A STRAIGHTFORWARD GENERALIZATION OF DILIBERTO AND STRAUS' ALGOR--ETC(U)

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A STRAIGHTFORWARD GENERALIZATION OF DILIBERTO AND STRAUS' ALGORITHM DOES NOT WORK

Nira Richter-Dyn
ABSTRACT

An algorithm for best approximating in the sup-norm a function \( f \in C[0,1]^2 \) by functions from tensor-product spaces of the form 
\( \pi_k \otimes C[0,1] \otimes C[0,1] \otimes \pi_2 \), is considered. For the case \( k = \ell = 0 \) the Diliberto and Straus algorithm is known to converge. A straightforward generalization of this algorithm to general \( k, \ell \) is formulated, and an example is constructed demonstrating that this algorithm does not converge for \( k^2 + \ell^2 > 0 \).

AMS (MOS) Subject Classifications: 41A50, 41A63

Key Words: Algorithm, Tensor-Product Spaces, Best Approximation by Polynomials

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SIGNIFICANCE AND EXPLANATION

Often it is desirable to approximate a given function as closely as possible by a member of a class of functions that are simpler to evaluate.

For a general continuous function of two variables \( f(x,y) \) a best approximating function of the simpler form \( h(y) + g(x) \) can be computed by the algorithm of Diliberto and Straus. Since such an approximation can be quite far from the approximated function, a better approximation of the form \[ \sum_{i=0}^{k} h_i(y)x^i + \sum_{j=0}^{k} g_j(x)y^j \] is considered. One way to try to construct such an approximation is to generalize the Diliberto and Straus algorithm to this more general setting. The generalized algorithm is simple in the sense that only one-dimensional best approximations by polynomials have to be computed. In this note it is shown by a simple example, that this "natural" generalization cannot be expected to converge, and therefore other methods should be developed.

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The algorithm of Diliberto and Straus for approximating a bivariate function by a
sum of univariate ones proposed in 1951 [1], has been recently investigated in several
works [2], [3], [4], where convergence and various properties of the algorithm are
studied.

The algorithm, designed for computing the best approximation to \( f \in C[0,1]^2 \) in
the sup-norm from the space

\[
M = \{ \phi \mid \phi(x,y) \in C[0,1]^2, \ \phi(x,y) = h(y) + g(x) \},
\]

is of the following form:

\[
f_0(x,y) = f(x,y)
\]

\[
f_{2n+1}(x,y) = f_{2n}(x,y) - \frac{1}{2} \left[ \max_{0 \leq \xi \leq 1} f_{2n}(\xi,y) + \min_{0 \leq \xi \leq 1} f_{2n}(\xi,y) \right],
\]

\[
f_{2n+2}(x,y) = f_{2n+1}(x,y) - \frac{1}{2} \left[ \max_{0 \leq n \leq 1} f_{2n+1}(x,n) + \min_{0 \leq n \leq 1} f_{2n+1}(x,n) \right],
\]

\[
\text{for } n = 0, 1, \ldots
\]

It is proved in [1], [3], [4] that \( \lim_{n \to \infty} \| f - f_n \| = \inf_{\phi \in M} \| f - \phi \| \), although the rate of convergence
might be extremely slow [2]. Algorithm (2) can be interpreted as a sequence of repeated
applications of the operator of one dimensional best approximation by constants to
\( f(x,y) \), regarded alternately as a function of \( x \) and as a function of \( y \). More specif-
ically, let \( J_x \) be the operator associating with \( f(x,y) \in C[0,1]^2 \) the function
\( (J_x f)(y) \in C[0,1], \) with \( (J_x f)(y_0) \) the constant of best approximation to \( f(x,y_0) \) in
the sup-norm on \( [0,1] \), and let \( J_y \) be defined similarly with the roles of \( x,y \) inter-
changed. Then (2) can be rewritten as

\[
f_0 = f, f_{2n+1} = f_{2n} - J^x f_{2n}, f_{2n+2} = f_{2n+1} - J^y f_{2n+1}, \text{ for } n = 0, 1, 2, \ldots
\]

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This formulation suggests a straightforward generalization of algorithm (3), namely
best approximating \( f(x,y) \) alternately in the \( x \) and \( y \) directions by polynomials
of degree \( k \) and \( l \) respectively, in order to obtain a best approximation to \( f(x,y) \)
from the tensor-product space
\[
M_{k,l} = \{ \phi(x,y) \mid \phi(x,y) \in \mathcal{C}[0,1]^2, \phi(x,y) = \sum_{j=0}^{k} h_j(y) x^j + \sum_{j=0}^{l} g_j(x) y^j \} = \\
\tau_k \otimes \mathcal{C}[0,1] \otimes \mathcal{C}[0,1] \otimes \tau_l
\]
(\( \tau \) denotes the space of all univariate polynomials of degree \( \leq k \).) With this nota-
tion the subspace \( M \) in (1) is the tensor-product space \( \mathcal{M}_{0,0} \). The generalization of
algorithm (3) to this more general setting is
\[
 f_{0}^{*} = f, f_{2n+1} = f_{2n}^{(k)} x, f_{2n+2} = f_{2n+1}^{(l)} y, n=0,1,2,\ldots,
\]
where \( (j^{(k)}_x f)(x,y) = \sum_{j=0}^{k} h_j(y) x^j \) is the polynomial of best approximation to \( f(x,y_0) \)
in the sup-norm on \([0,1]\) from \( \tau_k \), and where \( (j^{(l)}_y f)(x_0,y) \) is similarly defined.

In the following we present a simple example demonstrating that algorithm (5) for
general \( k,l \) cannot be expected to converge to a best approximation to \( f_0(x,y) \). We
construct a function \( f(x,y) \) such that \( \|f\| > \inf \|f-\phi\| \), while the functions \( \{f_n\} \)
generated from it by (5) with \( k=0, l=1 \) satisfy \( \|f_n\| = \|f\| \) for all \( n \).

Consider \( f(x,y) \in \mathcal{C}[0,1]^2 \) subject to the following conditions:
\[
f(\frac{1}{4^n}, \frac{1}{6}) = (-1)^{i+1}, \quad j=2i,2i+1,2i+2, \quad i=0,1,2
\]
\[
f(\frac{1}{4^n}, \frac{1}{2}) = (-1)^{i+1}, \quad j=0,5,6
\]
\[
f(\frac{1}{2^n}, \frac{1}{6}) = (-1)^{j+1}, \quad j=0,1,2
\]
\[
|f(x,y)| < 1 \quad \text{elsewhere in } [0,1]^2.
\]
As can be easily observed
\[
(j^{(0)}_x f)(x,\frac{1}{6}) = 0, \quad i=0,1,\ldots,6 \quad \text{and} \quad (j^{(1)}_y f)(\frac{1}{4^n},y) = 0, \quad i=0,1,2,3,4
\]
and both \( f-j^{(0)}_x f \) and \( f-j^{(1)}_y f \) satisfy (6). Thus algorithm (5) with \( k=0, l=1 \)
generates a sequence \( \{f_n\} \) of functions satisfying (6) whenever \( f_0 \) satisfies (6),
and therefore \( \|f_n\| = 1 \) for all \( n \geq 0 \).
In order to verify that \( \|f\| > \inf_{\phi \in M_{0,1}} \|f - \phi\| \), it is sufficient to show that there does not exist a bounded linear functional \( \phi \in (C[0,1]^2)' \), \( \phi \neq 0 \), such that

\[
\langle \phi, u \rangle = 0 \quad \text{for all} \quad \phi \in M_{0,1},
\]

\[
\langle f, u \rangle = \|f\|.
\]

Indeed any \( \mu \neq 0 \) with property (8) is necessarily of the form

\[
\langle \zeta, \mu \rangle = \sum_{j=0}^{r} a_j \zeta(x_j, y_j), \quad \zeta \in C[0,1]^2, \quad \text{with} \quad r > 0, \quad a_j f(x_j, y_j) = |a_j|, \quad j = 0, \ldots, r,
\]

namely a linear combination of function values at extremal points of \( f \). Moreover condition (7) implies that \( \mu \) can be written as a linear combination of first differences in the \( x \) direction so as to vanish on all functions of the form \( h(y) \), and as a linear combination of second order divided differences in the \( y \) direction, so as to vanish on all functions of the form \( g_0(x) + g_1(x)y \).

These characteristics of \( \mu \) are consistent with the special structure of the 15 extremal points of \( f \), as given in (6), only if \( r = 14 \) in (9). Then \( \mu \) can be written as

\[
\langle \zeta, \mu \rangle = \sum_{i=0}^{4} c_i [_{4} \zeta, _i \zeta] = \sum_{i=0}^{4} c_i [_{4} \zeta, _i \zeta],
\]

where \( [_{4} \zeta, _i \zeta] \) denotes the second order divided difference of \( \zeta(\frac{1}{x}, y) \) at the extremal points of \( f \) with \( x = \frac{i}{4} \). The sum (10) can be rewritten as a linear combination of first differences in the \( x \) direction only if \( c_0, \ldots, c_4 \) satisfy the following system of linear equations:

\[
\begin{align*}
 c_0 &= c_1 = c_2, \\
 c_3 &= \frac{c_4}{3}, \\
 c_1 &= \frac{c_4}{4}, \\
 c_0 &= \frac{c_2}{15}, \\
 c_2 &= \frac{2}{5} c_3 + \frac{c_4}{4},
\end{align*}
\]

which admits only the trivial solution.

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REFERENCES


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\[
\sum_{i=1}^{k} x^2_i + e^2 > 0.
\]