MICROCOPY RESOLUTION TEST CHART

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Let \( f \) be a continuously differentiable \( (C^1) \) map from a compact rectangular region \( Q \) in the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \) into \( \mathbb{R}^n \). Gale and Nikaido showed that if all the principal minors of the Jacobian of \( f: Q \rightarrow \mathbb{R}^n \) are positive everywhere in \( Q \) then \( f \) is univalent (one-to-one) on \( Q \). This result was recently strengthened and extended to a general case where \( f \) is defined on a compact convex polyhedron \( P \subset \mathbb{R}^n \) with nonempty interior by Mas-Colell and others. In this paper, we deal with a special case where \( f \) is separable with respect to a subset of variables, i.e.,

\[
  f(x) = \sum_{i=1}^{m} f_i(x_1) + f_{m+1}(x_{m+1}, \ldots, x_n)
\]

for all \( x = (x_1, \ldots, x_n) \in P = P_1 \times P_2 \).

Here \( P_1 \) is a compact rectangular region in \( \mathbb{R}_m \), \( P_2 \) a compact polyhedron in \( \mathbb{R}^{n-m} \), \( f_1 \) a \( C^1 \)-map from an interval of the real line into \( \mathbb{R}^n \) and \( f_{m+1}: P_2 \rightarrow \mathbb{R}^n \) a \( C^1 \)-map. We show that the sufficient condition given by Mas-Colell for a \( C^1 \)-map \( f:P \rightarrow \mathbb{R}^n \) to be univalent can be weakened in this case.

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SIGNIFICANCE AND EXPLANATION

When solving one equation in one unknown, \( f(x) = q \), it is obvious geometrically that if \( f(x) \) is continuously differentiable and \( f'(x) \neq 0 \) for all \( x \), then for each \( q \) the equation has at most one solution (\( f \) is then said to be univalent). Of course the univalence of \( f \) does not ensure the existence of a solution, for example, \( e^x = 0 \).

When solving a system of \( n \) equations in \( n \) unknowns,
\[
f_i(x_1, \ldots, x_n) = q_i (i = 1, \ldots, n),
\]
the analogue of \( f'(x) \) is the \( n \times n \) Jacobian matrix \( \frac{\partial f}{\partial x} \). It is interesting to investigate conditions on the Jacobian matrix which will ensure the univalence of the left hand of \( (*) \). Such conditions are of practical importance if they are combined with conditions which ensure the existence of solutions because if \( (*) \) has a solution and if the left hand of \( (*) \) is univalent then the solution is unique.

In this paper we deal with the case where \( f_i(x_1, \ldots, x_n) \) can be written in the form
\[
f_i(x_1, \ldots, x_n) = \sum_{j=1}^{m} f_{ij}(x_j) + f_i^{m+1}(x_{m+1}, \ldots, x_n) \quad (i = 1, \ldots, n),
\]
and give a condition on the Jacobian matrix which ensures the univalence. As a special case, it is shown that if \( m = n-1 \) and if the determinant of the Jacobian matrix \( \frac{\partial f}{\partial x} \) is nonzero for all \( (x_1, \ldots, x_n) \) then the left hand of \( (*) \) is univalent.
GLOBALLY UNIVALENT C¹-MAPS WITH SEPARABILITY
Masakazu Kojima and Michael J. Todd

1. Introduction

Let $\mathbb{R}^n$ denote the $n$-dimensional Euclidean space. We say that a subset $Q$ of $\mathbb{R}^n$ is a rectangular region if

$$Q = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : a_i \leq x_i \leq b_i \ (i = 1, \ldots, n) \},$$

where $-\infty \leq a_i < b_i \leq +\infty \ (i = 1, \ldots, n)$. We call a rectangular region in the real line $\mathbb{R}$ an interval. More than a decade ago Gale and Nikaido [1] showed that if all the principal minors of the Jacobian of a continuously differentiable ($C^1$) map $f$ from a rectangular region $Q$ in $\mathbb{R}^n$ into $\mathbb{R}^n$ are positive everywhere in $Q$ then $f$ is univalent (one-to-one) on $Q$. Recently, this result was strengthened and generalized by Garcia and Zangwill [2] and Mas-Colell [4]. This paper has a close relation with Mas-Colell's generalization [4].

We say that a convex set $C \subset \mathbb{R}^n$ spans a subspace $L$ of $\mathbb{R}^n$ if

$$L = \{ \lambda (x-x^0) : \lambda \in \mathbb{R}, x \in C \}$$

for some relative interior point $x^0$ of $C$. It should be noted that the set

$$\{ \lambda (x-x^0) : \lambda \in \mathbb{R}, x \in C \}$$

does not depend on the choice of an interior point $x^0$ of $C$.

Let $L$ be a nonempty subspace of $\mathbb{R}^n$. We denote the orthogonal projection map from $\mathbb{R}^n$ onto $L$ by $\Pi_L : \mathbb{R}^n \rightarrow L$, that is, $\Pi_L (x) \in L$ and $\| x - \Pi_L (x) \| = \min \{ \| x - y \| : y \in L \}$ for every $x \in \mathbb{R}^n$. Let $M$ be an $n \times n$ matrix. Then the composite map $\Pi_L \circ M : L \rightarrow L$ is linear. Suppose $\dim L = k$ and that the set of the columns of an $n \times k$ matrix $A$ forms a basis of $L$. Then we can write

$$\Pi_L (x) = A (A^T A)^{-1} A^T x \quad \text{for every} \ x \in \mathbb{R}^n,$$

where $A^T$ denotes the transpose of the matrix $A$. It is easily verified that the linear map $\Pi_L \circ M : L \rightarrow L$ has a positive (or negative) determinant if and only if the $k \times k$ matrix $A^T MA$ has a positive (or negative) determinant. The positivity (or negativity) of the determinant of the linear map $\Pi_L \circ M : L \rightarrow L$ does not depend on the choice of a basis of $L$. We denote the Jacobian matrix of a $C^1$-map $f$ from a subset of $\mathbb{R}^n$ into $\mathbb{R}^m$ by $Df(x)$.

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Theorem 1 (Theorem 1 in Mas-Colell [4]): Let $P$ be a compact convex polyhedron in $\mathbb{R}^n$ with nonempty interior, and $f: P \rightarrow \mathbb{R}^n$ be a $C^1$-map. Assume that for every $x \in P$ and subspace $L \subset \mathbb{R}^n$ spanned by a face $\sigma$ of $P$ which includes $x$, the linear map $\Pi_L \circ Df(x): L \rightarrow L$ has a positive determinant. Then $f$ is univalent on $P$.

We consider a special case where a $C^1$-map $f: P \rightarrow \mathbb{R}^n$ is separable with respect to some of the variables and show that the assumption of Theorem 1 can be weakened in this case.

Theorem 2: Let $P_1 \subset \mathbb{R}^m$ be a compact rectangular region and $P_2 \subset \mathbb{R}^{n-m}$ a compact convex polyhedron with nonempty interior. Let $f: P_1 \times P_2 \rightarrow \mathbb{R}^n$ be a $C^1$-map such that

$$f(x) = \sum_{i=1}^{m} f^i(x_1) + f^i(x_{m+1}, \ldots, x_n)$$

for every $x = (x_1, \ldots, x_n) \in P_1 \times P_2$, where $f^i$ is a $C^1$-map from an interval of $\mathbb{R}$ into $\mathbb{R}^n$ and $f^i$ is a $C^1$-map from $P_2$ into $\mathbb{R}^{n-m}$. Assume that for every face $\tau_2$ of $P_2$ and every $x \in P_1 \times \tau_2$, the linear map $\Pi_L \circ Df(x): L \rightarrow L$ has a positive determinant, where $L$ is the subspace spanned by $P_1 \times \tau_2$. Then $f$ is univalent on $P_1 \times P_2$.

We will derive Theorem 2 from Theorem 1 in Section 2. When $m = 0$, Theorems 1 and 2 are equivalent. Suppose $m \geq 1$. Note that each face of $P_1 \times P_2$ is of the form $\tau_1 \times \tau_2$, with $\tau_i$ a face of $P_i$ ($i=1,2$); we do not require that $\Pi_L \circ Df(x): L \rightarrow L$ have a positive determinant if the subspace $L$ is spanned by a face $\tau_1 \times \tau_2$, $\tau_1 \neq P_1$. Hence the hypothesis of Theorem 2 is weaker than that of Theorem 1.

It is well-known that the positivity of a $C^1$-map $f: P \rightarrow \mathbb{R}^n$ does not necessarily ensure the univalence of $f$. For example, the map $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by

$$f_1(x_1, x_2) = (\exp x_1)(\sin x_2)$$
$$f_2(x_1, x_2) = - (\exp x_1)(\cos x_2)$$

has a positive Jacobian at every $x = (x_1, x_2) \in \mathbb{R}^2$, but it is not univalent. When the dimension $n$ is equal to 1 or $f: P \rightarrow \mathbb{R}^n$ is affine, however, the positivity of the Jacobian at every $x \in P$ implies the univalence. These two exceptional cases are unified by the following result.

Corollary: Let $Q \subset \mathbb{R}^n$ be a rectangular region and $f: Q \rightarrow \mathbb{R}^n$ a $C^1$-map such that

$$f(x) = \sum_{i=1}^{m} f^i(x_1)$$

for all $x \in Q$, where $f^i$ is a $C^1$-map from an interval of $\mathbb{R}$ into $\mathbb{R}^n$. Assume that the Jacobian of the
map \( f \) is nonzero at every \( x \in \Omega \). Then \( f \) is univalent on \( P \).

**Proof.** Let \( x^0 \in \Omega \). Then \( f: \Omega \to \mathbb{R}^n \) is univalent if and only if the map \( g: \Omega \to \mathbb{R}^n \)
defined by

\[
g(x) = Df(x^0) f(x) \quad \text{for all } x \in \Omega
\]
is univalent. Obviously \( g: \Omega \to \mathbb{R}^n \) is separable with respect to all the variables and
\( \det Dg(x) > 0 \) for all \( x \in \Omega \). By Theorem 2, we see that \( g \) is univalent on every
compact rectangular region contained in \( \Omega \). This implies \( g \) is univalent on \( \Omega \).

Q. E. D.
2. Proof of Theorem 2.

Throughout this section \( e^i \) denotes the \( i \)-th unit vector in \( \mathbb{R}^n \), and \( I \) the identity matrix of appropriate dimension. For simplicity of notation, we assume that \( 0 \in \mathbb{R}^n \) is a vertex of \( P_1 \times P_2 \). Let \( M = Df(0) \). We partition the \( n \times n \) matrix \( M \) such that

\[
M = \begin{bmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{bmatrix}
\]

Let \( E \) denote the \( n \times m \) matrix \([e^1, \ldots, e^m]\). Since \( 0 \in \mathbb{R}^n \) lies in a face \( P_1 \times \{0\} \) of \( P_1 \times P_2 \) and the set of the columns of \( E \) forms a basis of the subspace \( \mathbb{R}^m \times \{0\} \subset \mathbb{R}^n \) spanned by \( P_1 \times \{0\} \), we have

\[
\det M_{11} = \det E^T Df(0) E > 0
\]

Define the \( n \times n \) matrix

\[
N = \begin{bmatrix}
-M_{11}^{-1} & 0 \\
-M_{11}^{-1} M_{12} & -M_{11}^{-1} M_{21} & I
\end{bmatrix}
\]

and the map \( g : P_1 \times P_2 \to \mathbb{R}^n \) by

\[
g(x) = N f(x) \quad \text{for all } x \in P_1 \times P_2.
\]

Obviously \( g : P_1 \times P_2 \to \mathbb{R}^n \) is also separable with respect to the variables \( x_1, \ldots, x_m \).

That is, we can write

\[
g(x) = \sum_{i=1}^{m} g_i(x_i) + g^*(x_{m+1}, \ldots, x_n) \quad \text{for all } x \in P_1 \times P_2,
\]

where \( g_i (i = 1, \ldots, m) \) and \( g^* \) are \( C^1 \)-maps. Since the \( n \times n \) matrix \( N \) is nonsingular, \( f \) is univalent on \( P_1 \times P_2 \) if and only if \( g \) is univalent on \( P_1 \times P_2 \). We shall establish that \( g \) is univalent on \( P_1 \times P_2 \).

In view of Theorem 1, it suffices to show that for every \( x \in P_1 \times P_2 \) and subspace \( L \subset \mathbb{R}^n \) spanned by a \((k+1)\)-dimensional face \( \tau_1 \times \tau_2 \) of \( P_1 \times P_2 \) which includes \( x \) and for an \( n \times (k+1) \) matrix \( A \) whose columns form a basis of \( L \), the determinant of \( A^T Dg(x) A \) is positive. Assume that \( \tau_1 \) and \( \tau_2 \) have dimensions \( k \) and \( \ell \) respectively. Since \( \tau_1 \) is a face of the compact rectangular region \( P_1 \subset \mathbb{R}^m \), we can choose \( k \) vectors...
from the m unit vectors $e_1, \ldots, e_m$ in $\mathbb{R}^n$ for a basis of the subspace $L_1$ spanned by $\tau_1 \times \{0\} \subset \mathbb{R}^n$. For simplicity of notation, we assume that the last $k$ unit vectors $e_{m-k+1}, \ldots, e_m$ form a basis of the subspace $L_1$. Choose a set of $\ell$ vectors $u_1, \ldots, u_\ell$ for a basis of the linear subspace spanned by $\{0\} \times \tau_2 \subset \mathbb{R}^n$. Define the $n \times (k+\ell)$ matrix

$$A = [e_{m-k+1}, \ldots, e_m, u_1, \ldots, u_\ell].$$

By the construction, the set of the columns of the $n \times (k+\ell)$ matrix forms a basis of the subspace $L$. The purpose of the remainder of the proof is to show

$$\det A^T A > 0.$$ 

Let $y = (0, \ldots, 0, x_{m-k+1}, \ldots, x_n) \in \mathbb{R}^n$. Then $y$ lies in a face $P_1 \times \tau_2$ of $P_1 \times P_2$, and the columns of the $n \times (m+\ell)$ matrix

$$\tilde{A} = [e^1, \ldots, e^m, u^1, \ldots, u^\ell]$$

form a basis of the subspace $\tilde{L}$ spanned by $P_1 \times \tau_2$. By assumption, we have

$$\det \tilde{A}^T A = 0.$$ 

By the construction of $g: P_1 \times P_2 \rightarrow \mathbb{R}^n$, we see

$$Dg(0) = NDf(0) = 
\begin{bmatrix}
I & M^{-1}M_{12} \\
-M_{11}^{-1}M_{12} + M_{11}^{-1} & 0
\end{bmatrix}.$$ 

Hence it follows from the separability of the map $g: P_1 \times P_2 \rightarrow \mathbb{R}^n$ that

$$Dg^i(0) = e^i \quad (i = 1, 2, \ldots, m).$$

Let $B$ denote the $n \times (m-k)$ matrix $[e^1, \ldots, e^{m-k}]$. Then

$$Dg(y) = [B, Dg_{m-k+1}(x_{m-k+1}), \ldots, Dg_m(x_m), Dg_{m+1}(x_{m+1}), \ldots, x_n],$$

and

$$\tilde{A}^T Dg(y) \tilde{A} = 
\begin{bmatrix}
B^T Dg(y) B & B^T Dg(y) A \\
A^T Dg(y) B & A^T Dg(y) A
\end{bmatrix}.$$ 

It is easily verified that $B^T Dg(y) B = I$, $A^T Dg(y) B = 0$ and $A^T Dg(y) A = A^T Dg(x) A$. Hence

$$\det \tilde{A}^T Dg(x) A = \det \tilde{A}^T Dg(y) \tilde{A}.$$ 

If we write the $n \times (m+\ell)$ matrix $\tilde{A}$ as

$$-5-$$
\[ \bar{A} = \begin{bmatrix} 1 & 0 \\ 0 & A_{22} \end{bmatrix}, \]

where \( A_{22} \) is an \((n-m) \times l\) matrix, then we see

\[ \bar{A}^T \text{Dg}(y) \bar{A} = A^T \text{Ndf}(y) A \]

\[ = \begin{bmatrix} M_{11}^{-1} & 0 \\ -A_{22}^T M_{21} M_{11}^{-1} & I \end{bmatrix}^{T} A^T \text{df}(y) \bar{A}. \]

Recalling that \( \det M_{11} > 0 \), we consequently obtain \( \det A^T \text{Dg}(x) A = \det \bar{A}^T \text{Dg}(y) \bar{A} = (\det M_{11}^{-1}) \det A^T \text{df}(y) \bar{A} > 0 \).

Q.E.D.
3. Concluding Remark.

Recently, Kojima and Saijai extended Theorem 1 to the case where the map $f: P \rightarrow \mathbb{R}^N$ is piecewise continuously differentiable ([Theorem 4.3, 3]). Theorem 2 can be also extended to a piecewise continuously differentiable case. The same proof as we have given in Section 2 is valid for the extension if we use Theorem 4.3 of [3] instead of Theorem 1.
REFERENCES


**Globally Univalent $C^1$ -Maps with Separability**

Let $f$ be a continuously differentiable ($C^1$) map from a compact rectangular region $Q$ in the $n$-dimensional Euclidean space $\mathbb{R}^n$ into $\mathbb{R}^n$. Gale and Nikaido showed that if all the principal minors of the Jacobian of $f: Q \to \mathbb{R}^n$ are positive everywhere in $Q$ then $f$ is univalent (one-to-one) on $Q$. This result was recently strengthened and extended to a general case where $f$ is defined on a compact convex polyhedron $P \subset \mathbb{R}^n$ with nonempty interior by Mas-Colell and others. In this paper, we deal with a special case where $f$ is separable with respect to a subset of variables, i.e.,
Abstract (continued)

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for all \( x = (x_1, \ldots, x_n) \in P = P_1 \times P_2 \).

Here \( P_1 \) is a compact rectangular region in \( R^m \), \( P_2 \) a compact polyhedron in \( R^{n-m} \), \( f^i \) a \( C^1 \)-map from an interval of the real line into \( R^n \) and \( f^{m+1} : P_2 \rightarrow R^n \) a \( C^1 \)-map. We show that the sufficient condition given by Mas-Colell for a \( C^1 \)-map \( f : P \rightarrow R^n \) to be univalent can be weakened in this case.