ANALYSIS OF AXISYMMETRIC SHEET-METAL FORMING BY THE RIGID-PLASTIC, FINITE-ELEMENT METHOD

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This report describes the development of a finite-element model for analyzing sheet-metal forming processes. Materials are assumed to be rigid-plastic with the view that the usefulness of an analysis method depends largely upon solution accuracy and computation efficiency. First, the variational formulation applicable to sheet-metal forming is described by considering solution uniqueness and the
effect of geometry change involved in the forming processes. From this variational formulation, a finite-element process model based on the membrane theory is developed. Then, three basic sheet-metal forming processes, namely, the bulging of a sheet subject to hydrostatic pressure, the stretching of a sheet with a hemispherical head punch, and deep drawing of a sheet with a hemispherical head punch, are solved. The solutions arrived at by the rigid-plastic, finite-element method are compared with existing numerical solutions and the experimental data. The agreement is generally excellent and it is concluded that the rigid-plastic, finite-element method is efficient for analyzing sheet-metal forming problems with reasonable accuracy.
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SECTION I
INTRODUCTION

The metal forming processes basically involve large amounts of elastic deformation, and, due to the complexities of plasticity, the exact analysis of a process is infeasible in most of the cases. Thus, a number of approximate methods have been suggested, with varying degrees of approximation and idealization. Among these, techniques using the finite-element method take precedence because of their flexibility, ability to obtain a detailed solution, and the inherent proximity of their solutions to the exact one.

A prime objective of mathematical analysis of metalworking processes is to provide necessary information for proper design and control of these processes. Therefore, the method of analysis must be capable of determining the effects of various parameters on metal flow characteristics. Furthermore, the computation efficiency, as well as solution accuracy, is an important consideration for the method to be useful in analyzing metalworking problems.

With this viewpoint in mind, successful efforts have been carried out in analyzing various deformation processes, such as compression, heading, piercing, extrusion and drawing by the rigid-plastic, finite-element method (matrix method) [1]-[7].

The formulation of the matrix method, however, cannot be extended to the sheet-metal forming analysis due to the following reasons:

(1) The classical variational formulation which is the basis of the matrix method does not necessarily determine a unique deformation mode. Physically, there is no inherent indeterminacy for work-hardening solids, but this indeterminacy is due rather to the fact
that the workhardening rate is not included in the mathematical formulation of the classical variational principle.

(2) The kinematic assumption in the matrix method is not longer valid for the sheet-metal forming process. As long as bulk deformation or in-plane stretching are concerned, this kinematic assumption that the magnitude of the rate of rotation is negligible compared to the strain rate does not deviate much from the real situation and yields solutions consistent with reality. Geometric nonlinearity in sheet-metal forming, however, invalidates such a simplification.

The objective of the present investigation is, therefore, to develop and establish a finite-element method for sheet-metal forming processes.

In Section II various forms of variational formulations are reviewed in the light of uniqueness and geometry change which leads to a realization of the necessity of new formulations. In Section III a new formulation is obtained and the development of the finite-element model from it is described. With the particular example of sheet-metal forming processes in mind, the idealization of plane stress state and membrane theory is implemented. Furthermore, the development is confined to the case of axisymmetrical problems.

To establish the validity of the proposed method, three basic sheet-metal forming processes are analyzed and the solutions are compared with other available experimental data and numerical solutions. Hydrostatic bulging is treated in Section IV. Punch stretching with a hemispherical punch is discussed in Section V. To make the problem tractable, one moving contact boundary is considered first by neglecting die profile; then the analysis is extended to include two moving boundaries. In Section VI deep drawing with a hemispherical punch is solved.
SECTION II
BACKGROUND

1. Uniqueness

We consider the quasistatic deformation of a rigid-plastic solid. On a portion $S_v$ of the surface $S$ of this body are prescribed given velocities, while the remainder $S_T$ of the surface $S$ is subjected to given surface tractions $T_i$. Assuming that these surface velocities and tractions are such that the entire body is in a state of plastic flow, we want to determine the stresses $\sigma_{ij}$ and strain rates $\dot{\varepsilon}_{ij}$ throughout the body.

The conventional formulation of variational principle for this problem is that among all kinematically admissible strain rate fields $\dot{\varepsilon}_{ij}^*$, the actual one minimizes the expression (Hill [8]),

$$
\pi_1 = \int \bar{\sigma} \dot{\varepsilon}^* \, dv - \int_{S_T} T_i \dot{v}_i^* \, dS,
$$

(1)

where $\bar{\sigma}$ is the effective stress, $\dot{\varepsilon}^*$ is the effective strain rate defined by

$$
\bar{\sigma} = \frac{\sqrt{3}}{2} \sqrt{\sigma_{ij}^* \sigma_{ij}^*},
$$

$$
\dot{\varepsilon}^* = \frac{\sqrt{2}}{3} \sqrt{\dot{\varepsilon}_{ij}^* \dot{\varepsilon}_{ij}^*},
$$

respectively, where $\sigma_{ij}^*$ is the deviatoric component of $\sigma_{ij}$. Here a strain rate field $\dot{\varepsilon}_{ij}^*$, defined throughout the body under consideration, is called kinematically admissible if it is derivable from a velocity field $v_i^*$ which satisfies the condition of incompressibility $v_{i,i}^* = 0^\dagger$ throughout the body.

---

The comma denotes the differentiation with respect to coordinates, e.g.,

$$
v_{i,i} = \frac{\partial v_i}{\partial x_i}.
$$
and the boundary conditions on $S_v$. The variational principle in this form has been successfully applied to the analysis of metal forming problems, such as extrusion [6]. As was found out later, and we will discuss this shortly, the success is related to the type of boundary conditions prescribed on the surface of the body undergoing deformation. In general, with the variational formulation of $\pi_1$ in Eq. (1), there is a question regarding uniqueness of deformation mode even though the stress field is uniquely determined [8], [9].

Consider an incipient flow in a rigid-plastic solid, workhardening or perfect plastic, governed by the following partial differential equations which are, of course, dual to the variational formulation $\pi_1$. With respect to Cartesian reference frame $x_1$ the following equations hold:

Equilibrium equations

$$\sigma_{ij,j} = 0 \text{ in the absence of body force} \quad (2a)$$

Strain rate-velocity relationship

$$\dot{\varepsilon}_{ij} = \frac{1}{2}(\nabla_v, j + \nabla_v, i) \quad (2b)$$

 Constitutive equation

$$\mu \sigma'_{ij} = \dot{\varepsilon}_{ij}, \mu \text{ being an arbitrary constant} \quad (2c)$$

Yield criterion

$$\dot{\sigma} = \sqrt{\frac{3}{2}} \sqrt{\sigma_{ij} \sigma'_{ij}} = H(\dot{\varepsilon}), \text{ where } \dot{\varepsilon} \text{ is the effective strain defined by}$$

$$\dot{\varepsilon} = \int \dot{d}\varepsilon \text{ if } d\varepsilon = \sqrt{\frac{2}{3}} \sqrt{\dot{d}\varepsilon_{ij}} \dot{d}\varepsilon_{ij} \quad (2d)$$

Boundary conditions

$$n_j \sigma_{ij} = \hat{T}_i \text{ on } S_T,$$

$$v_i = \hat{v}_i \text{ on } S_v,$$

where $n_j$ is the unit normal vector to the surface of the body; $\hat{T}_i$ and $\hat{v}_i$ are prescribed values.
Suppose that \((\sigma_{ij}^{(1)}, \varepsilon_{ij}^{(1)})\) is the solution to this boundary value problem.

Construct a different set of stress fields and strain rate fields \((\sigma_{ij}^{(2)}, \varepsilon_{ij}^{(2)})\),

where \(\sigma_{ij}^{(2)} = \sigma_{ij}^{(1)}, \varepsilon_{ij}^{(2)} = C \varepsilon_{ij}^{(1)}\). C is any arbitrary factor and may vary from point to point throughout the body. Then, it is easily shown that this set \((\sigma_{ij}^{(2)}, \varepsilon_{ij}^{(2)})\) satisfies all the governing equations except for the boundary conditions on \(S_v\). On \(S_v\), the velocity integrated from \(\varepsilon_{ij}^{(2)}\) should coincide with the prescribed value \(\hat{v}_i\). Since strain rate-velocity relation is linear, integrating \(\varepsilon_{ij}^{(2)}\) would yield \(C \hat{v}_i\) if \(\varepsilon_{ij}^{(1)}\) is integrated to give \(\hat{v}_i\), and therefore C must be unity on \(S_v\). With this and the compatibility requirement the deformation mode may or may not be uniquely determined. One example of a well-established unique kinematic mode is in the plane-strain problem. In the plane-strain condition, unless one family of the characteristics is straight, the governing equation of the velocity field becomes the telegraphy equation which is hyperbolic and, therefore, the solution is uniquely determined if the boundary curve is not along a characteristic.

It can be readily shown that under certain boundary conditions the set \((\sigma_{ij}^{(1)}, C \varepsilon_{ij}^{(1)})\) also satisfies the boundary conditions on \(S_v\) and therefore the deformation mode is clearly not unique. The following is a partial list of such boundary conditions.

1. \(S_v = 0\), i.e., all the boundaries are traction boundaries;
2. \(\hat{v}_i = 0\) on \(S_v\);
3. On \(S_v\), only the ratio between the velocity components are prescribed, e.g., \(\hat{v}_i / \hat{v}_j = \alpha\);
4. Mixed boundary condition; e.g., a normal component of \(\hat{v}_i\) and a tangential component of \(\hat{t}_i\) are prescribed over the surface, or vice versa. In this case, the additional condition of whether all the characteristics meet on a curve in the region should be checked [10].
Concrete examples are (1) the expansion of spherical shells [11] or cylindrical shells [12] under internal pressure, and (4) the indentation of a semi-infinite body by a flat punch under the plane-strain condition [13], torsion of a prismatic bar [10]. Among sheet-metal forming processes, hydrostatic bulging belongs to case (2) and punch stretching to case (4) or (3).

Note that the physical meaning of these boundary conditions is that the plastic flow is unconstrained and all or part of the body is free to deform. Mathematically, this nonuniqueness is due to the fact that the Levy-Mises theory, implied in the variational formulation \( \Pi_1 \) and also appearing in the differential equations (2c), does not include the "viscosity effect" (in Prager's terminology [9]) and, therefore, this indeterminacy would be resolved if the workhardening effect is taken into account. In fact, for the workhardening solid there is no inherent indeterminacy in general; the apparent nonuniqueness is due simply to an inadequate formulation of the problem. In proper formulation, traction rate \( \dot{T}_i \) must be specified on \( S_T \), and then from an infinite number of kinematically possible modes the actual mode can be singled out by the additional requirement that there must exist an equilibrium distribution of stress rate compatible with the implied rate of hardening everywhere in the body and with the given traction rate \( \dot{T}_i \) on \( S_T \). Besides, the workhardening effect is explicitly brought into the constitutive equation in the form of

\[
h_{ij}^* = \frac{\sigma_{ij}^*}{\dot{\sigma}}
\]

where \( \dot{\sigma} \) is the time rate of \( \sigma \), \( h \) the workhardening effect of the material being equal to \( \frac{2}{3} \frac{d\sigma}{d\varepsilon} \). It can be shown that the constitutive equation (3) can always be reduced to the constitutive equation (2c), but not necessarily
vice versa. Therefore, for a perfectly plastic solid, specifying the traction rate does not resolve the indeterminacy. Hill, then, showed that among all variational modes compatible with the boundary conditions for \( \hat{v}_i \) on \( S_v \) and the existing stress distribution \( \sigma_{ij} \), the actual mode minimizes the following expression when geometry changes are neglected (Hill [8]):

\[
\pi_2 = \frac{1}{2} \int h(\varepsilon_{ij}^*)^2 \, dv - \int_{S_T} \hat{T}_i \varepsilon_i \, dS. \tag{4}
\]

Note that the virtual mode \( \varepsilon_{ij}^* \) in \( \pi_2 \) should be normal to the yield surface at the existing stress point in the stress space due to the compatibility requirement with existing stress distributions. For statically indeterminate problems, however, there is a coupling between stress field and strain rate field and we have to solve these two sets of variables simultaneously.

2. Geometry change

When the effect of geometry change cannot be neglected during deformation, it is necessary to reconsider the specification of the loading on \( S_T \) and the stresses since the changes in shape and area of surface elements are themselves unknown.

Let \( x_i \) be the position vector in a Cartesian reference frame at time \( t \) and after an infinitesimal time \( \delta t \), \( \bar{x}_i \) be the position. Let us call the configuration at time \( t \) undeformed configuration and the one at time \( t + \delta t \) deformed configuration. When an actual force \( dP_i \) acts upon the area element \( da \) at time \( t + \delta t \), there are various ways of reckoning this force.

First, the actual force \( dP_i \) is referred to the deformed configuration, or

\[
dP_i = n_j \sigma_{ij} \, da, \tag{5a}
\]
where \( n_j \) is the unit normal vector to the surface element of area \( da \) in a deformed configuration. The stress tensor \( \sigma_{ij} \) defined in this manner is called Cauchy stress tensor, or sometimes, true stress tensor.

Second, the actual force \( dP_i \) is referred to the undeformed configuration, or

\[
dP_i = N_j S_{ij} dA,
\]

where \( N_j \) is the unit normal vector to the surface element of area \( dA \) in an undeformed configuration. The stress tensor \( S_{ij} \) defined in this manner is called the first kind of Kirchhoff stress tensor, or sometimes, nominal stress tensor. This tensor has the disadvantage of not being symmetric and therefore awkward to use in a constitutive equation with a symmetric strain tensor. Nonetheless, sometimes this stress tensor is used with nonsymmetric velocity gradients [14].

Third, to obtain a stress tensor, which is symmetric and referred to the undeformed configuration, we proceed as follows. Instead of the actual force \( dP_i \), consider a force \( \tilde{dP}_i \) related to the force \( dP_0 \) in the same way that a material vector \( dX_i \) is related by the deformation to the corresponding vector \( dx_i \). That is,

\[
\tilde{dP}_i = \frac{\partial X_i}{\partial x_j} dP_j.
\]

Refer this pseudo-force \( \tilde{dP}_i \) to the undeformed configuration to define the second kind of Kirchhoff stress tensor \( \tau_{ij} \):

\[
d\tilde{P}_i = N_j \tau_{ij} dA.
\]
\[ n_i \, da = \frac{\rho_0}{\rho} \, N_j \, \frac{\partial x_i}{\partial x_j} \, dA, \quad (6) \]

where \( \rho_0 \) and \( \rho \) are densities of the volume element before and after the deformation, the relationship between different stress measures is obtained. From Eqs. (6), (5a), and (5b),

\[ dP_i = n_j \sigma_{ij} \, da = \sigma_{ij} \frac{\rho_0}{\rho} \, N_k \, \frac{\partial x_i}{\partial x_j} \, dA \]

\[ = N_j S_{ij} \, dA \quad (7a) \]

or

\[ S_{ij} = \frac{\rho_0}{\rho} \, \frac{\partial x_i}{\partial x_k} \, \sigma_{ik} \quad (7b) \]

and from Eqs. (6), (5a), (5c), and (5d),

\[ \tau_{ij} = \frac{\rho_0}{\rho} \, \frac{\partial x_i}{\partial x_k} \, \sigma_{kj} \, \frac{\partial x_j}{\partial x_k} \quad (7c) \]

All these different stress tensors become exactly the same when we bring the deformed configurations to the undeformed configurations and make them identical in the limit. Stress rates, however, are not the same. Let \( \delta u_i \) be the increment of displacement of the element; then

\[ \delta S_{ij} = \delta \sigma_{ij} - \sigma_{kj} \, \frac{\partial}{\partial x_k} \delta u_i \quad \text{to the first order, neglecting plastic volume change. Or, in terms of rates,} \]

\[ S_{ij} = \dot{\sigma}_{ij} - \sigma_{kj} \, \dot{v}_{i,k} \quad (8) \]

Let us compare the magnitude of the second term with that of the first term in the right-hand side of Eq. (8). Since
\[
\frac{\partial v_i}{\partial x_k} = \frac{1}{2} \left[ \frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right] + \frac{1}{2} \left[ \frac{\partial v_i}{\partial x_k} - \frac{\partial v_k}{\partial x_i} \right]
\]

\[
= \frac{1}{2} \dot{\varepsilon}_{ik} + \frac{1}{2} \dot{\omega}_{ik},
\]

where \( \dot{\varepsilon}_{ik} \) is the rate of deformation and \( \dot{\omega}_{ik} \) the rate of rotation. Then

\[
\sigma_{kj} \frac{\partial v_i}{\partial x_k} = \frac{1}{2} \sigma_{kj} \dot{\varepsilon}_{ik} + \frac{1}{2} \sigma_{kj} \dot{\omega}_{ik}.
\]

Now, if workhardening characteristics are given by the relation

\[
\tilde{\sigma} = H(\tilde{\varepsilon}),
\]

then

\[
H' = \frac{d\tilde{\sigma}}{d\tilde{\varepsilon}}
\]

or

\[
d\tilde{\varepsilon} = \frac{d\tilde{\sigma}}{H'}.
\]

For an order of magnitude calculation we can write approximately

\[
\dot{\varepsilon}_{ik} = \frac{\sigma_{ik}}{H'}
\]

or

\[
\sigma_{kj} \dot{\varepsilon}_{ik} = \frac{\sigma_{kj} \sigma_{ik}}{H'}.
\]

Then, from Eqs. (9b) and (10d), Eq. (8) becomes

\[
\dot{S}_{ij} = \sigma_{ij} \left(1 - \frac{\sigma_{ij}}{H'}\right) + \frac{1}{2} \sigma_{kj} \dot{\omega}_{ik}.
\]
From Eq. (11) we conclude that if the order of the rate of rotation $\dot{\omega}_{ij}$ is the same as or less than the order of strain rate $\dot{\varepsilon}_{ij}$, and if the work-hardening rate $H'$ is greater than the stress level, then $\dot{S}_{ij} = \dot{\sigma}_{ij}$. Otherwise, geometry change should not be neglected.

It could be shown [16] that when geometry change is taken into account, the condition for continuing equilibrium requires that

$$\frac{\partial S_{ij}}{\partial x_1} = 0$$

in the absence of body force.

Using this condition, Hill subsequently derived the following variational formulation [17]:

$$\pi_3 = \frac{1}{2} \int h(\dot{\varepsilon}_{ij})^2 \, dv - \frac{1}{2} \int \sigma_{k,ij} v^*_{i,ik} v_{j,i} v \, dv - \int_{S_T} T_i v^*_i \, dS.$$  \hspace{1cm} (12)

Formulation $\pi_3$ follows essentially the same line of formulation $\pi_2$ except that now geometry change is considered. In the formulation $\pi_3$, as well as in $\pi_2$, virtual mode must be compatible with the existing stress distribution and the boundary condition on $S_v$. As has been discussed earlier for statically indeterminate problems this is not an appropriate formulation.

Summarizing the development so far, the kinematic mode in sheet-metal forming of a rigid-plastic solid is not uniquely determined by considering the first-order expansion of the potential alone. Consideration up to second-order expansion of the potential, or equivalent consideration of workhardening rate in a physical sense, needs stress rate terms explicitly in the variational formulation. When geometry change cannot be neglected, these stress rate- are related to stress distribution, which is not known for statically indeterminate problems. The approach of viewing the deformation
as determining the incipient flow by assuming the deformed configuration coincident with the undeformed configuration clearly does not lead to a workable variational formulation for sheet-metal forming of a rigid-plastic solid. In this respect, it is intended to develop an appropriate variational formulation in the next section.
SECTION III
FINITE-ELEMENT FORMULATION

1. Variational formulation

Let $x_i$ be the position vector in a Cartesian frame of reference at
time $t$, the moment under consideration. Let $\sigma_{ij}$ be the true stress at
time $t$ and $\sigma_{ij} + d\sigma_{ij}$ the true stress in the same material element after
an infinitesimal time $dt$, both tensors being associated with the same
Cartesian axes. Let $ds_{ij}$ be the increment in nominal stress in the same
element in time $dt$, based on the dimensions at time $t$. Let $du_i$ be the
increment of displacement of the element, then

$$ds_{ij} = d\sigma_{ij} - \sigma_j \frac{\partial (du_i)}{\partial x_k}.$$  (13)

Requiring continuing equilibrium of stresses, the virtual work principle
gives

$$\int_V \left( \sigma_{ij} + d\sigma_{ij} - \sigma_{kj} \frac{\partial (du_i)}{\partial x_k} \right) \delta (\sigma_{ij}) dV = \int_{S_F} (E_j + dF_j) \delta (du_j) ds,$$  (14)

where $T_j = \ell_i \sigma_{ij}$ and $dT_j = \ell_i ds_{ij}$, $\ell_i$ being the unit normal to the surface
at time $t$. The variational formulation is obtained from Eq. (14) as follows:

$$\delta \phi = \delta \left\{ \int_V \sigma_{ij} \delta \varepsilon_{ij} dV + \int_V \frac{1}{2} \frac{\partial (du_i)}{\partial x_k} \delta (\sigma_{ij}) dV - \int_V \frac{1}{2} \sigma_{kj} \frac{\partial (du_i)}{\partial x_k} \frac{\partial (du_j)}{\partial x_i} dV - \int_{S_F} (T_j + dT_j) du_j ds \right\} = 0,$$  (15)
where
\[ \text{de}_{ij} = \frac{1}{2} \left( \frac{\partial (du_i)}{\partial x_j} + \frac{\partial (du_j)}{\partial x_i} \right) \]

and \( h = \frac{2}{3} H' \), with \( H' \) the slope of the stress and strain curve. The first three terms of the functional \( \phi \) represent the energy dissipated during the time \( dt \) up to the second order. If it is assumed that the principal axes of true strain-rate keep the same directions in the element and the principal components of strain-rate maintain the constant ratios during the time \( dt \), the dissipated energy can be expressed directly [18] as

\[ \sum (\sigma \left( \text{dE}_p \right) + \frac{1}{2} h \text{dE}_p^2) \]

per unit volume, where \( \text{dE}_p \) is the logarithmic strain components. The final form of the functional becomes

\[ \phi = \int_V \delta \text{dE} \, dV + \frac{1}{2} \int H'(\text{dE})^2 \, dV - \int_{S_F} (T_j + dT_j) \, du_j \, dS, \]

where \( \text{dE} \) is defined by

\[ \text{dE} = \sqrt{\frac{2}{3} \sum (\text{dE}_p)^2} \]

2. Theory of the finite-element method

An important step in finite-element modeling is obtaining approximate state equations in a region. The weighted residual method derives the state equations directly from the governing differential equations. Let us write the governing differential equation as

\[ Lu - f = 0, \]
where \( L \) is the differential operator, \( f \) is the known function, and \( u \) is the solution. With the trial solution \( u^* \), Eq. (17) is not satisfied, but there remains an error or residual \( R \) such that

\[
R = Lu^* - f. \tag{18}
\]

This residual is multiplied by weight function \( w \) and integrated over the domain and the state equations are derived from the condition that this integral vanishes with a given choice of weight function \( w \):

\[
\int wR \, dv = 0. \tag{19}
\]

One well-known method among weighted residual methods is Galerkin's approach.

A more frequently used approach is the derivation from a variational principle which is a dual expression of the governing differential equation. Assume that a functional \( \Phi \), which is equivalent to the differential equation, has been established. Let a continuum be divided into a finite collection \( M \) of subdomains called elements interconnected at a finite number of nodes \( N \). If it is true that the total functional is equal to the sum of the contributions of each element \( \phi^{(m)} \), then we may write as follows:

\[
\Phi = \sum_{m=1}^{M} \phi^{(m)}(u). \tag{20}
\]

In each element let us approximate the solution with a linear combination of trial functions \( v_i \) such that

\[
u = \sum \alpha_i v_i \tag{21}\]

holds, where \( \alpha_i \) are unknown coefficients to be determined later. By substituting Eq. (21) into Eq. (20), we have
\[ \phi = \sum_{m=1}^{M} \varphi^{(m)}(\alpha_i v_i) \]

\[ = \sum_{m=1}^{M} \varphi^{(m)}(\alpha_i) \quad \text{since } v_i \text{'s are known} \]  \hspace{1cm} (22)

\[ = \varphi(\alpha_i). \]

The original \( \phi \) of \( u \) is now discretized with a function \( \varphi \) of parameters \( \alpha_i \), and the initial variational problem reduces to determining the \( \alpha_i \) that minimizes \( \varphi \). The minimization of \( \varphi \) with respect to \( \alpha_i \) may be written as

\[ \delta \varphi = \frac{\delta \varphi}{\delta \alpha_i} \delta \alpha_i = 0, \]  \hspace{1cm} (23)

where \( \delta \) denotes the first variation. Since \( \alpha_i \)'s are independent, expression (23) is equivalent to a set of simultaneous equations,

\[ \frac{\partial \varphi}{\partial \alpha_i} = 0. \]  \hspace{1cm} (24)

This is, in fact, the classical Ritz technique. It is the choice of trial functions that makes the finite-element method different from the Ritz method and renders it successful; they are piecewise polynomials. Besides, the coefficients \( \alpha_i \), called nodal values in the finite-element literature, do have a definite physical meaning, such as displacement or velocity.

The trial function \( v_i \) must satisfy certain requirements to enable convergence as the subdivision into ever smaller elements is attempted. First, as the element size decreases, the functions in the integral must tend to be single-valued and well behaved in physical problems. This is called the "completeness" requirement and is satisfied if the trial function is of class \( C^p \) when \( p \) is the highest order in the integrand of the
functional. Second, the validity of the summation implied in Eq. (20) must be preserved. This is called the "compatibility" requirement and is satisfied if $v_1$ is of class $C^{p-1}$ [19], [20]. When admissible trial functions are used, the functional converges monotonically with an increasing number of elements (or decreasing size) at a rate proportional to $h^2$ where $h$ is a characteristic element dimension.

3. Modeling of axisymmetric problems

The general outline of the finite-element modeling stated above will be expanded in detail for the case of axisymmetric thin shells subject to axisymmetric loading. This particular problem is of interest since some basic sheet-metal forming processes belong to this category. When the ratio of thickness to the radius of curvature is sufficiently small, bending moment and shearing forces may be neglected without serious error and the membrane theory may be justified [21]. Moreover, the state of stress can be treated as an approximate plane so long as $\frac{dt}{ds}$ is small compared with unity, where $t$ is the local thickness and $s$ is the distance in any direction parallel to the surface. We now may rewrite $\phi$ with the substitution of $t \, dA = dv$ to Eq. (16):

$$\phi = \int \delta(d\tilde{E})t \, dA + \frac{1}{2} \int H'(d\tilde{E})^2(t \, dA) - \int (T + dT)du_j \, dA$$

(25)

for the unit included angle of the element, where $A$ is the area of the element and $t$ is the sheet thickness.

From the symmetry of the problem it is easily shown that the circumferential direction and the meridian direction are the principal directions and if the friction between the shell and the external agent is negligible, the thickness direction will be the third principal direction. Within the
order of approximation taken in the formulation, the logarithmic strain increment may be used as the strain increment measure. Then the definitions of strain increments are

\[
\begin{align*}
\text{d}E &= \begin{bmatrix} \text{d}E'_r \\ \text{d}E'_\theta \end{bmatrix} = \begin{bmatrix} \frac{\ln s}{s_0} \\ \frac{\ln r}{r_0} \end{bmatrix},
\end{align*}
\]

if, during an incremental deformation, an element of undeformed length \( s_0 \) is stretched to the length \( s \) and the point currently at the radial distance \( r_0 \) moves to the deformed radial location \( r \). Subscripts \( r, \theta \) refer to the meridian and the circumferential direction, respectively.

To bring the model closer to reality in the present investigation, normal anisotropy is included and the corresponding stress-strain increment relation is obtained, using Hill's criterion [13], as

\[
\begin{align*}
\frac{\text{d}E'_r}{(1 + R)\sigma'_r - R\sigma'_\theta} &= \frac{\text{d}E'_\theta}{(1 + R)\sigma'_\theta - R\sigma'_r} = \frac{\text{d}E}{(1 + R)\bar{\sigma}},
\end{align*}
\]

where \( R \) is the planar isotropy parameter which is the ratio of width strain to the thickness strain in uniaxial tension. The effective stress and the effective strain are defined\(^\dagger\) as

\[
\bar{\sigma} = \sqrt{\sigma'_\theta^2 - \frac{2R}{1 + R} \sigma_r' \sigma'_\theta + \sigma_r'^2},
\]

\[
\text{d}\bar{E} = \frac{1 + R}{\sqrt{1 + 2R}} \sqrt{\text{d}E_r'^2 + \frac{2R}{1 + R} \text{d}E'_\theta \text{d}E'_r + \text{d}E'_\theta^2}.
\]

\(^\dagger\)Note that \( H' = \frac{\text{d}\bar{\sigma}}{\text{d}\bar{E}} \) must be consistent with these definitions.
The effective strain, \( \varepsilon_{e} \), may be written in matrix form as

\[
\varepsilon_{e} = \sqrt{\frac{2}{3}} \left[ \varepsilon_{e}^{T} D \varepsilon_{e} \right]^{1/2},
\]

where

\[
D = \frac{3(1 + R)}{2(1 + 2R)} \begin{bmatrix} 1 + R & R \\ R & 1 + R \end{bmatrix}.
\]

The sheet geometry is approximated by a series of conical frustra, as shown in Fig. 1. Linear trial functions, or shape functions, as they are often called in the finite-element literature, are enough since the integrand in the functional is of class \( C^1 \). The unknown coefficients, or nodal values, are taken to be the incremental displacement at nodes. Then we may write

\[
u^{(m)} = (dv_1, dw_1, dv_2, dw_2)^T
\]

for a representative element \( m \), where \( dv_i, dw_i \) are the radial and the axial components of incremental displacement of the \( i \)-th node. Then the incremental displacement field inside the element may be written as

\[
\begin{pmatrix} dv_1 \\ dw_1 \\ dv_2 \\ dw_2 \end{pmatrix} = \begin{bmatrix} \frac{1 + t'}{2} & 0 & \frac{1 - t'}{2} & 0 \\ 0 & \frac{1 + t'}{2} & 0 & \frac{1 - t'}{2} \end{bmatrix} \begin{pmatrix} dv_1 \\ dw_1 \\ dv_2 \\ dw_2 \end{pmatrix}
\]

\[
= N_u^{(m)}
\]

where \( t' \) is the local coordinate varying from the value of -1 at node 2.
to \( +1 \) at node 1. (See Fig. 1.) Due to this incremental displacement field, an element of length \( s_0 \),

\[
s_0 = \sqrt{(r_{01} - r_{02})^2 + (z_{01} - z_{02})^2},
\]

is stretched to a new length \( s \),

\[
s = \sqrt{(r_1 - r_2)^2 + (z_2 - z_1)^2},
\]

(32)

where \((r_{0i})_i, (z_{0i})_i\) are the radial and the vertical positions of the \( i \)-th node at the undeformed configuration and \((r)_i, (z)_i\) at the deformed configuration. Since the element is straight, any point of \( t' \) in the local coordinate is shown to have a global radial position \( r_0 \) determined by

\[
r_0 = \left(\frac{1 + t'}{2}\right)r_{01} + \left(\frac{1 - t'}{2}\right)r_{02}.
\]

(33)

The new position \( r \) of the same particle is given by

\[
r = r_0 + \frac{(1 + t')}{2} \, dv_1 + \frac{(1 - t')}{2} \, dv_2.
\]

(34)
We are now at the position of calculating the strain increment field.

Recall the equation (25) and substitute Eqs. (32), (33), and (34) into it to obtain

\[
\begin{aligned}
\frac{1}{2} \ln \left( \frac{(r_0)_1 - (r_0)_2 + dv_1 - dv_2}{s_0} \right)^2 + \left\{ (z_0)_1 - (z_0)_2 + dw_2 - dw_1 \right\}^2 \\
\ln \left( \frac{r_0 + (1 + t')}{2} dv_1 + \frac{(1 - t')}{2} dv_2}{r_0} \right)
\end{aligned}
\]

(35)

We may write \( \phi^{(m)} \), a contribution from the m-th element to the total functional \( \Phi \), in terms of nodal values, for unit angle included:

\[
\phi^{(m)} = \int \left( \delta \dot{dE} + \frac{1}{2} H'(dE)^2 t \right) dA - \int (T_1 + dT_1) \nu_1 dA \\
= \int \delta \left( \frac{\sqrt{2}}{3} t \right) \left[ \dot{dE}^T D \dot{dE} \right]^{1/2} dA + \frac{1}{2} \int H'(\frac{2}{3} t) \left[ \dot{dE}^T D \dot{dE} \right] dA - \int \frac{N^T N u^{(m)}}{2} dA ,
\]

(36)

where

\[
T = \left\{ T_1 + dT_1, T_2 + dT_2, T_3 + dT_3, T_4 + dT_4 \right\}
\]

Minimization gives a set of simultaneous equations:

\[
\frac{\partial \phi^{(m)}}{\partial u^{(m)}} = \int \left( \frac{\sqrt{2}}{3} t \right) \delta \left[ \dot{dE}^T D \dot{dE} \right]^{-1/2} \left[ \frac{\partial (dE)^T}{\partial u^{(m)}} \right] D \dot{dE} dA + \int \left( \frac{2}{3} t \right) H' \left[ \frac{\partial (dE)^T}{\partial u^{(m)}} \right] D \dot{dE} dA \\
- \int N^T dA .
\]

(37)

From Eq. (35),
Therefore, Eq. (37) becomes
\[
\frac{\partial \Phi}{\partial u^{(m)}} = \int \left( \frac{2}{3} t \right) \tilde{\delta} \left[ \frac{2}{3} dE^T D dE \right]^{-1/2} \tilde{Q} D dA + \int \left( \frac{2}{3} t \right) H' \tilde{Q} D dA - \int N^T dA = 0.
\]

These equations, being valid for an m-th element, are now to be combined under the condition of compatibility that the first-order derivative of nodal value may be discontinuous across element boundaries but the nodal value itself must be continuous,
\[
\frac{\partial \Phi}{\partial u} = \sum \frac{\partial \Phi}{\partial u^{(m)}} = 0.
\]

4. Linearization

Eqs. (39) and (40) are nonlinear equations and it is very difficult to solve them without linearizing. One way is to take an initial guess of the solution to the equation as \( u^* \) and rewrite Eq. (39) in terms of the differences between this initial guess and the correct solution \( \Delta u \), where \( u_{\text{correct}} = u^* + \Delta u \), and expand it. Where the initial guess is sufficiently close to the correct solution, we may neglect higher-order terms of \( \Delta u \) and
thereby linearize successfully. This can be done mathematically in a systematic way and is called the Newton-Raphson method [22]. Say we have a nonlinear equation \( \psi(u) = 0 \), then we may expand into a series with respect to the correct solution \( u_0 \) such that

\[
\psi(u) = \psi(u_0) + \left( \frac{d \psi}{du} \right)_{u=u_0} (u - u_0) + \frac{1}{2} \left( \frac{d^2 \psi}{du^2} \right)_{u=u_0} (u - u_0)^2 + \cdots
\]

\[
= \psi^* + \left( \frac{d \psi}{du} \right)^* \Delta u + \frac{1}{2} \left( \frac{d^2 \psi}{du^2} \right)^* (\Delta u)^2 + \cdots = 0.
\]

If \( u \) and \( u_0 \) are sufficiently close, we may neglect the higher-order terms and write

\[
\psi = \psi^* + \left( \frac{d \psi}{du} \right)^* \Delta u = 0. \tag{41}
\]

In our formulations the equations to be minimized are \( \frac{\partial \Phi^{(m)}}{\partial u} = 0 \), and, therefore, the expressions corresponding to Eq. (41) are

\[
\left| \frac{\partial^2 \Phi^{(m)}}{\partial u_i (m) \partial u_j (m)} \right| (\Delta u) = \left| \frac{\partial \Phi^{(m)}}{\partial u_i} \right|^* \tag{42}
\]

It may be shown that

\[
\frac{\partial^2 \Phi^{(m)}}{\partial u_i (m) \partial u_j (m)} = P^{(m)} = \frac{2}{3} \int \frac{1}{dE} \left( (\sigma + H' \ dE) (K - \frac{2}{3} \ bb^T) + \frac{2}{3} \ \frac{H'bb^T}{dE^2} \right) \ dA \tag{45a}
\]

where

\[
\begin{align*}
\sigma &= QD \ dE \\
K &= QQ^T
\end{align*}
\]
and that
\[ \frac{\partial \Phi^{(m)}}{\partial u^{(m)}} = \mathbf{H}^{(m)} - \mathbf{F}^{(m)}, \]  
(45b)

where
\[ \mathbf{H}^{(m)} = \frac{2}{3} \int \frac{1}{\text{d}E} (\tilde{\sigma} + \mathbf{H}'dE) \text{d}A, \]
\[ \mathbf{F}^{(m)} = \int \mathbf{N}^T \text{d}A. \]

By assembling the equations obtained for an element, we finally have
\[ \mathbf{P}^* \Delta \mathbf{u} = \mathbf{F} - \mathbf{H}^* \]  
(44)

We evaluate the integrals with the Gaussian quadrature formulation.

We have yet to introduce the boundary conditions for solving a physical problem. For an incremental displacement prescribed boundary, the corresponding perturbations should vanish and, for a traction prescribed boundary, the prescribed traction value will enter into the \( \mathbf{F} \) vector. The solution procedure is as follows:

1. Assume an initial guess \( \mathbf{u}_1 \), and compute \( \mathbf{P}, \mathbf{H}, \mathbf{F} \) corresponding to this guess.
2. Solve Eq. (3.31) and obtain \( \Delta \mathbf{u} \).
3. Obtain a new initial guess \( \mathbf{u}_2 = \mathbf{u}_1 + \Delta \mathbf{u} \).

Repeat this process until convergence is achieved. Convergence is checked by the fractional norm. A norm is defined by a square root value, i.e.,
\[ \| \mathbf{u} \| = \sqrt{u_1^2 + u_2^2 + \cdots} \]

and
\[ \| \Delta u \| = \sqrt{(\Delta u_1)^2 + (\Delta u_2)^2 + \cdots} \]

The fractional norm is the ratio \( \| \Delta u \| \) and when, for subsequent iterations, this value reaches the magnitude smaller than a predetermined value, say, \( 10^{-6} \), the iteration stops and the solution is thus obtained.
SECTION IV
HYDROSTATIC BULGING

1. Introduction

The ductility of sheet metal under biaxial stress is often examined by means of the so-called bulge test. A uniform plane sheet is placed over a die with an aperture and is firmly clamped around the perimeter. An increasing hydrostatic pressure is applied to one side of the sheet, causing it to bulge through the aperture. From the measured profile and thickness of the plastically deformed sheet near the pole, it is possible to calculate the local state of stress in terms of the applied pressure. If, in addition, the state of strain is measured by means of a grid, the stress-strain characteristics of the metal under biaxial tension are obtained. The advantage of this test over any other simple one is that a greater range of pre-instability strain can be obtained.

Hydrostatic bulging is not only important as a material property test, but also as a forming operation. Thus, a number of theoretical investigations, dealing with axisymmetric hydrostatic bulging (Fig. 2) has appeared in the literature.

The classical analysis of bulging is the one by Hill [23]. His solutions are, however, special ones. Instead of analyzing deformation with a given stress-strain characteristic, Hill first adopted special kinematic assumptions and from them deduced the necessary stress-strain characteristics which satisfy all the governing equations under the prescribed kinematic mode. The kinematic assumptions are first, that any material element describes a circular path which is, moreover, orthogonal to the
momentary profile, and second, that circumferential strain is numerically equal to the tangential strain. The required stress-strain characteristic is found to be an exponential type. Hill's other solution on a linear workhardening solid uses the method of successive approximation by adopting a yield criterion which is neither von Mises nor Tresca, for the purpose of mathematical simplicity.

Analyses of work by Woo [24], Yamada [25], and Wang [26] are based upon the realistic choices of stress-strain characteristics and the yield criterion. In applying the deformation theory of rigid plasticity, Wang experiences a mathematical difficulty and attributes this to the fact that the differential equations associated with the deformation theory possesses a singularity which has the effect of restricting the range of calculation within a certain value of the polar strain. Besides, the agreement of deformation theory predictions with the experiment is rather poorer than the incremental theory prediction [27].

In applying the incremental theory of rigid plasticity, researchers experience a difficulty in satisfying the boundary condition at the fixed edge, i.e., \( \dot{\epsilon}_0 = 0 \). To avoid this difficulty, Woo uses the deformation theory, while Yamada reasons that introducing an elastic strain component into the formulation will resolve this "mathematical difficulty" (in Yamada's terms) and turns to the elasto-plastic constitutive law. Another theoretical work of interest comes from Wang, using the parametric representation of the stresses.
The only published solution on hydrostatic bulging using the finite-element method is the one by Iseki et al. [28], with the incremental theory of elasto-plasticity.

2. Computational procedures

In adopting the finite-element model to hydrostatic bulging, it is necessary to reconsider the external work increment term, since the pressure is uniform over the entire surface of a closed shell. In this case the increment of external work may be written as [29], [30],

\[ \Delta w = p \bar{V} \bar{V} , \]

where \( \bar{V} \bar{V} \) is the increase of the volume enclosed by the deformed sheet and \( p \) is the pressure acting on the deformed configuration.

As an initial condition, Hill's special solution is utilized. In other words, the initial profile of the bulge is assumed to be a part of a sphere whose radius is given by \( r = \frac{1}{2} \left( \frac{a^2}{h} + h \right) \), where \( a \) is the radius of the original blank and \( h \) is the polar height at the moment. With this geometry, a pressure \( p \) is prescribed. This pressure should be greater, at least, than the pressure which makes the sheet having initial geometry everywhere plastic. The initial guess on the incremental displacement is also obtained from Hill's special solution by assuming normal trajectory of the element particle to the bulge profile. The program for computing the initial guess is given in Appendix A.

When a converged solution is obtained for the given pressure, a new bulge profile is determined from the initial bulge profile and incremental displacement grid. Then the pressure is assigned a higher value and the converged solution for the previous step is used as the initial guess for the incremental displacement field and the computation continues in this way. The program for the analysis of hydraulic bulge is given in Appendix B.
3. Results and discussion

To examine the validity of the present FEM for hydrostatic bulging, the solution is compared with those achieved by the elasto-plastic FEM and the experiment.

The following conditions were employed for the comparison with the elasto-plastic FEM:

Workhardening characteristics: \( \bar{\sigma} = 105(0.0019 + \bar{\varepsilon})^{0.2} \times 10^9 \text{kg/m}^2 \)
\[ = 1.036(0.0019 + \bar{\varepsilon})^{0.2} \times 10^9 \text{N/m}^2 \]

Thickness: \( 3.0 \times 10^{-4} \text{m} (= 0.3 \text{ mm}) \)

Radius of the sheet: \( 2.4 \times 10^{-2} \text{m} (= 24 \text{ mm}) \)

Anisotropy parameter: 1.0

An identical problem was also solved by Yamada [25], using the finite-difference method with the elastic-plastic theory. Fig. 3 shows the relationship between hydrostatic pressure and the polar thickness strain. The solid line represents the elasto-plastic FEM (and also the finite-difference method) and the points indicate the solution given by the rigid-plastic FEM. The deviation of the first point by the rigid-plastic FEM is thought to reflect the approximation involved in the initial condition that the sheet is everywhere plastic and that the initial geometry is a part of a sphere. The solution can be improved numerically by taking a smaller value of \( h \) in generating the initial condition. Nevertheless, the solutions after this first step are in extremely good agreement with the elasto-plastic FEM and any disturbance in the initial conditions does not matter after an initial deformation of a small magnitude. The pressure increment is raised by twice after some deformation and it is to be noted that the solutions
Figure 3. Hydrostatic pressure vs. polar thickness strain.
with the larger pressure increment size are still accurate. This means that the method is computationally economical with a reasonable accuracy. After the last point in the diagram the solution diverges and it is thought that the pressure maximum has been reached. The convergence is excellent; in every step, five to seven iterations seem to be sufficient. Fig. 4(a), (b) show the comparisons of strain distributions. The circumferential strain distributions are in good agreement. The tangential strain distribution by the rigid-plastic FEM deviates somewhat at the edge from that by the elasto-plastic FEM. The tangential strain is more sensitive to the method employed than the circumferential strain, but this deviation of tangential strain is not serious because the solution closely follows that by the finite-difference method and we may conclude that the strain distribution is accurately predicted. Fig. 5 shows the distributions of stresses when the polar thickness strain is (-0.4). Fig. 6 shows the bulge profile at some stages of deformation. A number of material elements are traced during deformation and are shown on each bulge profile.

Next, the solution is compared with Mellor's [31] experiment on half-hard aluminum.

Workhardening characteristics: \[ \sigma = 15,460(1 + 0.76\varepsilon) \text{ psi} \]
\[ = 1.066(1 + 0.76\varepsilon) \times 10^8 \text{N/m}^2 \]

Radius of the sheet: 5.0 inches = 1.27 m
Thickness: .035 inch = 8.89 \times 10^{-4} m
Anisotropy parameter: 1.0

One thing to be mentioned is that in the actual experiment, the die has a round profile of radius \( \frac{3}{8} \) in., but in the analysis this profile has been
Figure 4. (a) and (b) Distribution of Strains.
Figure 5. Distribution of Stresses

Figure 6. Bulge Profile
neglected. Fig. 7 shows that the agreement of the relation between pressure and polar height is good. The agreement in the bulge profile is also excellent, as in Fig. 8. As shown in Fig. 9(a), (b), the theoretical circumferential strain still closely predicts the experimental one, but there is some discrepancy in thickness strain distribution. As has been mentioned, the actual die has a round profile which has been neglected in the analysis, and it is thought that the thickness strain is more sensitive to the profile than is the circumferential strain. Initially, there is virtually no discrepancy, but increases at later stages. This may be explained by the fact that initially the sheet is not in contact with the profile, but as deformation continues, more of the sheet is brought into contact with the profile and makes the actual situation different from the one used in the analysis.

In general, the theoretical prediction by the rigid-plastic FEM is in good agreement with both the experiments and the analyses by the elastoplastic FEM and the finite-difference method.
Figure 7. Polar Height vs. Pressure
Figure 8. Bulge Profile
SECTION V
STRETCHING OF A SHEET WITH HEMISPHERICAL PUNCH

1. Introduction

Punch stretching is commonly used to assess the pressing quality of sheet metals. A circular sheet is clamped firmly along the periphery and is stretched by a rigid punch of hemispherical shape. The depth of the deformed sheet when it fractures is usually taken as a measure of ductility. See the schematic diagram in Fig. 10.

An experimental investigation of punch stretching as a forming problem dates back to Loxely and Freeman [32], who demonstrated that the interfacial friction between the punch and the sheet has a significant effect on the strain distribution in the sheet and, consequently, on the location of fracture and dome height when the sheet fractures. Keeler and Backofen [33], in characterizing the limit stretching, followed the strain history of each element with the progress of deformation and observed the occurrence of discontinuity in tangential strain at a certain element, which was subsequently interpreted as the onset of diffuse necking [34]. Based upon Hill's analysis [35], they believed that localized necking is not possible in punch stretching, but that only diffuse necking takes place, increasing the overall nonuniformity of straining.

The observation of localized necking in situations where Hill's analysis denies one has been well established in the case of in-plane stretching and has prompted the development of Marciniak and Kuczynski's theory [36], [37]. In punch stretching, Gosh and Hecker [38] observed localized necking and reported that local necking sets in even though
Figure 10. Schematic View of the Stretching of a Sheet with a Hemispherical Head Punch
the plane-strain condition, which is thought to be responsible for local
necking in in-plane stretching, is not achieved. This is attributed to the
fact that in punch stretching an increment in tangential strain is geometri-
cally tied to an increment in circumferential strain and, therefore, the
approach to plane-strain condition becomes slower. Another experimental
investigation of punch stretching is the one by Alexander and Kaftanoglu
[39]. They observed that the deformation is limited by the "strain
propagation instability" or, local necking in common terminology, and not
by "maximum load instability" or, diffuse necking.

From the viewpoint of the deformation analysis, punch stretching is a
complicated problem because a moving boundary separates the region in contact
with the punch head from the unsupported one. The friction over the punch
head gives rise to additional complications. One special solution is by
Chakrabarty [40]. Following the line of Hill's special solution on hydro-
static bulging he obtained an analytical solution for a special material
having exponential type stress-strain characteristics. For more general
materials the only solutions available are the numerical ones. Numerical
solutions of importance are those by Woo [41] and by Wang [42], [43].

Woo's and Wang's solutions were obtained by the finite-difference
method. The only solution by the finite-element method on punch stretching
is one by Wifi [44]. His elasto-plastic, finite-element model does not
neglect the bending moment nor the effect of shear stress and uses two-
dimensional triangular elements to take the thickness variation into
account. Friction, which is of primary significance compared with the
secondary effect of bending and thickness, is assumed to be perfect, meaning
that once the element touches the punch head, it does not slide over the
punch but sticks to it.
2. Computational procedures

In applying the finite-element method to punch stretching, a thought should be given to the implementation of boundary conditions. The boundary conditions in punch stretching are stated not only by prescribing tractions and incremental displacements but sometimes by their ratios. In this report, the problem is similar to the ball indentation problem (Lee et al. [45]).

The radial and vertical positions of the material elements in the contact region are not independent but they are related to each other through a mathematical expression for the geometrical requirement that they must be actually on the surface of the punch head. The expression is

\[(r_0 + v)^2 + (c + z_0 + w)^2 = r_p^2, \quad (46)\]

where \(r_0, z_0\) are radial and vertical positions of the element at the present undeformed configuration; \(v, w\) are the increments of horizontal and vertical displacements, and \(c\) is a parameter related to the punch height \(h\) by the expression

\[c = r_p - h.\]

See Fig. 11. Recall that the finite-element formulation in Chapter IV has already been linearized and what it really solves for are the perturbation terms. Therefore, we also linearize the boundary condition (46) to obtain

\[2(r_0 + v^*) \Delta v + 2(c + z_0^* + w^*) \Delta w = r_p^2 - (r_0 + v^*)^2 - (c + z_0^* + w^*)^2, \quad (47)\]

where starred (*) quantities are initial guesses, and \(\Delta v, \Delta w\) are perturbations. By rearranging (47), we have
Figure 11. Geometrical Requirement for the Node on the Contact Region
\[ \Delta v = \frac{1}{\alpha} \Delta w + \beta, \quad \text{(48a)} \]

where

\[ \alpha = \frac{(r_0 + v^*)}{(c + z_0 + w^*)} = \frac{1}{\tan \theta}, \quad \text{(48b)} \]

and

\[ \beta = \frac{r_0^2 - (c + z_0 + w^*)^2 - (r_0 + v^*)^2}{2(r_0 + v^*)}, \quad \text{(48c)} \]

When the finite-element model is implemented, all the tractions are transformed into generalized nodal forces. Therefore, it is convenient to write the boundary condition in terms of the generalized nodal forces \( \pi(r) \) and \( \pi(z) \), the horizontal and vertical components, respectively. See Fig. 11.

Now

\[ \pi(r) = N \cos \theta - S \sin \theta \]
\[ \pi(z) = N \sin \theta + S \cos \theta \quad \text{(49)} \]

where \( N \) and \( S \) are generalized forces normal and tangential to the punch head. We eliminate \( N \) through the relation

\[ \cos^2 \theta + \sin^2 \theta = 1 \]

and obtain

\[ \pi_z \cos \theta = \pi_r \sin \theta + k, \quad \text{(50)} \]

where \( k \) is the frictional force at nodes. However, from geometry we know that the following holds:
\[
\cos \theta = \frac{r_0 + v^*}{r_p}, \quad (51a)
\]
\[
\sin \theta = \frac{z_0 + w^* + c}{r_p}. \quad (51b)
\]

So, (50) may be written as
\[
\pi(z) + \frac{\pi(r)}{\alpha} = \frac{kr_p}{(r_0 + v^*)}. \quad (52)
\]

If the die has a round profile of the radius \(r_D\), then the requirement for a material element to lie geometrically on the profile is similar to the requirement to be satisfied on the punch head. Therefore, we have (similar to Eq. (46)),
\[
(a - r_0 - v)^2 + (r_D - z_0 - w)^2 = r_D^2, \quad (53)
\]
where \(a\) is the radius of the sheet. Linearization of Eq. (53) gives
\[
2(a - r_0 - v) \Delta v + 2(r_D - z_0 - w) \Delta w = -r_D^2 + (a - r_0 - v)^2 + (r_D - z_0 - w^*)
\]

or, rewriting,
\[
\Delta v = \frac{\Delta w}{\gamma} + \Omega, \quad (55a)
\]
where
\[
\gamma = \frac{(a - r_0 - v^*)}{(r_D - z_0 - w^*)}, \quad (55b)
\]
and
\[
\Omega = \frac{(a - r_0 - v^*)^2 + (r_D - z_0 - w^*)^2 - r_D^2}{2(a - r_0 - v^*)}. \quad (55c)
\]
The tractional boundary condition over the die profile can be written similarly as
\[
\pi(z) + \frac{\pi(r)}{\gamma} = -\frac{kY_D}{(a - r_0 - v^*)}.
\] (56)

For the portion of the sheet which is not in contact with the punch head nor with the die profile, the displacement increment in the radial direction and the displacement increment in the axial direction are not bound to each other, as is the case for the contact region, but remains as independent variables. Tractions are, however, given the value of zero.

With the advancement of the punch head, the portion of the sheet in contact with the punch or die profile increases and, consequently, the boundary separating this "contact region" from the "unsupported region" changes. The presence of this moving boundary is always a source of complications in the numerical analysis of punch stretching because it requires a basically trial-and-error approach. The treatment of the moving boundary used in the present analysis for punch stretching without round die corners is explained in detail as follows:

First, assume the position of the boundary in future configurations. In the FEM this means assuming which nodes will be in contact with the punch head in the future configuration. Then, obtain a converged solution based upon this assumption and check to see if it is true. Since the position of the boundary is already known in the current configuration, in practice we assume and check how much this boundary advances.

(1) Check whether the boundary is assumed to advance too fast.

Compute the normal component of the generalized nodal force
to the punch head for the nodes in contact with the punch head. If every generalized normal force is directed outward from the punch head, then all the nodes which are assumed to be in contact with the punch head actually do so. On the other hand, the generalized normal force in the direction toward the punch head for any node means that external force other than the one exerted by the punch is necessary for this particular node to conform with the punch geometry. Since there is physically no source of applied force other than the punch, the assumption that this particular node is in contact with the punch head is not correct and the position of the boundary should be re-assumed to exclude this node from the contact region.

(2) Check whether the boundary is assumed to advance too slowly. Compute the distance between the nodes in the unsupported region and the center of the hemisphere of the punch head. If this distance is shorter than the radius of the punch head for any node, it means that this particular node is inside the punch head. Since this is physically impossible, the assumption that this particular node is not in contact with the punch head is not correct and the position of the boundary should be re-assumed to include this particular node.

Although this basically trial-and-error approach seems to be very time consuming, in actual computation we can predict the movement of the boundary fairly accurately based upon the distance between the free nodes and the punch surface. Furthermore, since we already know the position of the boundary in the current configuration, it is enough to check the
boundary assumption for only a few nodes neighboring the previous position of the boundary, not for whole nodes.

The procedure described above for the contact region on a punch head is also applicable for the contact region at the die corner. Handling two moving boundaries simultaneously really does not invoke any new theoretical difficulties but only takes more computation time and may be impractical for inefficient numerical methods.

In order to implement Coulomb friction between sheet and punch or die, we first prescribe a tangential friction force \( S \) and obtain a converged solution and then compute generalized nodal forces. From Eqs. \((49)\) we then are able to compute the normal component \( N \) and the friction coefficient \( \mu = \frac{S}{N} \) corresponding to the initially prescribed value of \( S \). If the computed friction coefficient is not what is intended, then we modify the \( S \) value and repeat the process. It should be noted here that the correction of frictional force \( S \) needs the necessary modification only in the \( F \) matrix (Eq. \((44)\)), while the stiffness matrix \( P \), which is the most time-consuming part, remains the same.

The deformation step is controlled by the punch head increment, which is designed in the present codes to yield the maximum increment of effective strain roughly equal to a preset value. In the present work the optimum size is shown to be a 0.04 increment of effective strain. The solution generally converged after \( 10 \sim 15 \) iterations for a single step within the fractional norm of \( 10^{-6} \). The actual program is shown in Appendix C.

3. Results and discussion

The present rigid-plastic FEM is compared with the finite-difference methods by Wang [43] and Woo [41], and also with the experiment by Kaftanoglu and Alexander [39].
(1) Comparison with the finite-difference solution by Wang

The parameters used in Wang's example are as follows:

Material: copper

Stress-strain characteristics: \( \sigma = 30.5 \varepsilon^{0.326} \text{ ton/in.}^2 \)

\[ = 4.6361 \varepsilon^{0.326} \times 10^8 \text{ N/m}^2 \]

Anisotropy: \( R = 1.0 \)

Friction: \( \mu = 0.04 \)

Thickness: \( t = 0.035 \text{ in.} = 8.89 \times 10^{-2} \text{ m} \)

Punch radius: \( r_p = 1.0 \text{ in.} = 2.54 \times 10^{-2} \text{ m} \)

Radius of sheet: \( a = 1.15 \text{ in.} = 2.921 \times 10^{-2} \text{ m} \)

Initial radius is sometimes denoted by \( r_0 \).

The two methods are in excellent agreement in predicting the punch head for a given punch travel. See Fig. 12; the solid line represents Wang's solution and the points represent the rigid-plastic FEM. Fig. 13 shows the thickness strain distribution. Again, a good agreement between the two solutions is apparent.

The second example has the following parameters:

Stress-strain characteristics: \( \sigma = k \varepsilon^{0.2} \)

Anisotropy: \( R = 1.0 \)

Friction: \( \mu = 0.2 \)

Punch radius: \( r_p = 1.0 \)

Radius of sheet: \( r_0 = 1.0 \)

In Wang's work all the results are reported in the dimensionless number. Figs. 14 and 15 show the circumferential strain distribution and thickness strain distribution, respectively. The solid line represents Wang and points
Figure 12. Punch Head vs. Punch Travel
Figure 13. Thickness Strain Distribution
Figure 14. Circumferential Strain Distribution

Figure 15. Thickness Strain Distribution
represent the rigid-plastic FEM. Excellent agreement of the two solutions is demonstrated.

The step size has an important effect upon the accuracy and efficiency of the solution. The smaller the step size, the better the accuracy, although more computation time is required. Fig. 16 demonstrates that there is a limit to increasing efficiency while maintaining accuracy. For example, solutions with a step size of 0.08 in the effective strain increment deviates considerably from the solutions obtained with step sizes of .02 or .04. In the remainder of the work the step size of .04 is most often used.

Compared with this significant effect of step size, the mesh size does not exert a great influence upon the solution, as is demonstrated in Fig. 17. The solution with a coarse mesh (10 elements) is essentially the same as the one with a finer mesh (40 elements), even though the latter will be helpful in pinpointing the exact location of peak strain.

In the examples above, there is only one moving boundary, that between punch and sheet, since the presence of the round die profile is neglected. In practice, the die always has a round profile and as the radius of the profile gets larger, it becomes necessary to include the die profile in the analysis. In this case there are two moving boundaries, the second being the one between sheet and die. The only work reported which includes the die profile into the analysis is the one by Woo.

(2) Comparison with the finite-difference solution by Woo

The parameters in Woo's example are:

Stress-strain characteristics: \( \sigma = 5.4 + 27.8 \varepsilon^{0.504} \text{ ton/in.}^2 \)

For \( \varepsilon < 0.36: \)

\[ = (0.08208 + 0.422569 \varepsilon^{0.504}) \times 10^9 \text{N/m}^2 \]

For \( \varepsilon > 0.36: \)

\[ = (0.08208 + 0.37089 \varepsilon^{0.375}) \times 10^9 \text{N/m}^2 \]
Figure 16. Effect of Step Size

Figure 17. Effect of Mesh Size
Material: copper

Punch radius: 1 in. = 2.54 × 10^{-2} m

Die profile radius: 0.3 in. = 7.62 × 10^{-3} m

Radius of sheet: 1.3 in. = 3.302 × 10^{-2} m

Coefficient of friction: 0.04

Thickness of sheet: 0.035 in. = 8.89 × 10^{-4} m

Figs. 18 and 19 are the thickness strain distribution and the circumferential strain distribution. Solutions by Woo are represented by solid lines and the solutions by the rigid-plastic FEM are represented by points. Agreement between the two solutions is excellent for most of the deformation. However, at later stages of deformation, a discrepancy is observed around the edges. Re-examining Woo's computational procedure reveals that in order to avoid the difficulty of satisfying boundary conditions exactly along the fixed edge (\( \varepsilon_0 = 0 \)), he allowed a small increment of circumferential strain along the edge at each stage. In the present rigid-plastic FEM such difficulty does not exist, so there is no need to relax the boundary condition. The discrepancy observed at later stages of deformation may be attributed to this difference.

With regard to the instability, Woo stated that it occurs when the resultant tangential stress determined from the strain hardening characteristics cannot obtain the value required for the equilibrium and at that instant he stopped the computation. In the present rigid-plastic analysis such an instability is not observed at the point reported by Woo, and the computation continues.

(3) Comparison with the experiment by Kaftanoglu and Alexander

The parameters of Kaftanoglu and Alexander's experiment on soft copper are:
Figure 18. Distribution of Thickness Strain When Die Profile Is Considered

Figure 19. Distribution of Circumferential Strain When Die Profile Is Considered
Stress-strain characteristics: $\sigma = 68,394(0.0122 + \varepsilon)0.3789$ psi

$= 4.7156 \times 10^8 \text{N/m}^2$

Thickness: 0.048 in. = $1.219 \times 10^{-3}$ m

Friction condition: PTFE film lubricant

Radius of the sheet: 0.717 in. = $1.821 \times 10^{-3}$ m

Punch radius: 0.65 in. = $1.651 \times 10^{-3}$ m

Kaftanoglu reports that the friction condition changes with deformation and measures three different friction coefficients: $\mu = 0.2$ at stage 1, $\mu = 0.135$ at stage 2, and $\mu = 0.07$ at stage 3. To include the changing friction coefficient into the analysis, we need more information on the friction history, which is difficult to obtain experimentally. Therefore, as a representative value, we use the mean of three values of the friction coefficient, $\mu = 0.135$, for our computation. Figs. 20 and 21 show the distribution of the circumferential strain and the thickness strain. The agreement between the experimental data and the numerical solution is a reasonable one considering the fact that the exact friction condition is not known.

(4) Influence of formulation of constitutive relation

Various formulations have been given for plastic stress-strain relationships of workhardening materials. Among them, the parabolic hardening law has been used extensively for sheet metals because of the ease with which it characterizes workhardening properties of materials. However, it was suggested recently [46] that the Voce equation [47] is a better representation of materials behavior when solving plasticity problems involving workhardening rate. The forming limit curves were compared using the
Figure 20. Comparison of the Numerical Solution with the Experimental Data for Circumferential Strain Distribution

Figure 21. Comparison with the Experiment for Thickness Strain Distribution
parabolic hardening law and the Voce equation [48], and the result indicates
an importance of the choice of workhardening representation of materials.
Because the term containing the rate of workhardening appears in the
finite-element formulation of sheet-metal forming, it is of importance to
examine the influence of workhardening representation on the mechanics
computed by the finite-element method. The material is aluminum alloy
2036-T4. The parameters are as follows:

Stress-strain characteristics: $\sigma = 86,000(\varepsilon)^{0.222}$ psi for the parabolic
hardening law

$\sigma = 65,000(1 - (1 - 0.508)\exp(-8.51\varepsilon))$ psi
for the Voce equation

Fig. 22 shows the two stress-strain curves together with tension test data
from the specimens cut in the three directions ($0^\circ$, $45^\circ$, $90^\circ$).

\footnote{(1) The stress and strain values in tension tests in the three directions
were converted to values of the effective stress and effective strain
according to

(a) Tension in the $0^\circ$ (rolling) direction:

$$\bar{\sigma} = \frac{\sqrt{3}}{2} \sqrt{\frac{r_{90} + r_{0}r_{90}}{r_{0} + r_{90} + r_{0}r_{90}}} \sigma_{0}, \quad \bar{\varepsilon} = \frac{\sqrt{2}}{3} \sqrt{\frac{r_{0} + r_{90} + r_{0}r_{90}}{r_{90} + r_{0}r_{90}}} \varepsilon_{0};$$

(b) Tension in the $45^\circ$ direction:

$$\bar{\sigma} = \frac{\sqrt{3}}{2} \sqrt{\frac{(r_{0} + r_{90})(1 + r_{45})}{2(r_{0} + r_{90} + r_{0}r_{90})}} \sigma_{45}, \quad \bar{\varepsilon} = \frac{\sqrt{2}}{3} \sqrt{\frac{2(r_{0} + r_{90} + r_{0}r_{90})}{(r_{0} + r_{90})(1 + r_{45})}} \varepsilon_{45};$$

(c) Tension in the $90^\circ$ direction:

$$\bar{\sigma} = \frac{\sqrt{3}}{2} \sqrt{\frac{r_{0} + r_{0}r_{90}}{r_{0} + r_{90} + r_{0}r_{90}}} \sigma_{90}, \quad \bar{\varepsilon} = \frac{\sqrt{2}}{3} \sqrt{\frac{r_{0} + r_{90} + r_{0}r_{90}}{r_{0} + r_{0}r_{90}}} \varepsilon_{90};$$

(Footnote continued on next page)
Figure 22. Stress-strain Curve for Al. 2036-T4
r-value: $r_0 = 0.66$, $r_{45} = 0.69$, $r_{90} = 0.70$, and $r_a = 0.685$

Radius of die opening: (0.80 in.)

Blank thickness:

Radius of punch head: 0.75 in. and 0.45 in.

Coefficient of friction: 0 and 0.2

Punch stretching was performed on a horizontal hydraulic press. Tests were interrupted for strain measurements (thickness and circumferential strains) from the grids photoprinted on the specimen. Load-displacement relationships were also recorded. First, the experimental strain distributions were compared with computed results, using the parabolic hardening law in Fig. 23. In the experiment Johnson's wax was used as the lubricant and was applied at each stage. In comparison, two discrepancies are apparent: (i) the coefficient of friction does not stay constant; particularly, at the last stage, the experimental strain distributions indicate that the coefficient of friction is less than 0.2, which, however, gives good agreement for other stages, and (ii) the measured thickness and circumferential strains for a given punch depth do not follow the corresponding theoretical curves. This is attributed to the fact that the accurate strain measurements is extremely difficult for critical comparison between theory and experiment. The load values summarized in Table 1 show an excellent agreement between the two.

where $r_0$, $r_{45}$, $r_{90}$ are the r-values obtained from the tension of specimens cut in the $0^\circ$, $45^\circ$, and $90^\circ$ directions, respectively.

(2) The effective stress and effective strain defined in the formulation of this report differ from the definition above by a factor such as

$$
\bar{\sigma} = \sqrt{\frac{3}{2} \frac{1 + \frac{r}{r_a}}{2 + \frac{r}{r_a}}}, \quad \bar{\varepsilon} = \sqrt{\frac{2}{3} \frac{2 + \frac{r}{r_a}}{1 + \frac{r}{r_a}}},
$$

where $r_a$ is the average r-value defined by $r_a = \frac{r_0 + 2r_{45} + r_{90}}{4}$. 

60
### Table 1

**PUNCH LOAD AND DISPLACEMENT RELATIONS**

**Punch head radius = 19.05 mm (0.75 in.)**

<table>
<thead>
<tr>
<th>Displacement (in.)</th>
<th>Theoretical</th>
<th></th>
<th>Experimental</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Punch load</td>
<td></td>
<td>Punch load</td>
<td></td>
</tr>
<tr>
<td></td>
<td>mm (in.) N</td>
<td></td>
<td>(lb) N</td>
<td></td>
</tr>
<tr>
<td>4.06 (0.160)</td>
<td>6,330 (1,423)</td>
<td>(970)</td>
<td>4,315</td>
<td>2.79 (0.110)</td>
</tr>
<tr>
<td>6.10 (0.240)</td>
<td>10,889 (2,448)</td>
<td>(1,730)</td>
<td>7,695</td>
<td>4.83 (0.190)</td>
</tr>
<tr>
<td>7.54 (0.297)</td>
<td>14,483 (3,256)</td>
<td>(2,990)</td>
<td>13,300</td>
<td>7.11 (0.280)</td>
</tr>
<tr>
<td>9.80 (0.386)</td>
<td>22,059 (4,959)</td>
<td>(4,940)</td>
<td>21,974</td>
<td>10.08 (0.397)</td>
</tr>
<tr>
<td>12.45 (0.490)</td>
<td>30,301 (6,812)</td>
<td>(6,580)</td>
<td>29,269</td>
<td>12.45 (0.490)</td>
</tr>
</tbody>
</table>

**Punch head radius = 11.43 mm (0.45 in.)**

<table>
<thead>
<tr>
<th>Displacement (in.)</th>
<th>Theoretical</th>
<th></th>
<th>Experimental</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Punch load</td>
<td></td>
<td>Punch load</td>
<td></td>
</tr>
<tr>
<td></td>
<td>mm (in.) N</td>
<td></td>
<td>(lb) N</td>
<td></td>
</tr>
<tr>
<td>4.06 (0.160)</td>
<td>5,124 (1,152)</td>
<td>(920)</td>
<td>4,092</td>
<td>2.72 (0.107)</td>
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<tr>
<td>6.10 (0.240)</td>
<td>8,131 (1,828)</td>
<td>(2,000)</td>
<td>8,896</td>
<td>6.30 (0.248)</td>
</tr>
<tr>
<td>8.53 (0.336)</td>
<td>12,237 (2,751)</td>
<td>(2,770)</td>
<td>12,322</td>
<td>8.18 (0.322)</td>
</tr>
<tr>
<td>9.58 (0.377)</td>
<td>13,963 (3,139)</td>
<td>(3,130)</td>
<td>13,923</td>
<td>9.68 (0.381)</td>
</tr>
</tbody>
</table>
Figure 23. Experimental (Johnson's wax as lubricant) and Theoretical ($\mu = 0.2$) Strain Distributions for Punch Size ($r_p/r_0 = 0.75/0.80$). (a) Thickness Strains; (b) Circumferential Strains
Figure 24. Experimental (Johnson's wax as lubricant) and Theoretical ($\mu = 0.2$) Strain Distributions for Punch Size ($r_p/r_0 = 0.45/0.80$).
(a) Thickness Strains; (b) Circumferential Strains
For a smaller punch size, the strain distributions are compared in Fig. 24. The same observations as those in Fig. 23 apply. Again, the punch load is in good agreement.

The influence of workhardening representations on the detailed mechanics is examined in Figs. 25, 26, 27, and 28. Referring to Fig. 25, the general trend of strain distributions is not altered by the workhardening representation. However, the magnitude of strains, particularly, peak strains, differ. With the Voce equation, the peak strains are larger than those computed by the parabolic workhardening law. This difference becomes larger as the punch penetrates.

It appears that the difference of the two is more significant for higher friction in the larger punch size. However, in the smaller punch size, the difference of the two strain distributions is about the same for the two coefficients of friction, 0 and 0.2, as shown in Fig. 26.

It is rather surprising to find in Figs. 27 and 28 that the punch load for the same punch displacement is higher with the parabolic workhardening law than that with the Voce equation. The difference becomes significant for large punch penetration. From these results, it is concluded that the representation of the workhardening characteristics of the material does have an influence on the computed strain distributions and load-displacement relationships. The difference becomes critical for large punch displacement in predicting both peak strains and the punch load. In order to determine which representation is preferable, however, more experiments with improved accuracy and control are needed.
Figure 25. Comparison of Theoretical Thickness Strain Distributions Using (1) the Parabolic Work-hardening Law and (2) the Voce Equation for Punch Size ($r_p/r_0 = 0.75/0.80$) with (a) $\mu = 0$ and (b) $\mu = 0.2$
Figure 26. Comparison of Theoretical Thickness Strain Distributions Using (1) the Parabolic Workhardening Law and (2) the Voce Equation for Punch Size ($r_p/r_0 = 0.45/0.80$) with (a) $\mu = 0$ and (b) $\mu = 0.2$
Figure 27. Comparison of Theoretical Load Displacement Curves Using (1) the Parabolic Workhardening Law and (2) the Voce Equation for $\mu = 0$
Figure 28. Comparison of Theoretical Load Displacement Curves Using (1) the Parabolic Workhardening Law and (2) the Voce Equation for \( \mu = 0.2 \)
1. Introduction

In a deep drawing test a circular sheet of metal is placed between the blank holder and the die and then fully drawn into the shape of a cup. The formability is then measured by the maximum size of the blank which can be drawn without a failure, or, more often, by its ratio to the punch diameter. This ratio is called the limiting drawing ratio and this particular kind of test is called the Swift test.

Deep drawing is not only a useful method of material testing, but also one of the basic operations in sheet-metal stamping. In practice, various shapes are possible for the bottom of the punch; however, most past investigations are on deep drawing with a flat-bottomed punch [49]-[56].

Among the earlier works on deep drawing are those by Hill [13] and by Chung and Swift [52] using the incremental theory of plasticity. More refined analyses are the finite-difference solutions by Chiang and Kobayashi [57], Wang and Budiansky [51], and by Chakrabarty and Mellor [49]. Even though such a refinement improves the understanding of the deep drawing process, their works are not complete because they treat the deep drawing problem as an in-plane pure radial drawing and are concerned mostly with the deformation mechanics on the flange. However, it has been observed experimentally (Chung and Swift [52]) that the die profile and the punch profile significantly affect the punch load and the strain distributions and therefore a further refinement is necessary by considering these parameters in the analysis. Woo [53] performs such an analysis and then
is able to show that the solution obtained by extrapolating the strain distribution over the flange to the die throat predicts more straining than the one obtained by taking the profiles into consideration.

Contrary to these numerous investigations on deep drawing with a flat-bottomed punch, very few works are reported on the deep drawing of a sheet with a hemispherical head punch (Fig. 29). Woo [58] analyzes this problem by breaking down the deep drawing process into two component processes of the pure radial drawing over the flange and the punch stretching over the hemispherical punch head. He first obtains solutions for pure radial drawing in the flange and then uses this solution at a point initially situated near the die lip as the boundary condition for the stretching problem, and thereby essentially matched the punch stretching component with the pure radial drawing component at a particular point in the die profile region.

Instead of this tedious process of boundary matching, it is desirable to have a numerically efficient and reliable method which can treat the problem in a unified manner. The FEM is such an alternative. The finite-element model developed for the deep drawing problem is the one by Wifi [44] with a limited treatment of friction. Also, Levy et al. [59] developed the elasto-plastic finite-element program for cupdrawing based on the deformation theory of plasticity.

2. Computational procedure

The entire sheet undergoing the deep drawing process can be divided into four regions: the contact region with the punch head, the unsupported region, the contact region with the die profile, and the flange over the die. Different kinds of boundary restrictions are imposed depending upon
Figure 29. Schematic View of Deep Drawing of a Sheet with a Hemispherical Head Punch
the regions. For example, the flange is constrained to move only horizontally along the die face, while the contact region with the die profile or punch head should satisfy the kind of boundary conditions discussed in Section V.

The only difference in deep drawing with a hemispherical head punch from the punch stretching with a round die corner is the presence of the flange which is free to slide over the die. The addition of this moving flange is, in effect, equivalent to the addition of the third moving boundary, because, even though the boundary separating the flange from the die profile remains stationary in the space, it continues to move from the viewpoint of the deforming sheet. To treat this we make an assumption on this third moving boundary and see if it is true by checking the radial positions of the nodes. If the new radial position of any node which is assumed to lie on the flange or the die profile does not fall on the expected region after converged solution is obtained, then the boundary assumption is modified.

Another point to be mentioned is the blank holding condition of which there are two types: clearance holding and force holding. The idealization of the deformation state corresponding to the force blank holding is the plane stress state and the one corresponding to the clearance holding in the plane-strain state. The present rigid-plastic FEM is built to handle the plane stress state deformation and therefore a modification is necessary to handle the clearance blank holding. No reported work on deep drawing with a hemispherical head punch under clearance blank holding is available and therefore in the present work only the deep drawing with the force holding is analyzed. The blank holding force is implemented in the formulation as a tangential friction force acting on the last node located at the
rim of the sheet. The distribution of the blank holding force over a finite area near the rim can be handled without difficulty in the present FEM, but this distributional effect turns out to be insignificant [53]. Therefore, tangential frictional force is confined to the last node at the rim of the sheet. The increment of deformation is controlled by the punch head movement. The program is in Appendix D.

3. Results and discussion

The only available work on the complete analysis of deep drawing with the hemispherical head punch is one by Woo [58]. Along with the numerical solution by the finite-difference method, he also conducted an experiment. The parameters are:

Material: soft copper

Stress-strain characteristics: \( \sigma = 5.4 + 27.8 \varepsilon^{0.504} \text{ ton/in.}^2 \)

for \( \varepsilon < 0.36 \):

\[ = (0.08208 + 0.422569 \varepsilon^{0.504}) \times 10^9 \text{N/m}^2 \]

\[ = 5.4 + 24.4 \varepsilon^{0.375} \text{ ton/in.}^2 \]

for \( \varepsilon > 0.36 \):

\[ = (0.08208 + 0.37089 \varepsilon^{0.375}) \times 10^9 \text{N/m}^2 \]

Blank radius: 2.2 in. = 5.588 \( \times 10^{-2} \) m

Radius of the die throat: 2.123 in. = 5.392 \( \times 10^{-2} \) m

Radius of die profile: 0.5 in. = 1.27 \( \times 10^{-2} \) m

Radius of punch head: 1 in. = 2.54 \( \times 10^{-2} \) m

Blank holding force: 0.5 ton = 500 kg

The solution by the rigid-plastic FEM is in excellent agreement with the experiment for the flange part; however, over the punch head it predicts more straining than the experiment when the friction coefficient of 0.04 is assigned for the contact region over the punch head and over the die in the
numerical analysis. When the friction coefficient is increased to a value of 0.1 over the punch head, while the same friction coefficient of 0.04 is used for the flange, the analysis predicts less straining over the punch head than the experiment. See Figs. 30, 31, 32, and 33. The deviation of the numerical solution from the experimental data gets larger as deformation progresses, which is reflected in the punch load vs. punch depth relationship in Fig. 34.

The lubricant used in the experiment is graphite in tallow and Woo suggested the friction coefficient to be 0.04. In the analysis, the practical difficulty always lies in the assignment of a reasonable value of friction coefficient because friction coefficient under a real sheet-metal forming condition is hard to measure and it may even change during deformation.

Comparison of Woo's numerical solution with the experimental data does not yield any better agreement than the present rigid-plastic FEM. In comparing his numerical solution with the experiment, Woo made the correction on the circumferential strain based upon the argument that the strain value obtained from the analysis is the value at the neutral surface of the sheet, while experimental data are obtained from the outside surface and therefore a compensation for the thickness difference is necessary. There could be a question about Woo's correction because the ratio of the punch radius or die profile radius to the sheet thickness is sufficiently large in his experiment that the membrane theory is justifiable. Besides, it seems a more consistent way to consider the problem in the three-dimensional stress state instead of the plane stress condition, which is the case used in Woo's analysis, if the variation of the strain across the thickness is to be taken into account.
Figure 30. Distribution of Thickness Strain for $\mu_p = 0.04$, $\mu_d = 0.04$
Figure 31. Distribution of Circumferential Strain for $\mu_p = 0.04$, $\mu_d = 0.04$
Figure 32. Distribution of Circumferential Strain for $\mu_p = 0.1$, $\mu_d = 0.04$
Figure 33. Distribution of Circumferential Strain for $\mu_p = 0.1$, $\mu_d = 0.04$

Figure 34. Punch Load vs. Punch Depth
It is necessary to have more numerical solutions and experimental data with a known friction state to assess the validity of the present rigid-plastic FEM for deep drawing problems. However, the present rigid-plastic FEM had dealt with other sheet-metal forming problems in a unified and consistent manner and therefore it seems reasonable to expect its validity for deep drawing problems when it is established for other problems.
SECTION VII
SUMMARY AND DISCUSSION

It has been made clear that classical variational formulations for the rigid-plastic solid are not appropriate for solving the sheet-metal forming problems. This is due to the nonuniqueness of the deformation mode under certain boundary conditions. This nonuniqueness, however, can be resolved by taking the workhardening rate into consideration. Such an introduction of the workhardening rate into the formulation, on the other hand, necessitates the consideration on the geometry change. The available classical formulation in which these two aspects are considered is not, however, applicable to the statically indeterminate problems, sheet-metal forming being one, because it is formulated in such a way that knowledge of stress distribution is necessary.

Within the framework of Eulerian descriptions and the hypothetical identity of the deformed configuration with the undeformed configuration, further improvement in the applicability of the variational formulations to the statically indeterminate problems is not possible. Therefore, an incremental deformation at a generic stage is considered by separating the deformed configuration from the undeformed configuration. The relevant equations are expressed with the undeformed configuration at each step as the reference frame and the variational formulation is established.

From this variational formulation a finite-element model is developed for the sheet-metal forming problems. In many sheet-metal forming processes the membrane theory is justifiable and therefore this idealization is introduced in building the model.
Three basic sheet-metal forming processes, i.e., the bulging of a sheet subject to the hydrostatic pressure, the stretching of a sheet with a hemispherical head punch, and deep drawing of a sheet with a hemispherical head punch are solved by the proposed method and its solutions are compared with the existing numerical solutions and the experimental data. The agreement is generally excellent and therefore the prime objective of the present investigation has been achieved.

In hydrostatic bulging the strain distributions and the pressure vs. polar height relationship predicted by the present rigid-plastic FEM are in excellent agreement with the available numerical solution by the elasto-plastic FEM and experimental data. The difficulty of satisfying the boundary condition along the fixed periphery experienced in the finite-difference method does not appear in the present rigid-plastic FEM.

In punch stretching, to make the problem more tractable, the presence of the die profile is neglected first so that there is only one moving boundary. This problem is successfully solved. Taking the die profile into consideration is equivalent to introducing another moving boundary, and while handling two moving boundaries simultaneously could be time consuming, the present rigid-plastic FEM again proves to be efficient and reliable. The strain distributions and the punch load vs. punch depth relationship predicted by the present rigid-plastic FEM are in excellent agreement with the numerical solutions by the finite-difference method and the experimental data.

We then investigate the influence of workhardening representation by comparing solutions, computed by both the parabolic workhardening law and the Voce equation methods. The two workhardening representations result
in the difference of peak strains and load-displacement relationships, and the difference becomes increasingly significant as punch displacement increases. It is concluded, however, that the selection of a proper work-hardening representation requires more experiments with improved accuracy and control.

The present method is further extended to the deep drawing problem. The strain distribution predicted by the present rigid-plastic FEM is in excellent agreement with the experimental data over the flange of the sheet; however, over the punch head, agreement is not as good. By assigning two different values of the friction coefficient over the punch head, two strain distributions are obtained; one predicts more straining than the experimental data, and vice versa. Therefore, an improvement in the prediction seems possible by giving the friction coefficient a proper value which is between these two bounds; however, the validity of the present rigid-plastic FEM for deep drawing analysis remains inconclusive at this stage mostly because of the lack of comparable numerical solutions and experimental data. This is apparently due to the increased sophistication and accompanying computation time when three moving boundaries are treated simultaneously and to the practical difficulty of determining proper friction coefficients.

It is concluded that the present rigid-plastic FEM can treat the sheet-metal forming problems with efficiency and reasonable accuracy.
APPENDIX A

PROGRAM FOR THE INITIAL GUESS FOR HYDROSTATIC BULGING ANALYSIS

This program is to provide the initial guess and initial geometry for Appendix B. It is based upon the analysis by Hill [23].

(I) Data preparation

1. Read NUMNP (I5)
   NUMNP: Total number of nodal points to be generated

2. Read RADIUS, DIS1, DIS2 (3F 10.0)
   RADIUS: Radius of the sheet to be bulged
   DIS1: Polar height of the bulge in the initial geometry
   DIS2: Polar height of the bulge in the new configuration
GRID 1 **PROGRAM GRID1 (INPUT, OUTPUT, TAPE1=INPUT, TAPE2=OUTPUT, PUNCH)**
GRID 2 **C**********************************************************
GRID 3 **C THIS PROGRAM IS TO GENERATE THE INITIAL GEOMETRY AND VELOCITY**
GRID 4 **C FIELD FOR HYDRODYNAMIC BLODGE PROBLEM, FOLLOWING HILL**
GRID 5 **C**********************************************************
GRID 6 **C**********************************************************
GRID 7 **C**********************************************************
GRID 8 **C**********************************************************
GRID 9 **C**********************************************************
GRID 10 **C**********************************************************
GRID 11 **C**********************************************************
GRID 12 **C**********************************************************
GRID 13 **C**********************************************************
GRID 14 **C**********************************************************
GRID 15 **C**********************************************************
GRID 16 **C**********************************************************
GRID 17 **C**********************************************************
GRID 18 **C**********************************************************
GRID 19 **C**********************************************************
GRID 20 **C**********************************************************
GRID 21 **C**********************************************************
GRID 22 **C**********************************************************
GRID 23 **C**********************************************************
GRID 24 **C**********************************************************
GRID 25 **C**********************************************************
GRID 26 **C**********************************************************
GRID 27 **C**********************************************************
GRID 28 **C**********************************************************
GRID 29 **C**********************************************************
GRID 30 **C**********************************************************
GRID 31 **C**********************************************************
GRID 32 **C**********************************************************
GRID 33 **C**********************************************************
GRID 34 **C**********************************************************
GRID 35 **CALL GUES (AN1, AN2, AN4, AN5, AN6, AN7, AN8, AN9, AN10)**
GRID 36 **END**

GRID 38 **SUBROUTINE GUES (EF, FZ, CODE, SLCE, FZ, UR, UZ, UZ, NUMKE)**
GRID 39 **DIMENSION RA(I), ZF(I), CODE(I), KLOF(I), E(I), T(I), U(I), U(1), U(1)**
GRID 40 **1, OUTPUT)**
GRID 41 **C**********************************************************
GRID 42 **C**********************************************************
GRID 43 **C**********************************************************
GRID 44 **C**********************************************************
GRID 45 **C**********************************************************
GRID 46 **C**********************************************************
GRID 47 **C**********************************************************
GRID 48 **C**********************************************************
GRID 49 **C**********************************************************
GRID 50 **C**********************************************************
GRID 51 **C**********************************************************
GRID 52 **C**********************************************************
GRID 53 **C**********************************************************
GRID 54 **C**********************************************************
GRID 55 **C**********************************************************
GRID 56 **C**********************************************************
GRID 57 **C**********************************************************
GRID 58 **C**********************************************************
GRID 59 **C**********************************************************
GRID 60 **C**********************************************************
GRID 61 **C**********************************************************
GRID 62 **C**********************************************************
GRID 63 **C**********************************************************
GRID 64 **C**********************************************************
GRID 65 **C**********************************************************
GRID 66 **C**********************************************************
GRID 67 **C**********************************************************
GRID 68 **C**********************************************************
GRID 69 **C**********************************************************
GRID 70 **C**********************************************************
GRID 71 **C**********************************************************
GRID 72 **C**********************************************************
GRID 73 **C**********************************************************

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GRM 74   301 CONTINUE
GRM 75   C     IF (K .LT. 2) GO TO 50
GRM 76   C     NC 200 1*X1,NUMAP
GRM 77   URF {1}NUMAP = 1-LF{1}
GRM 78   U7{1}U7{1} = 6/7{1}
GRM 79   200 CONTINUE
GRM 80   C     CODE{1} = 3.0
GRM 81   C     CODE{NUMAP} = 1.0
GRM 82   C     SLCP{NUMAP} = 0.0
GRM 83   C     DC 500 1*X1,NUMAP
GRM 84   C     RFE{3}RFE{3} = RLF{3}
GRM 85   C     U7{3}U7{3} = 6/7{3}
GRM 86   C     WRITE{5,1000} RFE{1},22{1},U7{1},U7{1},SLCP{1}
GRM 87   C     WRITE{5,1000} RFE{1},22{1},U7{1},U7{1},SLCP{1}
GRM 88   C     PUN{1},LCC{1},EEE{1},22{1},U7{1},U7{1},SLCP{1}
GRM 89   500 CONTINUE
GRM 90   C     1017 FCWAT{4}F20.15
GRM 91   1011 FCWAT{15,F20.7}
GRM 92   1001 FCWAT{15,F20.7}
GRM 93   1000 FCWAT{15,F20.7}
GRM 94   1000 RETURN
GRM 95   ENC
APPENDIX B

PROGRAM FOR THE ANALYSIS OF HYDROSTATIC BULGING

This program is for the analysis of hydrostatic bulging.

(I) Data preparation

1. Read HED (A 12)
   Output title

2. Read RVALUE, T, ACOEF (SF 10.0)
   RVALUE: Normal anisotropy parameter
            Set 1.0 for isotropic material
   T: Initial thickness of blank
   ACOEF: Accelerating coefficient
          To start with, set 1.0

3. Read ITER, NREAD, ITCONT, NFORM, NPUNCH, NPRINT, FLIMIT (615, F 10.0)
   The program control card
   ITER: Number of iterations to be executed
   NREAD: 1, if new data are to be supplied;
          0, otherwise
   ITCONT: 0, if computation starts at the very beginning and first/
          second steps are included in the steps to be computed;
          1, otherwise
   NFORM: Number of steps to be computed
   NPUNCH: 1, if solution is to be punched at the end of each step;
          0, otherwise
   FLIMIT: Value of (error norm)/(solution norm) required for
          convergence. To start with, set this .000001

4. Read NUMNP (6 I 5)
   NUMNP: Number of nodal points

5. Read YVALUE, PRESTN, EXPNT, PRESTS (4F 10.0)
   Material characteristics are specified.
   Stress = YVALUE* (Strain + PRESTN)**EXPNT + PRESTS
6. Read PRES, DPRES (4F 10.0)
   PRES: Current pressure value
   DPRES: Increment of pressure

7. Read N, CODE(N), R(N), Z(N), UR(N), UZ(N), SLOP(N), (I5, F5.0, SF 10.0)
   Nodal information
   N: Node number. Node number 1 is at the rim of the blank and the last node is at the pole
   R(N): Radial position of the node
   Z(N): Axial position of the node
   UR(N): Increment of displacement in radial direction
   UZ(N): Increment of displacement in axial direction
   SLOP(N): Slope of the element
   Set this 0.0
   CODE(N): Type of boundary conditions:
   1.0, if magnitude of UR(N) is fixed;
   2.0, if magnitude of UZ(N) is fixed;
   3.0, if magnitudes of UR(N) and UZ(N) are fixed;
   0.0, if neither the magnitude of UR(N) nor UZ(N) are fixed

   In subroutine PRELIM the interpolation of data is built in.

8. If NREAD = 1, the input data is to be placed behind nodal information cards
PROGRAM PULGE (INPUT, OUTPUT, TAPES= INPUT, TAPES= OUTPUT, PUNCH)

C******************************************************************************
C THIS PROGRAM IS TO ANALYZE THE HYDROSTATIC PULGE
C******************************************************************************

C COMMON/PERCON/NPEK, NPEK, MPEK1(12), CMVEK, MVEK, YIELD, TEST, ITER.
C COMMON/NPRINT, NPRINT, VALUE, TMF, M, N, E, NP, NPSET, NPRINT, NPRES
C COMMON/NOUT, VALUE, NPRES, EVENT, PRERES
C

C******************************************************************************
C PROGRAM IS FOR CONTROLLING THE DIMENSIONS OF THE COMPLETE
C PROGRAM. ITS PURPOSE IS TO PREVENT ASSIGNING A LARGER THAN
C NEEDED DIMENSION FOR ANY ARRAY THROUGH THE USE OF THE
C******************************************************************************

C******************************************************************************
C FOLLOWING STATEMENTS
C******************************************************************************

C******************************************************************************
C NPRINT IS THE DIMENSION OF ARRAY A. ITS VALUE CAN BE DETERMINED
C PRECISELY BY RUNNING THE PROGRAM ONCE.
C******************************************************************************

C******************************************************************************
C TEST1.
C******************************************************************************

READS(1000), FOF
READS(1004), VALUE, TAOF
READS(1001), ITER, NPRINT, ICONT, MCCK, NPRINT, LIMIT
READS(1003), NUMP
READ(1004), VALUE, OPRES, EVENT, PRERES
READS(1004), OPRES, PRESS

C******************************************************************************
C WRITE/DOUT TITLE
C******************************************************************************

VALUE = VALUE OF THE ANISOTROPY PARAMETER
ACOF = ACCELERATING OR DECELERATING COEFFICIENT OF CONVERGENCE
ICONT = ITERATION. IF COMPUTATION STARTS AT THE VERY BEGINNING AND FIRST/
SECOND STEPS ARE INCLUDED IN THE STEPS TO BE COMPUTED
IOTHERWISE = ITH INDEX IS RELATED TO THE DETERMINATION OF STEP SIZE

VALUE = NUMBER OF STEPS ASSIGNED PER RUN
NPRES = IF DATA ARE TO BE PUNCH
NPRES = NO, OTHERWISE
NPRES = VALUE OF (ERROR NO%/SOLUTION NO%) REQUIRED
NPRES = IF CONVERGENCE
NPRES = IF MICAL PRINT DATA ARE TO BE PRINTED
NPRES = NO, OTHERWISE
NPRES = NUMBER OF MICAL POINTS
NPRES = PRESS CURRERNT PRESSURE
NPRES = IPRINT, INCREMENT OF THE PRESSURE
NPRES = VALUE, PRESET, EVENT, PRESET ARE TO EXPRESS THE WORKHARDENING
NPRES = CHARACTERISTICS OF THE PLAIN
NPRES = VALUE OF PRESET=STRAIN STRAIN=PRESET
NPRES = NUMBER OF EQUATIONS TO BE SOLVED
NPRES = NUMBER OF ELEMENTS
NPRES = MANDANCE WIDTH

C******************************************************************************

NPRINT = 1
VALUE = VALUE
NPRES = NUMBER=3
NPRES = NO, NOPRES = NUMBER=4
NPRES = NOPRES = NUMBER=9
NPRES = NPRE=4
PRESET = VALUE
NPRES = VARIABLES
BULGE 230

THE COMPUTATION IS INTERRUPTED AFTER A NUMBER OF STEPS AND RESTARTED, THEN, IF NECESSARY, CAN BE FEED

DETAIL OF THE PRESENT CONFIGURATION

CALL MCDI5YJ(N).call((N)ARRAY((N)),DIMES((N))

SCALE(RESP,CMRS)

CALL BACSS(P, VB*NT,ARRAY(N))
SUBROUTINE CONCEN(A,B,NEO,MAND,X,U)

DIMENSION R(NEO),JNEO,11

DO 250 I=1,6
M=(I-1)*I/2+1

DO 240 J=1,6
D(I,J)=M (I-J+1)*I/2+1

DO 230 K=1,6
D(I,J)=M (I-K+1)*K/2+1

DO 220 M=1,6
D(I,J)=M (I-M+1)*M/2+1

END

SUBROUTINE CONCEN(A,B,NEO,MAND,X,U)

DIMENSION R(NEO),JNEO,11

DO 250 I=1,6
M=(I-1)*I/2+1

DO 240 J=1,6
D(I,J)=M (I-J+1)*I/2+1

DO 230 K=1,6
D(I,J)=M (I-K+1)*K/2+1

DO 220 M=1,6
D(I,J)=M (I-M+1)*M/2+1

END
**SUBROUTINE MODIFY(C, D, V, W)**

**RULE 689**  
**C**  
**RULE 690**  
**DIMENSION CON(11), A(NEG,1), X(1)**

**RULE 733**  
**DO 121 I=1, NNUMP**

**RULE 734**  
**IL=IL-1**

**RULE 735**  
**IF (IL>IL-1) GO TO 101**

**RULE 736**  
**IF (IL>IL-1) GO TO 112**

**RULE 737**  
**C=CONE(11)**

**RULE 738**  
**IF (C.EQ. 1.) GO TO 101**

**RULE 739**  
**IF (C.EQ. 2.) GO TO 102**

**RULE 740**  
**IF (C.EQ. 3.) GO TO 103**

**RULE 741**  
**CALL CCONDE(A, P, NEG, W, RAND, IL, 1)**

**RULE 742**  
**GO TO 121**

**RULE 743**  
**101 CONTINUE**

**RULE 744**  
**CALL CONDE(A, P, NEG, W, RAND, IL, 1)**

**RULE 745**  
**GO TO 121**

**RULE 746**  
**102 CONTINUE**

**RULE 747**  
**CALL CONDE(A, P, NEG, W, RAND, IL, 1)**

**RULE 748**  
**GO TO 121**

**RULE 749**  
**103 CONTINUE**

**RULE 750**  
**CALL CONDE(A, P, NEG, W, RAND, IL, 1)**

**RULE 751**  
**GO TO 121**

**RULE 752**  
**121 CONTINUE**

**RULE 753**  
**CALL CONDE(A, P, NEG, W, RAND, IL, 1)**

**RULE 754**  
**102 CONTINUE**

**RULE 755**  
**CALL CONDE(A, P, NEG, W, RAND, IL, 1)**

**RULE 756**  
**CALL CONDE(A, P, NEG, W, RAND, IL, 1)**

**RULE 757**  
**CALL CONDE(A, P, NEG, W, RAND, IL, 1)**

**RULE 758**  
**121 CONTINUE**

**RULE 759**  
**RETURN**

**RULE 760**  
**END**

**SUBROUTINE TRIA(NA, WW)**

**RULE 802**  
**C**  
**RULE 803**  
**DIMENSION A(NA,1)**

**RULE 837**  
**100 NN=1**

**RULE 838**  
**IF (N.EQ.WW) RETURN**

**RULE 839**  
**IF (A(N,1).NE.0.) GC TO 150**

**RULE 840**  
**GO TO 150**

**RULE 841**  
**150 L=4**

**RULE 842**  
**W=4**

**RULE 843**  
**GO 260 L=2, WR**

**RULE 844**  
**I=1+1**

**RULE 845**  
**C=A(N,1)/A(N,N)**

**RULE 846**  
**IF (C.EQ.W) GC TO 260**

**RULE 847**  
**J=0**

**RULE 848**  
**GO 259 K=L, WR**

**RULE 849**  
**J=L+1**

**RULE 850**  
**GO 280 A(I,J)=A(I,J)-C*A(N,K)**

**RULE 851**  
**A(K,J)=C**

**RULE 852**  
**GO TO 169**

**RULE 853**  
**CALL TRIA(NA, WW)**

**RULE 854**  
**END**
SUBROUTINE BACK(MM,NN,A,B)

DIMENSION A(I),F(I)

MM=M-1

270

C=9(N)

IF(A(N),N.E,0,G.E,C(B(N))/A(N))

1F(N),C(N)GO TO 700

ILNN+1

1NNN+1(N,N,NNN)

M=K

DO 295 K=1,N

295 M=M+1

GO TO 270

C

360 ILNN

DO 220 K=1,N

220 K=K+1 RETURN

DO 223 K=1,N,NNN

223 M=M+1

GO TO 295

M=M+1

DO 400 I=1,N

400 M=M+1

GO TO 360

C

END

SUBROUTINE HARE(EPS,Y)

COMMON/WATERL,YVALUE,PRESN,EXPN,PRESTS

COMMON/WATERL,YVALUE,PRESN,EXPN,PRESTS

RETURN

END

SUBROUTINE HARP(EPS,Y)

COMMON/WATERL,YVALUE,PRESN,EXPN,PRESTS

RETURN

END

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APPENDIX C

PROGRAM FOR THE ANALYSIS OF PUNCH STRETCHING

This program is for the analysis of the stretching of a sheet with hemispherical punch, where the die profile is neglected.

(I) Data card preparation

1. Read HED (A 12)
2. Read RVALUE, T, ACOEF (5F 10.0)
3. Read ITER, NREAD, ITCONT, NFORM, NPUNCH, NPRINT, FLIMIT (6I5, F10.0)
4. Read NUMNP (6I5)
5. Read PNRAD, RADIUS, FRITN (4F 10.0)
   PNRAD: Radius of the hemispherical punch
   RADIUS: Radius of the blank
   FRITN: Friction coefficient between the punch head and the blank
6. Read YVALUE, PRESTN, EXPNT, PRESTS (4F 10.0)
7. Read ECONST, TDIST (4F 10.0)
   ECONST: Step size in terms of the maximum magnitude of the effective strain increment. To start with, set this 0.04
   TDIST: Criterion distance of the contact of the sheet with the punch head. To start with, set this 0.008
8. Read N, CODE(N), R(N), Z(N), UR(N), UZ(N), SLOP(N), (IS, F5.0, SF 10.0)
   Code (N) = 4.0 for the contact zone of the sheet with the punch head
9. If NREAD = 1, the new input data is to be placed behind the nodal information card
PROGRAM STOCH(INPUT, OUTPUT, TAPE=INPUT, TAPE=OUTPUT, PUNCH)

WHERE, THE RADIUS OF THE DIL PROFILE IS NEGLECTED.

COMMON/COMMON/INC, INX, INY, IC, ICER, ICER, YIELD, YIELD, 

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SUBROUTINE PRL(I, T, T0, U7, CTD, SLOD)

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
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**Notes:**
- Stock numbers are not clearly visible or interpretable due to the image quality.
SUBROUTINE TRIA(N,N,A)

DIMENSION A(N,N)

C***TRIANGULIZATION OF GAUSSIAN ELIMINATION FOR THE SOLUTION
C OF FULLY SYMMETRIC MATRIX

C***

N=0

100 N=N+1

IF(N .NE. N) RETURN

101 IF(A(N,N) .LE. 0.0) GO TO 160

102 GO TO 100

150 I=AH

190 DO 201 I2=1,N

201 A(I,I)=A(I,I) / A(I,I)

250 IF(I .LE. N) GO TO 250

260 CONTINUE

END

C This subroutine handles the mixed boundary condition.

DIMENSION A(IT,1,1)

IF (IT.EQ.0) RETURN

DO 200 IT = 1, 100

END
SUBROUTINE HARD3(EPS,Y)

COMPUTE WORK HARDENING RATE

RETURN

END

COMPUTE WORK HARDENING RATE

RETURN

END
APPENDIX D
PROGRAM FOR THE ANALYSIS OF DEEP DRAWING AND PUNCH STRETCHING WITH ROUND DIE CORNER

This program is for the analysis of deep drawing with a hemispherical punch head and stretching with a hemispherical punch head. In stretching, a round die profile is considered.

(I) Data preparation card
1. Read HED (A 12)
2. Read RVALUE, T, ACOEF (5F 10.0)
3. Read ITER, NREAD, ITCONT, NFORM, NPUNCH, NPRINT, FLIMIT (615, F10.0)
4. Read NUMNP, NDEX (615)
   NDEX: 2, if punch stretching is to be analyzed
          3, if deep drawing is to be analyzed
5. Read PNRAD, RADIUS, DIERAD, RTART (4F 10.0)
   DIERAD: Radius of the die profile
   RTHRT: Distance from the pole to the die throat
6. Read FRITNP, FRITND, BHFCE (4F 10.0)
   FRITNP: Friction coefficient between the punch head and the blank
   FRITND: Friction coefficient between the die and the blank
   BHFCE: Blank holding force
          Set 0.0 for punch stretching problem
7. Read YVALUE, PRESTN, EXPNT, PRESTS (4F 10.0)
8. Read TCONTC, TDIST, ECONST (4F 10.0)
   TCONTC: Criterion distance of the contact with the die profile
            To start with, set this 0.002
9. Read N, CODE(N), R(N), Z(N), UR(N), UZ(N), SLOP(N), (15, F5.0, 5F 10.0)
10. If NREAD = 1, the new input data is to be placed behind the nodal information card.
PROGRAM SHEET 1 (INPUT, OUTPUT, TAPES, INPUT, TAPES, OUTPUT, PUNCH)

COMMON/AC,C,NUMP, NOE, N,ENI(2), MLL, NEDC, NDEF, YIELD, TEST, IER,
INRD, INPUNCH, NPRINT, EVALUE, T, MRCN, RADIUS, FRING, FRINT,
RECONST, RNAME, RTYPE, ITREAD, ICOND, TOIST, MNAME

COMMON

C PROGRAM WITH FOR PUNCH STRETCHING WITH ROUND PROFILE AND
C FOR DRAWING BY J. L. KIM
C

COMMON/AC,SVALUE,RPRINT,EXRTN,PREST
COMMON/ISOTROPHY
C

C PROGRAM. OTHERWISE:
C FOR CONTROLLING THE DIMENSION OF THE COMPLETE
C FOR PREVENTING ASSIGNING A LARGER THAN
C NECESSARY DIMENSION FOR ANY ARRAY THROUGH THE USE OF THE
C FOLLOWING STATEMENT

C

COMMON/AE,5000
C

C NFIELD IS THE DIMENSION OF ARRAY A. ITS VALUE CAN BE DETERMINED
C PRECISELY BY RUNNING THE PROGRAM ONCE.

C

COMMON/AE,5000
C

C THIS IER IS RELATED TO THE DETERMINATION OF STEP SIZE
C INCREASING NUMBER OF STEPS ASSIGNED PER RUN
C INCREASING, IF DATA ARE TO BE PRINTED
C # =, OTHERWISE
C PLIMIT=VALUE OF ERROR NORM/LSOLUTION NORM REQUIRED
C FOR CONVERGENCE
C NOPRINT, IF NORMAL PRINT DATA ARE TO BE PRINTED
C #, OTHERWISE
C NUMP=NUMBER OF MESH POINTS
C RADIUS=ADIUS OF HEMISHERICAL HUNCH HEAD
C DIRECTION COEFFICIENT BETWEEN BLANK AND PUNCH
C DIRECTION COEFFICIENT BETWEEN BLANK AND PUNCH
C DIRECTION COEFFICIENT OF DT PROFILE
C OTHERS=ADIUS OF DT THREAT
C ECONST=STEP SIZE IN MAXIMUM EFFECTIVE STAIN INCREMENT
C TRN=3, IF PUNCH STRETCHING WITH ROUND PROFILE
C #, IF DEEP DRAWING
C

C

C VVALUE, RPRESSN, EXRTN, PREST ARE TO EXPRESS THE WORKHARDENING
C CHARACTERISTICS OF THE BLANK
C STRESSES=VALUE OF (PRESSN+STRAIN)*PREST

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IBM 370/145 User's Library

Chapter 21: User's Library

21.1 Description

This chapter describes the IBM 370/145 User's Library, which is a collection of programs and subroutines that can be used in IBM 370/145 FORTRAN programs. The library includes routines for arithmetic operations, scientific calculations, and other tasks.

21.2 Structure

The library is organized into sections, each containing a collection of related routines. The sections are:

- Section A: Programming Environment
- Section B: Mathematical Functions
- Section C: Specialized Routines
- Section D: System Routines

Each section contains a detailed description of the routines it contains, including input and output requirements, and examples of how to use them.

21.3 Example

Here is an example of how to use a routine from the library:

```fortran
      REAL A, B, C
      REAL, DIMENSION(3) X

      CALL SUBLIB(A, B, C, X)
```

This call to the subroutine SUBLIB would perform some calculations and store the results in the variables A, B, C, and the array X.

21.4 Conclusion

The IBM 370/145 User's Library is an invaluable resource for anyone using IBM 370/145 FORTRAN, providing access to a wide range of routines that can save time and effort in programming.

For more information, see the IBM 370/145 User's Library manual, which is available from IBM.

End of Chapter 21

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| SHEET 162 | RECONSTR, WIND, ETHER, DIFRA, TOCAM, POINT, SHEET |
| SHEET 163 | C                                                                 |
| SHEET 164 | C*********************************************************************MAIN010 |
| SHEET 165 | C READ AND PRINT DE CONTROL INFORMATION AND MATERIAL PROPERTIES MAIN0013 |
| SHEET 166 | C                                                                 |
| SHEET 167 | C                                                                 |
| SHEET 168 | C                                                                 |
| SHEET 169 | C                                                                 |
| SHEET 170 | C                                                                 |
| SHEET 171 | C 50 CONTINUE                                                                 |
| SHEET 172 | C WRITE (6,2003) HED, NUNAM, AMAP, NMAP, NAIN, ZAIN, UN, ZUN, SLPUN |
| SHEET 173 | C CALL HARG, VIELC)                                                                 |
| SHEET 174 | C WRITE(6,2010) YIELC                                                                 |
| SHEET 175 | C WRITE(6,2011)                                                                 |
| SHEET 176 | C WRITE(6,2012) READ(N, BREAD, DIFRA, ETHER, PRINT, PRINT) |
| SHEET 177 | C WRITE(6,2013) YIELD, DESSA, EXIS, RESSS |
| SHEET 178 | C WRITE(6,2014) RECON                                                                 |
| SHEET 179 | C WRITE(6,1000) ITER                                                                 |
| SHEET 180 | C*********************************************************************MAIN0030 |
| SHEET 181 | C READ AND PRINT LOCAL NODE DATA                                                                 |
| SHEET 182 | C*********************************************************************MAIN0031 |
| SHEET 183 | C                                                                 |
| SHEET 184 | C                                                                 |
| SHEET 185 | C                                                                 |
| SHEET 186 | C IF (NPRT, CO, CI) GO TO 60                                                                 |
| SHEET 187 | C WRITE (6,1114)                                                                 |
| SHEET 188 | C WRITE (6,12004)                                                                 |
| SHEET 189 | C 60 READ (5,1002) N, CODE(N), P(N), Z(N), U(N), ZU(N), SLP(N) |
| SHEET 190 | C                                                                 |
| SHEET 191 | C                                                                 |
| SHEET 192 | C                                                                 |
| SHEET 193 | C                                                                 |
| SHEET 194 | C                                                                 |
| SHEET 195 | C                                                                 |
| SHEET 196 | C                                                                 |
| SHEET 197 | C                                                                 |
| SHEET 198 | C                                                                 |
| SHEET 199 | C                                                                 |
| SHEET 200 | C                                                                 |
| SHEET 201 | C                                                                 |
| SHEET 202 | C IF (NPRT, CI) GO TO 60                                                                 |
| SHEET 203 | C WRITE (6,6004)                                                                 |
| SHEET 204 | C WRITE (6,6024)                                                                 |
| SHEET 205 | C WRITE (6,6044)                                                                 |
| SHEET 206 | C WRITE (6,6064)                                                                 |
| SHEET 207 | C WRITE (6,6084)                                                                 |
| SHEET 208 | C WRITE (6,6104)                                                                 |
| SHEET 209 | C WRITE (6,6124)                                                                 |
| SHEET 210 | C WRITE (6,6144)                                                                 |
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| SHEET 224 | C                                                                 |
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| SHEET 246 | C                                                                 |

121
SUBROUTINE PLASTEK, (T, UER, UI, CREF, SLOPE, YX, YX, SPM, C1, C2, SLT, STPS)
IERS, R, A, THICK, ALPHA, GAMMA, ETA, FPRNC, DHI, FF, TOUCH, CNTAC, LUR, UUZ,
NAG, NEL, LIMIT, ITCOUNT, ACPH, NDE

C********************************************************************************
C** PLASTEK IS THE CONTROLLING SUBROUTINE
C******************************************************************************
C********************************************************************************
C** C** COMMON/GCENS/C/NUM, NUCPL, HDO (12), DLL, NELEC, FIVEL, TEST, ITET,
C** COMMON/INCI/G/DECL, N3AT, 3015, N3AT, TEST, DDPL, MINT, RINT, RINTR,
C** COMMON/ECN/G/DECL, N3AT, 3015, N3AT, TEST, DDPL, MINT, RINT, RINTR,
C** COMMON/TU/DECL, N3AT, 3015, N3AT, TEST, DDPL, MINT, RINT, RINTR,
C** COMMON/DTU/DECL, N3AT, 3015, N3AT, TEST, DDPL, MINT, RINT, RINTR,
C** DIMENSION R(1), Z(1), UER, UI, C(31), SLT(15), YX(15), X(15),
C** THICK(15), GAMMA(15), ETA(15), DHI(15), FF(15), A(15),
C** C** COMMON/EC/DECL, N3AT, 3015, N3AT, TEST, DDPL, MINT, RINT, RINTR,
C** COMMON/ET/DECL, N3AT, 3015, N3AT, TEST, DDPL, MINT, RINT, RINTR,
C** COMMON/DT/DECL, N3AT, 3015, N3AT, TEST, DDPL, MINT, RINT, RINTR,
C** COMMON/NN/DECL, N3AT, 3015, N3AT, TEST, DDPL, MINT, RINT, RINTR,
C** COMMON/NNN/DECL, N3AT, 3015, N3AT, TEST, DDPL, MINT, RINT, RINTR,
C******************************************************************************
C******************************************************************************
C** C** THE FIRST NODE IS LOCATED AT THE END OF THE PLANK
C******************************************************************************
C******************************************************************************
C******************************************************************************
C******************************************************************************
C******************************************************************************
C******************************************************************************
C******************************************************************************
C******************************************************************************
SHEET 328 C

DO 459 N=1,NUNFFL
SHEET 329 459 TEP51(N),N=0,00001
SHEET 330 C

SHEET 331 C

IF (F(N),I.0) GC TC 440
SHEET 332 READ (1017)(I1,CI1),CODE(I1),I=1,NUNNP
SHEET 333 READ (1017)(I1,CI1,),I=1,NUNNP
SHEET 334 READ (1017)(I1,CI1),N=1,NUFFL
SHEET 335 READ (1017)(I1,CI1),NUMFL
SHEET 336 READ (1233),IMTEDKCI,NUMFL
SHEET 337 C

SHEET 338 C

IF (0.0 LE. 0.0) GC TC 440
SHEET 339 C

SHEET 340 C

SHEET 341 C

SHEET 342 C

SHEET 343 C

SHEET 344 C

SHEET 345 C

SHEET 346 C

SHEET 347 C

SHEET 348 C

SHEET 349 C

SHEET 350 C

SHEET 351 C

SHEET 352 C

SHEET 353 C

SHEET 354 C

SHEET 355 C

SHEET 356 C

SHEET 357 C

SHEET 358 C

SHEET 359 C

SHEET 360 C

SHEET 361 C

SHEET 362 C

SHEET 363 C

SHEET 364 C

SHEET 365 C

SHEET 366 C

SHEET 367 C

SHEET 368 C

SHEET 369 C

SHEET 370 C

SHEET 371 C

SHEET 372 C

SHEET 373 C

SHEET 374 C

SHEET 375 C

SHEET 376 C

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SHEET 378 C

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SHEET 391 C

SHEET 392 C

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SHEET 398 C

SHEET 399 C

SHEET 400 C

SHEET 401 C

SHEET 402 C

SHEET 403 C

SHEET 404 C

SHEET 405 C

SHEET 406 C

SHEET 407 C

SHEET 408 C

SHEET 409 C

SHEET 410 C

SHEET 411 C

SHEET 412 C

123
KIS!

II' C

\[
\begin{align*}
&\text{compute the yield stress and the work-hardening rate} \\
&\text{DC 320 N/m, N/m} \\
&\text{CALL HARDTENS(A), YD, XY) (1)} \\
&\text{CALL HARDTENS(EP, E), YX(N)} \\
&\text{320 CONTINUE} \\
&\text{WRITE(6, 1031)} \\
&\text{CONTINUE} \\
&\text{DETAIL OF THE PRESENT CONFIGURATION} \\
&\text{SOMEONE OF ANGLE PHI} \\
&\text{PHI IN (N), PHI IN (N), THICK (N), OL (N)} \\
&\text{PP1=VALUE,1} \\
&\text{KC=VALUE,1} \\
&\text{CALL MODIFY(CODE, A, ALPHA, GAMMA, ETA, FRACCE, FF, A, I03)} \\
&\text{CALL TRIANGLE(N, MBAND, A, P)} \\
&\text{CALL RACKS(N, MBAND, A, P)} \\
&\text{INTRODUCTION OF BOUNDARY CONDITION} \\
&\text{RANKED SYMMETRIC SOLUTION} \\
&\text{CALL TRIANGLE(N, MBAND, A, P)} \\
&\text{Perturbation of U1 is computed from perturbation of U1 for nodes} \\
&\text{TO OBTAIN AN EFFICIENT CONVERGENCE ACCTR IS COMPUTED}
\end{align*}
\]
SHEET 408  
C NUMERICAL TERN TIMES ACCELERATING COEFFICIENT IS NEVER
SHEET 409  
C GREATER THAN THE INITIAL VALUE, BUT A FRACTION OF IT.
SHEET 500  
C.G. #22 MEANS HALF
SHEET S01  
CFEED print print print print print print print print print print print print print print print print print print print
SHEET 502  
C=100,  
SHEET 503  
AP=2,  
SHEET 504  
CONCOP=0
SHEET 505  
NC 105 X 12, NUMBC
SHEET 506  
IF(ANS(1)(1) LT. 0.0000001)(U(1))=0.  
SHEET 507  
IF(ANS(1)(1) LT. 0.0000001)(U(1))=0.  
SHEET 508  
U(1)=U(1)
SHEET 509  
U(1)=U(1)
SHEET 510  
I=1+I
SHEET 511  
I=I+1  
SHEET 512  
IF(I=10) .GO TO 102
SHEET 513  
STOP
SHEET 514  
102 IF(I=1) .GO TO 103 TC 103
SHEET 515  
CON=CUM(1)(1)/RI
SHEET 516  
A1H=N(1)(1), C0F, CI0F
SHEET 517  
103 CONTINUE
SHEET S18  
C
SHEET 519  
C
SHEET 520  
105 CONTINUE
SHEET 521  
IF(CON=0) .EQ. 1.1A2#A#5.
SHEET 522  
IF(122#C0F#1/2)
SHEET 523  
IF(122#C0F#1/2)
SHEET 524  
A1C0F=1/A2
SHEET 525  
IF(A1#C0F#1) .GOES TO 2300
SHEET 526  
IF(A1#C0F#1) .GOES TO 1
SHEET 527  
A1=1.0
SHEET 528  
WRITE(5,1044)A1
SHEET 529  
C
SHEET 530  
C
SHEET 531  
C
SHEET 532  
C
SHEET 533  
C
SHEET 534  
C
SHEET 535  
DO 130 I=1, NUMBC  
SHEET 536  
I=I+1  
SHEET 537  
I=I+1  
SHEET 538  
SHEET 539  
SHEET 540  
SHEET 541  
SHEET 542  
SHEET 543  
SHEET 544  
130 CONTINUE
SHEET 545  
C
SHEET 546  
WRITE(5,1016)K
SHEET 547  
WRITE(5,1006)K
SHEET 548  
C
SHEET 549  
C
SHEET 550  
C
SHEET 551  
C
SHEET 552  
C
SHEET 553  
C
SHEET 554  
C
SHEET 555  
C
SHEET 556  
C
SHEET 557  
C
SHEET 558  
C
SHEET 559  
C
SHEET 560  
C
SHEET 561  
SHEET 562  
134 CONTINUE
SHEET 563  
C
SHEET 564  
C
SHEET 565  
C
SHEET 566  
C
SHEET 567  
C
SHEET 568  
C
SHEET 569  
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SHEET 570  
C
SHEET 571  
C
SHEET 572  
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SHEET 573  
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SHEET 574  
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SHEET 575  
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SHEET 576  
C
SHEET 577  
C
SHEET 578  
C
SHEET 579  
C
SHEET 580  
C
SHEET 581  
C
SHEET 582  
C
125
Sheet 543 C EPS(2,N)*INCREMENT OF TANGENTIAL STRAIN
Sheet 544 C EPS(3,N)*INCREMENT OF THICKNESS STRAIN
Sheet 545 C

Sheet 546 C

Sheet 547 C

Sheet 548 C

Sheet 549 C

Sheet 550 C

Sheet 551 C

Sheet 552 C

Sheet 553 C

Sheet 554 C

Sheet 555 C

Sheet 556 C

Sheet 557 C

Sheet 558 C

Sheet 559 C

Sheet 560 C

Sheet 561 C

Sheet 562 C

Sheet 563 C

Sheet 564 C

Sheet 565 C

Sheet 566 C

Sheet 567 C

Sheet 568 C

Sheet 569 C

Sheet 570 C

Sheet 571 C

Sheet 572 C

Sheet 573 C

Sheet 574 C

Sheet 575 C

Sheet 576 C

Sheet 577 C

Sheet 578 C

Sheet 579 C

Sheet 580 C

Sheet 581 C

Sheet 582 C

Sheet 583 C

Sheet 584 C

Sheet 585 C

Sheet 586 C

Sheet 587 C

Sheet 588 C

Sheet 589 C

Sheet 590 C

Sheet 591 C

Sheet 592 C

Sheet 593 C

Sheet 594 C

Sheet 595 C

Sheet 596 C

Sheet 597 C

Sheet 598 C

Sheet 599 C

Sheet 600 C

Sheet 601 C

Sheet 602 C

Sheet 603 C

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Sheet 606 C

Sheet 607 C

Sheet 608 C

Sheet 609 C

Sheet 610 C

Sheet 611 C

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Sheet 613 C

Sheet 614 C

Sheet 615 D

Sheet 616 D

Sheet 617 D

Sheet 618 D

Sheet 619 D

Sheet 620 D

Sheet 621 D

Sheet 622 D

Sheet 623 D

Sheet 624 D

Sheet 625 D

Sheet 626 D

Sheet 627 D

Sheet 628 D

Sheet 629 D

Sheet 630 D

Sheet 631 D

Sheet 632 D

Sheet 633 D

Sheet 634 C

Sheet 635 C

Sheet 636 C

Sheet 637 C

Sheet 638 C

Sheet 639 C

Sheet 640 C

Sheet 641 C

Sheet 642 C

Sheet 643 C

Sheet 644 C

Sheet 645 C

Sheet 646 C

Sheet 647 C

Sheet 648 C

Sheet 649 C

Sheet 650 C

Sheet 651 C

Sheet 652 C

Sheet 653 C

Sheet 654 C

Sheet 655 C

Sheet 656 C

Sheet 657 C

Sheet 658 C

Sheet 659 C

Sheet 660 C

Sheet 661 C

Sheet 662 C

Sheet 663 C

Sheet 664 C

Sheet 665 C

Sheet 666 C

Sheet 667 C

126
SHEET 666 C IFXK .GE. 1000 TO 2200
SHEET 667 C
SHEET 668 C DC 1900 N1:N1:N1
SHEET 669 C IFK(1) .LE. 1000 TO 1900
SHEET 670 C N1:1 = 1
SHEET 671 C N1=1:N1:N1
SHEET 672 C CODE(N1) = 1
SHEET 673 C CODE(N1) = 30
SHEET 674 C 1900 CONTINUE
SHEET 675 C GC TO 2001
SHEET 676 C 2000 CONTINUE
SHEET 677 C 2200 CONTINUE
SHEET 678 C
SHEET 679 C
SHEET 680 C IFSCREW .LT. LIMITICO TO 777
SHEET 681 C
SHEET 682 C
SHEET 683 C
SHEET 684 C
SHEET 685 C
SHEET 686 C
SHEET 687 C WRITE(6,2600)
SHEET 688 C WRITE(6,2605)
SHEET 689 C
SHEET 690 C
SHEET 691 C
SHEET 692 C
SHEET 693 C
SHEET 694 C
SHEET 695 C
SHEET 696 C
SHEET 697 C
SHEET 698 C
SHEET 699 C
SHEET 700 C
SHEET 701 C
SHEET 702 C
SHEET 703 C
SHEET 704 C
SHEET 705 C
SHEET 706 C
SHEET 707 C
SHEET 708 C
SHEET 709 C
SHEET 710 C
SHEET 711 C
SHEET 712 C
SHEET 713 C
SHEET 714 C
SHEET 715 C
SHEET 716 C
SHEET 717 C
SHEET 718 C
SHEET 719 C
SHEET 720 C
SHEET 721 C
SHEET 722 C
SHEET 723 C
SHEET 724 C
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SHEET 726 C
SHEET 727 C
SHEET 728 C
SHEET 729 C
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SHEET 731 C
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SHEET 735 C
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SHEET 737 C
SHEET 738 C
SHEET 739 C
SHEET 740 C
SHEET 741 C
SHEET 742 C
SHEET 743 C
SHEET 744 C
SHEET 745 C
SHEET 746 C
SHEET 747 C
SHEET 748 C
SHEET 749 C
SHEET 750 C
SHEET 751 C
SHEET 752 C

127
SHEET 783  C
SHEET 784
SHEET 785 GO TO 2101
SHEET 786  1500 CONTINUE
SHEET 787 IF (KIND EQ. 216G TC 100)
SHEET 788 IF (KIND EQ. 216G TC 3501)
SHEET 789 CONTINUE
SHEET 790 C
SHEET 791 C
SHEET 792 C
SHEET 793 C
SHEET 794 C
SHEET 795 C
SHEET 796 C
SHEET 797 C
SHEET 798 C
SHEET 799 C
SHEET 800 C
SHEET 801 C
SHEET 802 C
SHEET 803 C
SHEET 804 C
SHEET 805 C
SHEET 806 C
SHEET 807 C
SHEET 808 C
SHEET 809 C
SHEET 810 C
SHEET 811 C
SHEET 812 C
SHEET 813 C
SHEET 814 C
SHEET 815 C
SHEET 816 C
SHEET 817 C
SHEET 818 C
SHEET 819 C
SHEET 820 C
SHEET 821 C
SHEET 822 C
SHEET 823 C
SHEET 824 C
SHEET 825 C
SHEET 826 C
SHEET 827 C
SHEET 828 C
SHEET 829 C
SHEET 830 C
SHEET 831 C
SHEET 832 C
SHEET 833 C
SHEET 834 C
SHEET 835 C
SHEET 836 C
SHEET 837 C

GO TO 2101

CONTINUE

IF (KIND EQ. 216G TC 100)

IF (KIND EQ. 216G TC 3501)

IF (KIND EQ. 216G TC 100)

IF (KIND EQ. 216G TC 3501)

CONTINUE

C

C

C

C

C

C

C

C

C

C

C

C

C

C

C

C

TO CHECK WHETHER THE AXIOM IS ASSUMED TO WE'VE TOO FAST.
SHEET 330

*******************************************************************************

SHEET 326

C

IFCODE=INTOUCH .EQ. 3 .S160 TO 500

SHEET 324

C

N77 = 1

SHEET 323

C

IF(EXTST, LT., 0.1) GO TO 500

SHEET 324

C

IF(EXTST(N77), LT., 0.00001), FP., TINSE, .EQ. 0.1 TO 500

SHEET 325

C

WRITE(S,5),EDITTEST

SHEET 327

C

NTOUCH=INTOUCH

SHEET 328

C

TCHEK=1,0

SHEET 329

C

TCHEK

SHEET 330

CO

500 CONTINUE

SHEET 331

C

****************************************************************************

SHEET 332

C

IFCODE=INTOUCH .EQ. 3 .S160 TO 500

SHEET 333

C

IF(ACTNC .EQ. 0) ACFHR TO 350

SHEET 334

C

IF(PLN .LT. -0.00001) IFC TO 350

SHEET 335

C

NTOUCH=INTOUCH+

SHEET 336

C

WRITE(S,5),EDITTEST

SHEET 337

C

ACFHCNTC=INTOUCH

SHEET 338

C

C

CHECK

SHEET 339

C

****************************************************************************

SHEET 340

C

MAKE BOUNDARY ADEQUATION FOR NEW STEP BASED UPON

SHEET 341

C

THE DISTANCE AWAY FROM PUNCH OF DIE

SHEET 342

C

****************************************************************************

SHEET 343

C

DO 840 IC=1,ACFHC

SHEET 344

C

IF(ACFHCNTC(IC),LT.,1.562,85)

SHEET 345

C

851 ACFHCNTC=ACFHC

SHEET 346

C

IF(ACTNC=ACFHCNTC)

SHEET 347

C

TCHEK=TCHEK(1)

SHEET 348

C

WRITE(S,5),FRNCE(ACFHCNTC+1),FRNCE(ACFHC)

SHEET 349

C

ACFHCNTC=INTOUCH

SHEET 350

C

C

ACFHCNTC=NTOUCH

SHEET 351

C

DO 845 IC=1,ACFHC

SHEET 352

C

854 ACFHCNTC=ACFHC

SHEET 353

C

855 ACFHCNTC=ACFHC

SHEET 354

C

856 CONTINUE

SHEET 355

C

****************************************************************************

SHEET 356

C

WRITE(6,1043)

SHEET 357

C

DO 344 K=1,NUMEL

SHEET 358

C

DO 442 I=1, A

SHEET 359

C

443 TETE(I,1),TERE(I,1),FRSE(I,1),TST

SHEET 360

C

IF(ROTH, LT., FLUXT),THICK(I),THICK(I),THICK(I),THICK(I),THICK(I)

SHEET 361

C

WRITE(6,10033),TETE(I,1),TERE(I,1),FLUXT(I,1),THICK(I),THICK(I),THICK(I)

SHEET 362

C

444 CONTINUE

SHEET 363

C

****************************************************************************

SHEET 364

C

WRITE(S,5),FRSE(I,1)

SHEET 365

C

DO 755 K=1,NUMEL

SHEET 366

C

775 IF(THICK(I),LT.,FLUXT),FRSE(I,1),FRSE(I,1)
SUBROUTINE TRIV(NM,NN,M)

DIMENSION A(NM,NN)

C***************************************************************************
C TRANGULARIZATION OF GAUSSIAN ELIMINATION FOR THE SOLUTION
C***************************************************************************

SUBROUTINE RACKE(NM,NN,A,R)

DIMENSION A(1),R(1)

C***************************************************************************
C PACK SUBROUTINE FOR SOLUTION OF RANKED SYMNETIC MATRIX
C***************************************************************************

SUBROUTINE MAPP(SY,F)

COMMON/WATER/WW,Y1,Y2,Y3,Y4,Y5,Y6,Y7,Y8,Y9,Y10,Y11,Y12
COMMON/TEST/TEST,TEST,TEST,TEST,TEST

C***************************************************************************
C WORKHARDING CHARACTERISTIC CURVE
C***************************************************************************
```plaintext
| SHEET 1401 | C                                                                 |
| SHEET 1402 | C                                                                 |
| SHEET 1403 | VALUETVVAL                                                        |
| SHEET 1404 | EXPTN=EXPTN                                                      |
| SHEET 1405 | VALUEPRESTNAPS**PRESTNAPS**                                      |
| SHEET 1406 | RETURN                                                       |
| SHEET 1407 | END                                                             |

| SHEET 1501 | SUBROUTINE HARD(S,E,F,N)                                       |
| SHEET 1502 | COMMON W,ESY,TVVAL,PRESTN,EXPTN,PRESTN                       |
| SHEET 1503 | C                                                                 |
| SHEET 1504 | C                                                                 |
| SHEET 1505 | COMMON KEK VARLCLQ RAFE                                       |
| SHEET 1506 | C                                                                 |
| SHEET 1507 | C                                                                 |
| SHEET 1508 | EXP=EXPTN                                                  |
| SHEET 1509 | VALUE=TVVAL                                               |
| SHEET 1510 | VALUE=TVVAL=1=PRESTNAPS=(**FRONT+1++)                      |
| SHEET 1511 | RETURN                                                       |
| SHEET 1512 | END                                                             |

| SHEET 1513 | SUBROUTINE RCM(XA,NEO,WMF5,CONST,CCSTF,CCSTG,IF,12)              |
| SHEET 1514 | DIMENSION A(NEO,11,11)                                          |
| SHEET 1515 | C                                                                 |
| SHEET 1516 | C                                                                 |
| SHEET 1517 | C                                                                 |
| SHEET 1518 | C                                                                 |
| SHEET 1519 | C                                                                 |
| SHEET 1520 | C                                                                 |
| SHEET 1521 | I=12-I+1                                                      |
| SHEET 1522 | A(I,1)=A(I,1)+2.*W(AE,I)/CCSTF+A(I,1)/CCSTF/CCSTF            |
| SHEET 1523 | N(I)=N(I)+1*(T)+E(I)/CCSTF-CONSTA+1*E(I,1)/CCSTF/CCSTF/A(I,1) |
| SHEET 1524 | N(I)=N(I)+CCSTG                                              |
| SHEET 1525 | A(I,1)=1.0                                                  |
| SHEET 1526 | A(I,1)=1.0                                                  |
| SHEET 1527 | N(I)=N(I)+0                                                  |
| SHEET 1528 | N(I)=N(I)+0                                                  |
| SHEET 1529 | N(I)=N(I)+0                                                  |
| SHEET 1530 | DC 000 W+2,WMF5                                             |
| SHEET 1531 | K=17-W+1                                                  |
| SHEET 1532 | IF K =LT. 0.50 TO 300                                        |
| SHEET 1533 | IFK =GT. 0.50 TO 300                                        |
| SHEET 1534 | W=0.0                                                         |
| SHEET 1535 | A(K,M)=A(K,M)+A(K,M)/CCSTF                                 |
| SHEET 1536 | (K)=A(K)-A(K,M)+CCSTG                                    |
| SHEET 1537 | A(K,M)=W                                                     |
| SHEET 1538 | GC TO 200                                                  |
| SHEET 1539 | W=0.0                                                         |
| SHEET 1540 | 150 CONTINUE                                              |
| SHEET 1541 | W=0.0                                                         |
| SHEET 1542 | A(K,M)=A(K,M)+A(K,M)/CCSTF                                 |
| SHEET 1543 | (K)=A(K)-A(K,M)+CCSTG                                    |
| SHEET 1544 | A(K,M)=W                                                     |
| SHEET 1545 | 500 CONTINUE                                              |
| SHEET 1546 | W=0.0                                                         |
| SHEET 1547 | A(K,M)=A(K,M)+A(K,M)/CCSTF                                 |
| SHEET 1548 | (K)=A(K)-A(K,M)+CCSTG                                    |
| SHEET 1549 | A(K,M)=W                                                     |
| SHEET 1550 | 700 CONTINUE                                              |
| SHEET 1551 | W=0.0                                                         |
| SHEET 1552 | A(K,M)=A(K,M)+A(K,M)/CCSTF                                 |
| SHEET 1553 | (K)=A(K)-A(K,M)+CCSTG                                    |
| SHEET 1554 | A(K,M)=W                                                     |
| SHEET 1555 | 700 CONTINUE                                              |
| SHEET 1556 | W=0.0                                                         |
| SHEET 1557 | RETURN                                                       |
| SHEET 1558 | END                                                           |
```
REFERENCES


