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CRITICAL POINT THEOREMS FOR INDEFINITE FUNCTIONALS

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ABSTRACT

A variational principle of a minimax nature is developed and used to prove the existence of critical points for certain variational problems which are indefinite. The proofs are carried out directly in an infinite dimensional Hilbert space. Special cases of these problems previously had been tractable only by an elaborate finite dimensional approximation procedure. The main applications given here are to Hamiltonian systems of ordinary differential equations where the existence of time periodic solutions is established for several classes of Hamiltonians.

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SIGNIFICANCE AND EXPLANATION

Many functionals which arise in a variational formulation of physical problems are highly indefinite, i.e. are not bounded from above or below even modulo subspaces or submanifolds of finite dimension or codimension. E.g. in mechanics, this is often the case for a Lagrangian. In some recent research, finite dimensional approximation arguments have successfully handled some such questions. The present paper gives a direct infinite dimensional method for finding critical points of a class of such problems and applies it in particular to find periodic solutions of Hamiltonian's equations in mechanics under various hypotheses on the Hamiltonian.

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Introduction

In some recent papers, the existence of time periodic solutions has been established for a family of semilinear wave equations [1] and certain Hamiltonian systems of ordinary differential equations [2-4]. These equations have the common feature that formally they can be derived from variational problems. However these variational problems are indefinite in a very strong sense, in particular the corresponding functionals are not bounded from above or from below, even modulo subspaces or submanifolds of finite dimension or codimension. Due to the lack of an appropriate theory to obtain critical points directly, finite dimensional approximation arguments were employed in [1]-[4]. Critical point theorems in \( \mathbb{R}^n \) provided approximate solutions to the equations and suitable estimates and comparison arguments allowed passage to a limit to solve the problem of interest.

The main purpose of this paper is to prove some abstract existence theorems for critical points of a real valued functional on a Hilbert space. These theorems provide a framework for a direct infinite dimensional treatment of some of the problems mentioned earlier. When applicable they permit a simpler treatment of these questions and clarify their underlying common structure. In more concrete situations the abstract existence theorems given here yield critical points which are weak solutions of the associated differential equations. Hence regularity becomes a separate question for us, unlike [1]-[4] where existence and regularity were obtained simultaneously in the course of the approximation procedure.

\[ \text{Theorem 0.1:} \quad \text{Let } E \text{ be a real Hilbert space, } E_1 \text{ a closed subspace of } E, \text{ and } E_2 = E_1^\perp. \text{ Suppose that } f \in C^1(E, \mathbb{R}) \text{ and satisfies}
\]

\[ (f_1) \quad f(u) = \frac{1}{2} (I_1 u, u) + b(u) \quad \text{where } u = u_1 + u_2 \in E_1 \oplus E_2, \]

\[ I_1 u = I_1 u_1 + I_1 u_2 \text{ and } I_1 : E_1 \rightarrow E_1, \quad f = 1, 2 \text{ is a (bounded) linear self-adjoint mapping.} \]

\[ (f_2) \quad b \text{ is weakly continuous and is uniformly differentiable on bounded subsets of } E. \]

\[ (f_3) \quad \text{If for a sequence } (u_m), \quad f(u_m) \text{ is bounded from above and}
\]

\[ f(u_m) \to 0 \text{ as } m \to \infty, \text{ then } (u_m) \text{ is bounded.} \]

\[ (f_4) \quad \text{There are constants } r_1, r_2, \rho, \sigma, \omega \text{ with } r_1 > \rho, \sigma > \omega \text{ and}
\]

\[ r_1, r_2 > 0 \text{ and there is an } e \in \mathbb{R} \cap E_1 \text{ such that}
\]

\[ (1) \quad f \equiv \sigma \text{ on } S \equiv \mathbb{B}_\rho, \]

\[ (2) \quad f \equiv \omega \text{ on } \mathbb{Q} \text{ where } \mathbb{Q} = \{ x : 0 \leq r \leq r_1 \} \oplus (B_{r_2} \cap E_2). \]

Then \( f \) possesses a critical value \( c \leq \sigma \).

**Remarks 0.2:** In \((f_2)\), \( b \) weakly continuous means whenever \( u_m \) converges weakly to \( u, \ b(u_m) \to b(u) \). The uniform differentiability of
on bounded sets means for any $R, \varepsilon > 0$, there is a $\delta = \delta(R, \varepsilon) > 0$ (and independent of $u$) such that if $u, u + v \in B_R$ and $\|v\| \leq \delta$, then

$$|b(u + v) - b(u) - (b'(u), v)| \leq \varepsilon \|v\|$$

where $b'(u)$ denotes the Frechet derivative of $b$ at $u$. The form of $f$ and $(f_2)$ imply that $f$ is also uniformly differentiable on bounded subsets of $E$. Using $(f_2)$ and standard theorems [5], it follows that $B(u) = b'(u)$ maps weakly convergent to strongly convergent sequences and a fortiori is compact. (In the usual fashion we are interpreting $B$ as a mapping of $E$ to $F$). Hypothesis $(f_3)$ is a weakened version of the Palais-Smale condition (PS) which states that whenever $|f(u_m)|$ is bounded and $f'(u_m) \to 0$, then $u_m$ possesses a convergent subsequence.

In $(f_4)$ (11), $\partial Q$ refers to the boundary of $Q$ as a subset of span $\{e\} \subset E$.

Theorem 0.1 is the third in a line of related abstract results. Suppose $E$ is a real Banach space, $(f_1)$ - $(f_2)$ are omitted, and $(f_3)$ is replaced by (PS). Then if $E_2 = \{0\}$, we essentially have a theorem of Ambrosetti and Rabinowitz [6], [7] while if $E_2$ is finite dimensional we have a variant of results of Rabinowitz [1], [8] and Benci [9]. Unlike these earlier papers, here both $E_1$ and $E_2$ are permitted to be infinite dimensional. This is the case of greatest interest to us and several examples will be given in $\S 2$ to 5. As in [6] - [9], a minimax characterization of the critical value $c$ can be given and this will be done in $\S 1$.

The applications of the predecessors of Theorem 0.1 have been to problems involving functionals $b(u)$ which in an appropriate sense increase at a more rapid rate than quadratic. (Henceforth such problems will be called superquadratic). This behavior is hidden in our hypotheses in $(f_4)$ (11). When combined, the pair of hypotheses $(f_4)$ (1) - (11) imply a certain topological linking occurs between $S$ and $\partial Q$. This linking implies $c \equiv \alpha$ and in applications shows any critical point corresponding to $c$ is nontrivial. These cryptic remarks will be made more precise later.

In $\S 1$, Theorem 0.1 will be obtained as a consequence of a more general result, Theorem 1.4, in which $(f_4)$ is supplanted by an abstract linking condition. Stating matters thusly permits a simultaneous proof of Theorem 0.1 and another result, Theorem 1.33 suitable for functionals with subquadratic nonlinearities. Here subquadratic refers to less rapid growth than quadratic in an appropriate sense. A simpler version of the latter type of result was obtained in [10]. (See also [11] - [13]).

A model problem which gives a simple illustration of Theorem 0.1 for an elliptic system of partial differential equations is given in $\S 2$. Then the abstract theorems are applied to obtain existence results for superquadratic $(\S 3)$ and subquadratic $(\S 4)$ Hamiltonian systems of ordinary differential equations. Lastly in $\S 5$ we indicate how to apply the theory to semilinear hyperbolic partial differential equations.

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§1. The abstract theorems.

This section is devoted to the proof of a general critical point theorem and specializations of the result appropriate for sub- and superquadratic problems. We begin with a notion of linking.

Mappings $h : [0, 1] \times E \to E$ will be denoted by $h(t, u)$ or $h_0(u)$. In the sequel $E$ always denotes a real Hilbert space, $E = E_1 \oplus E_2$, and if $u \in E$, $u = u_1 + u_2$ with $u_1 \in E_1$, $i = 1, 2$. Further set $P_i u = u_i$, $i = 1, 2$.

Let $\Sigma$ denote the class of mappings $\varphi \in C([0, 1] \times E, E)$ for which $P_2 \varphi_t(u) = u_2 - W_t(u)$ with $W_t$ compact for $t \in [0, 1]$ and $\varphi_0(u) = u$. Let $S$ and $Q$ be Hilbert manifolds, $Q$ having a boundary, $\partial Q$. We say $S$ and $\partial Q$ link if whenever $\varphi \in \Sigma$ and $\varphi_1(\partial Q) \cap S \neq \emptyset$ for all $t \in [0, 1]$, then $\varphi_t(Q) \cap S \neq \emptyset$ for all $t \in [0, 1]$. A somewhat inaccurate but geometrically useful picture to keep in mind is of $S$ and $\partial Q$ linking if every Hilbert manifold modelled on $Q$ and sharing the same boundary intersects $S$. The following two lemmas provide examples of linking sets.

Lemma 1.1: Suppose $Q = B_2 \cap E_2$, $Q \subset Q$, and $S = Q + E_1$. Then $S$ and $\partial Q$ link.

Proof: Let $\varphi \in \Sigma$ be such that $\varphi_1(\partial Q) \cap S \neq \emptyset$ for all $t \in [0, 1]$. We claim $\varphi_t(Q) \cap S \neq \emptyset$ or equivalently there is a $q_t \in Q$ such that $P_2 \varphi_t(q_t) = q$ for all $t \in [0, 1]$. Observe that for $u \in E_2$,

$\varphi_t(u) = P_2 \varphi_t(u) = u - W_t(u)$

has the appropriate form to apply the theory of degree of Leray-Schauder. We denote the Leray-Schauder degree of a map $\varphi$ with respect to a bounded open set $\Omega$ and a point $a \in \partial \Omega$ by $d(\varphi, \Omega, a)$. Thus consider $d(\varphi_t, Q, q)$. Since $\varphi_1(\partial Q) \cap S \neq \emptyset$, this degree is well defined for all $t \in [0, 1]$. By the homotopy invariance property of degree, $d(\varphi_t, Q, q) = d(\varphi_0, Q, q)$. Since $\varphi_0(u) = u$ and $q \in Q$, $d(\varphi_0, Q, q) = 1$. Hence there exists a point $q_t$ as desired for each $t \in [0, 1]$.

Remark 1.2: The linking obtained in Lemma 1.1 is of the type which often occurs in problems having subquadratic nonlinearities as will be seen from the applications given in §4. The next lemma describes a more subtle type of linking arising in superquadratic problems. Examples will be given in §2-3.

Lemma 1.3: Let $\varphi \in \partial B_1 \cap E_1$ and $r_1 > r > 0$. If $S = B_2 \cap E_1$ and $Q = \{r \in [0, r_1] \cap (B_2 \cap E_2) \}$, then $S$ and $\partial Q$ link.

Proof: Let $\varphi \in \Sigma$ and suppose that $\varphi_1(\partial Q) \cap S \neq \emptyset$ for all $t \in [0, 1]$. We must show $\varphi_t(Q) \cap S \neq \emptyset$ or equivalently $P_2 \varphi_t(q_t) = 0$ and $\|P_2 \varphi_t(q_t)\| = \|\varphi_t(q_t)\| = \rho$ for some $q_t \in Q$ and each $t \in [0, 1]$. Let $u \in E_2$, $r \in R$, and set

$y_t(r, u) = (P_2 \varphi_t(u + ru) - \rho, P_2 \varphi_t(u + ru))$.

Then as in Lemma 1.1, $d(y_t, Q, 0)$ is well defined for $t \in [0, 1]$ and $d(y_t, Q, 0) = d(y_0, Q, 0) = 1$ since $y_0(r, u) = (r - \rho, u)$. Hence the result follows.
A general critical point theorem can now be stated.

**Theorem 1.4.** Let $E$ be a real Hilbert space, $E_1$ a closed subspace of $E$, and $E_2 = E_i^\perp$. Let $f \in C^1(E, \mathbb{R})$ and satisfy $(f_1) - (f_3)$ and

$(f_0)$ There exist Hilbert manifolds $S, Q \subset E$, $Q$ having a boundary, constants $\alpha > 0$, and $v \in E_2$ such that

(i) $S \subset v + E_1$ and $f = \alpha$ on $S$,
(ii) $f \leq \alpha$ on $Q$,
(iii) $S$ and $Q$ link.

Then $f$ possesses a critical value $c = \alpha$.

The proof of Theorem 1.4 will be accomplished in several steps. Roughly speaking, we obtain $c$ as a minimax of $f$ over a class of surfaces modelled on $Q$. To be a bit more precise, using $(f_1) - (f_2)$, we obtain Theorem 1.5, a variant of a standard deformation result. The properties of the mappings given by Theorem 1.5 suggest how to construct the appropriate class of surfaces and define $c$. Applying $(f_4)$, we show $c = \alpha$. Lastly Theorem 1.5 and $(f_3)$ or more properly a local version of $(f_3)$ relative to $c$ are employed to show $c$ is indeed a critical value of $f$.

To begin this program, let $\Gamma$ denote the set of mappings $h \in C^\infty([0, 1] \times E, E)$ such that:

$(f_1)$ $h_t(u) = U_t(u) + K_t(u)$ where $U_t, K_t \in C([0, 1] \times E, E), U_t$ is a homeomorphism of $E$ onto $E$ and $K_t$ is compact for each $t \in [0, 1],

$(f_2)$ $u_0(u) = u, \quad K_0(u) = 0$,

$(f_3)$ $P_t U_t(u) = U_t(P_t(u)), \quad i = 1, 2$

$(f_4)$ $h_t$ maps bounded sets to bounded sets.

For $h \in \Gamma$, let $h^j_t(u)$ denote the $j$-fold composite of $h$ with itself, i.e., $h^1_t(u) = h_t(u), \quad h^2_t(u) = h_t(h_t(u)), \quad$ and $h^{j+1}_t(u) = h_t(h^j_t(u))$ for $j > 1$. Let $\mathcal{B} = (B_1 \cap E_1) \oplus (B_2 \cap E_2)$.

**Theorem 1.5.** Let $f \in C^1(E, \mathbb{R})$ and satisfy $(f_1) - (f_2)$. Then for any $B, \gamma > 0$ and $\epsilon \in (0, \frac{1}{2})$, there is a $k \in \mathbb{N}$ and $\eta \in \Gamma$ such that

$f^k(\eta_k^\gamma(u)) = f(u) + \gamma$ for all $u \in \mathcal{B}$ and $t \in [0, 1]$.

If $c \in \mathbb{R}$ and $\|f'(w)\| < \sqrt{2\epsilon}$ for all $w \in \mathcal{B} \cap \Gamma^{-1}(\{c - \epsilon, c + \epsilon\})$, then $f(\eta_k^\gamma(u)) = c - \frac{\epsilon}{2}$ whenever $u \in \mathcal{B} \cap \Gamma^{-1}(\{c - \epsilon, c + \epsilon\})$.

Proof: Theorems like this result form a basic part of any minimax proof of the existence of critical points of a real valued functional. See e.g. [7]. The mapping $\eta$ is usually determined by solving an appropriate differential equation involving $-f'(\eta)$. Such an approach seems to fail here since it does not give an $\eta$ satisfying $(f_1') - (f_3')$ which are crucial for our later purposes.

Choose $x : C^\infty(\mathbb{R}, \mathbb{R})$ such that $x(s) = \gamma \phi(s)$ if $s \in R + 1$, $x(s) = 0$ if $s \in R + 2$, and $x(s) < 0$ if $s \in (R + 1, R + 2)$.

We can further assume $x(s) \in (R + 2 - s)^2$ for $s \in [R + \frac{1}{2}, R + 2], R + 2.5$.

For $u = u_1 + u_2 \in E_1 \oplus E_2$, set $V_t(u) = x(||u_1||)P_t f^t(u), \quad i = 1, 2$.
and \( V(u) = V_1(u) + V_2(u) \). Note that \( V(u) = f'(u) \) in \( B_{R+1} \) and by \((f_1) - (f_2)\), there is a constant \( M = M(R) \) such that
\[
\max_{u \in B_{R+2}} |f'(u)| \leq M.
\]
Set
\[
\varepsilon = M^{-1} \min \{ \gamma, \frac{R}{2} \}.
\]
Since \( f \) is uniformly differentiable, there is a \( \delta = \delta(\varepsilon, R) > 0 \) such that
\[
|f(u + v) - f(u)| \leq \delta \|v\|
\]
if \( u, u + v \in B_{R+2} \) and \( \|v\| \leq \delta \). We can assume \( \delta \leq 1 \).

Choose \( k \in \mathbb{N} \) such that
\[
\frac{1}{k} < \frac{\varepsilon}{2M}, \quad \frac{1}{k} < \frac{1}{R} \max_{1 \leq i \leq L} \{ L_1[i], \frac{1}{2} L_2[i] \}
\]
Now define \( \eta_t(u) = u - \frac{1}{k} V(u) \). We claim \( \eta \in \Gamma \).

Indeed observe that
\[
P_t \eta_t(u) = (I - \frac{1}{k} X(\|u\|) L_1)_t u + \frac{1}{k} X(\|u\|) P_t B(u)
\]
where \( I \) denotes the identity map in \( E \). Hence setting
\[
U_t(u) = \sum_{1 \leq i \leq L} \left( I - \frac{1}{k} X(\|u\|) L_1 \right)_t u
\]
and
\[
K_t(u) = \frac{1}{k} \sum_{1 \leq i \leq L} X(\|u\|) P_t B(u),
\]
it is clear that \( K_t \) is compact and \( \eta \) satisfies \((f_2)\) and \((f_3)\).

The form of \( \eta \) and \((f_1) - (f_2)\) imply that \((f_3)\) is satisfied. Lastly to complete the verification of \((f_1)\), we must show \( U_t \) in a homeomorphism of \( E \) onto \( E \). By \((f_3)\), it suffices to show that \( P_t U_t \) is a homeomorphism of \( E \) onto \( E \), \( i = 1, 2 \). Thus let \( u, v \in E \).

For any \( t \in [0, 1] \),
\[
\frac{1}{k} \| X(\|u\|) L_1 u - X(\|u\|) L_1 v \| \leq \frac{1}{k} \| u - v \|
\]
via \((5, 8)\) and some simple estimates. Hence given any \( w \in E \),
\[
L_w(u) = \frac{1}{k} X(\|u\|) L_1 u
\]
is a contraction on \( E \). It then follows from the contracting mapping theorem that \( L_w(u) \) has a unique fixed point \( u \).

This observation and \((5, 8)\) easily imply \( P_t U_t \) is a homeomorphism on \( E \) and \( \eta \in \Gamma \).

Next observe that \( \eta_t : B_{R+2} \rightarrow B_{R+2} \). In fact if \( u = u_1 + u_2 \in B_{R+2} \),
\[
P_t \eta_t(u) = \frac{1}{k} V_t(u) = \frac{M}{k} X(\|u_1\|).
\]
Since \( \operatorname{dist}(u, B_{R+2}) \), the distance from \( u \) to \( B_{R+2} \) satisfies
\[
\operatorname{dist}(u, B_{R+2}) = \min_{1 \leq i \leq 2} R + 2 - \|u_i\|
\]
if we show the right hand side of \((1, 10)\) does not exceed \( R + 2 - \|u_1\| \),
\[
R + 2 - \|u_1\| \leq (R + 2 - \|u_1\|)^2 \leq M k^{-1} X(\|u_1\|)
\]
via \((5, 8)\) while if \( \|u_1\| \leq R + \frac{3}{2} \),
\[
R + 2 - \|u_1\| \leq (R + 2 - \|u_1\|)^2 \leq M k^{-1} X(\|u_1\|)
\]
by the choice of \( x \) and \( M \).
Now $\phi$ of Theorem 1.5 can be verified. By (1.8) again,
\[ \| \eta_{i}(u) - u \| \leq \epsilon \quad \text{for all} \quad u \in \mathbb{R}^{n+2}. \]
Hence by (1.7)
\[ (1.11) \quad f(\eta_{i}(u)) = f(u) - \frac{1}{k} (f'(u), V(u)) + \frac{k}{k} \| V(u) \| . \]
Since $E_{k} = E_{k}^{*}$,
\[ (1.12) \quad \| f'(u), V(u) \| = \lambda (\| u_{2} \|) \| P_{2} f'(u) \|^{2} + \lambda (\| u_{2} \|) \| P_{2} f'(u) \|^{2} = 0. \]
Thus by (1.6) and (1.11),
\[ (1.13) \quad f(\eta_{i}(u)) = f(u) + \frac{\lambda}{k} . \]
Since $\mathbb{R}^{n+2}$ is invariant under $\eta_{i}$, $\phi$ obtains by iterating (1.13) $k$ times.

Finally to prove $\phi^{0}$, three cases are considered:

**Case I:** $\eta_{i}^{m}(u) \in \mathbb{R}^{n+2} \cap \Omega^{*}([c - \epsilon, c + \epsilon])$ for $1 \leq i \leq m$. Since $V(u) = f'(u)$ in $\mathbb{R}^{n+2}$, by (1.11) and (1.6):
\[ (1.14) \quad f(\eta_{i}^{m}(u)) = f(u) + \frac{\lambda}{k} \sum_{j=1}^{m} \| f'(\eta_{i}^{j-1}(u)) \|^{2} + \frac{\epsilon}{k} \leq -2\epsilon + \frac{\epsilon}{k} = -\frac{\epsilon}{2} \]
which implies $\phi^{0}$.

**Case II:** $\eta_{i}^{m}(u) \in \mathbb{R}^{n+2} \cap \Omega^{*}([c - \epsilon, c + \epsilon])$ for $1 \leq i \leq m - 1$ but $\eta_{m}^{m}(u) \notin \Omega^{*}([c - \epsilon, c + \epsilon])$.

Estimating as in (1.14) shows that $f(\eta_{i}^{m}(u)) < f(\eta_{i}^{m-1}(u))$ for $1 \leq i \leq m$. Thus $\eta_{i}^{m}(u) \notin \Omega^{*}([c - \epsilon, c + \epsilon])$ implies that
\[ f(\eta_{i}^{m}(u)) < c - \epsilon . \]
Therefore
\[ f(\eta_{i}^{m}(u)) = f(\eta_{i}^{m-1}(u)) + \frac{\lambda}{k} \sum_{j=m+1}^{k} (f(\eta_{i}^{m-1}(u)) - f(\eta_{i}^{j-1}(u))) \]
\[ \leq c - \epsilon + \frac{\lambda}{k} \frac{m}{2} \leq c - \frac{\epsilon}{2} \]
via (1.11), (1.12), and (1.6).

**Case III:** $\eta_{i}^{1}(u) \in \mathbb{R}^{n+2} \cap \Omega^{*}([c - \epsilon, c + \epsilon])$ for $1 \leq i \leq m - 1$ but $\eta_{i}^{m}(u) \notin \mathbb{R}^{n+2}$.

Then
\[ (1.15) \quad 1 \leq \| \eta_{i}^{m}(u) - u \| \leq \| \eta_{i}^{m}(u) - u \| = \sum_{j=1}^{m} \| \eta_{i}^{j-1}(u) - \eta_{i}^{j-1}(u) \| \]
\[ \leq \frac{1}{k} \sum_{j=1}^{m} \| f'(\eta_{i}^{j-1}(u)) \|^{2} \frac{\epsilon}{k} \]
Employing (1.11) and (1.6) in (1.15) yields
\[ 1 \leq \frac{1}{k} \sum_{j=1}^{m} k (f(\eta_{i}^{j-1}(u)) - f(\eta_{i}^{j-1}(u))) \]
\[ \leq \frac{1}{k} k \left( f(\eta_{i}^{m}(u)) - f(\eta_{i}^{m}(u)) + \frac{\epsilon}{2} \right) \]
which implies $\phi^{0}$.
Hence \( f(h_1(u)) \leq c - \frac{S}{2} \) by Case II. The proof is complete.

Motivated by the properties of \( f \), we introduce a class of maps, \( \Lambda \), that will be used to determine \( c \). Let \( \mathcal{G}_s = \{ u \in E | f(u) \leq s \} \). Let \( \Lambda \) denote the set of \( h \in C([0, 1] \times E, E) \) for which there is a \( \beta \in (0, \frac{S}{2}) \), \( m \in \mathbb{N} \), and \( h^{(1)}, \ldots, h^{(m)} \in \Gamma \) such that
\[
h = h^{(1)} \cdots h^{(m)} \quad \text{and} \quad h(Q) \subseteq \mathcal{G} \quad \text{for all} \quad \beta \leq \frac{S}{2} - \beta.
\]
Observe that \( \Lambda \neq \emptyset \) since \( h(u) = u \in \Lambda \) by \((T_4)\) (ii).

Now define
\[
(1.16) \quad \ c = \inf_{h \in \Lambda} \sup_{u \in Q} f(h(u)).
\]
We will show \( c \) is a critical value of \( f \). As a preliminary result we require:

**Proposition 1.17:** If \( f \) satisfies \((T_4)\), then \( c = \alpha \).

**Proof:** It suffices to show that
\[
(1.18) \quad h_1(Q) \cap S \neq \emptyset
\]
for all \( h \in \Lambda \). For then if \( y \in h_1(Q) \cap S \),
\[
(1.19) \quad \sup_{u \in Q} f(h_1(u)) = f(y) = \inf_{w \in S} f(w) = \alpha
\]
via \((T_4)\) (i). Since \((1.19)\) holds for all \( h \in \Lambda \), \((1.18)\) and \((1.19)\) imply \( c = \alpha \).

The proof of \((1.18)\) follows from the stronger assertion that
\[
(1.20) \quad h_1(Q) \cap S \neq \emptyset
\]
for all \( h \in \Lambda \) and \( t \in [0, 1] \). Since \( S - v \subseteq E_1 \) by \((T_4)\) (i), \((1.20)\) is equivalent to finding, for each \( t \in [0, 1] \), \( u \in \mathcal{G} \) such that
\[
(1.21) \quad \left\{ \begin{array}{l}
\quad P_1 h_t(u) = S - v \\
\quad P_2 h_t(u) = v 
\end{array} \right.
\]
We will solve \((1.21)\) by converting it to an equivalent problem to which the linking hypotheses can be applied. Suppose first that \( h \in \Lambda \) with the corresponding \( m = 1 \). Letting \( u = u_1 + u_2 \in E_1 \oplus E_2 \) as usual, by \((T_1)\) and \((T_1)\), \((1.21)\) becomes
\[
(1.22) \quad \left\{ \begin{array}{l}
\quad P_1 h_t(u) = S - v \\
\quad P_2 h_t(u) = U_1(u_2) + P_2 K_1(u) = v 
\end{array} \right.
\]
More generally, suppose \((1.22)\) (ii) is replaced by
\[
(1.23) \quad P_2 h_t(u) = P_2 Z_t(u)
\]
where \( Z_t(u) \) is compact and \( Z_0(u) = v \). By \((T_1)\) and \((T_1)\) \((1.23)\) is equivalent to
\[
(1.24) \quad u_2 = U_1^{-1}(P_2(K_1(u) - Z_t(u))) = P_2 Y_t(u)
\]
where \( Y_t \) is compact and \( Y_0(u) = v \). Suppose we have shown by induction that whenever \( h \in \Lambda \) with \( m = n - 1 \), \((1.23)\) is equivalent to
\[
(1.25) \quad u_2 = P_2 Y_t(u)
\]
with \( Y_t \) compact and \( Y_0(u) = v \). Let \( h \in \Lambda \) with \( m = n \) so \( h = h^{(1)} \cdots h^{(n)} \). Let \( h = h^{(1)} \cdots h^{(n)} \). Hence
h = h^{(1)} + h and as above, the equation \( P_2 h_t(u) = v \) is equivalent to
\[
(1.26) \quad P_2 h_t(u) = (U^{(1)}_t)^{-1} (P_2 F^{(1)}_t (h_t(u)) + v)
\]
where \( h^{(1)} = U^{(1)}_t + K^{(1)} \) and the right hand side of (1.26) is compact
and equals \( v \) when \( t = 0 \). Hence by induction hypothesis there is a
compact \( Y \) such that (1.26) is equivalent to solving (1.25).

Now set \( \Phi_t(u) = P_1 h_t(u) + u_2 - P_2 Y_t(u) + v \). It is evident that
\( \Phi \in \Sigma \), \( P_1 \Phi_t = P_1 h_t \) and by our above remarks, \( P_2 \Phi_t = v \) is equivalent
to \( P_2 h_t = v \). Therefore \( \Phi_t(u) \in S \) if and only if \( h_t(u) \in S \) and
to complete the proof we need only show
\[
(1.27) \quad \phi_t(Q) \cap S \neq \emptyset
\]
for all \( t \in [0, 1] \). Since \( \Phi \in \Sigma \) and \( S \) and \( \partial Q \) link by (1.27)
(1.17) follows if
\[
(1.28) \quad \phi_t(\partial Q) \cap S = \emptyset.
\]
Suppose the contrary that there is a \( u \in \partial Q \) and \( t \in [0, 1] \) such
that \( \phi_t(u) \in S \). Then \( h_t(u) \in S \). But \( h_t(\partial Q) \subset G_{1/4,1/4} \) since
\( h \wedge \wedge \) while \( S \cap \bigcup_{\beta} S_{\frac{3}{2},\frac{1}{2}} = \emptyset \) by (1.27) (1.17), a
contradiction. Thus \( (1.28) \) is satisfied and the proof is
complete.

Now we can prove that \( c \) is a critical value of \( f \).

**Theorem 1.29:** Let \( E \) be a real Hilbert space, \( E_1 \) a closed subspace
of \( E \), \( E_2 = E_1^\perp \), and \( f : C^1(E, R) \) with \( f \) satisfying \( (f_1) \), \( (f_2) \),
and \( (f_3) \). If \( c \) is defined by (1.36) and
\( f'(u_m) \to 0 \) implies that \( (u_m) \) is bounded
also holds, then \( c \) is a critical value of \( f \).

**Proof:** Observe first that since \( h(u) = u \wedge \wedge \), \( c \) is finite via \( (f_2) \).
Moreover \( c > g \) by Proposition 1.17. If \( c \) is not a critical value
of \( f \), \( f'(u) \neq 0 \) for all \( u \in f^{-1}(c) \). We claim there is an \( \varepsilon > 0 \)
such that \( \|f'(u)\| > \sqrt{2}\varepsilon \) for all \( u \in f^{-1}([c - \varepsilon, c + \varepsilon]) \). If not,
there is a sequence of positive \( \varepsilon_n \to 0 \) and \( u_n \in f^{-1}([c - \varepsilon_n, c + \varepsilon_n]) \)
such that \( \|f'(u_n)\| < \sqrt{2}\varepsilon \). By \( (f_3) \), \( (u_n) \) is a bounded
sequence. Hence it possesses a weakly convergent subsequence having
limit \( u \). By \( (f_2) \), \( B(u_n) \to B(u) \) along this subsequence. Since
\( L u_n \) also converges weakly to \( L u \) along the subsequence and \( f'(u_n) \to 0 \),
it follows that \( L u_n = f'(u_n) - B(u_n) \to B(u) \). Hence \( L u_n \to L u \) and
\( f'(u) = 0 \). By \( (f_2) \) again,
\[
f(u) = \frac{1}{2} (L u_n, u_n) + b(u_n) - f(u) = c.
\]
But then \( c \) is a critical value of \( f \), contrary to assumption. Thus there
exists an \( \varepsilon \) as desired above. It can further be assumed that \( \varepsilon < 1/4 \).

Choose an \( h \in A \) with \( h \) corresponding \( \beta \) such that
\[
(c) \quad c = \sup_{u \in \partial Q} f(h_t(u)) \leq c + \varepsilon
\]
and \( h_t(\mathcal{Q}) \subset G, \frac{G+\alpha}{2} - \frac{\beta}{2} \). By (\( \Gamma_4 \)), \( h_t(\mathcal{Q}) \) is bounded. Therefore there is an \( R > 0 \) such that \( h_t(\mathcal{Q}) \subset B_R \). By Theorem 1.5 with \( \gamma = \frac{1}{2} \min(\beta, \epsilon) \), there exists an \( \eta \subset \Gamma \) and \( k \in \mathbb{N} \) such that \( h_t^k \) satisfies \( I^0 \) and \( Z^0 \) of that theorem. Let \( q_t(u) = h_t^k(h_t(u)) \). Since \( h_t(\mathcal{Q}) \subset G, \frac{G+\alpha}{2} - \frac{\beta}{2} \) \( B_R \), \( q_t(\mathcal{Q}) \subset G, \frac{G+\alpha}{2} - \frac{\beta}{2} \).

Hence \( q \subset \eta \) and by (1.3)
\[
(1.31) \quad c \leq \sup_{u \in \mathcal{Q}} f(q_t(u)).
\]

We know from (1.30) that \( f(h_t(u)) \leq c + \epsilon \) for all \( u \in \mathcal{Q} \). Thus if \( h_t(u) \in f^{-1}([c-\epsilon, c+\epsilon]) \), then \( q_t(u) \in \mathcal{Q} \) via \( Z^0 \) of

Theorem 1.5. On the other hand if \( h_t(u) \in G, c-\epsilon \), then \( q_t(u) \in \mathcal{Q} \) via \( I^0 \) of Theorem 1.5 and our choice of \( \gamma \).

Consequently
\[
(1.32) \quad \sup_{u \in \mathcal{Q}} f(q_t(u)) \leq c - \frac{\epsilon}{2}
\]

contrary to (1.31) and the theorem is proved.

**Proof of Theorem 1.4:** Immediate from Theorem 1.20 since (\( f_1 \)) implies (\( f_2 \)) with any choice of \( a > 0 \).

**Proof of Theorem 0.1:** Immediate from Theorem 1.4 with \( v = 0 \) in (\( \Gamma_4 \)) since (\( f_4 \)) and Lemma 1.1 imply (\( \Gamma_4 \)).

We conclude this section with the subquadratic analogue of Theorem 0.1.

**Theorem 1.33:** Let the hypotheses of Theorem 0.1 be satisfied with (\( f_4 \)) replaced by

\( (f_4) \) There are constants \( R > 0 \) and \( a > \omega \) and \( v \in \mathcal{Q} \) such that

(1) \( f \equiv \alpha \) on \( S = v + \mathcal{E}_1 \)

(2) \( f \equiv \omega \) on \( \partial \mathcal{Q} \).

Then \( f \) possesses a critical value \( c \geq \alpha \).

**Proof:** Immediate from Theorem 1.4 since (\( f_4 \)) and Lemma 1.1 imply (\( \Gamma_4 \)).

**Remark 1.34:** Theorem 1.33 can be generalized slightly by replacing \( \mathcal{Q} \) by \( \mathcal{Q} = e + \mathcal{Q} \) where \( e \in \mathcal{E}_1 \) and requiring that \( f \equiv \omega \) on \( \partial \mathcal{Q} \).

The proof of Lemma 1.1 then shows \( S \) and \( \partial \mathcal{Q} \) link so Theorem 1.4 is again applicable.
§2. An example.

In this section we give a simple model problem which illustrates the indefinite nature of functionals to which Theorem 0.1 can be applied.

Let \( \Omega \subset \mathbb{R}^3 \) be a bounded domain with a smooth boundary. To avoid distinguishing between cases later, assume \( n = 2 \). In what follows we denote \((v, w) \in \mathbb{R}^2 \) by \( u \) and \((v^2 + w^2)^{\frac{1}{2}} \) by \( |u| \). Consider the elliptic system of partial differential equations:

\[
\begin{aligned}
\Delta v &= |u|^{s-1}v && \quad x \in \Omega \\
\Delta w &= -|u|^{s-1}w && \quad x \in \Omega \\
v &= w = 0, \quad \Omega \end{aligned}
\]

(2.1)

where \( 1 < s < (n+2)(n-2)^{-1} \). Let \( W^{1,2}_0(\Omega) \) denote the space of functions having square integrable first derivatives in \( \Omega \) and vanishing in a generalized sense on \( \partial \Omega \). Let \( E = (W^{1,2}_0(\Omega))^2 \) with

\[
\|u\|_E^2 = \|u\|^2 = \int_\Omega |\nabla u|^2 \, dx.
\]

For \( u \in E \), set

\[
f(u) = \int_\Omega \left[ \frac{1}{2} (|\nabla u|^2 - |\nabla w|^2) - \frac{1}{s+1} |u|^{s+1} \right] \, dx.
\]

(2.2)

The indefinite nature of \( f \) is exceedingly clear.

We will show

Theorem 2.1: \( f \) as defined in (2.2) possesses a positive critical value in \( E \).

Remark 2.4: The corresponding critical point \( u \) will be a weak solution of (2.1) and standard regularity arguments then show \( u \in C^2(\Omega)^2 \).

These regularity considerations will not be discussed further here. Note that \( 0 \) is a critical value of \( f \) so \( c \neq 0 \), implies the corresponding critical point \( u \) is a nontrivial solution of (2.1). If \( |u|^{s+1} \) is replaced by \( |u|^{s+1} + q(x, u) \) in (2.2) where \( q \in C^2(\Omega \times \mathbb{R}^2, \mathbb{R}) \) and satisfies

\[
\begin{align*}
(q_1) & \quad q(x, u) = 0 \quad u \in \Omega \times \mathbb{R}^2 \\
(q_2) & \quad q(x, u) = 0 \quad |u|^2 \quad u = 0 \\
(q_3) & \quad |q(x, u)| \leq a_1 + a_2 |u|^\rho \quad \text{for all } u \in \mathbb{R}^2 \quad 1 \leq \rho \leq s,
\end{align*}
\]

the same theorem obtains with the critical point \( u \) a classical solution of (2.1). However we prefer to keep this example as simple as possible. In (q) and elsewhere in this paper, \( a_1 \) and \( M_j, j \in \mathbb{N} \) repeatedly denote nonnegative constants.

Proof of Theorem 2.1: Setting \( E_1 = W^{1,2}_0(\Omega) \times \{0\}, E_2 = \{0\} \times W^{1,2}_0(\Omega) \)

\[
\begin{align*}
u_1 &= (v, 0), \quad u_2 = (0, w), \quad L = \frac{1}{2} L_1 \, u_1, \quad i = 1, 2, \quad \text{and} \quad \\
b(u) &= \int_\Omega \frac{1}{s+1} |u|^{s+1} \, dx
\end{align*}
\]

for \( u \in E \), we see that \( E = E_1^\perp \) and \( f \) satisfies (f1). That (f2) is satisfied and \( f \in C^1 \) follows readily from the form of \( b \) and the fact that \( E \) is compactly embedded in \((L^1(\Omega))^2 \) for \( 1 \leq t \leq \frac{2n}{n-2} \).

Since a more complicated such result will be proved in §3, we omit further
details here. (See also [7]).

It remains to verify (f_2) and (f_4). Suppose \( f(u_m) \equiv M \) and \( f'(u_m) \to 0 \) as \( m \to \infty \). Then for large \( m \),

\[
|f'(u_m) v| \leq |v|
\]

for all \( v \in \mathcal{V} \). Choosing \( v = u_m \) gives

\[
M + \frac{1}{2} |u_m|^2 = f(u_m) - \frac{1}{2} f'(u_m) u_m = \int_\Omega \left( \frac{1}{2} - \frac{1}{s+1} \right) |u_m|^{s+1} \, dx.
\]

Next successively choosing \( v = (v_m, 0) \) and \( v = (0, w_m) \) in (2.5) and adding yields:

\[
|u_m|^2 \leq a_\gamma \left( \int_\Omega |u_m|^{s+1} \, dx + 2 |v_m| \right).
\]

Combining (2.6) and (2.7) gives

\[
|u_m|^2 \leq a_\gamma |u_m| + a_3
\]

from which the boundedness of \( (u_m) \) and \( (f_2) \) follow.

Next suppose \( u = (v, 0) \in \mathcal{I}_1 \). Since

\[
|z(u)| \leq a_\gamma |u|^{s+1}
\]

via the Sobolev embedding theorem,

\[
f(u) = \frac{1}{2} |u|^2 - a_\gamma |u|^{s+1} = \frac{1}{2} |u|^2 \left( 1 - 2a_\gamma |u|^{s-1} \right).
\]

Choosing \( \rho \) so that \( 1 + 4a_\gamma \rho^{s+1} \), we have

\[
f(u) \leq \frac{1}{2} \rho^2 \equiv a \frac{1}{2} \rho^2 \equiv a
\]

on \( S = \mathcal{B}_\rho \cap \mathcal{I}_1 \), i.e., \((f_4)(1)\) holds. Lastly to obtain \((f_4)(1)\),

consider \( f(u + re) \) where \( r > 0 \), \( e = (\tilde{v}, 0) \in \mathcal{B}_1 \) \( u = (0, w) \in \mathcal{B}_2 \cap \mathcal{I}_2 \), and \( f_2 \) is free for the moment. Thus

\[
f(u + re) = \frac{1}{2} |u + re|^2 - \frac{1}{s+1} \int_\Omega |u + re|^{s+1} \, dx
\]

so \( f \equiv 0 \) if \( r = 0 \). By the Hölder inequality,

\[
\frac{1}{s+1} \int_\Omega |u + re|^{s+1} \, dx \leq a_\gamma \left( \int_\Omega |u + re|^2 \, dx \right)^{\frac{s+1}{2}}
\]

\[
= a_\gamma \left( \int_\Omega |u|^2 + r^2 |e|^2 \, dx \right)^{\frac{s+1}{2}}.
\]

Hence by (2.12) - (2.13),

\[
f(u + re) \leq \frac{1}{2} |u|^2 - a_\gamma r^{s+1} \left( \int_\Omega |e|^2 \, dx \right)^{\frac{s+1}{2}}.
\]

Choose \( r_1 \) so that

\[
\psi(r) = \frac{1}{2} |u|^2 - a_\gamma r^{s+1} \left( \int_\Omega |e|^2 \, dx \right)^{\frac{s+1}{2}} \leq 0
\]

for \( r = r_1 \). If \( M = \max_{r \in [0, r_1]} \psi(r) \), choose \( r_2 \) so that \( M \leq \frac{r_2}{2} \).

It then follows from \( \psi(0, r_2) \) (2.12) - (2.14) that \( f \equiv 0 \equiv \mathcal{I}_1 \) on \( \partial \Omega \) and the proof of Theorem 2.3 is complete.
13. Applications to superquadratic Hamiltonian systems

This and the following section contain applications of the results of 11 to Hamiltonian systems of ordinary differential equations. We treat the superquadratic case now and the subquadratic case in 14. We will be concerned in particular with the existence of periodic solutions of such systems.

Let $H \in C^1(\mathbb{R}^{2n}, \mathbb{R})$ and consider the Hamiltonian system of ordinary differential equations:

$$\dot{p} = -H_q(p, q), \quad \dot{q} = H_p(p, q)$$

where $p$ and $q$ are $n$-tuples and $\cdot$ denotes $\frac{\partial}{\partial \cdot}$. Letting

$$z = (p, q) \quad \text{and} \quad \mathcal{J} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

where $I$ denotes the identity matrix in $\mathbb{R}^n$, (3.1) becomes

$$\dot{z} = \mathcal{J} H_z(z)$$

As our first result for (3.2) we apply Theorem 0.1 to give a simplified proof of Theorem 2.1 of [2]. Below $(\cdot, \cdot)$ denotes the usual inner product in $\mathbb{R}^n$.

**Theorem 3.1:** Suppose $H \in C^1(\mathbb{R}^{2n}, \mathbb{R})$ and satisfies

$$H(z) = 0 \quad \text{for} \quad z = 0,$$

$$H(z) = o(|z|^2) \quad \text{at} \quad z = 0,$$

$$H(z) \neq 0 \quad \text{at} \quad z \neq 0.$$

Then for any $T > 0$, (3.2) possesses a nonconstant $T$ periodic solution.

Before applying Theorem 0.1 to obtain Theorem 3.1, a few remarks and preliminaries are necessary. By making the change of variables

$$t = 2\pi t' \equiv \lambda^{-1} t,$$

(3.2) transforms to

$$\dot{x} = \lambda \mathcal{J} H_x(x)$$

and we seek a $2\pi$ periodic solution of (3.4) with $\lambda$ prescribed. Next, integrating $H_z$ shows

$$H(z) = a_1 |z|^2 - a_2,$$

for all $z \in \mathbb{R}^{2n}$. Hence $H$ is a bounded quadratic nonlinearity. The lack of an upper bound for the rate of which $H(z) \to -\infty$ as $|z| \to \infty$ creates technical difficulties which can be avoided by replacing (3.4) by a modified problem as follows. Let $k > 0$ and $x \in C^0(\mathbb{R}, \mathbb{R})$ such that

$$x(z) = 1 \quad \text{if} \quad z \in X, \quad x(z) = 0 \quad \text{if} \quad z \in X^0, \quad \text{and} \quad \dot{x}(z) < 0 \quad \text{if} \quad z \in (X, X^0).$$

Set

$$H_k(z) = x(|z|) H(z) + (1 - x(|z|)) \frac{1}{|z|^k},$$

where $k = \nu(X) > 0$. Then $H_k \in C^1(\mathbb{R}^{2n}, \mathbb{R})$ and it is easy to verify (see e.g., [2]) that for $k$ sufficiently large, $H_k$ satisfies

$$H_k - H_\nu$$

with $\nu$ replaced by $\nu = \max(0, \frac{1}{2}).$ Moreover $H_k(z)$ grows at a controlled superquadratic rate as $|z| \to \infty$.

Consider the new Hamiltonian system

$$\dot{x} = \lambda \mathcal{J} H_k(x).$$

If (3.6) has a nonconstant $2\pi$ periodic solution $x_k$ such that

$$\|x_k\|_\infty = k,$$

Then for any $T > 0$, (3.2) possesses a nonconstant $T$ periodic solution.
where $L^n$ refers to $(L^2(S^1))^{2n}$, then $x_k$ satisfies (3.4).

Ultimately $K$ will be chosen so as to satisfy (3.7).

The space in which we will treat (3, 8) is $E$, the space of 2n-tuples of $Z^n$ periodic functions which possess a derivative of order \( \frac{1}{2} \). Perhaps the simplest way to introduce this space is as follows. Let $z \in C^n(R, R^{2n})$ be $Z^n$ periodic. Then $z$ has a Fourier expansion $z = \sum a_k e^{ikt}$ with $a_k \in R^{2n}$ and $a_{-k} = \overline{a_k}$. $E$ is the closure of the set of such functions under the (Hilbert space) norm

$$||z||^2 = \sum_{k \in Z} (1 + |k|) |a_k|^2.$$

$E$ can also be identified with Sobolev space $(W^{1,2}(S^1))^{2n}$ obtained by interpolating between $(L^2(S^1))^{2n}$ and $(W^{1,2}(S^1))^{2n}$. For our later purposes, it is convenient to put another norm on $E$ equivalent to the above one. For $z \in E$, let $p$ and $q$ denote respectively the first and last n components of $E$. If $z$ is smooth, the action integral of $z$ is defined by

$$A(z) = \int_0^{2\pi} (p, q) p^n dt.$$

Writing $p = \sum_{k} p_k e^{ikt}$, $q = \sum_{k} q_k e^{ikt}$,

$$A(z) = 2\pi \sum_{k \in Z} k(p_k, q_k) a_k^2$$

so in fact $A$ extends to a continuous quadratic form on $E$. It is easy to verify -- see e.g. [2] -- that if $E^0$, $E^+$, $E^-$ are the (closed) subspaces of $E$ on which $A$ is null, positive definite, and negative definite,

$$E = E^0 \oplus E^+ \oplus E^-.$$  Moreover if $z = (p, q)$ and $\zeta = (\varphi, \psi)$ belong to distinct such subspaces,

$$\int_0^{2\pi} (p, \varphi) p^n + (q, \psi) q^n dt = 0$$

and

$$\int_0^{2\pi} (z, \zeta)_{2n} dt = 0.$$

Hence for $z = z^0 + z^+ + z^- \in E$, we can and will take as norm in $E$:

$$||z||^2_E = ||z^0||^2 + A(z^+) - A(z^-) = ||z||^2$$

and the associated inner product makes $E$ a Hilbert space with $E^0$, $E^+$, $E^-$ orthogonal subspaces of $E$. Finally we note that for any $\beta \in [1, \infty)$ and $z \in (L^\beta(S^1))^{2n}$,

$$||z||_{L^\beta} \leq C \beta ||z||$$

with the embedding of $E$ in $L^\beta$ being compact [3, 14].

Let

$$f(z) = A(z) - \lambda \int_0^{2\pi} H_k(z) dt.$$

The definition of $H_k$ and (3.8) imply that $f$ is defined for $z \in E$. Moreover $f$ is indefinite via our above discussion of $A(z)$. We will show that $f$ satisfies the hypotheses of Theorem 0.1 (with $\omega = 0$).

Let $E_1 = E^+$ and $E_2 = E^0 \oplus E^-$. Then $E_1 = E_2^\perp$ and $f$ satisfies (3.1) with $L_1 z$ being defined for $z \in E_1$ by

$$\langle f \rangle_{L_1} = A'(z) \zeta$$

\(\zeta\ of which $A$ is null, positive definite, and negative definite,
for all \( \zeta \in E \), \( i = 1, 2 \). (In (3.9), \( A'(z) \zeta \) denotes the Frechet derivative of \( A \) at \( z \) acting on \( \zeta \).) Hence
\[
\frac{1}{2} \langle L(x), z \rangle = A(z) \quad \text{and} \quad b(z) = -\lambda \int_0^{2\pi} H_X(z) \, dt.
\]
It remains to prove that \( f \in C^1(E, R) \) and satisfies \( (f_2) - (f_4) \). The verification of the smoothness of \( f \) follows similar lines as does \( (f_2) \) but is simpler. Hence we will prove the latter and leave the former to the reader.

**Lemma 3.10** If \( H \in C^1(R^{2n}, R) \), \( b \) is weakly continuous.

**Proof:** Since \( H \) is continuous, so is \( H_X \). Let \( (z_m) \subset E \) and suppose \( z_m \) converges weakly to \( z \). Then by (3.8) and the remark following it, \( z_m \rightarrow z \) in \( L^2 \) for all \( \beta \in [1, \infty) \). A theorem of Krasnoselski [5] implies that if \( g \in C([0, \pi]) \) and satisfies
\[
(g_1) \quad |g(t, z)| \leq a_3 + a_4(z) \sqrt{|t|},
\]
then
\[
\frac{1}{2} \int_0^{2\pi} g(t, z(t)) \, dt
\]
is continuous from \( (L^2([0, \pi]))^2 \) to \( L^2([0, \pi]) \) for \( 1 \leq \gamma, \delta < \infty \). The definition of \( H_X \) shows that it satisfies (3.11) with \( \gamma = 4 \) and \( \delta = 1 \). Hence choosing \( \beta = 4 \), we see \( z_m \rightarrow z \) in \( L^1 \) and therefore \( b(z_m) \rightarrow b(z) \).

Next we present a result which implies the uniform differentiability of \( b \) and is also useful in other contexts.

**Proposition 3.12** Let \( D \subset R^4 \) be a bounded domain and for some \( s, \hat{s} \), \( g \in C^1(D \times R^n, R) \) where \( g = g(x, z) \) and
\[
(g_1) \quad |g_z(x, z)| = a_1 + a_2|z|/\sqrt{s}, \quad 1 < s < \hat{s}.
\]
If \( \hat{z} \) is a subspace of \( (L^1(D))^k \) with
\[
(z, \hat{z}) \leq a, \|z\| \leq \hat{a},
\]
for all \( z \in \hat{z} \) and all \( r \in [2, \hat{r} + 1] \), then
\[
b(z) \leq \int_D g(x, z(x)) \, dx
\]
is uniformly differentiable on bounded subsets of \( \hat{z} \).

**Proof:** For notational convenience the \( x \) dependence of \( g \) will be suppressed in what follows. The verification that \( b(z), \ b'(z) \) are defined for \( z \in \hat{z} \) will be left to the reader.

Choose \( \hat{z} \) and \( R > 0 \) and suppose that \( \|z\| \leq R \) and suppose that \( \|z+h\| \leq R \). We must find a \( \delta = \delta(c, R) \) such that
\[
\|b(z+h) - b(z) - b'(z)h\| \leq \epsilon \|h\|_E
\]
whenever \( \|h\|_E \leq \delta \). Observe that
\[
|b(z+h) - b(z) - b'(z)h| = \int_D g(x+h) - g(x) - g_z(z)h \, dx = \int_D g(x+h) - g(x) - g_z(z)h \, dx.
\]
Let
\[
I = \{ x \in \hat{z} \mid |h(x)| = \beta \},
\]
\[
II = \{ x \in \hat{z} \mid |h(x)| = \gamma \},
\]
\[
III = \{ x \in \hat{z} \mid |h(x)| \leq \gamma \text{ and } |z(x)| \leq \beta \}.
\]
where $\beta$ and $\gamma$ are free for the moment. The right hand side of (3.15) is dominated by the sum of the integrals of $G$ over these three sets. These integrals will be estimated next. First

$$\int G \, dx = \int \left( |g(z+h) - g(z)| + |q(z) h| \right) \, dx.$$  

Hence by the mean value theorem, $(q_1)$, and the Hölder inequality,

$$\int G \, dx = a_2 \left( \int I \frac{1}{\sqrt{\sigma}} \, dx \right)^{\frac{1}{2}} \left( \int I \left| z \right| \, dx \right)^{\frac{1}{2}} \left( \int I \frac{\left| h \right| \left| r \right| \, dx \right)^{\frac{1}{2}} \right.$$  

$$+ \left( \int I \frac{\left| h \right| \, dx} {\left| r \right|} \right)^{\frac{1}{2}} \left( \int I \frac{\left| h \right| \left| r \right| \, dx \right)^{\frac{1}{2}} \right) \right.$$  

where $c \leq (1, *)$ is defined by

$$\frac{1}{c} = \frac{1}{s+1} = 1$$

and such a $c$ exists since $s < \hat{c}$. Hence by (3.13),

$$\int G \, dx = a_2 M^{1/\sigma} \left( \frac{R^s + \frac{1}{l} \|h\|_E^s} {\|r\|_E} \right) \|h\|_E$$

where $M$ denotes the measure of $I$. Since

$$R = \int I \left| z \right| \, dx = a_2 \left( \int I \left| z \right| \, dx \right)^{1/3} \geq a_3 \beta M^2,$$

we see

$$M^{1/\sigma} \leq \left( \frac{R}{a_3 \beta} \right)^{1/\sigma} \leq M_1$$

and (3.18) becomes

$$\int G \, dx \leq a_3 M_1 \left( R^s + \frac{1}{l} \|h\|_E^s \right) \|h\|_E.$$  

Next as in (3.16) - (3.17),

$$\int G \, dx \leq a_2 \left[ \left( \int I \left| z \right| \, dx \right)^{1/2} \left( \int I \left| h \right| \, dx \right)^{1/2} \right]$$

$$\gamma^{1-\sigma} \left( \int I \frac{\left| h \right| \, dx} {\left| r \right|} \right)^{\frac{1}{2}} \leq a_4 \left( \frac{R^s + \frac{1}{l} \|h\|_E^s} {\|r\|_E} \right) \|h\|_E \|r\|_E$$

where $\alpha = (s+1)(s+1)^{-1} > 1$ and $a_2$ depends on $\gamma$.

Finally to estimate the remaining term, observe that since $g \in C^1$ given any $\hat{\varepsilon}, \beta > 0$, there is a $\gamma = \gamma(\hat{\varepsilon}, \beta)$ such that $|z| = \beta$ and $|h| \leq \hat{\varepsilon}$ imply that

$$|g(z+h) - g(z) - q(z) h| \leq \hat{\varepsilon} |h|.$$  

Hence if $\beta \leq \beta$ and $\gamma \leq \hat{\varepsilon}$, (3.21) shows

$$\int G \, dx \leq \hat{\varepsilon} \int \left| h \right| \, dx \leq a_5 \hat{\varepsilon} \|h\|_E$$

via (3.13) and the Hölder inequality.

Now choose $\hat{\varepsilon}$ so that $a_5 \hat{\varepsilon} \leq \frac{1}{2}$ and choose $\beta = \beta$.

This determines $\gamma$ in terms of $\varepsilon$ and $\beta$. Combining (3.19), (3.20), and (3.22) yields

$$\int G \, dx \leq a_4 M_1 \left( R^s + \|h\|_E^s \right) \|h\|_E + a_2 \left( \frac{R^s + \frac{1}{l} \|h\|_E^s} {\|r\|_E} \|h\|_E \right)^{1-\sigma} \|h\|_E \|r\|_E$$

The proof is now completed by choosing $\beta$ so large that $a_4 M_1 \beta^s \leq \frac{1}{2}$, taking $\gamma = \hat{\varepsilon}$ and $\delta$ so small that $a_4 M_1 \delta^s + a_2 (\frac{R^s + \frac{1}{l} \|h\|_E^s} {\|r\|_E} \|h\|_E \right)^{1-\sigma} \|h\|_E \|r\|_E \leq \frac{1}{3}$.

**Corollary 3.24:** If $H \in C^1(R^n, R)$, $b$ is uniformly differentiable on bounded subsets of $E$. 

Proof: Immediate from Proposition 3.12, the definition of \( H_K \), and (3.8).

Lemma 3.25: If \( H \) satisfies (H), \( f \) satisfies (f).

Proof: As was noted earlier, if \( H \) satisfies (H), so does \((H_K)\) with \( \theta \) replaced by \( \mu = \max(0, \frac{1}{K}) \). Suppose \((z_m)\) is a sequence with \( f(z_m) \rightarrow M \) and \( f'(z_m) \rightarrow 0 \) as \( m \rightarrow \infty \). Then for large \( m \) with \( z = z_m \), we have

\[
M + \frac{1}{2} \| z \| = f(z) - \frac{1}{2} f'(z)z = \lambda \int_0^{2\pi} \left[ H_{K}(z, H_K(z)) \right] z \, dt.
\]

Applying (H) for \((H_K)\) gives

\[
M + \frac{1}{2} \| z \| = \lambda \int_0^{2\pi} \left[ H_{K}(z, H_K(z)) \right] z \, dt - M_1
\]

where the constant \( M_1 \) is independent of \( K \). The form of \( H_K \) then implies

\[
\| z \|_4^2 \leq M_2 \| z \|_2
\]

where \( M_2 \) and \( M_3 \) depend on \( K \) (as well as \( \lambda \) and \( \mu \)). Moreover by (3.27), (H), (3.5) for \( H_K \), and the Hölder inequality,

\[
\lambda \int_0^{2\pi} \left[ H_{K}(z, H_K(z)) \right] z \, dt = M_2 \int_0^{2\pi} \| z \|^{1/2} dt = M_3 \int_0^{2\pi} \| z \|^{1/2} dt^3/2
\]

Writing \( z = z^0 + z^1 + z^2 \), substituting in the right hand side of (3.29) and using the \( L^2 \) orthogonality of the components of \( z \) yields

\[
\| z^0 \| \leq M_2 \| z \|^{1/2}
\]

Moreover since

\[
(3.31) \quad |f'(z_m)\| \leq M_2 \| z \|^{1/2}
\]

for large \( m \) and all \( z \in E \), choosing \( \zeta = z_m^* \) gives

\[
(3.32) \quad 2 \| z^0 \| \leq A^0(\zeta) A^0(\zeta) \leq \epsilon \int_0^{2\pi} |H_{K}(z, H_K(z))| |z^0| \, dt + \| z^0 \| \leq M_2^2 \| z \|^{3/2} + \| z^0 \|
\]

Hence by (3.32), (3.8), and (3.28),

\[
(3.33) \quad \| z^0 \| \leq M_2(1 + \| z \|^{3/2})
\]

Taking \( \zeta = -z_m^* \) in (3.31) yields a similar inequality for \( z_m^* \).

Adding the two to (3.30) shows

\[
(3.34) \quad \| z \| \leq M_2(1 + \| z \|^{3/2} + \| z \|^{3/4})
\]

from which the boundedness of \((z_m)\) follows.

Next we will verify (f).

Lemma 3.35: If \( H \) satisfies (H), \( f \) satisfies (f) (i).

Proof: By (H), for any \( \epsilon > 0 \), there is a \( \delta > 0 \) such that

\[
|H(x)| = |H_K(x)| \leq \epsilon |x|^2
\]

if \( |x| \leq \delta < K \). The form of \( H_K \) then implies there is a constant \( A_2 > 0 \) so that

\[
(3.36) \quad \| H_K(x) \| \leq \epsilon |x|^2 + A_2 \| x \|^{1/2}
\]

for all \( x \in K \). Consequently for \( x \in E \), (3.36) and (3.8) show that

\[
(3.37) \quad \int_0^{2\pi} |H_K(x)| \, dt = \epsilon |z|^2 + A_2 \| z \|^{1/2} \leq M_2 \| z \|^{1/2} + A_2 \| z \|^{1/2}
\]

\[
= M_2 \| z \|^{3/4} + \| z \|^{1/4}
\]

\[
= A_2 \| z \|^{1/4} + \| z \|^{1/4}
\]

\[
= M_2 \| z \|^{3/4} + A_2 \| z \|^{1/4}
\]

\[
= M_2 \| z \|^{3/4} + A_2 \| z \|^{1/4}
\]
Thus for \( z \in E_1 \),
\[
(3.38) \quad f(z) = \|z\|^2 - \lambda a_3 \left( \|z\|^2 + A_3 \|z\|^4 \right) = \|z\|^2 \left( 1 - \lambda a_3 - \lambda a_3 A_3 \|z\|^2 \right).
\]
Choosing \( \epsilon = (3\lambda a_3)^{-1} \) and \( \rho \), so that \( \lambda a_3 A_3 \rho^2 \leq 3^{-1} \) gives
\[
(3.39) \quad f(z) = \frac{1}{3} \rho^2 = \alpha
\]
on \( S = \partial_B \cap \Sigma_1 \).

Remark 3.40: Note that \( \rho \) and \( \alpha \) depend on \( K \) via \( A_3 \).

Lemma 3.41: If \( H \) satisfies (\( H_1 \)) and (\( H_2 \)), there exists a set \( Q \) with \( r_1 \) and \( r_2 \) independent of \( K \) such that \( f \) satisfies (\( f_3 \)) (ii) (with \( \omega = 0 \)).

Proof: Let \( z = z^0 + z^\ast \in B_{r_2} \cap \Sigma_2 \) where \( r_2 \) is free for now and consider
\[
(3.42) \quad f(z + re) = r^2 - \|z^\ast\|^2 - \lambda \int_0^{2\pi} H_K(z + re) \, dt.
\]
Note that for \( r = 0 \), \( f(z) \leq \alpha \) via (\( H_1 \)). By (3.5) for (\( H_K \)) and the Hölder inequality,
\[
(3.43) \quad \int_0^{2\pi} H_K(z + re) \, dt = a_3 \left( \int_0^{2\pi} |z + re|^2 \, dt \right) - a_4 \geq 0
\]
for all \( \epsilon \) \in \( E \). For smooth \( \xi \), (3.47) implies that
\[
(3.48) \quad \int_0^{2\pi} \left[ \left( \frac{1}{2} \rho \left( \int_0^{2\pi} |e^2| \, dt \right)^{\frac{1}{2}} \right)^2 + \left( \int_0^{2\pi} |e^2| \, dt \right) \right] \, dt = \lambda \int_0^{2\pi} H_K(z) \, dt.
\]
For \( w \in (L^2(\mathbb{R}))^j \), let
\[
(3.44) \quad f(z + re) = r^2 - \|z^\ast\|^2 - a_6 r^p + \alpha \gamma.
\]
Choose \( r_1 \) so that
\[
(3.45) \quad \varphi(r) = r^2 - a_6 r^p + \alpha \gamma \leq 0
\]
for \( r = r_1 \). Let \( M = \max_{r \in [0, r_1]} \varphi(r) \). Since
\[
\int_0^{2\pi} \left( \|z\|^2 + a_6 \|z^0\|^{2p} \right) \, dt = a_8 \min \left( \|z\|^2, \|z\|^{2p} \right)
\]
by choosing \( r_2 \) large enough we have
\[
(3.46) \quad M \leq \|z\|^2 + a_6 \|z^0\|^{2p}
\]
for \( \|z\| \geq r_2 \). Then follows from (3.44) - (3.46) and (\( H_1 \)) that \( f \leq 0 \) \( \omega \) on \( \partial Q \). Note also that \( r_1 \) and \( r_2 \) are independent of \( K \).

Proof of Theorem 3.3: The above lemmas show \( f \) satisfies the hypotheses of Theorem 0.1. Therefore \( f \) has a positive critical value \( c \) and a corresponding critical point \( z \). It remains to show that \( z = z(K) \) is a nonconstant solution of (3.6) and that \( \|z\| = K \) for appropriately chosen \( K \). Proof of the first statement requires a simple regularity argument.
\[ [w] = \frac{1}{2\pi} \int_0^{2\pi} w(t) \, dt. \]

i.e. \([w]\) is the mean value of \(w\). By (3.5) and the form of \(H_k\),

\(H_k(z) \in L^2(\mathbb{S}^1)^{2n}\) and (3.48) with \(\zeta = 1\) shows that \([H_k(z)] = 0\).

Observe also that if \(v \in L^2(\mathbb{S}^1)\) and \([v] = 0\), there exists an \(x \in W^{1,2}(\mathbb{S}^1)\) such that \(x = y\) and \(x\) is uniquely determined if its mean value is specified. Thus we can find a unique \(\hat{\mathbf{z}} = (\hat{\phi}, \hat{\psi}) \in (W^{1,2}(\mathbb{S}^1))^{2n}\)

such that \([\hat{\mathbf{z}}] = [\mathbf{z}]\) and

\((3.49)\)

\[ \hat{\mathbf{z}} = \lambda \mathcal{F} H_k(z). \]

Taking the inner product of (3.49) with \(\mathcal{F} \zeta\) where \(\zeta \in \mathcal{E}\) is smooth yields

\((3.50)\)

\[ \int_0^{2\pi} [\mathbf{z}, \mathbf{v}]_{\mathbb{R}^{2n}} \, \mathbf{v} \mathbf{d}t = \int_0^{2\pi} \left( H_k(z), \zeta \right)_{\mathbb{R}^{2n}} \, \mathbf{d}t. \]

Comparing (3.50) to (3.48) and noting that \([\mathbf{z}] = [\hat{\mathbf{z}}]\) and smooth \(\zeta\) are dense in \(\mathcal{E}\) shows that \(z = \hat{\mathbf{z}}\) a.e. and \(z \in (W^{1,2}(\mathbb{S}^1))^{2n}\).

Hence by (3.48), (3.6) is satisfied in the a.e. sense. But since \(z \in (W^{1,2}(\mathbb{S}^1))^{2n}\), the Sobolev embedding theorem implies \(z\) and \(H_k(z)\) are continuous. Thus (3.6) implies \(z \in C^1\) and we have a classical solution of the equation. Moreover \(z\) is nonconstant since \(f(z) = c \neq 0\) by Theorem 0.1 while if \(z\) were constant,

\[ f(z) = -\lambda \int_0^{2\pi} H_k(z) \, dt \neq 0 \]

via \((H_j)\).

Lastly to obtain the \(L^\infty\) upper bound for \(z\), recall that by (1.16),

\[ c = \inf_{h \in \mathcal{A}} \max_{z \in \mathcal{C}} f(h(z)). \]

Since \(h(t, x) = x \in \mathcal{A}\),

\[ c = \max_{z \in \mathcal{C}} f(z). \]

Writing \(z = z^0 + z^+ + r e\) for \(z \in \mathcal{C}\) where \(r \in [0, r_1]\) and \(\|z^0 + z^+\|^2 \leq r_1^2\), we have

\[ f(z) = r^2 - \|z^+\|^2 - \lambda \int_0^{2\pi} H_k(z) \, dt \leq r_1^2. \]

Hence \(c \leq r_1^2\) and by Lemma 3.41, this estimate is independent of \(K\).

Now (3.26) - (3.27) with \(M\) replaced by \(c\) and the \(\|z\|\) term omitted because of (3.47) give

\((3.51)\)

\[ r_1^2 + M_1 \leq c + M_1 = \lambda (1 - \mu) \int_0^{2\pi} (z, H_k(z))_{\mathbb{R}^{2n}} \, dt \]

where \(M_1\) is independent of \(K\). By \((H_j)\),

\((3.52)\)

\[ H_k(z) \equiv \mu (z, H_k(z))_{\mathbb{R}^{2n}} + M_2 \]

for all \(z \in \mathbb{R}^{2n}\). Letting \(\zeta = z(t)\), integrating (3.52) over \([0, 2\pi]\) and using the fact that \(H_k(z) \equiv constant\) since (3.6) is a Hamiltonian system yields

\((3.53)\)

\[ 2\pi \int_0^{2\pi} H_k(z) \, dt \equiv \mu \int_0^{2\pi} (z, H_k(z))_{\mathbb{R}^{2n}} + 2\pi M_2 \]

Now combining (3.51), (3.53), and (3.5) for \(H_k\) gives the desired \(L^\infty\) estimate for \(z\) and the proof of Theorem 3.3 is complete.
Remark 3.53': If $H$ depends explicitly on $t$ in a $T$ periodic fashion and satisfies $(H_1)-(H_3)$, Theorem 0.1 can again be applied to get a $T$ periodic solution of (3.6). A further growth condition on $H_2$ such as was used in [2] is then required to go from (3.6) to (3.4).

Corollary 3.54: Suppose $H(x) = Q(x) + \hat{H}(x) \in C^1(M^{2n},\mathbb{R})$ where $Q$ is a quadratic form and $\hat{H}$ satisfies $(H_1)-(H_3)$. Then the conclusion of Theorem 3.3 obtains.

Proof: For $x \in E$, let

$$\hat{A}(x) = A(x) - \int_0^{2\pi} Q(x) dt,$$

where $A$, $\hat{A}$ are as earlier. If $L, \hat{L}$ are respectively the (bounded) self-adjoint linear operators associated with $A$ and $\hat{A}$, i.e.

$$(Lz, z) = A(z), \quad (\hat{L}z, z) = \hat{A}(z),$$

it is easily verified that $\hat{L} = L + C$ where $C$ is compact. Therefore the essential spectrum of $\hat{L}$ is the same as that of $L$, namely $[1, -1]$. Hence $\ker \hat{L}$ is finite (and possibly zero) dimensional.

Moreover $E$ has a splitting into orthogonal subspaces $E = E^+ \oplus E^- \oplus E^0$ with

$$\hat{A}(x^+) \leq x^+ I_{E^+}, \quad \hat{A}(x^-) \leq x^- I_{E^-}$$

for $x^+ \in E^+, x^- \in E^-$, where $x^+, x^-$ are respectively the smallest positive and largest negative eigenvalue of $L$ and $E^0 = \ker \hat{L}$.

The proof of Theorem 3.3 now carries over to this situation with only minor changes.

Remark 3.54': A result similar to Corollary 3.54 with $Q$ positive definite was obtained in [2]. A somewhat deeper result in which $H$ is only required to satisfy $(H_3)$ can be found in [3].

For our final two results for (3.2) in this section we prescribe the energy of the solution rather than the period and study the relationship between superquadratic growth at $\pm \infty$ and at 0 and the resulting period.

Theorem 3.55: Suppose $H \in C^1(M^{2n},\mathbb{R})$ and satisfies $(H_3)$. Then for all large $a > 0$, (3.2) has a $T = T(a)$ periodic solution on $H^{-1}(a)$ and $T(a) \to 0$ as $a \to \infty$.

Proof: Suppose $H^{-1}(a)$ is exterior to $B_r$ in $M^{2n}$ for all $a \geq a_0$. Fix such an $a$. Define a new function $\bar{H}(z)$ as follows: Let $\psi : S^{2n-1} \to H^{-1}(a)$ be the radial map and $\tau = 2$ be fixed for now. Let $\bar{H}(0) = 0$ and for $z \neq 0$,

$$\bar{H}(z) = a \frac{|z|^2}{|\psi(z)/|z||}.$$

Since $\psi \in C^1$, $H \in C^1(M^{2n},\mathbb{R})$ and is homogeneous of order $\tau$.

Thus $\bar{H}$ satisfies the hypotheses of Theorem 3.3. By that theorem with $T = 2\tau$, (3.2) with $H$ replaced by $\bar{H}$ possesses a nonconstant $2\tau$ periodic solution $\omega(t)$. Since (3.2) is a Hamiltonian system,

$$\bar{H}(\omega(t)) \equiv \text{constant} \equiv K.$$ 

By the homogeneity of $\bar{H}$, $\bar{H}(\beta \omega) = a$ where $\beta = (K^{-1}a)^{1/\tau}$. Then $\zeta = \beta \omega$ satisfies
\( \zeta = \beta \mathcal{P}_z(w) = \beta^{2-\tau} \mathcal{P}_z(\xi) \).

Thus \( \zeta \) is a \( 2\pi \) periodic solution of (3.4) with \( \lambda = \beta^{2-\tau} \) and \( H \) replaced by \( \mathcal{H} \), and \( \zeta \) lies on \( H^{-1}(a) \). By reparametrizing \( \zeta \), we obtain a periodic solution of (3.2) on \( H^{-1}(a) \). To see this, observe that since \( H^{-1}(a) \) is a common level set for \( H \) and \( \mathcal{H} \),

\[
\mathcal{H}_z(\xi) = \sigma(\xi) H_z(\xi)
\]

for all \( \xi \in H^{-1}(a) \) and \( \sigma(\xi) \neq 0 \) on this set. Consider \( z(t) = \zeta(\delta(t)) \). Then \( z \) satisfies

\[
z = \beta^{2-\tau} \sigma(z) \mathcal{H}_z(z) \delta.
\]

Choosing \( \delta \in C^1 \) so that \( \delta(0) = 0 \) and

\[
\delta = \beta^{1-2} \sigma(z) (\delta(t))^{-1}
\]

\( z \) is a solution of (3.2) on \( H^{-1}(a) \). Since the right hand side of (3.59) is positive, \( \delta \) is monotone increasing and there is a \( T = T(a) \) such that \( \delta(T) = 2\pi \). Hence \( z \) is \( 2\pi \) periodic in \( t \).

It remains to estimate \( T \). By (3.57) and (H3),

\[
\sigma(\xi) = \frac{\tau a}{(\xi, H_z(\xi))^{2n-2}} \leq \theta \tau
\]

for \( \xi \in H^{-1}(a) \). Integrating (3.59) then gives

\[
2\pi = \int_0^T \delta \, dt = T(\theta \tau)^{-1} \beta^{2-2} = T(\theta \tau)^{-1} \left( \frac{T \tau}{\beta^{2-2}} \right).
\]

To get a bound for \( K \), recall \( w \) is a critical point of \( f \) so

\[
c = f(w) - \frac{1}{2} f'(w)w = \int_0^{2\pi} \left[ \frac{1}{2} f(w, H_z(w)) - \mathcal{H}(w) \right] \, dt
\]

\[
= (\frac{T}{\tau} - 1) \int_0^{2\pi} \mathcal{H}(w) \, dt = 2\pi \left( \frac{T}{\tau} - 1 \right) K \equiv a_3 K
\]

Thus by (3.61) - (3.62),

\[
2\pi \tau \frac{T}{\tau} - \frac{T}{\tau} \mathcal{H}\left( \frac{T}{\tau} \right) \quad \tau \frac{T}{\tau} - 1 = a_3 \left( \frac{T}{\tau} \right)^{-2}.
\]

To finish, we need an upper bound for \( c \). For \( z \in \mathbb{R}^{2n} \setminus \{0\} \),

\[
\mathcal{H}(z) = |z|^T \mathcal{H}(\frac{z}{|z|}) = |z|^T \min_{\xi \in g_{2n-1}} \mathcal{H}(\xi).
\]

Since \( \Phi(\xi) = \rho(\xi) \) where \( \rho(\xi) > 0 \), \( a = \mathcal{H}(\rho(\xi) \xi) = \rho(\xi)^T \mathcal{H}(\xi) \) so by (3.64),

\[
\mathcal{H}(z) = \frac{a|z|^T}{\max_{g_{2n-1}} \rho(\xi)^T}.
\]

For \( |z| \leq \tau \), \( \mathcal{H}(z) = a_3 |z|^{1/\theta} \) via (H3). In particular,

\[
a = \mathcal{H}(\rho(\xi) \xi) = a_3 \rho(\xi)^{1/\theta}.
\]

Hence \( \rho(\xi) \leq \left( \frac{a_3}{a_5} \right) \) and

\[
\mathcal{H}(z) = a \left( \frac{a_3}{a_5} \right)^{-\theta} |z|^T = a_1^{-\theta} a_6 \rho(\xi)^T.
\]

Choose \( \tau \in (2, \theta^{-1}) \) so \( \theta \tau < 1 \) and

\[
\mathcal{H}(z) = a_1^{-\theta} a_6 \xi |z|^T = a_7 |z|^T
\]

for \( a = a_0 \) with \( a_7 \) independent of \( a = a_0 \). Thus \( \mathcal{H} \) satisfies (3.5) with constants independent of \( a = a_0 \). The proof of
Lemma 3.41 shows that $r_1$ depends only on $a_0$ for $a = a_0$.

Thus

$$c = \sup_{\Omega} f(z) \leq r_1^2$$

as in the proof of Theorem 3.3. Consequently

$$r_1 = a_0 \left( \frac{r_1}{a_0} \right)^{1 - \frac{1}{2}} \rightarrow 0$$

as $a \rightarrow a_0$ and the proof is complete.

Remark 3.71: Theorem 3.55 can also be proved using Theorem 1.1 of [12] and the estimates contained there. However that theorem uses deeper symmetry properties of (3.2) than the structure we have exploited here.

The arguments used in proving Theorem 3.55 apply equally well to other super and subquadratic problems at 0 and $+\infty$. E.g.

Theorem 3.72: Suppose $H \in C^1(\mathbb{R}^{2n}, \mathbb{R})$ and satisfies

($H'$) There is a $\Omega \subset (0, \frac{1}{2})$ and $L > 0$ such that

$$0 < \tilde{H}(x, H_0(x)) \in \mathbb{R}^{2n} \leq H(x) \text{ for } |x| \leq L.$$  

Then for all small $a > 0$, (3.2) has a $T = T(a)$ periodic solution on $H^{-1}(a)$ and $T(a) \rightarrow \infty$ as $a \rightarrow 0$.

Proof: By ($H'$), there is an $a_0 > 0$ such that $H^{-1}(a)$ is a compact $C^1$ manifold which bounds a star shaped neighborhood of 0 and lies in $\mathbb{R}_{L}$ in $\mathbb{R}^{2n}$ for $0 < a \leq a_0$. Define $\Psi$ as in Theorem 3.55 and argue in exactly the same fashion up to (3.60) which is replaced by the reverse inequality via ($H'_1$). Reversing (3.61) and applying (3.62) yields

$$T = a_0 \left( \frac{a}{a_0} \right)^{1 - \frac{1}{2}} \rightarrow 0$$

Now we need a lower bound for $c$. By ($H'_3$) and (3.64) - (3.66)

$$\Pi(z) = a^{1 - \Theta_T} a_0 |z|^T a^{1 - \Theta_T} a_0 |z|^T \leq a_0 a_0 |z|^T$$

for $a = a_0$. Using this in (3.64), the proof of Lemma 3.35 gives a independent of $a$ for $a = a_0$. Hence

$$T = a_0 \left( \frac{a}{a_0} \right)^{1 - \frac{1}{2}} \rightarrow 0$$

as $a \rightarrow 0$.

Remark 3.76: We have not shown that $T(a)$ is a minimal or primitive period for the corresponding solution $z_a$. Thus there is a $k = k(a) \in \mathbb{N}$ such that $Tk^{-1}$ is the minimal period of $z_a$ and it is conceivable that $Tk^{-1} \neq 0$ as $a \rightarrow 0$. 
4. Applications to subquadratic Hamiltonian systems

This section contains several applications of Theorem 1.33 to forced and autonomous Hamiltonian systems.

Theorem 4.1: Suppose $H \in C^1(\mathbb{R} \times \mathbb{R}^{2n}, \mathbb{R})$ and satisfies

\begin{enumerate}[(H_1)]
    
    
    \item\textbf{$(H_1)$} $H(t, z)$ is $T$ periodic in $t$
    
    \item\textbf{$(H_2)$} There is a constant $M > 0$ such that $|H_z(t, z)| \leq M$ for all $t \in [0, T]$ and $z \in \mathbb{R}^{2n}$
    
    \item\textbf{$(H_3)$} $H(t, z) \to -\infty$ as $|z| \to \infty$ uniformly for $t \in [0, T]$.
\end{enumerate}

Then

\begin{equation}
(4.2) \quad \dot{z} = JH_z(t, z)
\end{equation}

possesses a $T$ periodic solution.

Proof: For convenience we take $T = 2\pi$. Let $E$ be as in Theorem 3.3, $E_1 = E^0 \oplus E^*$, $E_2 = E^*$, and

\begin{equation}
q(z) = \int_0^{2\pi} H(t, z)\, dt = A(z).
\end{equation}

Since $(H_3)$ implies that $|H(t, z)| \leq M_1 + M_2|z|$ for $z \in \mathbb{R}^{2n}$, it follows as in Theorem 3.3 that $q \in C^1(E, R)$ and satisfies $(f_1) - (f_2)$ with $L_1, L_2$ and $b$ as earlier aside from a factor of $(-1)$. To check $(f_1)$, suppose $q(z_m) \leq H$ and $q'(z_m) = 0$ as $m \to -\infty$. Then dropping subscripts and writing $z = z^0 + z^k + z^\pi$, we have

\begin{equation}
(4.3) \quad \|z^\pi\|^2 \leq \int_0^{2\pi} H_z(t, z)z^\pi\, dt + \|z^\pi\|^2 \leq M_2\|z^\pi\|^2
\end{equation}

as in Lemma 3.25 via $(H_3)$ and $(3.8)$. Hence $(z^-_m)$ and similarly $(z^+_m)$ are uniformly bounded. In order to get an estimate for $(z^0_m)$, note by the bounds already established,

\begin{equation}
(4.4) \quad \bar{M} = g(z) = \int_0^{2\pi} H(t, z)\, dt + \|z^-\|^2 - \|z^+\|^2
\end{equation}

\begin{equation}
\leq \int_0^{2\pi} H(t, z^0)\, dt + \int_0^{2\pi} (H(t, z) - H(t, z^0))\, dt \leq M_2
\end{equation}

\begin{equation}
\leq \int_0^{2\pi} H(t, z^0)\, dt - M\int_0^{2\pi} |z^+ + z^-|\, dt \leq M_3.
\end{equation}

Therefore $(4.4)$, $(3.8)$, and $(H_3)$ imply $(z^0_m)$ are uniformly bounded and $(f_1)$ is satisfied.

Lastly we verify $(f_3)$. If $z = z^0 + z^- \in E_1$,

\begin{equation}
q(z) = \int_0^{2\pi} H(t, z)\, dt + \|z^-\|^2 - M_2\|z^-\|^2
\end{equation}

as in $(4.4)$. Hence $q$ is bounded from below on $S = E_2$ by some constant $\sigma$. Similarly for $z \in E_2$,

\begin{equation}
q(z) \leq M_3\|z\|^2 - \|z\|^2
\end{equation}

so if $z \in Q = \delta_{E_2} \cap E_2$ with $\delta$ sufficiently large, $q(z) \leq \sigma - 1 \equiv \omega$. Hence Theorem 1.33 (with $v = 0$ in $(f_3)$) implies $q$ has a critical value $c \approx \omega$. As in the proof of Theorem 3.3, any corresponding critical point is a classical solution of $(4.2)$.

Corollary 4.5: If $(H_3)$ is replaced by $H(t, z) \to -\infty$ as $|z| \to \infty$ uniformly in $t$, then the conclusion of Theorem 4.1 obtains.

Proof: Let $f(z) = -q(z)$, $E_1 = E^0 \oplus E^*$, $E_2 = E^*$, and argue as above.
Remark 4.6 (1): Theorem 4.1 and Corollary 4.5 are the analogues for Hamiltonian systems of results obtained earlier for elliptic equations by Ahmad, Lazer, and Paul [11]. See also [10], [12], [13].

(11) If H is independent of t, Theorem 4.1 and Corollary 4.5 are valid with any choice of T. However (H_5) implies that H_2(x) has a zero in B_M in \( \mathbb{R}^{2n} \). Thus for these results to be of interest in the autonomous case, one must further show the solution is nonconstant. An example of such a result will be given next.

Theorem 4.7: Suppose \( H \in C^1(\mathbb{R}^{2n}, \mathbb{R}) \), satisfies \((H_3)-(H_6)\) and 
\( (H_7) \quad H(0) = 0 \) and \( |H_2(z)|, H(z) > 0 \) for \( z \neq 0 \).

Then for all \( T \) sufficiently large, \((3.2)\) possesses a nonconstant \( T \) periodic solution.

Proof: Let 
\[ g(z) = \lambda \int_0^{2\pi} H(z) \, dt - A(z) . \]

It follows from \((H_3)-(H_6)\) and Theorem 4.1 that \( g \) has a critical value \( \varepsilon \) and \((3.2)\) a corresponding solution for any \( \lambda = (2\pi)^{-1} T \). By 
\( (H_7) \) and \((H_7) \), \( H_2(0) = 0 \) and \( 0 \) is the only constant solution of \((3.2)\). To complete the proof it suffices to show that \( c > 0 \) if \( \lambda \) is sufficiently large. By Theorem 1.33 and \((H_7)\), 
\[ c = \rho = \inf \{ g(z) : \rho \}
\]
since \( S = E_1 + E_2 = E' \). Thus a better estimate than \((4.8)\) is necessary. Let \( v \in Q, \nu \neq 0 \). By Lemma 1.1, \( v + S \equiv S_w \) and

\( SQ \) link. If we can show that 
\[ (4.9) \quad 0 < \rho = \inf \{ g(z) : v \}
\]
then by Proposition 1.17 with \( S_w \) replacing \( S, \nu \) and \( v \) we are through. To verify \((4.9)\), note that \( z \in S_w, z = z_0 + z' + v \).

Three cases are considered:

Case (1): \( z'^2 > v + 1 \)

Then 
\[ g(z) = \lambda \int_0^{2\pi} H(z) \, dt + z'^2 - v^2 \leq 0 . \]

Case (II): \( z'^2 < v + 1 \) and \( |z_0| > a \)

Then by \((H_7)\),
\[ (4.10) \quad \int_0^{2\pi} H(z_0 + z' + v) \, dt = 2\pi H(z_0) = -M \| z'^2 - v \|
\]
as in \((4.4)\). For a sufficiently large, which we make, 
\[ \int_0^{2\pi} H(z_0 + z' + v) \, dt \geq 1 \]
by \((H_7)\) and \((4.10)\). Hence 
\[ g(z) = \lambda - v^2 \geq 0 \]
if \( \lambda > \| v \|^2 + 1 \).

Case (III): \( z'^2 < v + 1 \) and \( |z_0| \leq a \)

Let \( \Omega = \{ z \in S_w, z'^2 < v + 1 \} \) and \( |z_0| \leq a \). Then \( \Omega \) is convex and weakly compact. Since \( \int_0^{2\pi} H(z) \, dt \) is weakly continuous,
it achieves its infimum, \( c \), on \( \Omega \) at \( \hat{z} = \hat{z}_0 + \hat{z}^* + v \) and
\( \hat{z} \neq 0 \) because \( v \neq 0 \). Moreover \( c > 0 \) by \((H_3)\). Therefore
\[
g(z) = \lambda \sigma + \|z^*\|^2 - \|v\|^2 = 1
\]
for \( \lambda = \sigma^{-1}(\|v\|^2 + 1) \).

Combining the three cases, we get \( \sigma = 1 \) and the proof is complete.

Our next result is of the same nature as Theorem 4.1. In a sense we weakens \((H_5)\) but strengthen \((H_6)\).

**Theorem 4.11**: Suppose \( H \in C^1(\mathbb{R}^{2n}, \mathbb{R}) \), satisfies \((H_5)\), and \((H_6)\).

There is an \( T > 0 \) and \( \theta \in (\frac{1}{2}, 1) \) such that
\[
H(t, z) = \theta(z, H_z(t, z))_{\mathbb{R}^{2n}} > 0 \quad \text{for} \quad |z| = \pi \quad \text{and} \quad t \in [0, T]
\]
and
\[
(\pi)
\]
are constants \( a_0 > 0, \ a_2 > 0 \), and \( s \in (1, 2^{1/2}) \) such that
\[
H(t, z) = a_2 |z|^s + a_2 \quad \text{for} \quad z \in \mathbb{R}^{2n} \quad \text{and} \quad t \in [0, T].
\]

Then \((4.2)\) possesses a \( T \)-periodic solution.

**Proof**: Again we will obtain the result from Theorem 1.35. For convenience set \( T = 2\pi \). Integrating \((H_3)\) shows that
\[
H(t, z) = a_2 |z|^s + a_2
\]
for \( z \in \mathbb{R}^{2n} \). Thus with \( E, E_1, E_2 \) and \( g \) as in Theorem 4.1, we have \( g \in C^1(\mathbb{R}, \mathbb{R}) \) and \( g \) satisfies \((f_1) - (f_3)\). To verify \((f_4)\), suppose \( g(z_m) = M \) and \( g'(z_m) = 0 \). Dropping subscripts as usual, for large \( m \):

\[
(4.13) \quad M + \frac{1}{2} \int |z| \leq g(z) - \frac{1}{2} g'(z)z = \int \left[H(t, z) - \frac{1}{2} g'(z, H_z(t, z))_{\mathbb{R}^{2n}}\right] dt
\]

\[
= (1 - \frac{1}{2\theta}) \int H(t, z) dt - M_1 = \frac{1}{2} \int |z|^s dt - M_2.
\]

via \((H_5)\) and \((H_{10})\). Define \( \sigma \) by \( s^{-1} + \sigma^{-1} = 1 \). Since \( E \subset (L^\infty(\mathbb{R}^3)^{2n}, \mathbb{R}^2) \), the embedding being continuous, the negative norm dual \( \mathbb{E} = (w^0, \cdot^{1/2}(s^1)^{2n}) \) of \( E, [15] \), contains \((L^\infty(s^1)^{2n}) \) with continuous embedding. Therefore by \((4.13)\),

\[
(4.14) \quad M_3(1 + \|z\|^2) \leq \|z\|_E^s.
\]

Since

\[
(4.15) \quad \|z\|_E^s \leq \sup \{w, w\} = \sup \{w^0, w^{00}\} \sup \{s^{00}, s^{000}\}
\]

it follows that

\[
(4.16) \quad \|z\|_E^s \leq 2s |z|^0.
\]

Therefore by \((4.14)\) and \((4.16)\),

\[
(4.17) \quad M_4(1 + \|z\|^2) \leq |z|^s.
\]

Now \((H_5)\) and \((H_3)\) show that for any \( z \in (s, 1/2) \), there is an \( M_5 = M_5(E) \) such that...
(4.18) \[ |z^+|^2 \leq \|z^+\| (\max_{[0,2\pi]} |H_n(t, \xi)| + \mathcal{E} \|z\| + 1). \]

Combining (4.18) with the similar expression obtained for \( z^- \) gives

\[ |z^+|^2 + |z^-|^2 = M_0 (1 + |z^0|^2). \]

Hence by (4.14) and (4.19)

\[ |z^0|^2 \leq M_0 (1 + |z^0|^2) \]

which yields a bound for \( |z^0| \) and then for \( (x_m) \) via (4.19). Thus \( (\xi_j) \) holds.

It remains to verify \( (\xi_j) \). Let \( z = z^0 + z^- \in E_1 \). Then as in (4.15) and (4.17),

\[ g(z) = |z^-|^2 + M_0 |z^0|^2 - M_0 \]

which implies there is an \( \alpha \in \mathbb{R} \) such that \( g = \alpha \) on \( B = B_1 \). If \( x \in E_2 \),

\[ g(z) = \int_0^{2\pi} H(t, z) \mathcal{E} |z|^2 = \nu_0 (|z^0|^{1/2} + 1) - |z|^2 \]

via (4.12) and (3.8). Since \( \theta > \frac{1}{2} \), (4.22) shows that \( g(z) \leq \alpha - 1 = \alpha \) on \( \mathcal{H}_0 = B_2 \cap E_1 \) provided that \( R \) is chosen sufficiently large. The proof is complete.

Remark 4.21: In the subquadratic situations treated in Theorems 4.1 and 4.11, we chose \( E_1 = \mathbb{D} \oplus E^- \) and \( E_2 = E^+ \) for their proofs. It is possible to reverse the roles of these sets and give alternate proofs obtaining possibly different critical values in the process. By doing so, however, we must work with Theorem 1.29 instead of Theorem 1.33. To illustrate we briefly sketch a second proof of Theorem 4.11. Choose \( \mathcal{E} \) as usual, \( f(\cdot) = -\eta(\cdot), E_1 = \mathbb{D}, E_2 = \mathbb{D} \oplus \mathbb{E} \) with corresponding \( L_1, L_2 \) as in Theorem 3.3. The earlier proof of Theorem 4.11 shows that \( f \in C^1(\mathbb{E}, \mathbb{R}) \) and satisfies \( (T_1) - (T_2) \). Moreover (4.22) and (4.21) with \( f \) replacing \( g \) and inequalities reversed give \( (T_3) \) and therefore \( (T_3) \) via Lemma 1.1. Lastly \( f \) satisfies \( (T_4) \). Indeed let \( c \) be defined by (1.8) and e.g. \( a = 1 \). Suppose \( |f(x)| - c \leq m \) and \( f'(z_n) \to 0 \) as \( m \to -\infty \). Then for large \( n \),

\[ (4.24) |f(z) - f(z_n)| - \frac{1}{2} |z|^2 \leq f'(z_n)|z^2| \mathcal{E} \int_0^{2\pi} H(t, z) \mathcal{E} \int_0^{2\pi} H(t, z) \mathcal{E} \]

The boundness of \( (x_m) \) now follows from (4.13) - (4.20).

Remark 4.25: (1) As was the case with Corollary 4.5, the conclusions of Theorem 4.11 remain valid if the inequalities involving \( H \) in \( (H_8) \) and \( (H_9) \) are reversed.

(2) When \( H \) is independent of \( t \), \( (z, H_2)_{\mathbb{E}^2} > 0 \) on \( |z| = F \) implies that \( H_2 \) has a zero in \( B_2 \) of \( \mathbb{R}^2 \). Thus, like Theorem 4.1, Theorem 4.11 per se gives no information about the autonomous case. The following theorem, related to Theorem 4.7, addresses this point.

Theorem 4.26: Suppose \( H \in C^1(\mathbb{E}^2, \mathbb{R}) \) and satisfies \( (H_8) - (H_9) \). Then for all \( F \) sufficiently large, \( (3.2) \) possesses a nonconstant \( T \)

periodic solution.

Proof: By \( (H_8) \), \( H \neq 0 \) if \( |z| = F \). Thus if \( \xi \) is a constant solution of \( (3.2) \),
\[ g(\zeta) = \kappa \int_0^{2\tau} H(\xi) \, dt \leq 2\kappa \lambda \max_{|x| \leq \tau} |H(x)| = \kappa M_1. \]

Therefore, arguing as in Theorem 4.7, it suffices to show

\[ (4.27) \quad \inf_{B_\tau} g(x) > \kappa M_1. \]

Greater care must exercised in choosing \( v \) since \((H_\tau)\) is no longer at our disposal. For \( z \in B_\tau \),

\[ (4.28) \quad g(z) = \kappa \int_0^{2\tau} H(z) \, dt + \|z^0 + v\|^2 - \|v\|^2 \]

\[ = \kappa \int_0^{2\tau} \|z_0 + z + v\|^2 \, dt - 2\kappa \|z_0\|^2 + \|v\|^2 \]

via \((H_\tau)\). By making appropriate choices for \( w \) in \((4.15)\), we obtain

\[ (4.29) \quad \int_0^{2\tau} \|z_0 + z + v\|^2 \, dt = \kappa \left[ \|z_0\|^2 + \left( \frac{\|v\|^2}{1 + \tau^2} \right)^2 \right]. \]

Setting \( v = w \), where \( w \in \partial B_1 \cap H_\tau \), \((4.28)\) \(\to\) \((4.29)\) yield

\[ (4.30) \quad g(z) = \kappa \left[ a_4 \left( \|z_0\|^2 + \tau^2 \right) - 2\kappa \|z_0\|^2 \right] + \|v\|^2 - \|v\|^2. \]

Selecting \( \tau \) so large that \( a_4 \tau^2 - 2\kappa \|z_0\|^2 < 3M_1 \) makes

\[ g(z) = \kappa M - \tau^2. \]

Finally taking \( \kappa \) so large that \( \kappa M_1 \geq \tau^2 \) shows \( g(z) = 2\kappa M \) for \( z \in B_\tau \). Hence \((4.27)\) is satisfied and the proof is complete.

Under certain additional hypotheses, the restriction to large \( \tau \) in Theorem 4.26 can be removed. We present one such result which will be useful later.

**Corollary 4.12:** Suppose \( H \in C^1(\mathbb{R}^{2n}, \mathbb{R}) \) and satisfies \((H_\tau)\) \(-\) \((H_\Theta)\), \((H_\phi)\) holding for all \( z \neq 0 \) and \((H_\Theta)\) with \( a_2 = 0 \). Then for all \( \tau > 0 \), \((3.2)\) has a nonconstant \( T \) periodic solution.

**Proof:** Arguing as in Theorem 4.26, we need only choose \( v = v(T) \) so that \((4.27)\) holds with \( M_1 = 0 \). Under our present hypotheses \((4.30)\) becomes

\[ (4.32) \quad g(z) = a_4 \lambda \tau - \tau^2. \]

Choosing e.g. \( \tau^2 = \frac{1}{2} a_4 \lambda \), \( g(z) = \frac{1}{2} a_4 \lambda \tau = 0 \)

Thus \((4.27)\) is verified and the corollary is proved.

We conclude this section with the subquadratic analogues of Theorems 3.55 and 3.72. Since there are several common arguments in their proofs, we will be somewhat sketchy here.

**Theorem 4.13:** Suppose \( H \in C^1(\mathbb{R}^{2n}, \mathbb{R}) \) and satisfies \((H_\tau)\) \((H_\Theta)\) \((H_\phi)\) There is a \( \theta \in \left( \frac{1}{2}, 1 \right) \) \(\text{ and } \lambda > 0 \) such that \((H_\Theta)\) \(0 < H(z) \leq \theta (z, H(z)) \) \(\mathbb{R}^{2n} \) \(0 < |z| \leq \tau \).

Then for all \( a > 0 \), \((3.2)\) has a \( T = T(a) \) \( \text{ periodic solution on } H^{-1}(a) \) \(\text{ and } T(a) \to 0 \text{ as } a \to 0. \)

**Proof:** By \((H_\Theta)\), there is an \( a_0 > 0 \) \(\text{ such that } \, 0 < a = a_0, \)

\(H^{-1}(a)\) \(\text{ is a compact } C^1 \text{ manifold which bounds a star shaped neighborhood of } \, 0 \).

Define \( H(z) \) as in Theorem 3.55 where \( \tau \) now lies in \((1, 2)\). Then \( H \) satisfies the hypotheses of Corollary 4.12, and that result provides a nonconstant \( 2 \tau \) periodic solution \( w \) of \((3.2)\) with \( H \) replaced by \( H \). The proof is now essentially identical to that of Theorem 3.55 up to \((3.69)\), the exceptions being the replacement of \( f \) by \( g \), the search
for a lower bound for $c$, and the choice of $\tau \in (0^{-1}, 2)$. Also $a = a_0$
here. To complete the proof we observe that (3.68) and (4.32) with
$k = 1$ give the lower bound for $c$
.

Similar ideas readily establish

**Theorem 4.34:** Suppose $H \in C^1(\mathbb{R}^{2n}, \mathbb{R})$ and satisfies

(H_{12}) There is a $G \in (\frac{1}{2}, 1)$ and $T > 0$ such that

$$0 < G(z, H) \in C^0(\mathbb{R}^{2n}, \mathbb{R})$$

for $|z| = T$.

Then for all large $a > 0$, (3.2) has a $T = T(a)$ periodic solution on

$H^{-1}(a)$ and $T(a) \to a$ as $a \to \infty$.

We will not carry out the details.

**Remark 4.35:** Under the assumptions that $H$ is globally convex and
appropriately subquadratic, Clarke and Ikeland [56] have recently obtained
some existence results for (3.2) for solutions having a prescribed minimal
period.

§5. A semilinear wave equation

The theory of §1 can be used to prove the existence of time periodic
solutions of one dimensional semilinear wave equations. A sketch of how to
do this will be given in this section. Consider the problem

$$\begin{cases}
u_{tt} - \nu_{xx} + g(x, u) = 0 & 0 < x < T \\ u(0, t) = u(T, t) = 0 & t \in \mathbb{R}
\end{cases}$$

(5.1)

where we seek time periodic solutions of (5.1). For technical reasons –
see [1] – we have only been able to treat the case in which the period, $T$,
is a rational multiple of the length of the spatial interval, i.e. $T \in \mathbb{Q}$.

For definiteness, suppose $t = \tau$ and $T = 2\tau$. Letting $G(x, u)$
denote the primitive of $g$, formally solutions of (5.1) are critical points of

$$\int_{0}^{2\tau} \int_{0}^{\tau} \left[ \frac{1}{2} (u_t^2 - u_x^2) - G(x, u) \right] dx dt .$$

(5.2)

The indefinite nature of this functional is fairly evident. The associated
action integral is

$$A(u) = \frac{1}{2} \int_{0}^{2\tau} \int_{0}^{\tau} \left( u_t^2 - u_x^2 \right) dx dt .$$

(5.3)

Writing

$$u = \sum_{j \in \mathbb{N}, k \in \mathbb{R}} a_{jk} \sin jx \sin \omega t ,$$

where $a_{-k} = a_{k}^*$, we find

$$A(u) = \sum_{j \in \mathbb{N}, k \in \mathbb{R}} (k^2 - j^2) |a_{jk}|^2 .$$

Thus $A$ is indefinite and moreover has an infinite dimensional null space
in contrast to the Hamiltonian case considered earlier. This further degeneracy
greatly complicates the treatment of (5.1).
Let $\mathbb{Z} = [0, 2\pi] \times [0, 2\pi]$. Then as in §3, $L^2(\mathbb{Z})$ splits into three mutually orthogonal subspaces with bases 

$$\{\sin jx e^{ikt} | j \in \mathbb{N}, k \in \mathbb{Z}, k > j\}, \quad \{\sin jx e^{ikt} | j \in \mathbb{N}, k \in \mathbb{Z}, |k| < j\}.$$ 

Let $N^0$, $N^+$, $N^-$ denote respectively the subspaces of $L^2(\mathbb{Z})$ spanned by these bases. Writing $a \in L^2$ as $u = u^0 + u^+ + u^- \in N^0 \oplus N^+ \oplus N^-$, we introduce a new norm on smooth such functions via

$$\|u\|^2 = A(u^+) - A(u^-) + \int_0^{2\pi} \int_0^{2\pi} (u_x^2) \, dx \, dt.$$

The associated inner product defines a Hilbert space topology on $E = \mathbb{L}^0 \oplus \mathbb{L}^+ \oplus \mathbb{L}^-$ where $\mathbb{L}^i = \{u \in N^i | \|u\| < \infty\}$, $i \in \{0, +, -\}$. Thus the $E^i$ are mutually orthogonal. Note that $E^0$ corresponds to the null space of $A$ (or the wave operator). On $E^0$, $\|\cdot\|$ is equivalent to the $W^{1,2}$ norm while on $E^+ \oplus E^-$ (and therefore on $E$), $\|\cdot\| = \|\cdot\|_{W^{0,1}}$. The choice of norm for $E^0$ is somewhat arbitrary.

With $E$ so defined, the functional given in (5.2) does not yet fit into the framework of §1, the main difficulty being the verification of $(f_5)$. However this can be overcome by suitably modifying (5.1) and (5.2). Thus let $\beta > 0$ and for $u = u^0 + u^+ + u^- \in E$, let $v = u^0$. We replace (5.1) by

$$\begin{cases} 
\frac{\partial u}{\partial t} - u_{xx} - \beta u + g(x, u) = 0 & 0 < x < \pi, \quad t \in \mathbb{R} \\
u(0, t) = 0 = u(\pi, t) 
\end{cases}$$

and (5.2) by

$$f(u) = A(u) - \int_0^{2\pi} \int_0^{2\pi} \left(\frac{\beta}{2} u_t^2 + G(x, u)\right) \, dx \, dt.$$
References


A variational principle of a minimax nature is developed and used to prove the existence of critical points of certain variational problems which are indefinite. The proofs are carried out directly in an infinite dimensional Hilbert space. Special cases of these problems previously had been tractable only by an elaborate finite dimensional approximation procedure. The main applications given here are to Hamiltonian systems of ordinary differential equations and the existence of time periodic solutions is established for several classes of Hamiltonians. 

Critical Point Theorems for Indefinite Functionals

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Supplementary Notes

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