irreducible. Then \( B \) in the decomposition (3.22) is also irreducible, since the inverse of block triangular matrix is also a block triangular one. As in the proof of Theorem 4.1 strict inequality holds in (4.23) unless \( D_0A \) and \( DA \) have the same positive eigenvector. So
D = aD₀ for some a > 0. This shows that we have strict inequality in (4.18) unless (4.19) holds provided that D₀ (which is either D₁ or D₂) have positive diagonal. The proof of the theorem is completed.

We conclude this section by pointing out that the convexity of r(D_A) on Dᵣ⁺ is a stronger result than the convexity of log r(e₀A) on Dᵣ. Indeed, let

\[(4.25) \quad D₀ = e₀, \quad Q₀ = \{q₁^{(0)}, \ldots, qₙ^{(0)}\}, \quad qᵢ^{(0)} = \log dᵢ^{(0)}, \quad i = 1, \ldots, n .\]

Suppose that log r(e₀A) is convex at Q = Q₀. This means

\[(4.26) \quad \log r(e₀A) \geq \log r(D₀A) + r(D₀A)^{-1} \sum_{i=1}^{n} \frac{\partial r(e₀A)}{\partial qᵢ} |_{Q=Q₀} (qᵢ^{(0)} - qᵢ^{(0)}), \quad Q = \text{diag}(q₁, \ldots, qₙ) .\]

As in the proof of Theorem 4.3

\[(4.27) \quad \frac{\partial r(e₀A)}{\partial qᵢ} |_{Q=Q₀} = \eta^T \frac{\partial e₀}{\partial qᵢ} |_{Q=Q₀} \eta A = r(D₀A) \eta qᵢ, \quad i = 1, \ldots, n .\]

where \(\eta, \xi\) given by (4.21).

Thus (4.26) is equivalent

\[(4.28) \quad r(e₀A) \geq r(D₀A) \prod_{i=1}^{n} \left( \frac{qᵢ^{(0)}}{dᵢ^{(0)}} \right)^{\eta qᵢ} = r(D₀A) \prod_{i=1}^{n} \left( \frac{qᵢ}{dᵢ^{(0)}} \right)^{\eta qᵢ}, \quad qᵢ = \log dᵢ, \quad i = 1, \ldots, n .\]

Using the relation between the arithmetic and the geometric means from (4.23) we get

\[(4.29) \quad r(e₀A) \geq r(D₀A) \prod_{i=1}^{n} \left( \frac{qᵢ}{dᵢ^{(0)}} \right)^{\eta qᵢ} \geq r(D₀A) \prod_{i=1}^{n} \left( \frac{qᵢ}{dᵢ^{(0)}} \right)^{\eta qᵢ} .\]

That is the convexity of r(D_A) at D₀ ∈ D⁺ᵣ implies the convexity of log r(e₀A) at Q₀ = log D₀. This demonstrates that the convexity of r(D_A) on D⁺ᵣ implies the convexity of log r(eᵣA) on Dᵣ. On the other hand if A is a permutation matrix ≠ I then r(DF) is not convex on D⁺ᵣ (for details see [5], Section 3).
5. Convex functions on the spectrum of totally positive matrices

A real valued $n \times n$ matrix is called a totally (strictly totally) positive matrix of order $k$ if all minors of $A$ of order less or equal to $k$ are non-negative (positive). We denote these matrices by $TP_j(STM_j)$. For $j = n$ we call these matrices simply by $TP(STM)$. A matrix $A$ is called oscillating if $A$ is TP and some power of $A$ is STP. It is known that a TP matrix if oscillating if and only if

\[ a_{ii} > 0, a_{i(i+1)} > 0, i = 1, \ldots, n, A = (a_{ij})_{1}^{n} \geq 0 \]

In that case $A$ is totally indecomposable.

If $A$ is $TP_j$ then

\[ \lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_j(A) \geq \lambda_k(A), k = j + 1, \ldots, n \]

If $A$ is $STM_j$ then we have strict inequalities in (5.2). See [6] and [9] for proofs of these results and more properties of these matrices. Let $A$ be $TP_j$. Define $\phi : A \mapsto R (A \subseteq D_n)$ as follows

\[ \phi(D) = F(\log \lambda_1(e^{DA}), \ldots, \log \lambda_j(e^{DA})) \]

As in Section 2 we were looking for necessary and sufficient conditions on $F$ which imply that $\phi$ is a convex function on $A \subseteq D$ for any $A$ which is $TP_j$. It turns out that we have an analogous result to Theorem 2.2. To do so we need few notations and definitions. Let $\overline{a} = (a_1, \ldots, a_j)$ and $\overline{b} = (b_1, \ldots, b_j)$ and $j < n$. We define $\overline{a} \ll \overline{b}$ if (2.12) holds for $k = 1, \ldots, j$. Thus if $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$ and $a < b$ then $\overline{a} \ll \overline{b}$. Conversely, if $\overline{a} \ll \overline{b}$ we can extend $\overline{a}$ to $a$ and $\overline{b}$ to $b$ such that $a < b$. A set $X \subseteq R^2$ is called a super convex if $\overline{X}$ is convex and

\[ \text{if } \overline{b} \in \overline{X}, \overline{a} \ll \overline{b}, \text{ then } \overline{a} \in \overline{X} \]

Clearly $\overline{X}$ is super convex in $R^2$ if and only if it could be extended to $X \subseteq R^n$ such that $X$ is strongly convex in $R^n$. Using the above arguments and Ostrowski's result (Theorem 2.3) we get

Lemma 5.1. Let $\overline{X}$ be a super convex set in $R^2$. Let $F : \overline{X} \mapsto R$. Assume that $F \in C^{(1)}(\overline{X})$. 

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Then

\( F(\overline{a}) \leq F(\overline{b}) \) if \( \overline{a} \ll \overline{b} \)

if and only if

\[
\frac{\partial F}{\partial x_1}(\overline{a}) \geq \frac{\partial F}{\partial x_2}(\overline{a}) \geq \cdots \geq \frac{\partial F}{\partial x_j}(\overline{a}) \geq 0
\]

for any \( \overline{a} \in \mathbb{R}^n \). Moreover strict inequality in (5.5) holds for \( \overline{a} \neq \overline{b} \) if and only if

\[
\frac{\partial F}{\partial x_i}(\overline{a}) > \frac{\partial F}{\partial x_{i+1}}(\overline{a}) \quad \text{if} \quad a_i > a_{i+1} \quad \text{and} \quad \frac{\partial F}{\partial x_j}(\overline{a}) > 0 \quad \text{if} \quad a_j > 0.
\]

Assume that \( A \) is TP. Denote

\[
\lambda^{(j)}(A) = (\lambda_1(A), \ldots, \lambda_j(A)), \quad \log \lambda^{(j)}(A) = (\log \lambda_1(A), \ldots, \log \lambda_j(A)).
\]

**Theorem 5.1.** Let \( A \) be an \( n \times n \) non-singular TP matrix. If \( j < n \) then

\[
\log \lambda^{(j)}(e^{D_1 + D_2})/2 < A << \frac{1}{2} \log \lambda^{(j)}(e^{D_1 A}) + \log \lambda^{(j)}(e^{D_2 A}).
\]

If \( j = n \) then

\[
\log \lambda(e^{D_1 + D_2}/2) < A << \frac{1}{2} [\log \lambda(e^{D_1 A}) + \log \lambda(e^{D_2 A})].
\]

If in addition \( A \) satisfies (5.1), or more generally \( A \) is totally indecomposable, then

\[
\log \lambda^{(j)}(e^{D_1 + D_2}/2) = \frac{1}{2} [\log \lambda^{(j)}(e^{D_1 A}) + \log \lambda^{(j)}(e^{D_2 A})]
\]

for any \( 1 \leq j \leq n \) if and only if (4.7) is satisfied for some \( a \).

**Proof.** Denote by \( C^k(A) \) the \( k \)-th compound of \( A \). Thus

\[
C^k(e^D) = e^{\varphi_k(D)}
\]

where \( \varphi_k \) is well defined map \( \varphi_k : D_n \to D_n \). It is easy to see using the properties of the compound matrices that \( \varphi_k \) is a linear map. According to Theorem 4.2 \( \log r(e^{D^k C^k(A)}) \) is convex on \( D_n \) for \( k = 1, \ldots, j \). Note that the non-singularity of \( A \) implies that \( r(C^k(A)) > 0 \).

Thus \( \log r(e^{\varphi_k(D)} C^k(A)) \) is convex on \( D_n \). Let

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It is well known that

\[ R_k(D) = \sum_{i=1}^{k} \log \lambda_i(e^D) \]  

Therefore \( R_k(D) \) is convex on \( D_n \) for \( k = 1, \ldots, j \). This is equivalent to (5.9) for \( j < n \).

For \( j = n \), \( R_n(D) \) is linear on \( D \) as

\[ R_n(D) = \log \det(e^D) = \sum_{i=1}^{n} d_i + \log \det(A) \]  

This verifies (5.10) if \( A \) is a TP matrix. Suppose that in addition \( A \) is totally indecomposable. According to Theorem 4.2 we have a strict inequality in (4.17) unless (4.7) holds.

Thus (5.11) can be satisfied if only (4.7) holds. Trivially (4.7) implies (5.11). The proof of the theorem is completed.

Theorem 5.2. Let \( \bar{X} \) be a super convex set in \( \mathbb{R}^j \) for \( 1 \leq j < n \) (a strongly convex containing a point \( a_1 > \ldots > a_n \), if \( j = n \). Let \( F : \bar{X} \to \mathbb{R} \). Assume that \( F \in \mathcal{C}_1(\bar{X}) \). Let \( A \) be a given \( n \times n \) non-singular TP matrix. Consider a spectral function \( \phi : A \to \mathbb{R} \), given by (5.3), where \( A \) is a convex set in \( D_n \) such that

\[ \log \lambda^{(j)}(e^D) \leq \bar{X}, \quad D \in A \]  

Then, for all such \( A, \phi \) is convex if and only if \( F \) is convex on \( \bar{X} \) and satisfies (5.6) in case that \( 1 \leq j < n \). Moreover, if \( A \) is totally indecomposable then \( \phi \) is strictly convex if and only if \( F \) is strictly convex and satisfies (5.7). In case \( j = n \), \( \phi \) is convex (strictly convex provided that \( A \) is totally indecomposable) if and only if \( F \) satisfies the assumptions of Theorem 2.2.

Proof. A proof of this theorem can be achieved by modifying in the obvious way the proof of Theorem 2.2. In fact, all the arguments of the proof of Theorem 2.2 carry over if one notices that the identity matrix is TP.
REFERENCES


* I would like to thank Professor Karlin for pointing out to me references [1] and [2] which inspired this work.

SF/jvs
Convex functionals, Schur preserving order, characterizations of the spectral radius.

In this paper we characterize all convex functionals defined on certain convex sets of hermitian matrices and which depend only on the eigenvalues of matrices. We extend these results to certain classes of non-negative matrices. This is done by formulating some new characterizations for the spectral radius of non-negative matrices, which are of independent interest.