A COMPARISON OF BJERHAMMAR'S METHODS AND COLLOCATION IN PHYSICAL GEODESY.

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In 1963 A. Bjerhammar solved the geodetic boundary value problem by applying Poisson's integral equation for a finite set of observed free-air gravity anomalies. Due to the relation between the number of observations \((m)\) and the number of chosen unknowns \((N)\) different solutions are obtained: non-singular \((m \geq N)\), least squares \((m > N)\) and minimum norm solutions \((m < N)\). In the special case \(N = m\) it is shown that the Bjerhammar solution with Poisson's...
kernel and a solution by collocation with the corresponding kernel are identical. Bjerhammar's method is generalized by using other kernel functions, and each minimum norm solution is shown to correspond to one specific set of degree variances in collocation.

The impulse approaches (reflexive prediction, Dirac method) of Bjerhammar are presented. The kernel function of the non-singular Dirac method is obtained from that in collocation by substituting \( c_n / (2n+1) (r_n^2/r_1^1)^n + 2 \) by \( c_n / (2n+1) (r_n^2/r_1^1)^n + 2 \). Hence the Dirac kernel is not symmetric in contrast to the one in collocation. Moreover, it is shown that for a given radius \( r_n \) of the Bjerhammar sphere, the Dirac method gives a better conditioning of the equation system. It is demonstrated that the two solutions are identical for Poisson's kernel, if the depth of the Bjerhammar sphere in collocation is half of that in the application of the Dirac approach. The solution by collocation is therefore twice as sensitive to the choice of radius as is the latter method.

In the theoretical case with a continuous coverage of observations at the surface of the earth, it is shown that both the Dirac method and collocation give a unique solution for any choice of positive degree variances of the kernel functions, whenever the solutions exist. However, the intermediate solutions for \( \Delta g^* \) and \( X \) at the Bjerhammar sphere do not exist in general. If collocation is applied by solving the Wiener-Hopf integral equation, a convergent solution is proved outside a sphere. However, inside the bounding sphere of the earth the convergence is still not proved.
Foreword

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# Table of Contents

Abstract ........................................................................................................ ii 
Foreword ......................................................................................................... iii 
Acknowledgments ............................................................................................ iv 

1. Introduction .................................................................................................. 1  

2. Bjerhammar Solutions at the Internal Sphere .................................................. 3 
   2.1 Generalization .......................................................................................... 4 
   2.2 Minimum Norm Solutions ...................................................................... 7 
   2.3 Least Squares Solutions for \( u^* \) .......................................................... 11 
   2.4 Impulse Approach ................................................................................... 12 

3. Stability of the Solutions .............................................................................. 15 
   3.1 The Effect of Smoothing (Noise) ............................................................ 21 

4. On the Convergence of an Iterative Solution ............................................... 22 

5. Predictions According to Bjerhammar ......................................................... 30 
   5.1 Minimum Norm Solutions ...................................................................... 33 
   5.2 Model Studies ........................................................................................ 35 

6. Integral Formulas as Limiting Cases ........................................................... 40 
   6.1 The Uniqueness of the Solutions ............................................................ 40 
   6.2 The Existence of the Solutions ................................................................ 44 
   6.3 The Effect of Smoothing (Noise) ............................................................ 54 
   6.4 The Limiting Case of a Least Squares Solution ....................................... 58 

7. On the Choice of Radius .............................................................................. 60
1. Introduction

In 1963 A. Bjerhammar formulated the boundary value problem of physical geodesy in the following way: "A finite number of gravity data (gravity anomalies) is given for a non-spherical surface and it is required to find such a solution that the boundary values are satisfied in all given points". (Discrete boundary value problem.)

For the solution of this problem Bjerhammar used a spherical reference surface completely embedded in the earth with its center at the earth's center of gravity and rotating with the same angular velocity as the real earth. The observed gravity anomalies ($\Delta g_j$) are reduced to the internal sphere (the Bjerhammar sphere) by means of Poisson's integral equation for the harmonic function $r \Delta g$ (see Figure 1):

![Diagram](image)

**Figure 1.** The observation $\Delta g$ at the surface of the earth is reduced to the fictitious field $\Delta g^*$ on the internal sphere by means of Poisson's integral equation.

† for the height anomalies (see section 5)
\[ r_j \Delta g_j = \frac{r_j^2 - r_8^2}{4\pi r_8} \int \frac{r_8 \Delta g^*}{r_{ji}^3} \, dS \]

where

\[ \Delta g_j = \text{real gravity} - \text{theoretical gravity} \]

\[ \Delta g^* = \text{(fictitious) gravity anomaly at the Bjerhammar sphere} \]

\[ r_j = \text{geocentric distance to the point } P_j \]

\[ r_8 = \text{radius of the Bjerhammar sphere} \]

\[ r_{ji}^2 = r_j^2 + r_8^2 - 2r_j r_8 \, t \]

\[ t = \cos \psi, \quad \psi = \text{geocentric angle between } P_1 \text{ and } P_j \]

\[ dS = \text{surface element at the internal sphere} \]

\[ S = 4\pi r_8^2 \]

Note. The gravity anomaly \( \Delta g_j \) at the point \( P_j \) is defined as the difference between the real gravity at \( P_j \) and the theoretical gravity at a point \( P_j' \), which is located along reference ellipsoid normal through \( P_j \) with theoretical potential at \( P_j' \) equal to the geopotential at \( P_j \). (See Heiskanen and Moritz, 1967, p. 83.)

Due to the irregular topography of the earth there is rigorously no solution of (1.1). However, for the Bjerhammar's problem (discrete boundary value problem) it is always possible to find a fictitious function \( \Delta g^* \) that fits (1.1) for the observations. Once \( \Delta g^* \) is determined it may be applied in Poisson's, Stokes' or Vening Meinesz' integral formula for the prediction of gravity anomalies, geoidal undulations or vertical deflections, respectively. It should be stated, that in order to be consistent with Stokes' formula, the harmonics of degree 0 and 1 should be removed from (1.1).

The method of least squares collocation was introduced in the principal presentations by T. Krarup (1969) and H. Moritz (1970, 1972). In general this method is defined as consisting of two parts: a least squares adjustment and a Wiener prediction. In this study we will consider only the latter part of collocation. If the relevant covariance functions are known, any physical quantity \( v_j \) in an arbitrary position \( P_j \) on or outside the surface of the earth may be estimated in an optimal way from a finite number of observations \( \Delta g \) by the following formula:
\[(1.2)\]  
\[v_1 = c_1 (C + D)^{-1} \Delta g\]

where

\[c_1 = \text{cov}(v_1, \Delta g)\]

\[C = \text{cov}(\Delta g, \Delta g)\]

\[D = \text{error covariance matrix.}\]

In particular we predict gravity anomalies by:

\[(1.3)\]  
\[\Delta g_1 = c_1 (C + D)^{-1} \Delta g\]

where the elements of \(c_1\) and \(C\) are given by the spatial covariance function for the gravity anomalies:

\[(1.4)\]
\[C_{ij} = \sum_{n=0}^{\infty} c_n \left(\frac{r_{ij}^2}{r_i r_j}\right)^{n+2} P_n(\cos \psi_{ij})\]

where \(c_n\) are the so-called degree variances of the gravity anomalies defined according to the definition in Heiskanen and Moritz (1967, p. 259):

\[(1.5)\]  
\[c_n = \frac{1}{4\pi} \iint \Delta g_n^2 d\sigma\]

where \(\Delta g_n\) is the anomaly Laplace harmonic on the Bjerhammar sphere.

The purpose of this paper is to study the relations between some of Bjerhammar's solutions and collocation. We start with solving Poisson's integral equation (1.1) in different ways according to Bjerhammar.

2. Bjerhammar Solutions at the Internal Sphere

In the numerical application of (1.1) Bjerhammar used a finite set of blocks on the internal sphere with a constant value of \(\Delta g^*\) over each block. In this way the following matrix equation is obtained from (1.1) for \(m\) observations and \(N\) surface blocks:

\[(2.1)\]  
\[A_{mn} \Delta g^* = \Delta g\]
where

\[ \Delta g = \text{vector of observed gravity anomalies} \]
\[ \Delta g^* = \text{vector of unknowns} \]
\[ A = \text{coefficient matrix with elements } A_{\ell k} \]

\begin{equation}
A_{\ell k} = \frac{r_L^2 - r_B^2}{4\pi r_L} \int \int_{\Delta S_k} \frac{dS}{r_L^3} \tag{2.2} \end{equation}

\[ \Delta S_k = \text{the surface of block } k \]

Applying the mean value theorem of integral calculus formula (2.2) may be rewritten:

\begin{equation}
A_{\ell k} = \frac{r_L^2 - r_B^2}{4\pi r_L r_B} \Delta S_k \tag{2.2a} \end{equation}

where \( k \) refers to a certain mean value point \( P_k \) inside \( \Delta S_k \). In practice we may approximate \( P_k \) by the center of the block \( k \). If the matrix \( A \) has full rank there is always a solution of (2.1) for \( N \geq m \). For \( N = m \) we obtain the unique solution:

\begin{equation}
\Delta g^* = A^{-1} \Delta g \tag{2.3} \end{equation}

where \( A^{-1} \) is the Cayley inverse of \( A \).

For \( N > m \) the solution is not unique unless an additional condition is satisfied (condition adjustment, Section 2.2). Finally, for \( N < m \) and full rank of \( A \) the least squares solution minimizes the square sum of the residuals (see Section 2.3).

2.1 Generalization

By expanding the Poisson kernel into a series of Legendre's polynomials, formula (1.1) becomes:

\begin{equation}
\Delta g_f = \frac{1}{S} \int \int S \Delta g^* \sum_{n=0}^{\infty} (2n+1) \left( \frac{r_B}{r_f} \right)^{n+2} P_n(\cos \psi) \, dS \tag{2.4} \end{equation}
We may also expand $\Delta g^*$ into a series of Laplace's harmonics (we assume that the expansion is convergent):

$$\Delta g^* = \sum_{n=0}^{\infty} \Delta g_n$$

Due to the orthogonality of the spherical harmonics when integrated over a sphere:

$$\int \int \Delta g_n \ P_n \ dS = 0 \quad \text{for} \quad n \neq n'$$

(2.4) may be rewritten in the following way:

$$\Delta g_j = \frac{1}{S} \int \int u^* \sum_{n=0}^{\infty} \sqrt{(2n+1) \ c_n^*} \ \left( \frac{r_s}{r_j} \right)^{n+2} \ P_n(\cos \psi) \ dS$$

(2.5)

$$u^* = \sum_{n=0}^{\infty} \frac{\Delta g_n}{\sqrt{c_n^*/(2n+1)}}$$

(2.6)

In contrast to the degree variances in collocation ($c_n$), which are a priori defined by formula (1.5), the parameters $c_n^*$ introduced in formulae (2.5) and (2.6) of the generalized Bjerhammar method, are more or less arbitrary. Formally the conversion from (2.4) to (2.5) is valid for any $c_n^*$ such that (2.6) and the kernel of (2.5) converge. However, from a practical point of view it will hardly be advisable to solve (2.5) for a $u^*$ that is less smooth (with less attenuating higher degree terms) than $\Delta g^*$. Already the solution for $\Delta g^*$ is questionable in the continuous case. See Section 5, Molodensky et al. (1962) and Pick (1965). In the discrete case a solution is always possible. By changing $c_n^*$ a variety of solutions to Bjerhammar's problem are obtained. These solutions for $u^*$ are completely in accordance with the solutions for $\Delta g^*$ in (2.1) - (2.2).

Example: The inverse Stokes function. We insert:

$$\sqrt{c_n^*} = \frac{n-1}{r_s} \sqrt{2n+1}$$
into (2.5) and (2.6) (with lower limit \( n = 2 \) of the summation). Then we obtain (for \( dS = r_\theta^2 d\theta \sigma \)):

\[
\Delta g_3 = \iint M(r_1, \psi) u^* d\sigma
\]

where

\[
M(r_1, \psi) = \frac{1}{4\pi r_1} \sum_{n=2}^{\infty} (2n+1)(n-1) \left( \frac{r_\theta}{r_1} \right)^n P_n(\cos \psi)
\]

and

\[
u^* = \sum_{n=2}^{\infty} \frac{r_\theta \Delta g_n}{n-1} = \sum_{n=2}^{\infty} T_n = T^*
\]

where we have introduced the notation (Heiskanen and Moritz, ibid., p. 97):

\[
T_n = \frac{r_\theta \Delta g_n}{(n-1)}
\]

\( T_n \) being the Laplace harmonic of degree \( n \) of the (fictitious) disturbing potential \( T^* \) at the Bjerhammar sphere.

\( M \) may be written in a closed form (Sjöberg, 1975, p. 107):

\[
M(r_1, \psi) = \frac{s}{8\pi r_1} \left[ \frac{3(1-s^2)^2}{\ell^5} + \frac{s^2-5}{\ell^3} + 2 \right]
\]

where

\[
\ell^2 = 1 - 2st + s^2
\]

\[
s = \frac{r_\theta}{r_1}
\]

\[
t = \cos \psi
\]

Note. In Sjöberg (ibid.) the term \( n = 0 \) is included in \( M \).

Further examples of \( u^* \) are given in Table 2.1.
2.2 Minimum Norm Solutions

As previously stated, a matrix equation

\[(2.7) \quad A u^* = \Delta g, \quad m < N\]

does not have a unique solution unless an additional condition is imposed. We may define this solution as the one satisfying the following condition:

\[(2.8a) \quad \|u^*\|^2 = \frac{1}{S} \sum_{k=1}^{N} (u_k^*)^2 \Delta S_k = \text{minimum}\]

or, with matrix notations

\[(2.8b) \quad (u^*)^T Q^{-1} u^* = \text{minimum}\]

where

\[Q^{-1} = \frac{1}{S} \left( \begin{array}{c} \Delta S_1 \\ \Delta S_2 \\ \vdots \\ \Delta S_N \end{array} \right)\]

From least squares adjustment we obtain the following solution of (2.7) with the condition (2.8) (see for instance, Bjerhammar, 1973, Ch. 12):

\[(2.9) \quad u^* = Q A^T (A Q A^T)^{-1} \Delta g\]

with the minimum norm

\[(2.9a) \quad (u^*)^T Q^{-1} u^* = (A Q A^T)^{-1} \Delta g\]

Subsequently, different solutions are obtained by changing the number of blocks at the internal sphere. In the original paper by Bjerhammar (1964) most studies were restricted to the non-singular case (\(N = m\)). Condition adjustment (\(N > m\)) was treated in 1968 and 1969. It is obvious that these solutions from (2.9) are generally more cumbersome to compute than (2.3). However, the computational effort is drastically reduced to that in the non-singular case, if we let \(N\) approach infinity for well-behaving surface elements. This we show next.
We rewrite (2.9) in the following way:

\[(2.10)\]
\[u^* = QA^t X\]

where

\[(2.11)\]
\[X = (QA^t)^{-1} \Delta g\]

The coefficients \((A)_{jk}\) are given by the generalization of (2.1) and (2.2a) by (2.5):

\[(2.12)\]
\[(A)_{jk} = \frac{1}{S} \sum_{n=0}^{\infty} \frac{\sqrt{(2n+1)}}{c_n^*} \left( \frac{r_a}{r_1} \right)^{n+2} P_n (\cos \psi) \Delta S_k\]

By letting \(N\) go to infinity in such a way that the largest diameter of all \(\Delta S_k\) approaches zero it is shown in the Appendix, Proposition A.1, that \((AQ^t)_{11}\) becomes:

\[(2.13)\]
\[C_{11} = \lim_{n \to \infty} (AQ^t)_{11} = \sum_{n=0}^{\infty} c_n^* \left( \frac{r_a}{r_1} \right)^{n+2} P_n (\cos \psi_1)\]

Thus we obtain in the limit:

\[(2.13a)\]
\[X = C^{-1} \Delta g\]

where the elements of \(C\) are given by (2.13). Applying (2.7) for the prediction of new anomalies from the original set \(\Delta g\) we obtain:

\[\Delta g_1 = c_1 X\]

or

\[(2.14)\]
\[\Delta g_1 = c_1 C^{-1} \Delta g\]
where

\[ c_1 = \lim_{N \to \infty} A_1 Q A_1^T \]

The elements of \(c_1\) are also given by (2.13). Furthermore it follows from (2.9a) that the minimum norm in the limit is given by:

\[ \|u^*\|^2 = \Delta g^T C^{-1} \Delta g \]

We notice that (2.14) and (2.13) are exactly the solutions obtained with the collocation formulae (1.3) - (1.4) with \(c_n\) substituted by \(c_n^{*}\). Hence, for each type of degree variances \((c, c^{*})\) in collocation, there is an identical minimum norm solution in the generalized Bjerhammar approach. Some of these relations are given in Table 2.1. The derivations are given in Sjöberg (1975).

### Table 2.1

The Relations Between the Degree Variances in Collocation and Some Minimum Norms in the Generalized Bjerhammar Method

<table>
<thead>
<tr>
<th>(\sqrt{c_n^{*}/(2n+1)})</th>
<th>Norm</th>
<th>Explanation of Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(\Delta g^{*})</td>
<td>gravity anomaly</td>
</tr>
<tr>
<td>((n-1)/r_b)</td>
<td>(T^{*})</td>
<td>disturbing potential</td>
</tr>
<tr>
<td>(4 \pi (n-1)/(2n+1))</td>
<td>(\varphi^{*})</td>
<td>density layer</td>
</tr>
<tr>
<td>(4 \pi n (n-1)/r_b(2n+1))</td>
<td>(\mu^{*})</td>
<td>double layer</td>
</tr>
<tr>
<td>((n-1)/(n+1))</td>
<td>(\delta^{*})</td>
<td>gravity disturbance</td>
</tr>
<tr>
<td>(\gamma (n-1)/n(n+1))</td>
<td>(\theta^{*})</td>
<td>deflection of the vertical</td>
</tr>
<tr>
<td>((n-1)/(2n+1)(n+1))</td>
<td>(\delta^{(n)} T^{*})</td>
<td>potential gradient</td>
</tr>
<tr>
<td>(r_b^{n-1} n! (n-1)/(n+\nu)!)</td>
<td>(r^{\nu})</td>
<td>the (\nu) th derivative of (T)</td>
</tr>
</tbody>
</table>

\(\gamma = \) normal gravity at the reference ellipsoid

† from Sjöberg, (1975).
Subsequently there is a duality between the choice of degree variances in
collocation and the minimum norm in the generalized Bjerhammar technique.
Such an approach with a minimum norm was also emphasized by Krarup (1969).

Remark. $c_n$ of formula (1.4) was replaced by $(2n+1) c_n$ in Sjöberg (ibid.) and
by $(2n+1) \sigma_n^2$ by Lauritzen (1973, p. 73) and Bjerhammar (1974 and 1977a,b).
See also Krarup (ibid.).

Also in the generalized Bjerhammar technique the observation errors
can be taken into account [cf. formula (1.2)]. In this case we start with the model:

$$ A u^* = \Delta g - \epsilon $$

where

$$ E \{ \epsilon \epsilon^T \} = D $$

Rearranging this equation as:

$$ [A, I] \begin{bmatrix} u^* \\ \epsilon \end{bmatrix} = \Delta g $$

we readily obtain the solution:

$$ (2.9b) \quad \begin{bmatrix} u^* \\ \epsilon \end{bmatrix} = \begin{bmatrix} QA^T \\ D \end{bmatrix} \begin{bmatrix} [AQA^T + D]^{-1} \Delta g \end{bmatrix} $$

which minimizes

$$ (u^*)^T Q^{-1} u^* + \epsilon^T D^{-1} \epsilon $$

The predictions are given by

$$ (2.16) \quad v_1 = A_1 QA^T [AQA^T + D]^{-1} \Delta g $$

and for $N \to \infty$ this solution and (1.2) are identical. In this case the minimum
norm becomes:

$$ (2.17) \quad \Delta g^T C^{-1} \Delta g + \epsilon^T D^{-1} \epsilon $$
2.3 Least Squares Solutions for $u^*$

If we apply formula (2.7) with $m > N$, this matrix equation will not in general be consistent. Adding a residual vector, we obtain the following model:

\[
(2.18) \quad A \, u^* = \Delta g - \epsilon \quad , \quad m > N \quad , \quad E \{ \epsilon \epsilon^T \} = P \, \sigma^2
\]

This system may be solved with ordinary least squares adjustment by elements with the solution:

\[
(2.19) \quad \hat{u}^* = (A^T P A)^{-1} A^T P \, \Delta g
\]

which minimizes the square sum of the residuals ($\epsilon' P \epsilon$). The variance of unit weight ($\sigma^2$) is estimated by:

\[
(2.20) \quad s^2 = \Delta g^T P (\Delta g - A \, \hat{u}^*) / (m - N)
\]

and the covariance matrix of $u^*$ after adjustment is estimated by:

\[
(2.21) \quad Q_{uu} = s^2 (A^T P A)^{-1}
\]

All these derivations follow from elementary least squares adjustment (see for instance, Bjerhammar, 1973, Ch. 11).

The solution $\hat{u}^*$ can now be used for the prediction of derived quantities ($v_i$) in an arbitrary position $P_i$:

\[
(2.22) \quad \hat{v}_i = A_i \, \hat{u}^*
\]

where $A_i$ is the operator (vector) that relates $v_i$ to $u^*$.

The prediction variance may be determined in the following way:

\[
\epsilon_i = v_i - \hat{v}_i = v_i - A_i (u + \epsilon_u)
\]
\[ \epsilon_i^2 = v_i^2 + A_1 (u u^T + \epsilon_u \epsilon_u^T) A_1^T - 2 A_1 (u + \epsilon_u) v_i^T \]

Taking the expectation of both members we obtain the prediction variance:

\[ m_1^2 = \sigma_{v_1}^2 + A_1 (C_{uu} + Q_u \sigma_u^2) A_1^T - 2 A_1 C_{uv} \]

where

\[ \sigma_{v_1}^2 = E \{ v_1^2 \} \]

\[ C_{uu} = E \{ uu^T \} \]

\[ C_{uv} = E \{ uv^T \} \]

Furthermore we have assumed that \( E \{ \epsilon_u v_1^T \} = 0 \). The prediction variance may be written:

\[ (2.23) \]

\[ m_1^2 = \sigma^2 A_1 Q_u A_1^T + m_\lambda^2 \]

where

\[ m_\lambda^2 = \sigma_{v_1}^2 + A_1 C_{uu} A_1^T - 2 A_1 C_{uv} \]

The first term of (2.23), which is caused by the error of \( u^* \), is easily determined in the adjustment. The second term \( (m_\lambda^2) \) is due to the error of \( A_1 \). The determination of \( m_\lambda^2 \) requires that the covariance relations are known.

The least squares solutions are favorable first of all when a large number of observations \( m \) are available. It should be noted that the solutions for \( m > N \) do not include pure prediction. A filtering of the data is always present in this type of prediction.

2.4 Impulse Approach

Bjerhammar (1974) introduced a method, where the unknowns \( u^* \) are located in discrete points at the internal sphere. This method, called the Dirac method, gives the following model:
\[ u^* (\bar{r}) = \sum_{k=1}^{N} u^*_k \delta (\bar{r} - \bar{r}_k) \]

where
\[ \bar{r} = (r_0, \varphi, \lambda) = \text{coordinates of a current point at the internal sphere} \]
\[ \bar{r}_k = (r_k, \varphi_k, \lambda_k) = \text{coordinates of a selected carrier point } P_k \text{ at the internal sphere} \]
\[ u^*_k = \text{unknown associated with } P_k \]
\[ N = \text{number of unknowns} \]
\[ \delta() = \text{Dirac's delta function, defined by} \]

\[ \frac{1}{4\pi} \iiint u^* (\bar{r}) \delta (\bar{r} - \bar{r}_k) \, d\sigma = u^* (\bar{r}_k) \]

Inserting (2.24) into (2.5) we arrive at

\[ \Delta g_j = \sum_{k=1}^{N} u^*_k \, K (\bar{r}_j, \bar{r}_k) \]

where

\[ K (\bar{r}_j, \bar{r}_k) = \sum_{\lambda=\emptyset}^{\infty} \sqrt{(2n+1) c_n^* \left( \frac{r_0}{\rho_j} \right)^n} \, P_n (\cos \psi_k) \]

For \( m \) observations \( (\Delta g_j) \) (2.25) gives an exact matrix equation which may be solved by means of condition adjustment, least squares adjustment or direct solution dependent on whether \( m < N, m > N, \) or \( m = N. \) A generalization of the method is obtained if we also allow for carrier points located outside the Bjerhammar sphere (reflexive prediction, Bjerhammer, 1974).

It is obvious that this method is a generalization of the commonly used buried mass point method. In fact, the latter is obtained in the special case (cf. Table 2.1).

\[ \sqrt{c_n^*} = \text{const.} \, (n-1)/\sqrt{(2n+1)} \]
The non-singular Dirac method with carrier points located at the intersections of the internal sphere with the radius vectors of the observations is very similar to collocation. The auto-covariance function for \( \Delta g \) in collocation (with internal sphere radius \( r_0 \)):

\[
(2.27) \quad C(j, i) = \sum_{n=2}^{\infty} c_n (r_0^2/r_j r_i)^n \times P_n(\cos \psi_j)
\]

corresponds to the following kernel function in the Dirac method:

\[
(2.28) \quad C(j, i) = \sqrt{(2n+1) c_n^*} \times (r_0^2/r_j r_i)^n \times P_n(\cos \psi_j)
\]

where \( r_0 \) is the selected internal sphere radius (not necessarily the same as \( r_0 \)). Hence, \( C(j, i) \) of (2.27) and (2.28) are identical for:

\[
c_n = c_n^* = 2n+1
\]

and

\[
(2.29) \quad r_0 = r_0^2/r_i
\]

Consider the case \( r_i = r \) = radius of the mean earth sphere:

\[
r_0 = r - h_0
\]

and

\[
r_0 = r - h_0
\]

where \( h_0 \) and \( h_0 \) are the depths of the internal spheres from the mean earth sphere. By inserting these expressions for \( r_0 \) and \( r_0 \) into (2.29) we obtain:

\[
h_0 = 2h_0 - h_0^2/r
\]

or (after omitting the last term)

\[
(2.30) \quad h_0 = 2h_0
\]
Thus we have shown that in the special case $c_n = c_n^* = 2n+1$ and the radii of the observation points are constant ($r_i$), the kernel function $C(i, i)$ of the Dirac method and collocation are identical whenever the depth to the Bjerhammar sphere in the previous method ($h_0$) is twice that in the latter method ($h_a$). This result holds also for the predictions of the two methods (see Sections 5 and 8). See also Bjerhammar (1977, a, b).

From numerical point of view it is of importance that the kernel function (2.28) is unsymmetric in contrast to (2.27).

3. Stability of the Solutions

It is of great importance for the prediction results that the matrix equations are well conditioned. In this section we are going to compare the stability of the following systems in different cases:

\[(3.1) \quad A u^* = \Delta g \quad z = z_1\]

and

\[(3.2) \quad C X = \Delta g \quad z = z_1\]

These equations were introduced in (2.7) and (2.13a). Formulae (3.1) ($m = N$) and (3.2) ($N = \infty$) may be regarded as the extreme cases of (2.7). From the derivations in Section 2.2 we may also consider (3.2) as an intermediate system of equations in collocation.

A suitable measure of the stability of a matrix system is the condition number defined as:

\[(3.3) \quad \chi = \lambda_{\text{max}} / \lambda_{\text{min}}\]

where $\lambda$ refers to the eigen values of the coefficient matrix of the system ($A$ and $C$). The condition number has the following bounds:

\[1 \leq \chi \leq \infty\]

where the lower bound is the ideal situation and $\infty$ is a completely unstable (singular) system. Our main problem is therefore to determine the eigen values $\lambda_{\text{min}}$ and $\lambda_{\text{max}}$. 

-15-
We start with a two-dimensional example. We assume that $\Delta g$ is observed at $m$ regularly distributed points on a circle of radius $r$. The unknowns ($u^*$) are located on a circle of radius $r$. Applying the impulse approach [formula (2.25) with $m = N$] the system of equations becomes:

$$
(3.4) \quad \sum_{k=1}^{n} K(\theta_j - \theta_k) u^*_k = \Delta g_j ; \quad j = 1, 2, \ldots, N
$$

where $K(\theta_j - \theta_k)$ is the two-dimensional kernel function corresponding to $u^*$ and

$$
\theta_p = p \frac{2\pi}{N}
$$

The Fourier series of $K$ is:

$$
(3.5) \quad K(\theta) = \sum_{n=-\infty}^{\infty} b_n e^{i n \theta}
$$

where

$$
(3.6) \quad b_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} K(\theta) e^{-i n \theta} d\theta
$$

Furthermore, the eigenvalues ($\lambda_\ell$) of (3.4) are given by:

$$
(3.7) \quad \sum_{k=1}^{N} K(\theta_j - \theta_k) X_\ell^j = \lambda_\ell X_\ell^j ; \quad \ell, j = 1, 2, \ldots, N
$$

where $X_\ell^j$ is the $j$th element of the $\ell$th eigen vector. Let us try:

$$
(3.8) \quad X_1^\ell = \cos x = \frac{e^{i x} + e^{-i x}}{2}
$$

where

$$
x = j \ell \frac{2\pi}{N}
$$

Using the relation

$$
(3.9) \quad \sum_{j=1}^{N} e^{i \frac{2\pi}{N} (\ell - q)} = \begin{cases} 
N & \text{if } \ell = q + N p ; \quad p = 0, \pm 1, \pm 2, \ldots \\
0 & \text{otherwise}
\end{cases}
$$
we can easily arrive at:

\[
\sum_{j=1}^{N} X_{j}(\ell) X_{j}(q) = \begin{cases} 
N/2 & \text{if } \ell = q \\
0 & \text{otherwise}
\end{cases}
\]

Thus (3.8) satisfies the necessary property of eigen vectors being orthogonal to each other. We have also shown that (3.8) should be normed by \( \sqrt{2}/N \) in order to be the eigen vectors of (3.7). Finally, \( X_{j}(\ell) \) must satisfy (3.7) for some eigen values \( \lambda_{\ell} \). Inserting (3.8) and (3.5) into (3.7) we obtain from the left member:

\[
\sum_{n=-\infty}^{\infty} b_{n} e^{in\pi \frac{2\ell}{N}} \sum_{k=1}^{N} e^{i(k-\ell)\frac{2\pi}{N} k} = N e^{i\ell \frac{2\pi}{N}} \sum_{p=-\infty}^{\infty} b_{\ell + Np}
\]

Here we have used (3.9). Thus we obtain:

\[
\lambda_{\ell} = N \sum_{p=-\infty}^{\infty} b_{\ell + Np}
\]

where \( b_{n} \) is given by (3.6). If \( K \) is Poisson’s kernel, then (Seeley, 1966, p. 14):

\[
b_{n} = s^{\frac{n}{2}} \frac{n!}{(\Gamma(n)/\Gamma(n+1))^{\frac{1}{2}}}
\]

and

\[
\lambda_{\ell} = N \sum_{p=-\infty}^{\infty} s^{\ell + \frac{n}{2} + p}
\]

The sum is given in a closed form in the Appendix, Corollary A.2. We obtain:

\[
\lambda_{\ell} = N (s^{\ell} + s^{N-\ell})/(1-s) , \quad \ell = 1, 2, \ldots, N
\]

The condition number is finally given by:

\[
x_{1} = (1+s^{\beta})/(s^{\beta}+s^{N-\beta})
\]
where

\[ \beta = \left( \frac{N}{2} \right) \text{ = integer part of } N/2 \]

An analogous derivation of the condition number of (3.2) gives:

\[ x_2 = \frac{(1 + s^{2n})}{s^2 \beta + s^{2n-2\beta}} \]

so that for large \( N \)

\[ x_2/x_1 \approx s^{-N/2} \]

It is obvious that the system (3.1) is more stable than (3.2) for a given depth to the Bjerhammar circle.

Next we proceed to the planar cases of (3.1) and (3.2). We are going to use an approximate similarity transformation of the coefficient matrix to diagonal form.

The operator \( C \) is said to be unitarily equivalent to \( S \), if:

\[ S = U \cdot C \cdot U^{-1} \]

where \( S_{ij} = \lambda_{ij} \delta_{ij} \), \( \delta_{ij} \) is the Kronecker's symbol and \( U \) and \( U^{-1} \) are unitary transformations, inverse to one another. For a matrix it is a similarity transformation to diagonal form. According to Moritz (1966) and Schwarz (1971) such a unitary equivalence exists between \( c(x,y) \) and its Fourier transform:

\[
(3.10) \quad S(u,v) = U \cdot c(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} c(x,y) e^{-i(xu+vy)} dx dy
\]

This formula may therefore be used for an approximate determination of the eigen values of (3.1) - (3.2). If \( m_o \) and \( n_o \) are the numbers of blocks in the \( x \)- and \( y \)-directions, we obtain (cf. Schwarz, ibid.,):

\[
(3.11) \quad \lambda_{nm} = a \cdot S\left( \frac{2n}{2n_o + 1}, \frac{2m}{2m_o + 1} \right)
\]
where

\[ 0 \leq n \leq n_0 \quad \quad 0 \leq m \leq m_0 \]

and

\[ a = \| C \| \]

The covariance functions will be used in the planar approximation. Furthermore, equal area blocks are assumed with \( n_0 = m_0 \).

We are now going to determine the condition number for the Poisson kernel. This kernel corresponds to \( c_n = 2n + 1 \) and the minimum norm of \( \Delta g^* \) (see Table 2.1). The planar equivalence to the operator \( A \) of (2.1) is:

\[
A = \iint_{\mathbb{R}^2} a(x,y) \, dx \, dy
\]

where

\[
a(x,y) = \frac{h}{2\pi} / \left( x^2 + y^2 + h^2 \right)
\]

\[ h = z + b/2 \]
\[ b = 2(R - r_B) \]
\[ R = \text{mean earth radius} \]
\[ r = \text{radius of the computation point} \]
\[ r_B = \text{radius of the Bjerhammar sphere} \]
\[ z = r - R \]

Formula (3.12') has the following Fourier transform:

\[
S(u,v) = \iint_{\mathbb{R}^2} a(x,y) \exp \left\{ -i (ux + vy) \right\} \, dx \, dy
\]

\[
= 2\pi h \int_0^\infty \frac{r}{(r^2 + h^2)^3/2} J_0(\omega r) \, dr = 2\pi \exp\left\{ -h\omega \right\}
\]

-19-
where
\[ r = \sqrt{x^2 + y^2}, \quad \omega = \sqrt{u^2 + v^2} \]
and
\[ J_0 (\omega r) = \text{zero order Bessel function}. \]

Hence
\[ \lambda_{sa} = 2 \pi a \exp \left\{ -2 \pi h \sqrt{\frac{r^2 + m^2}{(2n_0 + 1)}} \right\} \]

and
\[ (3.14) \quad x_s = \exp \left\{ 2 \pi \sqrt{h} \frac{n_0}{(2n_0 + 1)} \right\} \approx \exp \left\{ \frac{\pi}{2} h \right\} \]

The planar approximation to the Poisson covariance function is (Moritz, 1976, p. 41):

\[ (3.15) \quad C(x, y) = C(r) = \frac{B/Z^2}{(1 + r^2/Z^2)^{\alpha/2}} \]

where \( Z = z_1 + z_2 + b \) and \( B = 2 R^2 \).

The Fourier transform of the covariance function is:

\[ (3.16) \quad S(\omega) = 2 \pi B \exp \left\{ -Z \omega \right\} \]

Thus we obtain:
\[ x_c = \exp \left\{ 2 \pi \sqrt{Z} \frac{n_0}{(2n_0 + 1)} \right\} \]

Assuming that \( z_1 = z_2 + z \) we have \( Z = 2h \) and

\[ (3.17) \quad x_c = \exp \left\{ 4 \sqrt{2} \pi h \frac{n_0}{(2n_0 + 1)} \right\} \approx \exp \left\{ 2 \sqrt{2} \pi h \right\} \]

so that
\[ (3.18) \quad x_c \approx x_s^2 \]

-20-
From formula (3.18) it is obvious that the condition number \((x_e)\) of the "original" equation (3.1) is squared by letting the number of blocks on the internal sphere approach infinity \((N\to\infty)\) (and then solving the system (3.2)). A considerable loss of stability is achieved. Again we emphasize that the equation system (3.1) is generally unsymmetric in contrast to (3.2), which fact has to be considered in the numerical solution of the system. It is not recommended to form normal equations in (3.1) in order to obtain a symmetric system, because this procedure will increase the condition number to that of system (3.2). (3.1) should be solved directly, for instance, by Gauss elimination or in an iterative way by successive approximations.

3.1 The Effect of Smoothing (Noise)

We assume that the matrices \(A\) and \(C\) of formulae (3.1-2) are substituted by:

\[
\bar{A} = A + D
\]

and

\[
\bar{C} = C + BD
\]

where the elements of \(D\) are given by:

\[
d_{ij} = \begin{cases} k & \text{if } i=j, \quad k > 0 \\ 0 & \text{otherwise} \end{cases}
\]

and \(B = 2R^2\) (see formula (3.15)).

Then \(\bar{A}\) and \(\bar{C}\) correspond to the kernel functions:

\[
\bar{a}(x,y) = a(x,y) + k \delta(x,y)
\]

and

\[
\bar{c}(x,y) = c(x,y) + Bk \delta(x,y)
\]

with the spectra

\[
\bar{S}_a(u,v) = S_a(u,v) + k \\
\bar{S}_c(u,v) = S_c(u,v) + Bk
\]

-21-
and the eigen values

\[ \lambda_{ns}^{(a)} = \lambda_{ns} + a \]

\[ \lambda_{ns}^{(c)} = \lambda_{ns} + b \]

Here \( \delta \) is the delta function and \( S, \lambda \) and \( a \) are the same as in the previous section. In this case we obtain for the Poisson kernel (cf. (3.14) and (3.17)):

\[
X_n \approx \frac{k + 2\pi}{k + 2\pi \exp \{-\pi h/2\}} < \exp \{\sqrt{2} \pi h\}
\]

and

\[
X_0 \approx \frac{k + 2\pi}{k + 2\pi \exp \{-2\pi h\}} < \exp \{2\sqrt{2} \pi h\}
\]

We notice that the smoothing stabilizes both systems of equations (cf. Section 6.3).

4. On the Convergence of an Iterative Solution

In this section we are going to investigate a condition for convergence of an iterative solution of the matrix equation (3.2):

\[
(4.1) \quad C X = \Delta g
\]

Using the ordinary method of successive approximation, we obtain:

\[
(4.2) \quad X^{(k)} = \Delta g + (I - C) X^{(k-1)}; \quad k = 1, 2, \ldots
\]

and

\[
\Delta X^{(k)} = X^{(k)} - X^{(k-1)} = (I - C)(X^{(k-1)} - X^{(k-2)}) = (I - C)^{k-1} (X^{(i)} - X^{(0)})
\]
Hence
\[ \| \Delta X^{(k)} \| \leq \| I - C \|^{k-1} \| X^{(1)} - X^{(0)} \| \]

A sufficient condition for the convergence of (4.2), i.e.
\[ \lim_{k \to \infty} \| \Delta X^{(k)} \| = 0 \]
is therefore
\[ \| I - C \| < 1 \]
or
\[ \max_{i} \sum_{j} \| \delta_{ij} - c_{ij} \| < 1 \]

Now
\[ \| \delta_{ij} - c_{ij} \| = \begin{cases} |1 - c_{11}|, & \text{if } i = j \\ |c_{ij}|, & \text{if } i \neq j \end{cases} \]

so that the condition may be written:

(4.3a) \[ \sum_{j=1}^{\infty} |c_{ij}| < 2, \text{ if } c_{11} > 1 \]

and

(4.3b) \[ 2 c_{11} > \sum_{j=1}^{\infty} |c_{ij}|, \text{ if } c_{11} \leq 1 \]

For finite \( c_{11} \) we may always divide each row of the equation system with a sufficiently large number to ensure that the diagonal elements of \( C \) are equal to or less than unity. Hence we can limit our study to the condition (4.3b).

First, we study this condition for Poisson's integral equation for the circle (Sealey, 1966, p. 14):
(4.4) \[ \Delta g_j = \int_{-\pi}^{\pi} a(j, i) \Delta g^* \, d\theta_i \]

where

\[ a(j, i) = \frac{1}{2\pi} \frac{1 - s^2}{1 + s^2 - 2s \cos \psi_{ij}} = \frac{1}{2\pi} \frac{1 - s^2}{|s - e^{i\psi_{ij}}|^2} \]

\[ s = r_\theta / r_j \quad (< 1) \]

\[ \psi_{ij} = \theta_j - \theta_i \]

Suppose that we apply (4.4) in a discrete approach with equal spacing between the \( N \) unknowns (\( \Delta g^s \)). We obtain:

(4.5) \[ A \Delta g^* = \Delta g \]

where

(4.6) \[ A_{jk} = a(j, k) \Delta \theta = (1 - s^2) / N |s - e^{i(\theta_j - \theta_k)}|^2 \]

\[ \Delta \theta = 2\pi / N \]

\[ \theta_k = 2\pi k / N \quad ; \quad k = 1, 2, \ldots, N \]

In the non-singular Dirac approach (\( N = m \)) we arrive at the following condition from (4.3b) and (4.6) (for \( \theta_j = 0 \)):

\[ \frac{2}{m} \frac{1 - s^2}{(1-s)^2} > \frac{1}{m} \sum_{k=1}^{m} \frac{1 - s^2}{|s - e^{i\theta_k}|^2} \]

or

(4.7) \[ S(m, s) = \frac{1}{m} \sum_{k=1}^{m} \frac{1 - s^2}{|s - e^{i\theta_k}|^2} - \frac{2(1-s)}{m(1-s)} < 0 \]

The Poisson kernel has the following Fourier series:
Furthermore

\[ \sum_{k=1}^{\infty} e^{i \pi k} / s = \begin{cases} m & \text{if } n = m \cdot j, \ j = 0, \pm 1, \pm 2, \ldots \\ 0 & \text{otherwise} \end{cases} \]

so that (4.7) becomes

\[
S(m, s) = 1 + 2 \sum_{j=1}^{\infty} s^{i\pi j} = \frac{2(1+s)}{m(1-s)} - \frac{1+s}{1-s} - \frac{2(1+s)}{m(1-s)}
\]

Subsequently, for large values of \( m \) the critical relation between \( m \) and \( s \) with

\[ S(m_0, s) = 0 \]

is given by

\[ m_0 = 2 \frac{1+s}{1-s} \] (4.8a)

or

\[ s = \frac{m-2}{m+2} \] (4.8b)

The corresponding relation for collocation is obtained by substituting \( s \) by \( s^2 \) in the previous formulae. The results are shown in Table 4.1.

For comparison we also derive the relation between \( m \) and \( s \) in the case of condition adjustment (\( N > m \)). Then we arrive at the following condition from (4.3b):

\[ S(m, N, s) = \max_{1}^{n} \sum_{j=1}^{N} (AQA^T)_{ij} - 2(AQA^T)_{ij} < 0 \]

where

\[ (AQA^T)_{ij} = N \sum_{k=1}^{N} A_{ik} A_{jk} \]
Table 4.1

The Relation Between the Number of Observations \((m_0)\) and the Ratio \(h/a\) Compared to the Dirac Method and Collocation*

<table>
<thead>
<tr>
<th>(m_0)</th>
<th>(s)</th>
<th>(h = \frac{1+s}{a\pi s})</th>
<th>(s)</th>
<th>(h)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.667</td>
<td>0.796</td>
<td>0.816</td>
<td>0.708</td>
</tr>
<tr>
<td>100</td>
<td>0.961</td>
<td>0.650</td>
<td>0.961</td>
<td>0.643</td>
</tr>
<tr>
<td>1000</td>
<td>0.996</td>
<td>0.638</td>
<td>0.998</td>
<td>0.637</td>
</tr>
<tr>
<td>10000</td>
<td>0.9996</td>
<td>0.637</td>
<td>0.9998</td>
<td>0.637</td>
</tr>
<tr>
<td>∞</td>
<td>1</td>
<td>2/\pi</td>
<td>1</td>
<td>2/\pi</td>
</tr>
</tbody>
</table>

* \(h = r - r_0\) and \(a = 2\pi r/m_0\)

\[
A_{ik} = \frac{1-s^2}{N(1-2s \cos \psi_{ik} + s^2)}
\]

\[
\psi_{ik} = \theta_i - \theta_k = \frac{2\pi i}{m} - \frac{2\pi k}{N} = 2\pi (ip-k)/N
\]

\(p = N/m\) (integer)

The sum \(S(m, N, s)\) is given in the Appendix, Proposition A.3. For large \(N\) we have, approximately:

\[S(m, s) \approx 2 - m + \frac{2m}{1-s^2} - \frac{4}{1-s^2} < 0\]

or

\[m < 2 \frac{1-s^2}{1+s^2} \frac{1+s^2}{1-s^2}\]

-26-
For large $m$ we finally arrive at:

$$m < m_0 = \frac{1+s^2}{1-s^2}$$

and

$$s > \sqrt{\frac{m-2}{m+2}}$$

These are the limiting relations between $s$ and $m$ when using collocation [cf. (4.8 a-b)].

What is the relation between $s$ and $m$ that satisfies the condition (4.3 b) for the generalized integral equation:

$$\Delta g_j = \int_0^{2\pi} a(j, k) u^* d\theta_k$$

where

$$a(j, k) = \sum_{n=-\infty}^{\infty} \sqrt{c_n^k} s^{\text{ln}} e^{i\text{ln}(\theta_j - \theta_k)}$$

Following the previous derivations, we obtain in the non-singular case of the Dirac method [cf. (4.7)]:

$$S(m, s) = \sum_{j=-\infty}^{\infty} \sqrt{c_j^m} s^{\text{ln}} - \frac{2}{m} \sum_{n=-\infty}^{\infty} \sqrt{c_n^m} s^{\text{ln}} < 0$$

Thus the sufficient relation between $s$ and $m$ for the convergence according to (4.3b) is very much dependent on $c_n^m$.

Example: $\sqrt{c_n^m} = |n| + 1$

$$S(m, s) = \sum_{j=-\infty}^{\infty} |j| m s^{\text{ln}^2} + \sum_{j=-\infty}^{\infty} s^{\text{ln}^2} - \frac{2}{m} \left( \sum_{n=-\infty}^{\infty} |n| s^{\text{ln}} - \sum_{n=-\infty}^{\infty} s^{\text{ln}} \right) =$$

$$= (s \frac{d}{ds} + 1) \left( \sum_{j=-\infty}^{\infty} s^{\text{ln}^2} - \frac{2}{m} \sum_{n=-\infty}^{\infty} s^{\text{ln}} \right)$$
\[
\left( s \frac{d}{ds} + 1 \right) \left( 1 + 2 \frac{s}{1-s} - \frac{2}{m} \frac{s}{1-s} \right) =
\]

\[
= 2m \frac{s}{(1-s)^2} - \frac{4}{m} \frac{s}{1-s} + 1 + 2 \frac{s}{1-s} - \frac{2}{m} \frac{s}{1-s} < 0
\]

For large \( m \) we have approximately

\[
m - 2 - \frac{4s}{(1-s)^2} - \frac{4s}{1-s} < 0
\]

or

\[
m < m_0 = 2 + \frac{4s(2-s)}{(1-s)^2}
\]

and

\[
s > s_0 = 1 - \left\{ 1 - \left( \frac{m_0 - 2}{m_0 + 2} \right)^{1/2} \right\}
\]

For \( m_0 \to \infty, s_0 \to 1 \) and \( h/a \to 2/\pi \). Again we obtain \( 2\pi \) as a sufficient ratio between the height of the observations and the spacing of the observations for convergence in the continuous case (cf. Table 4.1).

We now investigate Poisson’s integral equation in the planar approximation (see formula (3.12)). We assume that a square of side \( B \) is divided into \( N \) surface elements of side \( b = B/N' \), where \( N' = \sqrt{N} \). Then we obtain the following coefficients:

\[
A_{jk} = \frac{b^2 h}{2\pi [(x_j - x_k)^2 + (y_j - y_k)^2 + b^2]^{3/2}} =
\]

\[
= c/2\pi [(j_x - k_x)^2 + (j_y - k_y)^2 + c^2]^{3/2}
\]

where

\[
c = h/b \]

\[
x_j = j_x, b \]

\[
y_j = j_y, b \]
In the non-singular case \( N = m' \) the condition (4.3b) becomes (for \( j_s = j_r = m'/2 \)):

\[
S (m', c) = \frac{c}{2\pi} \sum_{k_x = 1}^{m'} \sum_{k_y = 1}^{m'} \frac{1}{[(m'/2 - k_x)^2 + (m'/2 - k_y)^2 + c^2]^{3/2}} - \frac{1}{\pi c^2} < 0
\]

The critical relation between \( m' \) and \( c \) is shown in Table 4.2.

**Table 4.2**

The Ratio \( c = h/b \) Satisfying \( S (m', c) = 0 \) as Given for Different \( m' \).

<table>
<thead>
<tr>
<th>( m' )</th>
<th>Dirac Method</th>
<th>Collocation</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>0.57</td>
<td>0.285</td>
</tr>
<tr>
<td>100</td>
<td>0.54</td>
<td>0.270</td>
</tr>
<tr>
<td>500</td>
<td>0.52</td>
<td>0.260</td>
</tr>
<tr>
<td>1000</td>
<td>0.52</td>
<td>0.260</td>
</tr>
<tr>
<td>5000</td>
<td>0.51</td>
<td>0.255</td>
</tr>
<tr>
<td>10000</td>
<td>0.50</td>
<td>0.250</td>
</tr>
</tbody>
</table>

The result of the computations is that the ratio \( h/b \) should be less than 0.5 for an uncritical convergence of the non-singular Dirac solution with Poisson's kernel (see also Koch, 1968). If the method of collocation is used, the ratio should be less than 0.25. In this study we have based the results on the condition (4.3b) for a solution by successive approximations. However, Schwarz (1971) emphasized that more generous ratios (\( h/b \)) may be allowed by using other numerical methods. The figures given in this section should therefore, first of all, be used to compare the stability of different methods (Dirac method, condition adjustment, collocation) with each other.

In conclusion, the Dirac method with \( (m = N) \) gives a more stable solution than collocation for a given radius \( r_s \). In the case of condition adjustment \( (N > m) \) the stability of the Dirac method is roughly the same as for collocation. This study was restricted to the Poisson kernel functions, and the conditioning changes with the choice of degree variances \( (c_\kappa \text{ and } c_\kappa^*) \). In general, however, the above tendency can be expected for an arbitrary type of kernel function.
5. Predictions According to Bjerhammar

Using the collocation formula (1.2) any geophysical quantity $v_1$ can be predicted from a vector $\Delta g$ of gravity anomalies by:

\[(5.1a) \quad v_1 = c_1 X\]

where

\[(5.1b) \quad X = (C + D)^{-1} \Delta g\]

In the generalized Bjerhammar methods the unknowns $X$ of collocation are replaced by the unknowns $u^*$ at the internal sphere (see section 2). The prediction of $v_1$ becomes accordingly:

\[(5.2) \quad v_1 = F_1 u^*\]

or

\[(5.2') \quad v_1 = \sum_{k=1}^{N} F_{1k} u_{k}^*\]

where $F_1$ is the relevant coefficient matrix (with elements $F_{1k}$) relating $u^*$ to $v_1$. Formula (5.2') is the discrete form of the integral formula:

\[(5.2'') \quad v_1 = \frac{1}{S} \iiint f_{1k} u^* \, dS\]

and the kernel function $f_{1k}$ and the coefficient $F_{1k}$ are related by (cf. formulae (2.2) and (2.2a)):

\[(5.3) \quad F_{1k} = \frac{1}{S} \iiint f_{1k} \, dS = f_{1k} \Delta S_k / S\]

where $k$ corresponds to some mean value point inside $\Delta S_k$. In practice this point is approximated by the center of $\Delta S_k$. The function $f_{1k}$ can be derived from formula (2.5) for each specific quantity $v_1$. For $v_1 = \Delta g_1$ we have:

\[(5.4) \quad f_{1k} = \sum_{n=0}^{\infty} \sqrt{(2n+1)c_n} \left( \frac{r_k}{r_1} \right)^{n+2} P_n(\cos \psi_{1k})\]
and $F_{ik} = A_{ik}$ according to (2.12). Further examples are given below.

**Prediction of Disturbing Potentials**

In the spherical approximation the disturbing potential ($T_1$) is related to the gravity anomaly ($\Delta g$) by (Heiskanen and Moritz, ibid., p. 89):

\[
\Delta g_1 = - \frac{3T_1}{3r_1} - \frac{2T_1}{r_1^2}.
\]

(5.5)

By replacing $T_1$ of (5.5) by:

\[
f_{ik} = r_8 \sum_{n=2}^{\infty} \frac{\sqrt{(2n+1)c_*^*}}{n-1} \frac{\left( \frac{r_8}{r_1} \right)^{n+1}}{P_n (\cos \psi_{ik})}
\]

(5.6)

the right hand side of (5.5) becomes (5.4) (we exclude terms of degree less than 2). This result implies that $f_{ik}$ according to (5.6) is the kernel function in (5.2') for $v_1 = T_1$.

**Prediction of Height Anomalies**

From Bruns's theorem (Heiskanen and Moritz, ibid., p. 293) the kernel function for the height anomaly ($v_1 = \xi_1$) becomes:

\[
f_{ik} = \frac{r_8}{\gamma} \sum_{n=2}^{\infty} \frac{\sqrt{(2n+1)c_*^*}}{n-1} \frac{\left( \frac{r_8}{r_1} \right)^{n+1}}{P_n (\cos \psi_{ik})}
\]

(5.7)

where $\gamma$ is the theoretical gravity at the reference ellipsoid. In the special case $r_1 = r_8$ and $c_*^* = 2n+1$ ($u^* = \Delta g^*$) (5.7) inserted into (5.2') yields Stokes' formula (Heiskanen and Moritz, ibid., p. 94).

**Prediction of Deflections of the Vertical**

The two components of the deflections of the vertical ($\xi$, $\eta$) are related to the height anomaly $\xi$ according to (Heiskanen and Moritz, ibid., p. 312):

\[
\begin{align*}
\{\xi\} &= - \frac{1}{r_1} \left\{ \frac{\partial}{\partial \phi_1} \right\} \\
\{\eta\} &= - \frac{1}{r_1} \left\{ \frac{\partial}{\partial \lambda_1} \right\} \\
\zeta_1 &= - \frac{1}{r_1} \left\{ \frac{\partial}{\partial \phi_1} \right\} \\
\end{align*}
\]

\[
\begin{align*}
\{\xi\} &= - \frac{1}{r_1} \left\{ \frac{\partial}{\partial \phi_1} \right\} \\
\{\eta\} &= - \frac{1}{r_1} \left\{ \frac{\partial}{\partial \lambda_1} \right\} \\
\zeta_1 &= - \frac{1}{r_1} \left\{ \frac{\partial}{\partial \phi_1} \right\} \\
\end{align*}
\]

\[
\begin{align*}
\{\xi\} &= - \frac{1}{r_1} \left\{ \frac{\partial}{\partial \phi_1} \right\} \\
\{\eta\} &= - \frac{1}{r_1} \left\{ \frac{\partial}{\partial \lambda_1} \right\} \\
\zeta_1 &= - \frac{1}{r_1} \left\{ \frac{\partial}{\partial \phi_1} \right\} \\
\end{align*}
\]

-31-
where $\varphi_1$ and $\lambda_1$ are the geodetic latitude and longitude. By using the following relations from Heiskanen and Moritz (ibid., p. 113) and Abramowitz and Stegun (1964, p. 334):

$$\frac{\partial \psi}{\partial \varphi} = -\cos \alpha \quad \frac{\partial \psi}{\partial \lambda} = -\sin \alpha \cos \varphi$$

and

$$\frac{\partial P_n(\cos \psi)}{\partial \psi} = -\sin \psi \cdot \frac{d P_n(\cos \psi)}{d \cos \psi} =$$

$$= \frac{n(n+1)}{(2n+1)\sin \psi} \{ P_{n+1}(\cos \psi) - P_{n-1}(\cos \psi) \}$$

where $\alpha$ is the azimuth from the fixed (computation) point to the moving point, we obtain the following kernel functions for $\xi_1$ and $\eta_1$:

$$f_{ik} = \left\{ \begin{array}{l}
\cos \alpha_{ik} \\
\sin \alpha_{ik}
\end{array} \right\} \frac{1}{\gamma \sin \psi_{ik}} \times$$

$$\times \sum_{n=2}^{\infty} \frac{n(n+1)}{n-1} \sqrt{\frac{c_n^*}{2n+1}} \left( \frac{r_i}{r_1} \right)^{n+2} \left\{ P_{n+1}(\cos \psi) - P_{n-1}(\cos \psi) \right\}$$

where $\cos \alpha_{ik}$ refers to $\xi_1$ and $\sin \alpha_{ik}$ to $\eta_1$. Formula (5.8) inserted into (5.2) gives a generalization of Vening Meinesz' formula (Heiskanen and Moritz, ibid., p. 114).

**Prediction of the Vertical Gradient of Gravity**

The kernel function for $v_1 = \partial \Delta g_1 / \partial r_1$ is easily obtained from (5.4). The result is:

$$f_{ik} = -\frac{1}{r_i} \sum_{n=0}^{\infty} \sqrt{\frac{(2n+1)c_n^*}{(n+2)(r_i/r_1)^{n+2}}} P_n(\cos \psi_{ik})$$

For certain sets of $c_n^*$, the above kernel functions may be given in closed forms (cf. below and Sjöberg, 1975, section A.2).

**Note.** In the impulse methods (section 2.4) $F_{ik} = f_{ik}$.
5.1 Minimum Norm Solutions

The prediction of a quantity $v_1$ according to any of the methods of Bjerhammar described in section 2 is given by formulae (5.2) and (5.3) for any finite number of unknowns ($N$). For example, in the minimum norm solutions (section 2.2) the prediction formula becomes from (5.2) and (2.9):

$$v_1 = F_1 Q A^t (A Q A^t)^{-1} \Delta g$$

From this formula we obtain in the limit ($N \to \infty$)(cf. 2.14):

$$v_1 = k_1 C^{-1} \Delta g$$

(5.10)

where the elements of the matrix $C$ are given by formula (2.13):

$$C_{ij} = \sum_{n=0}^{\infty} c_\nu^n \left( \frac{r_0}{r_1} \right)^{n+2} P_n (\cos \psi_{ij})$$

and

$$k_1 = \lim_{N \to \infty} k_1$$

The following elements $k_{1k}$ of $k_1$ are obtained in accordance with formulae (2.13), (5.4) and (5.6)-(5.9). The coefficients $c_\nu^n$ are given by the selected minimum norm (see Table 2.1).

(5.4')

$$k_{1k} = \sum_{n=0}^{\infty} c_\nu^n s^{n+2} P_n (t) \quad \text{for } v_1 = \Delta g$$

(5.6')

$$k_{1k} = \frac{r_1}{\gamma} \sum_{n=2}^{\infty} \frac{c_\nu^n}{n-1} s^{n+2} P_n (t) \quad \text{for } v_1 = \zeta_1$$

(5.8)

$$k_{1k} = \frac{1}{\gamma \sin \psi_{ik}} \sum_{n=2}^{\infty} \frac{c_\nu^n n(n+1)}{n-1} \left\{ P_{n+1}(t) - P_{n-1}(t) \right\}$$

$$\times s^{n+2} \left\{ \cos \alpha_{ik} \right\} \quad \text{for } v_1 = \left\{ \xi_1 \right\}$$

$$\sin \alpha_{ik} \quad \text{for } v_1 = \left\{ \eta_1 \right\}$$

-33-
\begin{align*}
\text{(5.9') } \quad k_{ik} &= -\frac{1}{r_1} \sum_{n=0}^{\infty} c_n^k (n+2) s^{n+2} P_n(t) \quad \text{for } v_1 = \frac{\partial \Delta g_1}{\partial r_1} \\
\text{where} \\
s &= \frac{r_1^2}{r_j r_k} \quad \text{and} \quad t = \cos \psi_{ik}
\end{align*}

The above predictions from formula (5.10) are identical with the predictions using collocation (without noise). Erroneous observations can be considered by adding a noise covariance matrix to $C$ of (5.10) before inversion [see section 2, formula (2.16)]. The only type of observations considered in Bjerhammar's methods are free-air gravity anomalies. However, as in collocation, heterogeneous data might be included in the predictions.

Finally, we mention that for many norms the above kernel functions can be written in closed forms. As an example we give the kernel functions for $\| \Delta g^* \| = \min. (c = 2n+1)$:

\begin{align*}
\text{(5.4') } k_{ik} &= \frac{s^2 (1-s^2)}{\ell^3} \quad \text{for } v_1 = \Delta g_1 \\
\text{(5.6') } k_{ik} &= r_1 s^2 \left\{ \frac{2}{\ell} + 1 - 3s - st(5+3s^2/2) \right\} \quad \text{for } v_1 = T_1 \\
\text{(5.8') } k_{ik} &= \frac{s^3}{\gamma} \sin \psi_{ik} \left\{ \frac{\cos \alpha_{ik}}{\sin \alpha_{ik}} \right\} \left\{ - \frac{2}{\ell^3} - \frac{3}{\ell} + 5 - \frac{3st(1+\ell)}{\ell \phi} + \\
&\quad + \frac{3s^2 \phi}{2} \right\} \quad \text{for } v_1 = \left\{ \xi_1 \right\} \\
\text{(5.9') } k_{ik} &= -\frac{s^2}{2r_1 \ell^3} \left\{ 1 - 5s^2 + 3(1-s^2)^2/\ell^2 \right\} \quad \text{for } v_1 = \Delta g_1 / \partial r_1 \\
\text{where} \\
s &= \frac{r_1^2}{r_j r_k} \quad \text{and} \quad t = \cos \psi_{ik}
\end{align*}

More examples with derivations are given in Sjöberg (ibid., Appendices A.2 and A.3).
5.2 Model Studies

In Sjöberg (1975, section 19) some minimum norm solutions (formula (5.10)) for ξ, Δg and θ (=$\sqrt{\xi^2 + \eta^2}$) are computed for two simple earth models. Some of these results are reported below. For details we refer to Sjöberg (ibid.).

The following norms are used:

<table>
<thead>
<tr>
<th>Method</th>
<th>Norm</th>
<th>$c_*/(2n+1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Δg*</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>T*</td>
<td>$(n-1)^2/r_a^2$</td>
</tr>
<tr>
<td>3</td>
<td>Lauritzen$^1$</td>
<td>$(n-1)/((2n+1)(n-2))$</td>
</tr>
<tr>
<td>4</td>
<td>$\theta^*$</td>
<td>$(n-1)^2/(n(n+1))$</td>
</tr>
</tbody>
</table>

$^1$ empirical norm from Lauritzen (1973, section 10.3)

$^2$ $(\theta^*)^2 = (\xi^* r + (\eta^*)^2$)

5.2.1 Bjerhammar’s Model

Bjerhammar’s model consists of a homogeneous sphere with a spherical and homogeneous mass disturbance $M$ (Figure 2), where $M = 8.37758 \times 10^{25}$ kg and $a = 6362$ km. The radius of the main sphere is 6370 km.

![Figure 2. Bjerhammar's Model. The disturbing sphere $M$ is located with its center inside the main sphere.](image)
Table 5.1

RMS Prediction Errors for Bjerhammar's Model

<table>
<thead>
<tr>
<th>Method</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Depth*</td>
<td>ζ</td>
<td>Δg</td>
<td>θ''</td>
<td>ζ</td>
</tr>
<tr>
<td>km</td>
<td>m</td>
<td>mgal</td>
<td>m</td>
<td>mgal</td>
</tr>
<tr>
<td>0.01</td>
<td>11.90</td>
<td>34.2</td>
<td>11.90</td>
<td>34.2</td>
</tr>
<tr>
<td>0.1</td>
<td>12.22</td>
<td>35.0</td>
<td>8.48</td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td>12.55</td>
<td>36.0</td>
<td>7.92</td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td>15.02</td>
<td>44.0</td>
<td>6.39</td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td>16.47</td>
<td>47.2</td>
<td>7.61</td>
<td></td>
</tr>
<tr>
<td>2.0</td>
<td>17.88</td>
<td>49.0</td>
<td>9.16</td>
<td></td>
</tr>
<tr>
<td>2.6</td>
<td>0.32</td>
<td>6.4</td>
<td>1.44</td>
<td></td>
</tr>
<tr>
<td>2.8</td>
<td>0.19</td>
<td>3.3</td>
<td>0.71</td>
<td></td>
</tr>
<tr>
<td>3.0</td>
<td>0.08</td>
<td>2.2</td>
<td>0.32</td>
<td></td>
</tr>
<tr>
<td>3.2</td>
<td>0.06</td>
<td>3.5</td>
<td>0.77</td>
<td></td>
</tr>
<tr>
<td>3.4</td>
<td>0.15</td>
<td>5.1</td>
<td>1.29</td>
<td></td>
</tr>
<tr>
<td>4.0</td>
<td>48.6</td>
<td>8.02</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5.0</td>
<td>34.0</td>
<td>6.56</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6.0</td>
<td>28.6</td>
<td>6.25</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10.0</td>
<td>1.40</td>
<td>31.0</td>
<td>9.42</td>
<td></td>
</tr>
<tr>
<td>11.0</td>
<td>0.91</td>
<td>33.5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12.0</td>
<td>0.37</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>14.0</td>
<td>5.12</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* from surface of mean earth sphere to surface of Bjerhammar sphere.

No. of observations = 15, no. of prediction points = 24 (all points located within the spherical distance 7°30' from the axis through M of Figure 2). Reference: Table A.6 of Sjöberg (1975).
The "observed" $\Delta g$ values (due to the disturbance M) were selected at the surface of the model. The first set of observations (15 observations) was selected for $\omega \leq 7'30''$, the additional set of 35 observations was selected for $8' \leq \omega \leq 25'$ (all 35 points were located at the surface of the main sphere). The predictions of $\zeta$, $\Delta g$ and $\theta$ were carried out at the surface of the model (between the observation points) for different depths to the Bjerhammar sphere (Table 5.1). The RMS errors at the optimal depths are summarized in Table 5.2. Methods 1 and 4 give good agreements with the theoretical values. The methods 2 and 3 are not suitable for this local model. The importance of the choice of radius of the Bjerhammar sphere is obvious from Table 5.1.

Table 5.2

RMS Prediction Errors at the Optimal Depths to the Bjerhammar Sphere

<table>
<thead>
<tr>
<th>Method $^1$</th>
<th>Depth $^2$</th>
<th>$\zeta$</th>
<th>$\Delta g$</th>
<th>$\theta''$</th>
<th>Depth $^2$</th>
<th>$\zeta$</th>
<th>$\Delta g$</th>
<th>$\theta''$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>0.04</td>
<td>22.7</td>
<td>0.33</td>
<td>3</td>
<td>0.03</td>
<td>2.27</td>
<td>0.33</td>
</tr>
<tr>
<td>2</td>
<td>8-12</td>
<td>0.37</td>
<td>27.5</td>
<td>6.20</td>
<td>--</td>
<td>--</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>3</td>
<td>--</td>
<td>11.9</td>
<td>34.2</td>
<td>6.25</td>
<td>--</td>
<td>--</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>0.05</td>
<td>2.23</td>
<td>0.32</td>
<td>3</td>
<td>0.03</td>
<td>2.21</td>
<td>0.33</td>
</tr>
</tbody>
</table>

$^1$ Norm used
$^2$ Difference of the radii of main sphere and Bjerhammar sphere
$^3$ Not computed

No. of predicted points = 24, RMS (anomaly - mean value) = 182 mgal.
Reference: Sjöberg (1975, Table 19.1)

5.2.2 Molodenskii's Model

The surface of Molodenskii's model is a cone. Two spheres, whose centers are located on the axis of the cone, are taken as the anomalous masses (see Figure 3). The attractions of the spheres $m_2$ and $m_1$ on the axis of the cone 4050 m above the reference plane are 150 and 100 mgal, respectively ($m_2 \approx 1.458 \times 10^{15}$ kg, $m_1 \approx 6.334 \times 10^{13}$ kg). As there is a disturbing mass ($m_1$) above the reference plane,
it is clear that this model is much rougher than Bjerhammar's model. The computations revealed that for all methods (norms) the depth to the Bjerhammar sphere should be selected as close to the reference plane as possible (10 meters was used in the computations).

All observation and prediction points were selected on the surface of the cone. A summary of the RMS prediction errors is given in Tables 5.3 and 5.4. We notice that the predictions are essentially better for a more regular distribution of the observations (Table 5.4). Again, methods 2 and 3 are less favorable than 1 and 4. We also notice that these RMS errors are of a magnitude larger than those obtained for the smoother model of Bjerhammar.

A comparison between a minimum norm solution and the Dirac method for Molodenskii's model is given in section 8.

Figure 3. Molodenskii's Model
Table 5.3

Molodenskii’s Model. RMS Prediction Errors at the Optimal Depths to the Bjerhammar Sphere

<table>
<thead>
<tr>
<th>Method</th>
<th>Depth (m)</th>
<th>$\zeta$ (m)</th>
<th>$\Delta g$ (mgal)</th>
<th>$\xi''$</th>
<th>$\theta''$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>0.67</td>
<td>25.0</td>
<td>7.58</td>
<td>13.78</td>
</tr>
<tr>
<td>2</td>
<td>200-3500</td>
<td>0.23</td>
<td>33.3</td>
<td>5.48</td>
<td>6.25</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>13.5</td>
<td>45.5</td>
<td>14.24</td>
<td>28.25</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>0.63</td>
<td>25.0</td>
<td>7.58</td>
<td>13.76</td>
</tr>
</tbody>
</table>

1 Norm used
(10 m is the minimum depth used in the computations.) No. of observations = 20 (irregular distribution), no. of predictions = 19 [RMS (anomaly - mean) = 81.7 mgal]. From Sjöberg (1975, Table 19.2).

Table 5.4

Molodenskii’s Model. RMS Prediction Errors at the Depth 10 m to the Bjerhammar Sphere

<table>
<thead>
<tr>
<th>Method</th>
<th>$\zeta$ (m)</th>
<th>$\Delta g$ (mgal)</th>
<th>$\xi''$</th>
<th>$\theta''$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.09</td>
<td>15.4</td>
<td>2.45</td>
<td>2.70</td>
</tr>
<tr>
<td>3</td>
<td>5.2</td>
<td>18.9</td>
<td>3.11</td>
<td>5.80</td>
</tr>
<tr>
<td>4</td>
<td>0.06</td>
<td>15.4</td>
<td>2.45</td>
<td>2.70</td>
</tr>
</tbody>
</table>

1 Norm used
No. of observations = 24 (regular distribution), no. of predictions = 19 [RMS (anomaly - mean) = 81.7 mgal]. From Sjöberg (1975, Table 19.3).
6. Integral Formulas as Limiting Cases

In this section we are going to study the solutions by collocation and the Dirac method for a continuous field of observations at the surface of the earth. Two problems are discussed: the uniqueness of the solutions and the existence of the solutions.

6.1 The Uniqueness of the Solutions

The non-singular Dirac method was presented in (2.25) for a finite set of observations (we exclude terms of degree less than 2):

\[ \Delta g \rho = \sum_{k=1}^{n} u^* \alpha (P, k) ; \quad P = 1, 2, \ldots, \]

where

\[ A (P, k) = \sum_{n=2}^{\infty} \frac{(2n+1)\cos \psi P}{r^2} \left( \frac{r_b}{r_f} \right)^{n+2} P_n (\cos \psi P) \]

Using the solutions for \( u^* \) of (6.1) the disturbing potential \( (T) \) may be estimated by formulae (5.2) and (5.6) (for \( F_{1k} = f_{1k} \)):

\[ T_\rho = r_\rho \sum_{k=1}^{n} u^* \beta (P, k) \]

where

\[ S (P, k) = \sum_{n=2}^{\infty} \frac{(2n+1)\cos \psi P}{n-1} \left( \frac{r_b}{r_f} \right)^{n+2} P_n (\cos \psi P) \]

and the height anomaly is given by [cf. (5.7)]:

\[ \zeta \rho = \frac{T_\rho}{\gamma} \]

Let \( N \) go to infinity with a well-behaving distribution of the carrier points. Then we arrive at the following integral formulas from (6.1) and (6.2) in the continuous case:
\[(6.1') \quad \Delta g (P) = \frac{1}{4\pi} \int \int A (P, Q) u^* (Q) \, d\sigma_Q \]

and

\[(6.2') \quad \zeta (P) = \frac{1}{4\pi} \int \int S (P, Q) u^* (Q) \, d\sigma_Q \]

In the special case \( c_s^* = 2n + 1 \) \((u^* = \Delta g^*)\) (6.1') and (6.2') become Poisson's and the generalized Stokes' formula, respectively.

We now study the solution by collocation in the same way. From (1.2) we get the following intermediate step (for \( D=0 \)):

\[(6.3a) \quad \Delta g_P = C X, \quad P = 1, 2, \ldots, N \]

or

\[(6.3b) \quad \Delta g_P = \sum_{k=1}^{N} c (P, k) X_k, \quad P = 1, 2, \ldots, N \]

where

\[(6.3c) \quad c (P, k) = \sum_{k=2}^{n} \frac{c_n (r_0^2 / r_P r_k)^{k-2}}{\cos \psi_k} P_n (\cos \psi_k) \]

and

\[X = \text{vector of unknowns} \quad (X_k)\]

The prediction of \( \zeta_P \) is given by:

\[(6.4) \quad \zeta_P = \frac{r_P}{\gamma} \sum_{k=1}^{N} X_k \, S' (P, k) \]

where

\[S' (P, k) = \sum_{n=2}^{\infty} \frac{c_n}{n-1} \left( \frac{r_0^2}{r_P r_k} \right) P_n (\cos \psi_k) \]

Formula (6.3b) and (6.4) may be written:
\[ \Delta g_P = \frac{1}{4\pi} \int \int c(P, Q) \times (Q) \, d\sigma_Q, \quad P = 1, 2, \ldots, N \]

and

\[ \zeta_P = \frac{r_P}{4\pi} \int \int S^*(P, Q) \times (Q) \, d\sigma_Q \]

where

\[ X(Q) = \sum_{k=1}^{N} X_k \delta(Q - Q_k) \]

\[ \delta = \text{Dirac's delta function} \quad \text{see (2.24a)} \]

\[ Q_k = \text{the foot point at the internal sphere of the normal through the observation in} \ P_k \]

In the limit \( N \to \infty \) (with the maximum distance among the observations approaching 0), these equations become the following integral equations:

(6.3') \[ \Delta g(P) = \frac{1}{4\pi} \int \int c(P, Q) \times (Q) \, d\sigma_Q \]

and

(6.4') \[ \zeta(P) = \frac{r_P}{4\pi} \int \int S^*(P, Q) \times (Q) \, d\sigma_Q \]

Formulae (6.2') and (6.4') are estimates of \( \zeta(P) \). Furthermore, the estimates seem to differ for various choices of \( c, \alpha \), and \( c, \alpha \) of the kernel functions. However, next we are going to show that these differences are merely apparent whenever the solutions exist.

First, we expand \( c(P, Q) \) of (6.3c) into spherical harmonics \{cf. formula (A.1)\}:

\[ c(P, Q) = \sum_{n=2}^{\infty} \sum_{s=-n}^{n} \frac{C_n}{2n+1} Y_{ns}(P) Y_{ns}(Q) \left( \frac{r_P}{r_s} \right)^{n+s} \]

By inserting this expansion into (6.3') we obtain:

\[ \Delta g(P) = \sum_{n=2}^{\infty} \sum_{s=-n}^{n} B_{ns} \left( \frac{r_P}{r_s} \right)^{n+s} Y_{ns}(P) \]

-42-
where
\[ B_{ns} = \frac{c_n}{4\pi(2n+1)} \iint \left( \frac{r_s}{r_p} \right)^{n+2} Y_{ns}(Q) X(Q) \ d\sigma \]

In the same way (6.4) and \( \Delta g(P) \) may be developed into the following series:
\[ \hat{\zeta}(P) = \sum_{n=2}^{\infty} \sum_{s=-n}^{n} B_{ns} \frac{1}{n-1} \left( \frac{r_s}{r_p} \right)^{n+2} Y_{ns}(P) \]

and
\[ \Delta g(P) = \sum_{n=2}^{\infty} A_{ns} \left( \frac{r_s}{r_p} \right)^{n+2} Y_{ns}(P) \]

where
\[ A_{ns} = \frac{1}{4\pi} \iint \Delta g^*(Q) Y_{ns}(Q) \ d\sigma \]

From the above expansions we obtain for \( c_n > 0 \):
\[ B_{ns} = A_{ns} \]

and
\[ \hat{\zeta}(P) = \frac{r_p}{r_s} \sum_{n=2}^{\infty} \frac{1}{n-1} \left( \frac{r_s}{r_p} \right)^{n+2} A_{ns} Y_{ns}(P) = \]
\[ = \frac{r_s}{4\pi r_p} \iint S(r_p, \psi_{\infty}) \Delta g^*(Q) \ d\sigma \]

where
\[ S(r_p, \psi_{\infty}) = \sum_{n=2}^{\infty} \frac{2n+1}{n-1} \left( \frac{r_s}{r_p} \right)^{n+1} P_n(\cos \psi_{\infty}) \]

\( S(r_p, \psi_{\infty}) \) is the generalized Stokes' function. Thus we have shown that each collocation solution for \( \zeta \) (for \( c_n > 0 \)) in the continuous case equals Stokes' integral formula.
The same result may be shown for Poisson's and Vening Meinesz' integrals. Moreover, the same proof holds for the non-singular Dirac method in the continuous case (formula (6.1') and (6.2')). The uniqueness of all solutions may be regarded as a consequence of Stokes' theorem (Heiskanen and Moritz, 1967, p. 17), which states that a function harmonic outside a closed surface $S$ is uniquely determined by its values on $S$.

However, the conversion from one set of $c_z (c_z^*)$ to another is, of course, not valid unless the unknowns $(X$ and $u^*)$ exist and the kernel functions are bounded. This question is discussed in the next section.

6.2 The Existence of the Solutions

In sections 3 – 4 we have studied the stability of the coefficient matrix for different methods, which is of importance for the practical application with a finite set of observations. Now we ask under which circumstances there exists a solution in the continuous case (infinite number of observations). As will be shown, the existence of a solution to any of the integral equations is considerably dependent on the smoothness of the observed field.

In Moritz (1975), a proof of the convergence of least squares interpolation is presented for an element $T$ of a certain Hilbert space (with a given kernel function). However, Tscherning (1977a) showed that the disturbing potential of the earth ($T$) is not an element of the Hilbert space, associated with the empirical covariance function. Subsequently, Moritz' proof is not applicable in this case.

In our study we start with Poisson's integral equation for the exterior of a circle of radius $r_B$. All observations are assumed to be located on a circle of radius $r$. Then we have [cf. formula (4.4)]:

$$\Delta g (\theta_j) = \int_{-\pi}^{\pi} k (\theta_j, \theta_k) \Delta g^* (\theta_k) \, d \theta_k$$

where

$$k (\theta_j, \theta_k) = \frac{1}{2\pi} \frac{1 - s^{2}}{|s - e^{i(\theta_j - \theta_k)}|^2} = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} s^{n+1} e^{i(n\theta_j - \theta_k)}$$

Now we assume that $\Delta g$ may be expanded into a Fourier series and we try a corresponding series for $\Delta g^*$:

$$\Delta g (\theta_j) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta_j}$$
and

\[(6.7b) \quad \Delta g^* (\theta_k) = \sum_{\ell=-\infty}^{\infty} b_\ell e^{i\ell \theta_k}\]

Furthermore, we assume that both the unknowns ($\Delta g^*$) and the observations ($\Delta g$) are uniformly distributed with:

\[\theta_j = \frac{2\pi j}{m} \quad j = 1, 2, \ldots, m\]

where $m$ is the number of observations (Dirac method). This implies that $\Delta g^*$ may be written:

\[\Delta g^*(\theta) = \sum_{k=1}^{m} \Delta g^*(\theta_k) \delta(\theta - \theta_k)\]

and

\[(6.8) \quad \Delta g (\theta_j) = \sum_{k=1}^{m} k (\theta_j, \theta_k) \Delta g^*(\theta_k) \quad j = 1, 2, \ldots, m\]

Let us now regard (6.8) as a linear filter with each "tone" $b_\ell e^{i\ell \theta}$ of $\Delta g$ as input and $a_\ell e^{i\ell \theta}$ of $\Delta g$ as output. Then we obtain from (6.6) - (6.8):

\[a_\ell e^{i\ell \pi j/2} = \frac{b_\ell}{2\pi} \sum_{k=1}^{m} \sum_{n=-\infty}^{\infty} s^{1+2\pi (j-k)/a} e^{i\ell 2\pi k/a}\]

\[= \frac{b_\ell}{2\pi} \sum_{n=-\infty}^{\infty} s^{1+2\pi a\pi/j} \sum_{k=1}^{m} e^{i(\ell - n)2\pi k/a}\]

\[= \frac{b_\ell}{2\pi} e^{i\ell 2\pi \pi /a} m u_\ell, s (8)\]
where (see Appendix, Corollary A.2):

$$u_{k,s}(s) = \sum_{\ell = -\infty}^{\infty} s^{\lfloor \ell \rfloor + \alpha} = \left( s^{\lfloor \ell \rfloor + \alpha} + s^{\lfloor \ell \rfloor + 1 + \alpha} \right) / (1 - s^\alpha)$$

$$\alpha = \left\lfloor \frac{\ell}{m} \right\rfloor = \text{integer part of } \frac{\ell}{m}$$

Hence

$$b_\ell = 2\pi a_\ell / (m u_{k,s}(s))$$

Moreover we notice

$$\lim u_{k,s}(s) = s^{\lfloor \ell \rfloor}, \ m \to \infty$$

so that for large $m$ we have, approximately:

$$b_\ell \approx 2\pi a_\ell / (m s^{\lfloor \ell \rfloor})$$

We conclude that $\Delta g^*$ does not generally exist for a dense distribution of the observations, and the convergence for $\Delta g^*$ as $m \to \infty$ is very much dependent on the behavior of $a_\ell$ as $\ell \to \infty$.

Let us assume that the solution for $\Delta g^*$ exists. The predictions of $\Delta g(R, \theta)$ are then given by:

$$\Delta g^* (R, \theta) = \sum_{\ell = -\infty}^{\infty} \sum_{k=1}^{2\pi} k (\theta, \theta_k) \Delta g^*_\ell (\theta_k)$$

$$= \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} b_\ell \sum_{k=1}^{2\pi} s^{\lfloor \ell \rfloor} e^{i(\theta - \theta_k)} e^{i\ell \theta_k}$$

$$= m \sum_{\ell = -\infty}^{\infty} b_\ell e^{i \ell \theta} v_{\ell, s} (s, \theta)$$

-46-
where

\[ s_\ell = r_\ell / R \]

and

\[ v_{\ell, s}(s, \theta) = \sum_{l=-\infty}^{\infty} s_\ell v_{\ell, s}(s, \theta) e^{i s \theta} \]

This sum is given in a closed form in the Appendix, Proposition A.2. Inserting (6.9) we finally obtain:

\[ (6.10) \quad \Delta g(R, \theta) = \sum_{\ell=-\infty}^{\infty} a_\ell \frac{v_{\ell, s}(s, \theta)}{u_{\ell, s}(s)} e^{i \ell \theta} \]

Furthermore it follows from Corollary A.3 that:

\[ v_{\ell, s} \rightarrow s_\ell^{\mu_1}, \quad m \rightarrow \infty \]

so that

\[ \lim_{s \rightarrow \infty} \Delta g(R, \theta) = \sum_{\ell=-\infty}^{\infty} a_\ell \left( \frac{r}{R} \right)^{\mu_1} e^{i \ell \theta} \]

Thus, \( \Delta g \) is convergent in the limit for \( R \geq r \). (The convergence for \( R = r \) was proved by Hörmander. See Bjerhammar, 1974.) Subsequently, we have shown that although the predictions are uniformly convergent for \( R \geq r \), the intermediate solution \( \Delta g^* \) does not generally exist in the continuous case \( (m \rightarrow \infty) \). This result is of great practical importance.

Let us now substitute (6.5) by the following general integral equation [cf. formula (2.5)]:

\[ (6.11) \quad \Delta g(\theta_j) = \int_{-\pi}^{\pi} k(\theta_j, \theta_k) u^*(\theta_k) \, d\theta_k \]
where
\[ k(\theta_2, \theta_k) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \sqrt{c_n^*} \, \text{sgn}(n) \, e^{i(n \theta_2 - \theta_k)} \]

To ensure that the kernel \( k \) is bounded, we suppose that the magnitude of \( c_n^* \) satisfies:

\[ (6.12) \quad \sqrt{|c_n^*|} \, s^{d+1} < 1/n \text{ for } n > M \]

where \( M \) is an integer. The solution of (6.11) becomes:

\[ u^*(\theta) = \sum_{\ell=-\infty}^{\infty} b_{\ell} \, e^{i\ell \theta} \]

where [cf. (6.9) and (6.9')]:

\[ b_{\ell} = \frac{2\pi a_{\ell}}{m \sqrt{c_{\ell}^*} \, u_{\ell,1}(8)} \]

or, approximately for large \( m \):

\[ (6.13) \quad b_{\ell} \approx \frac{2\pi a_{\ell}}{m \sqrt{c_{\ell}^*} \, s^{d+1}} \]

From (6.12) and (6.13) it is obvious that it is generally not possible to select a set \( c_n^* \) such that \( u^* \) is convergent for arbitrary \( a_{\ell} \).

Next, we proceed to the global equivalence of the above derivations. We assume that the gravity field of the earth can be expanded into a series of spherical harmonics \([Y_{ns}(P)]\) for each point \((P)\) on the surface of the earth (cf. section 6.1):

\[ \Delta g(P) = \sum_{n=2}^{\infty} \sum_{m=-n}^{n} A_{ns} \left( \frac{r_2}{r_1} \right)^{n+2} Y_{ns}(P) \]

where \( Y_{ns}(P) \) is defined in the Appendix, formula A.1'. Furthermore, we expand \( u^* \) into a corresponding series:

\[ (6.14) \quad u^*(Q) = \sum_{n=2}^{\infty} \sum_{m=-n}^{n} u_{ns} \, Y_{ns}(Q) \]
where $u_{ns}$ are unknown coefficients. By inserting these two series into (6.1'), we arrive at the following identity for each spherical harmonic:

$$\sqrt{\frac{c_{ns}^*}{2n+1}} u_{ns} \equiv A_{ns}$$

or

$$(6.15) u_{ns} = A_{ns} \sqrt{\frac{2n+1}{c_{ns}^*}}$$

Here we have expanded $P_n (\cos \psi_{PQ})$ of the kernel function $A (P, Q)$ according to the Addition theorem (A.1):

$$P_n (\cos \psi_{PQ}) = \frac{1}{2n+1} \sum_{n=-n}^{n} Y_{nm} (P) Y_{nm} (Q)$$

and we have also used the orthogonality property for the spherical harmonics.

The same technique may be used to solve for the intermediate solution $X (Q)$ in collocation. We insert the expansion for $\Delta g (P)$ and:

$$(6.16) X (Q) = \sum_{n, s} X_{ns} Y_{ns} (Q)$$

into (6.3'), and we generalize the auto-covariance function by substituting the empirical set $c_s$ of (1.5) by an arbitrary set of positive definite parameters $c_s^*$. The result is:

$$\sum_{n_s} \sum_{n_{s'}} \frac{c_{ns'}^*}{2n+1} \left( \frac{r_n}{r_P} \right)^{n+2} Y_{ns} (P) X_{n's'} \int \left( \frac{r_n}{r_Q} \right)^{n+2} Y_{ns} (Q) Y_{n's'} (Q) d \sigma_Q =$$

$$(6.17) = \sum_{n,s} A_{ns} \left( \frac{r_n}{r_P} \right)^{n+2} Y_{ns} (P)$$
In this case we can not simply apply the orthogonality property of the spherical harmonics to identify the unknowns, because $r_q$ is a variable. However, if we limit ourselves to the special case $r_q = r = \text{constant}$ (all observations on a sphere) we obtain:

$$\left(\frac{r_\theta}{r}\right)^{n+2} \frac{c_n^*}{2n+1} X_{n \alpha} = A_{n \alpha}$$

or

$$X_{n \alpha} = \frac{2n+1}{c_n^*} A_{n \alpha} \left(\frac{r}{r_\theta}\right)^{n+2}$$

(6.18)

In summary, the solutions for $u^*$ and $X$ are:

$$u^*(Q) = \sum_{n, \alpha} \sqrt{\frac{2n+1}{c_n^*}} A_{n \alpha} Y_{n \alpha}(Q)$$

(6.19)

and

$$X(Q) = \sum_{n, \alpha} \frac{(2n+1)}{c_n^*} A_{n \alpha} \left(\frac{r}{r_\theta}\right)^{n+2} Y_{n \alpha}(Q)$$

(6.20)

If we now consider that the choice of $c_n^*$ is restricted by the condition of bounded kernel (covariance) functions, the convergence of (6.19) and (6.20) is essentially dependent on the smoothness of $A_{n \alpha}$ for higher degrees.

Let us for the moment assume that (6.19) and (6.20) converge. By inserting these solutions into the integrals (6.1') and (6.3'), we obtain in both cases:

$$\Delta^* g(P) = \sum_{n=2}^{\infty} \sum_{\alpha=-n}^{n} A_{n \alpha} \left(\frac{r_\theta}{r}\right)^{n+2} Y_{\alpha}(P)$$

(6.21)

This series is identical with the exterior gravity anomaly of the earth. Thus the predictions converge to the true values, whenever the unknowns $u^*$ and $X$ exist in the continuous case (cf. section 6.1).

According to the Hilbert-Schmidt theorem (Chambers, 1976, p. 50), the series solution (6.21) for $\Delta^* g(P)$ of the left member of the integral equation (6.3') is valid if and only if:
1) \[ c(P, Q) = \sum_{n=2}^{\infty} c_n^* \left( \frac{r_0}{r} \right)^{2(n+2)} P_n(\cos \psi_{PQ}) < \infty \]

and

2) \[ \|X\|^2 = \frac{1}{4\pi} \int \int X^2(Q) \, d\sigma_q < \infty \]

where \( r \) is the minimum radius to any point at the surface of the earth. Using (6,20), the condition 2) may be written:

\[ 2') \|X\|^2 = \sum_{n=2}^{\infty} \left[ \frac{(2n+1)s_n}{c_n^*} \left( \frac{r}{r_0} \right)^{2(n+2)} \right]^2 < \infty \]

where we have used the notation:

\[ s_n^2 = \sum_{n=-1}^{\infty} A_n \left( \frac{r_0}{r} \right)^{2(n+2)} P_n \left( \frac{r}{r_0} \right)^{2(n+2)} \]

\( s_n^2 \) is the anomaly degree variance at the sphere of radius \( r \) (cf. formula (1.5)). From 1) and 2') we arrive at the following inequality for the magnitude of \( c_n^* \):

\[ (6.22) \quad n^n \sigma_n < c_n^* \left( \frac{r_0}{r} \right)^{2(n+2)} < n^{-1} \]

In the same way we obtain the following conditions from the integral equation (6.1') with the solution (6.19):

1) \[ A(P, Q) = \sum_{n=0}^{\infty} \sqrt{(2n+1)c_n^*} \left( \frac{r_0}{r} \right)^{n+2} P_n(\cos \psi_{PQ}) < \infty \]

and

2) \[ \|u^*\|^2 = \sum_{n=0}^{\infty} (2n+1) \frac{s_n^2}{c_n^*} \left( \frac{r}{r_0} \right)^{2(n+2)} < \infty \]

and the corresponding magnitude bounds for \( c_n^* \) are given by:

\[ (6.23) \quad n^n \sigma_n^2 < c_n^* \left( \frac{r_0}{r} \right)^{2(n+2)} < n^{-3} \]
The inequalities (6.22) and (6.23) should be regarded as necessary bounds for \( c_n^* \) in a convergent solution of collocation and the Dirac method, respectively. We notice that both (6.22) and (6.23) are satisfied for:

\[
(6.23') \quad \sigma_n < n^{-5/2}
\]

However for the gravity field of the earth, we have according to Kaula's rule of thumb for the variation of the potential coefficients, that \( \sigma_n \) is on the order of \( n^{-1/2} \). Thus we conclude that the solutions for \( u^* \) and \( X \) are not convergent in the continuous case. The same negative result is obtained if we replace the observations \( \Delta g \) of (6.1) and (6.3) by the disturbing potential. In this case, we arrive at the same inequalities (6.22) and (6.23). The degree variances of \( T (\sigma_n^2 (T)) \) are of magnitude \( n^{-3} \) according to Kaula's rule, and it is clear that (6.23') is not satisfied.

In all the methods of Bjerhammar, the idea is to determine a fictitious field \( u^* \) (or \( \Delta g^* \)) on the internal sphere, and then to use \( u^* (\Delta g^*) \) in the classical integral equations for estimating geophysical quantities such as \( \Delta g, T, \xi, \) and \( \zeta \). In collocation, we are not restricted to the determination of the corresponding intermediate solution \( X \) as was suggested previously. For example, the prediction of \( \Delta g \) may be expressed directly as a linear combination of the observations (discrete case):

\[
(6.24) \quad \Delta g_P = \sum_{i=1}^{s} h_{ri} \Delta g_i
\]

where \( h_{ri} \) are the weights, which are given by the following discrete Wiener-Hopf equations:

\[
(6.25) \quad c_{rk} = \sum_{i=1}^{s} h_{ri} c_{ik} ; \quad k = 1, 2, \ldots, m
\]

where \( c \) is the covariance function of \( \Delta g \). In the continuous case (6.24) and (6.25) become:

\[
(6.24') \quad \Delta g (P) = \frac{1}{4\pi} \iint h(P, Q) \Delta g (Q) \, d\sigma_q
\]

and

\[
(6.25') \quad c (P, Q) = \frac{1}{4\pi} \iint h(P, Q') c(Q', Q) \, d\sigma_q
\]
If \( P \) is a point at the surface of the earth, (6.25) has the solution:

\[
h(P, Q) = \delta(P - Q)
\]

where \( \delta \) is Dirac's delta function defined in (2.24a). If \( P \) is a point outside the surface of the earth, we may assume that the covariance function \( c \) is spatially homogeneous and isotropic:

\[
c(P, Q) = \sum_{n=2}^{\infty} c_n \left( \frac{r_P}{r_Q r_Q'} \right)^{n+2} P_n (\cos \psi_{r_Q})
\]

In order to solve for \( h \) we try the following expansion:

\[
h(P, Q) = \sum_{n=2}^{\infty} h_n P_n (\cos \psi_{r_Q})
\]

where \( h_n \) are unknown coefficients to be determined. Let us restrict ourselves to the spherical approximation of the earth, i.e. \( r_Q = r_Q' = r = \text{constant} \). Then we obtain from (6.25):

\[
h_n = (2n+1) \left( \frac{r}{r_P} \right)^{n+2}
\]

and the solution for \( h \) becomes a modified Poisson's kernel function:

\[
h(P, Q) = \sum_{n=2}^{\infty} (2n+1) \left( \frac{r}{r_P} \right)^{n+2} P_n (\cos \psi_{r_Q}) =
\]

\[
= \frac{r^2}{r_P} \frac{(r_P^2 - r^2)}{(r_P^2 + r^2 - 2rr_P \cos \psi_{r_Q})^{3/2}} - \left( \frac{r}{r_P} \right)^3 \cos \psi_{r_Q}
\]

and (6.24') becomes Poisson's integral (without the terms of degrees less than 2). Thus we have found that by carrying out the collocation solution by solving the Weiner-Hopf integral equation, the correct gravity anomaly is recovered on and outside the surface of a spherical earth.
In the same way we can solve for the disturbing potential in the exterior of a sphere. Let us use the estimator:

\[ T(P) = \frac{1}{4\pi} \iint h(P, Q) \Delta g(Q) \, d\sigma \]  

where \( h(P, Q) \) is given by:

\[ k(P, Q) = \frac{1}{4\pi} \iint h(P, Q') c(Q', Q) \, d\sigma' \]

and

\[ k(P, Q) = \sum_{n=0}^{\infty} c_n \left( \frac{r_\theta}{r_\phi} \right)^{n+2} P_n(\cos \psi_Q) \]

\[ c(P, Q) = \sum_{n=0}^{\infty} \frac{c_n}{n-1} \left( \frac{r_\theta}{r_\phi} \right)^{n+2} P_n(\cos \psi_Q) \]

By trying an expansion for \( h \) in (6.26b) (cf. the previous example), the coefficients \( h_n \) are easily identified and the solution for \( h \) becomes:

\[ h(P, Q) = r S(r_\phi, \psi_Q) \]

where

\[ S(r_\phi, \psi_Q) = \sum_{n=0}^{\infty} \frac{2n+1}{n-1} \left( \frac{r_\phi}{r_\phi} \right)^{n+1} P_n(\cos \psi_Q) \]

Thus the solution for \( T/\gamma \) from (6.26a) with \( r_\phi = r \) is Stokes' formula (cf. section 6.1).

6.3 The Effect of Smoothing (Noise)

The solutions for \( u^* \) and \( X \) in the methods of Bjerhammar and collocation may be smoothed by adding a noise covariance function \( d(j, k) \) to the auto-covariance function \( c(j, k) \) [see formula (1.2) and (2.16)]. Consider the example of the previous section with a continuous field of observed \( \Delta g \) on a sphere of radius \( r \). Let us replace the covariance function of (6.3°) by:
(6.27) \[ \sigma(P, Q) = c(P, Q) + d(P, Q) \]

where \[ d(P, Q) = d \delta(\psi_{r_0}), \quad d > 0 \]

\( \delta(\psi_{r_0}) = \) Dirac's delta function, defined in (2.24a). This noise covariance function corresponds to pure white noise. By inserting (6.27) into (6.3'), we obtain in accordance with (6.17) and (6.18) (for \( r_p = r_q = r \)):

\[ (6.28) \]

\[ X_{n} = \frac{A_{n} (r_{\delta}/r)^{n+2}}{d + (r_{\delta}/r)^{2(n+2)}c_{n}/(2n+1)} \]

For large degrees \( n \) the coefficients \( X_{n} \) can be approximated by:

\[ X_{n} \approx \frac{1}{d} A_{n} \left( \frac{r_{\delta}}{r} \right)^{n+2} \]

Subsequently

\[ X(Q) = \sum_{n, s} X_{n} Y_{n}(Q) \]

is convergent whenever the spherical harmonic expansion of \( \Delta g \) has the radius of convergence \( r \).

The prediction of new anomalies from the intermediate solution \( X \) is given by (6.16) and (6.3'). The result is:

\[ \Delta^g(P) = \sum_{n, s} X_{n} \left( \frac{r_{\delta}}{r_{\gamma}} \right)^{n+2} \frac{c_{n}}{2n+1} Y_{n}(P) \]

By inserting (6.28) we arrive at the following predictor and prediction error:

\[ (6.29) \]

\[ \Delta^g(P) = \sum_{n, s} \frac{c_{n} (r_{\delta}/r)^{2(n+2)}}{(2n+1)d + c_{n} (r_{\delta}/r)^{2(n+2)}} A_{n} \left( \frac{r_{\delta}}{r} \right)^{n+2} Y_{n}(P) \]
and

\[(6.30) \quad \Delta g^\prime (P) - \Delta g (P) = - \sum_{n=1}^{\infty} \frac{d}{d + (r_0/r)^{2(n+2)}c_n/(2n+1)} \Lambda_n \left( \frac{r_0}{r} \right)^{n+2} Y_n (P) \]

Subsequently, by selecting a sufficiently small \(d > 0\), the prediction error becomes arbitrarily small. However, in practice the constant \(d\) has to exceed a certain minimum value due to the requirement of \(X\) to be within the range of the computer.

In a similar way we may approximate the solution for the Dirac method.

The smoothing is first of all justified when the observations are erroneous. [Cf. formula (1.2) and (1.3).] These predictions are statistically biased, which we conclude from the following application of (1.3). Let:

\[ \Delta g^\prime = c \phi (C + D)^{-1} \Delta \tilde{g} \]

be an estimator of the anomaly \(\Delta g^\prime\), which is included in the vector of observations \((\Delta g)\). Furthermore, \(\Delta \tilde{g}\) consists of a signal \(\Delta g\) and its error \(\epsilon\):

\[ \Delta \tilde{g} = \Delta g + \epsilon \]

where \(\Delta g\) is the true anomaly and:

\[ E \{ \epsilon \} = 0 \quad \text{and} \quad E \{ \epsilon \epsilon^\prime \} = D \]

\[ E \{ \cdot \} = \text{the probabilistic expectation} \]

Now it follows immediately that:

\[ E \{ \Delta g^\prime \} = c \phi (C + D)^{-1} E \{ \Delta \tilde{g} \} = c \phi (C + D)^{-1} \Delta g \]

Hence

\[ E \{ \Delta g^\prime \} \neq c \phi C^{-1} \Delta g = \Delta g^\prime \]

and we have proved that \(\Delta g^\prime\) is biased.
In the continuous case we may prove the bias in the following way. Let the observed anomaly field \( \Delta g \) consist of the signal \( \Delta g \) and the noise \( \epsilon \):

\[
\Delta g = \Delta g + \epsilon
\]

where

\[
\Delta g(P) = \sum_{n, s} A_{n s} \left( \frac{r_0}{r} \right)^{n+2} Y_{n s}(P)
\]

\[
\epsilon(P) = \sum_{n, s} \epsilon_{n s} \left( \frac{r_0}{r} \right)^{n+2} Y_{n s}(P)
\]

\[
E\{\epsilon\} = E\{\epsilon_{n s}\} = 0
\]

The estimator of \( \Delta g(P) \) when including the noise in the observations is in accordance with (6.29):

\[
\Delta g^\ast(P) = \sum_{n, s} \frac{c_n(r_0/r)^{2(n+2)}}{(2n+1)d + c_n(r_0/r)^{2(n+2)}} (A_{n s} + \epsilon_{n s}) \left( \frac{r_0}{r} \right)^{n+2} Y_{n s}(P)
\]

and the bias becomes

\[
E\{\Delta g^\ast(P)\} - \Delta g(P) = - \sum_{n, s} \frac{d}{d + (r_0/r)^{2(n+2)}} c_n/(2n+1) A_{n s} \left( \frac{r_0}{r} \right)^{n+2} Y_{n s}(P)
\]

The bias is due to the fact that \( \Delta g \) itself is not a random stochastic process, only its errors, \( \epsilon \). (The expectation should not be interchanged with the global average.) A way to diminish the bias is to subtract a low degree spherical harmonic reference expansion from the observations prior to the prediction. Even though we cannot correct for all the bias, the resulting residuals are, hopefully, more random than the original observations. The general prediction formula (1.2) should therefore be modified in the following way [cf. Sjöberg, 1975, formula (16.1) and Koch, 1977, formula (18)]:

\[
v_1 = v_1 + c_1 (C + D)^{-1} (\Delta g - \Delta g^\ast)
\]
where

\[ \hat{v}_i, \Delta g = \text{low degree spherical harmonic expansions for } v_i \text{ and } \Delta g. \]

This is the well-known collocation with the inclusion of systematic terms represented by \( \hat{v}_i \) and \( \Delta g \). We conclude that this formula should be used not only in local studies, but also in global applications of collocation to reduce the bias in the model.

6.4 The Limiting Case of a Least Squares Solution

Finally, we are going to study the convergence of the least squares solution of Poisson's equation for the circle. We assume that we are using the Dirac method with \( m > N \). The residuals may be written:

\[
\epsilon_j = \Delta g_j - \sum_{k=1}^{m} k(\theta_j, \theta_k) \Delta g_k^*, \quad j = 1, 2, \ldots, m
\]

The minimum of the square sum of the residuals is obtained for the normal equations:

\[
\sum_{j=1}^{n} k(\theta_j, \theta_q) \Delta g_j = \sum_{j=1}^{n} \sum_{k=1}^{n} k(\theta_j, \theta_k) k(\theta_j, \theta_k) \Delta g_k^*
\]

where

\[ q = 1, 2, \ldots, N \]

Inserting

\[
\Delta g(\theta_j) = \sum_{n=-\infty}^{\infty} a_n e^{i n \theta_j}, \quad \theta_j = 2\pi j/m = 2\pi r j/N
\]

and

\[
\Delta g^*(\theta_k) = \sum_{\ell=-\infty}^{\infty} b_{\ell} e^{i \ell \theta_k}, \quad \theta_k = 2\pi k/N
\]

where \( r \) is an integer (\( m \) is a whole multiple of \( N \)) into the normal equations it can be shown that:

\[
b_{\ell} = a_{\ell} \sum_{p} s^{\ell + n_p} / N \sum_{q} s^{\ell + n_q + n_d + n_p} - 58-
\]
where all summation intervals are $[-\infty, +\infty]$. Specially, for $m = \infty$ we arrive at:

$$b_\ell \approx a_\ell s^{-\ell I/N}$$

(6.27)

where the sums are given in a closed form in the Appendix, Corollary A.2. It follows from (6.32) that $\Delta g^*$ will generally not exist when $N$ approaches infinity, because for large $N$ we have approximately:

$$b_\ell \approx a_\ell s^{-\ell I/N}$$

The prediction of an arbitrary point $\Delta g(\theta, R)$ from $\Delta g^*$ and (6.32) gives:

$$\Delta g(\theta, R) = \sum_\ell a_\ell e^{i\ell \theta} \frac{\sum_{s=1}^{\infty} s^{1/2+u_\ell}}{\sum_{s=1}^{\infty} s^{1/2+u_\ell}}$$

where

$$v_{\ell,n}(\theta, R) = \sum_{s=1}^{\infty} s^{1/2+u_\ell} e^{i\eta_p \theta}, \quad s_R = \Gamma \pi / R$$

Specially for $\theta = \theta_j = 2\pi j/N$ and $R = r$ we obtain:

$$\Delta g(\theta_j, r) = \sum_\ell a_\ell e^{i\ell \theta_j} \frac{\left(\sum_{s=1}^{\infty} s^{1/2+u_\ell}\right)^2}{\sum_{s=1}^{\infty} s^{1/2+u_\ell}}$$

and finally

$$\lim_{N \to \infty} \Delta g(\theta_j, r) = \sum_\ell a_\ell e^{i\ell \theta_j} = \Delta g(\theta_j, r)$$

Hence, the least squares prediction converges to the true value when $N$ approaches infinity.
7. On the Choice of Radius

In the studies of the Bjerhammar problem little attention has been paid to the choice of radius of the Bjerhammar sphere \( r_B \). However, in Sjöberg (1975) the importance of this choice is demonstrated for different covariance functions. In this section we are going to use the following one-dimensional prediction estimate on a circle as a basis (cf. the previous section):

\[
(7.1) \quad \Delta g(r, \theta) = \sum_{\ell = -\infty}^{\infty} a_{\ell} \frac{v_{\ell, z}}{u_{\ell, z}} e^{i \ell \theta}
\]

where

\[
\begin{align*}
v_{\ell, z} &= v_{\ell, z}(s, \theta) = \sum_{j = -\infty}^{\infty} e^{i j z} e^{i j \theta} \\
u_{\ell, z} &= v_{\ell, z}(s, 0)
\end{align*}
\]

and \( m \) is the number of observations. The corresponding prediction errors are:

\[
(7.2a) \quad \epsilon(\theta) = \Delta g(r, \theta) - \Delta g(r, \theta) = \sum_{\ell = -\infty}^{\infty} \epsilon_{\ell}(\theta)
\]

where

\[
(7.2b) \quad \epsilon_{\ell}(\theta) = a_{\ell} \left[ \frac{v_{\ell, z}}{u_{\ell, z}} - 1 \right] e^{i \ell \theta}
\]

We define the mean prediction error in the following way:

\[
(7.3) \quad \bar{\epsilon} = \frac{1}{2\pi} \int_{0}^{2\pi} \epsilon(\theta) \, d\theta
\]

First we prove the following proposition.
Proposition 7.1: If $a_0 = 0$ then

$$0 < |\bar{\epsilon}| < \left| \sum_{j=-\infty}^{\infty} a_{j\pi} \right|$$

and

$$\bar{\epsilon} \to 0, \quad s \to 1$$
$$\bar{\epsilon} \to \sum_{j} a_{j\pi}, \quad s \to 0$$

where $m$ is the number of observations.

Proof: Using

$$\frac{1}{2\pi} \int_{0}^{2\pi} g(\theta) d\theta = a_0 = 0$$

it follows from (7.1):

$$\bar{\epsilon} = \frac{1}{2\pi} \int_{0}^{2\pi} g^{*}(\theta) d\theta = \sum_{j} \frac{a_{j\pi}}{u_{j\pi,\pi}} \sum_{i} s^{i+\pi} \left| \sum_{j} a_{j\pi} \right|$$

$$\sum_{j} \frac{a_{j\pi}}{u_{j\pi,\pi}} = \frac{1-s^{\pi}}{1+s^{\pi}} \sum_{j} a_{j\pi}$$

Here we have used Corollary A.2 with $\ell = jm$:

$$u_{j\pi,\pi} = \frac{1+s^{\pi}}{1-s^{\pi}}$$

From this expression, the proposition readily follows.

Although $s^{-1}$ gives the mean prediction error 0 and has also proved to give the most stable solutions (section 3), the "best" choice of radius $r_s$ must be determined in some mean square sense. Let us therefore study the prediction of the signal:
\begin{equation}
\Delta g_r(\theta) = b \cos (k\theta) + c \sin (k\theta)
\end{equation}

at the circle of radius $r$. We define the relative variance in the following way:

\begin{equation}
R_\ell = \frac{\int_0^{2\pi} \epsilon^2(\theta) \, d\theta}{\int_0^{2\pi} \Delta g_r^2 \, d\theta}
\end{equation}

where

\[
\epsilon(\theta) = \Delta g_r(\theta) - \Delta g_r(\theta) \quad \text{(prediction error)}
\]

It is shown in the Appendix, Proposition A.4, that (7.5) may be written (for $k > 0$):

\begin{equation}
R_\ell \left\{ \begin{array}{ll}
S_\ell & \text{if } \ell \neq \frac{m}{2} \quad \text{p, } p = 1, 2, 3, \ldots \\
1+2\beta \over 2(1+\beta) + \frac{1}{2(1+\beta)} S_\ell & \text{if } \ell = \frac{m}{2} \quad \text{p}
\end{array} \right.
\end{equation}

where

\begin{equation}
S_\ell = 1 + \frac{1-s^n}{1+s^n} \left( \frac{1+8^{2n-4n}}{(1+s^{2n-4n})^2} - \frac{1-s^n}{1+s^n-2s} \right) 2^{\ell-n}
\end{equation}

\[
n = \ell - \left[ \frac{\ell}{m} \right] m
\]

\[
\beta = (c/b)^{\ell}
\]

From (7.6) and (7.6') we draw the following conclusions:

\begin{itemize}
  \item[a)] \( \lim R_\ell = 0 \), \( m \to \infty \)
  \item[b)] \( \lim S_\ell = 1 \), \( s \to 1 \)
  \item[c)] \( \text{if } \ell < m/2 \text{ then } \lim R_\ell = 0 \), \( s \to 0 \)
  \item[d)] \( \text{if } \ell = m/2 \text{ then } S_\ell > 1/2 \text{ and } \lim S_\ell = 1/2 \), \( s \to 0 \)
  \item[e)] \( \text{if } \ell > m/2 \text{ then } S_\ell > 1/2 \)
  \item[f)] \( \text{if } \ell = mp \text{ then } S_\ell < 2 \text{ and } \lim S_\ell = 2 \), \( s \to 0 \)
  \item[g)] \( \text{if } \ell = m(p + 1/2) \text{ then } S_\ell < 3/2 \text{ and } \lim S_\ell = 3/2 \), \( s \to 0 \)
  \item[h)] \( \text{if } \ell > m/2 \text{ and } \ell \neq mp/2 \text{ then } \lim S_\ell = \infty \), \( s \to 0 \)
\end{itemize}
Figure 4. The relative variance $S_p$ in the Dirac method for the circle, using Poisson's kernel, is given for signals of various frequencies $f_1$ as a function of $h/a$, where $h$= distance from the Bjerhammar circle to the circle of observations and $a$= spacing of the data $(2\pi r/m)$. $m=50$.

Figure 5. The optimum ratio $h/a$ and the corresponding relative variance $(R_1)$ given as functions of $1/m$, where $1$ is the frequency of the signal and $m$ is the number of observations $(1 \geq 0.5 m)$. 

-63-
These statements are illustrated in Figures 4-5. It is obvious that for low-frequency observations ($k \leq m/2$), the radius $r_B$ should be chosen as small as possible in accordance with the numerical difficulties discussed in sections 3-4. (See also section 8.) For higher frequencies ($k > m/2$), the relative variance $R_k$ is at least 50% (see also Bjerhammar, 1977a). Accordingly, signals of frequency higher than $m/2$ cannot be predicted in a satisfactory way.

In a real prediction case we do not have the above ideal situation with a uniform distribution of data of one single frequency, and the most favorable distance $r_B$ is a function of the unknown coefficients $(a_j)$ and of the mean spacing of the data. In most cases empirically derived covariance functions are associated with a specific radius of the Bjerhammar sphere. Several tests in Sjöberg (1975) revealed that the solutions by collocation are rather unsensitive to the choice of kernel functions, if the optimal depth to the sphere is chosen for each of them. However, the choice of radius (changing for each distribution of the observations) is important for a good result. In conclusion, a study of the relation between the optimum radius $r_B$ and the spacing of the observations is recommended for each specific kernel function. For instance, in the numerical example of the next section, the optimum depth to the sphere in the Dirac method is approximately half the distance between neighboring observations.

8. Computations

The iterative method described in section 4 will be used to demonstrate the rate of convergence of the solutions for $\Delta g^*$ and $X$ in the Dirac method and collocation, respectively [see formulae (3.1), (3.2) and (4.2)]. The stability of each method is reflected by the number of necessary iterations for the solution of the vector $\Delta g^*$ or $X$.

First, the case with the degree variances equal to $2n + 1$ will be considered. The observations consist of 87 free air gravity anomalies regularly distributed with a spacing of approximately 0.5 (Table 8.1). The 50 prediction points and anomalies are given in Table 8.2. The mean value ($\bar{\Delta g}$) of these 50 anomalies is $-2.8$ mgal and the RMS value of $\Delta g - \bar{\Delta g}$ is $\pm 13.4$ mgal. In the computations we assume that the mean sea level radius is 6370 km. The prediction results are given in Table 8.3 and Figure 6. From the table we notice that the number of necessary iterations increases with the depth to the Bjerhammar sphere and that this increase is more pronounced for collocation than for the Dirac method. The solution by the Dirac method for one specific depth is identical with the collocation solution for half that depth. This result implies that the latter method is twice as sensitive to changes of the depth as the former.

For comparison we study also the predictions using collocation with the empirical covariance function implied by subroutine COVA (Tscherning and Rapp, 1974). The computations are performed for different lower bounds ($N_{\text{sn}}$) of the covariance function. In each case the RMS prediction error 10.1 mgal was obtained. The number of necessary iterations are given in Table 8.4.
### Table 8.1

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Observation points. 87 free-air gravity anomaly stations in Manitoba, Canada.
Table 8.2

Prediction points. 50 free air gravity anomaly stations in Manitoba, Canada.

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Figure 6. RMS prediction errors compared for various depths to the Bjerhammar sphere. 
$c_n^* = 2n + 1$, no. of observations = 87, no. of predictions = 50.
### Table 8.3

Comparison Between the Predictions of 50 Free Air Gravity Anomalies From 87 Observed Anomalies Using the Dirac Method and Collocation*

<table>
<thead>
<tr>
<th>( r_s ) (km)</th>
<th>depth (km)</th>
<th>No. of iterations</th>
<th>RMS error [mgal]</th>
<th>No. of iterations</th>
<th>RMS error [mgal]</th>
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<tr>
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<td>1</td>
<td>13.5</td>
<td>1</td>
<td>13.5</td>
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</table>

* The iterations of \( \Delta g^* \) and \( X \) are interrupted whenever the RMS residuals are less than 0.25 mgal, the maximum residuals are less than 0.5 mgal, or the number of iterations exceed 30. Degree variances = \( 2n+1 \).

### Table 8.4

Number of Necessary Iterations of the Vector \( C^{-1} \Delta g \) to Satisfy Either of the Conditions RMS Residual < 0.25 mgal or \( \max \text{ Residual} < 0.5 \text{ mgal} \).*

<table>
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<th>( N_{\text{init}} )</th>
<th>3</th>
<th>5</th>
<th>3</th>
<th>13</th>
<th>20</th>
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<tr>
<td>No. of iterations</td>
<td>18</td>
<td>16</td>
<td>14</td>
<td>13</td>
<td>10</td>
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</table>

* Subroutine COVA is used for different minimum degrees \( (N_{\text{init}}) \)
The empirical covariance function in collocation gives a slightly better prediction result than the Dirac method for its optimum radius of $r_8$. This gain is obtained at the cost of at least twice as many iterations. For a denser spacing of the data, the number of necessary iterations will increase and the difference between the two methods becomes more pronounced. With this reasoning, the non-iterative solution may fail, when using collocation (due to numerical singularity), while a solution by the Dirac method may still be useful. For example, the prediction result reported in Table 8.5 shows small differences between collocation and the Dirac method (93 well-distributed observations). However, when a few more observation points were included (99 points) between the previous ones (with minimum spacing 675 meters), the RMS prediction error of $\xi$ increased to 11.4' for the collocation type of solution, while the RMS error of the Dirac method was still useful (1.3'). (From Sjöberg, 1975.)

### Table 8.5

RMS Prediction Errors for Molodenskii’s Model

<table>
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<th>Dirac Method</th>
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<td>$\Delta g$</td>
<td>$\xi''$</td>
</tr>
<tr>
<td></td>
<td>/m/</td>
<td>/mgal/</td>
</tr>
<tr>
<td></td>
<td>/m/</td>
<td>/mgal/</td>
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<tr>
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</table>

* Depth to the Bjerhammar Sphere = 10m. No. of prediction points = 37. $c_{\pi} = 2n+1$. Reference: Sjöberg (1975, section 19.2.3).

### 9. Conclusions

The purpose of this report has been to compare some methods of A. Bjerhammar with collocation for the solution of the boundary value problem in physical geodesy. In practice, the number of observations are finite ("discrete boundary value problem"). In the present application collocation is most frequently identified as Wiener-Hopf prediction of stochastic processes, in which case the covariance functions are assumed to be known (homogeneous and isotropic). The main problem is therefore to find the appropriate covariance functions. Rigorously this has been proved to be an impossible task, because of the non-ergodicity of the empirical covariance functions (Lauritzen, 1973).
Bjerhammar's methods are based on Poisson's integral equation and Stokes' formula. For a finite set of observations there is always a fictitious field \( (\Delta g^*) \) at an internal sphere that satisfies the integral equation. In Bjerhammar's applications \( \Delta g \) has originally been considered as mean anomalies over blocks of certain sizes at the internal sphere (the Bjerhammar sphere). Depending on the number of such blocks, different types of solutions of the problem are obtained. If the number of blocks \( N \) are less than the number of observations \( m \), a unique solution is given from adjustment by elements. For \( N > m \) a unique solution is given by condition adjustment, which solution minimizes the norm of the unknowns \( (\Delta g^*) \). In the special case \( N = m \) and well-behaving surface elements we have shown that the minimum norm solution of Bjerhammar \( ||\Delta g^*|| = \text{minimum} \) approaches the solution by collocation with the degree variances equal to \( 2n+1 \). Furthermore, this proof has been generalized to yield, that for each set \( c_z \) in collocation, there is a corresponding minimum norm \( ||u^*|| \) in the generalized Bjerhammar approach (section 2.2). Thus the problem of selecting the degree variances in collocation is identical with the problem of selecting the minimum norm in the Bjerhammar method. It should be stated that already Krarup (1969) regarded collocation as a generalized approximation for a specified minimum norm.

A different type of solution, called reflexive prediction, was introduced by Bjerhammar (1974). In this method the external gravity field is assumed to be generated at a priori selected fictitious carrier points, on or outside the Bjerhammar sphere. Once the carrier points and the sphere are defined, Poisson's integral equation can be applied rigorously, and the result is a set of linear equations. If all carrier points are located at the internal sphere, the method is called the Dirac approach. Due to the arbitrary location and number of carrier points, a wide variety of solutions are possible. Of special interest are filtering (less number of carrier points than observations) and the non-singular Dirac method (see below).

In this report we have, first of all, compared collocation and the non-singular Dirac approach with carrier points located at the intersections of the internal sphere with the radius vectors of the observations. The coefficient matrix of the latter method is non-symmetric in contrast to the former. Moreover it was shown in section 2 that for a constant geocentric radius of all observations in an area (and \( c_z = 2n+1 \)) the two methods give identical prediction results for:

\[
(9.1) \quad h_0 = 2 h_8
\]

where \( h_0 \) and \( h_8 \) are the depths to the Bjerhammar sphere in the Dirac method and collocation, respectively. This result has been verified to hold also for approximately constant geocentric radius of the observations (section 8).
It was demonstrated in sections 3 and 4 that the numerical stability of the solutions differs for the two methods (collocation and Dirac). For a given radius of the Bjerhammar sphere the Dirac method is more stable. In the numerical example of section 8 it was found that for Poisson's kernel \((c_n^* = 2n+1)\), the conditionings of the two methods are equal provided that formula (9.1) is satisfied. For this kernel function the solution by collocation is twice as sensitive to the choice of Bjerhammar sphere as the solution by the impulse method, a fact that is important for the solution of a large system with a dense distribution of the observations.

In the continuous case (with observations covering all the surface of the earth) we have found (section 6) that all predictions with the generalized Dirac method and collocation are unique for various sets of positive \(c^*_n (c_n)\), whenever the solutions exist. This result may be regarded as a consequence of Stokes' theorem. In general, however, the intermediate solutions \(u^*\) and \(X\) do not exist in the continuous case. The existence of the solutions requires that the degree variances of the observations are at most of magnitude \(n^{-5}\), a condition which is not satisfied for the gravity field of the earth according to Kaula's rule. Approximate solutions may be found, for example, by adding a positive constant to the kernel function. In the same way it was found in section 6.3 that a solution may exist when considering that the observations are erroneous. However, these solutions are statistically biased. If collocation is carried out by solving the Wiener-Hopf integral equation, a convergent solution is obtained outside a sphere (all observations on the sphere). However, inside the bounding sphere of the real earth, the convergence is still not proved.

It should be noted that the deterministic approaches by Bjerhammar through Poisson's and Stokes' formulae do not provide estimated prediction errors, as is the case in the stochastic process approach (i.e. collocation according to Moritz). However, as the uncertainty of the covariance functions has a direct impact on the error estimates, these estimates might be of limited value.

For large systems the free choice of carrier points in reflexive prediction might be advantageous (filtering is possible). However, experiences by Bjerhammar (1977b) indicate that numerical difficulties may occur in the filtering process, due to ill-conditioning.

Finally, we like to mention that the original approaches of Bjerhammar are designed to solve problems in physical geodesy, first of all the geodetic boundary problem, for gravity anomalies as the only source of data. However, it is not difficult to modify the methods in order to take other types of geophysical information into account. Thus, both collocation and Bjerhammar's methods possess a flexibility for the processing of heterogeneous data.
10. Extensions and Recommendations

This study has shown that reflexive prediction (the Dirac method) is less sensitive to changes of the radius of the internal sphere than collocation for the particular kernel and covariance functions with $c_n = c_{n+1} = 2n+1$. The prediction results and the stability of the two methods are the same, if collocation is applied with half the depth to the Bjerhammar sphere used in reflexive prediction. We recommend these comparisons to be carried out for other covariance functions. Of special interest would be to compare an empirical covariance function with a best fitting kernel function (cf. Sjöberg, 1975, section 18).

Special attention should be paid to the relation between the distribution of the observations and the optimum radius of the Bjerhammar sphere for various covariance (kernel) functions.

A procedure to estimate the prediction errors in reflexive prediction is of interest for the user of the method. In order to reduce the bias of the predictions, we recommend the use of formula (6.31) in all applications of collocation to the geodetic boundary problem and related problems in physical geodesy. A corresponding formula should be used in reflexive prediction, if the kernel function is modified to take noise into account.

Theoretically, it is of interest to reveal whether the Wiener-Hopf type of predictions [formulae (6.24), (6.25), (6.26 a–b)] converges when applied to a continuous field of observations at the surface of the earth.
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Appendix

The Addition Theorem for Spherical Harmonics:

\[ P_n \left( \cos \psi_{ik} \right) = \frac{1}{2n+1} \sum_{m=-n}^{n} Y_{nm} \left( \theta_i, \lambda_i \right) Y_{nm} \left( \theta_k, \lambda_k \right) \]

where

\[ Y_{nm} \left( \theta, \lambda \right) = \overline{P}_{nm} \left( \cos \theta \right) \left\{ \begin{array}{ll}
\cos m\lambda & \text{if } m \geq 0 \\
\sin m\lambda & \text{if } m < 0
\end{array} \right. \]

\( (\theta, \lambda) \) = spherical coordinates (colatitude and longitude)

\[ \frac{1}{4\pi} \int\int Y_{nm} Y_{n'm'} \, d\sigma = \delta_{nn'} \delta_{mm'} \]

\( \sigma \) = the unit sphere
\( \delta \) = Kronecker's delta

**Corollary A.1:**

\[ \frac{1}{4\pi} \int\int P_n \left( \cos \psi_{ik} \right) P_{n'} \left( \cos \psi_{jk} \right) \, d\sigma_k = \frac{1}{2n+1} P_n \left( \cos \psi_{ik} \right) \Delta \sigma_k \]

The corollary follows from (A.1) and (A.2).

**Proposition A.1:**

If A is a matrix of dimensions (m x N), where

\[ (A)_{ik} = \frac{1}{4\pi} \sum_{n=0}^{\infty} \sqrt{(2n+1)} c_n \left( \frac{r_i}{r_k} \right)^{n+2} P_n \left( \cos \psi_{ik} \right) \Delta \sigma_k \]
and

\[ Q = 4\pi \left( \begin{array}{c}
\Delta \sigma_1 \\
\Delta \sigma_2 \\
\vdots \\
\Delta \sigma_n
\end{array} \right)^{-1} \]

and

\[ \sum_{k=1}^{N} \Delta \sigma_k = 4\pi \]

then

\[ \lim_{N \to \infty} (AQA^T)_{11} = \sum_{n=0}^{\infty} c_n \left( \frac{r_1}{r_1 r_1'} \right)^{n+2} P_n(\cos \psi_{1k}) \]

\[ \frac{x}{\sqrt{n+1}} \frac{x}{\sqrt{n+1}} P_{n+1}(\cos \psi_{1k}) \Delta \sigma_k \]

Proof:

\[ (AQA^T)_{11} = 4\pi \sum_{k=1}^{N} (A)_{1k} (A)_{k1} \Delta \sigma_k^{-1} = \]

\[ = \frac{1}{4\pi} \sum_{k=1}^{N} \sum_{n=0}^{\infty} \left( \frac{r_1}{r_1 r_1'} \right)^{n+2} P_n(\cos \psi_{1k}) \sum_{n=0}^{\infty} \left( \frac{r_1}{r_1} \right)^{n+2} P_{n+1}(\cos \psi_{1k}) \Delta \sigma_k \]

For \( N \to \infty \) in such a way that \( \max \Delta \sigma_k \to 0 \) the summation over \( k \) becomes the corresponding integral, so that:

\[ \lim_{N \to \infty} (AQA^T)_{11} = \frac{1}{4\pi} \sum_{n=0}^{\infty} \sqrt{(2n+1)(2n+1)} \sqrt{c_n c_n'} \left( \frac{r_1}{r_1} \right)^{n+2} \left( \frac{r_1}{r_1'} \right)^{n+2} \]

\[ \times \int \int P_n(\cos \psi_{1k}) P_{n+1}(\cos \psi_{1k}) d\sigma_k \]

\[ -76- \]
Corollary A.1 finally yields:

$$\lim_{n \to \infty} (AQA_1)^n = \sum_{n=0}^{\infty} c_n \left( \frac{r_0^2}{r_1 r_2} \right)^{n+2} P_n (\cos \psi_1)$$

Proposition A.2

(A.4) \[
\sum_{j=-\infty}^{\infty} S_t \theta \sum_{j=0}^{\infty} e^{i(s+Nj)\theta} = (se^{i\theta}) \left( s - \alpha \right) \left[ \frac{1}{1-s_n e^{-\theta\phi}} + \frac{s^{n-1} e^{i\theta} e^{-\theta\phi}}{1-s_n e^{-i\theta\phi}} \right]
\]

for \(0 \leq s < 1\) and \(\alpha = \left\lfloor \frac{m}{N} \right\rfloor\) = integer part of \(\frac{m}{N}\), and \(\phi = \begin{cases} \theta & \text{if } m \geq 0 \\ -\theta & \text{if } m < 0 \end{cases}\)

Proof:

Let us substitute \(j\) of (A.1) by \(k - \alpha\). Then we obtain in the left member:

$$\sum_{k=-\infty}^{\infty} S_t \sum_{k=-\infty}^{\infty} e^{i(s+Nk-\alpha)\theta} = S_1 + S_2$$

where

$$S_1 = (se^{i\theta}) \left( s - \alpha \right) \sum_{k=0}^{\infty} (se^{i\theta})^{\alpha} = (se^{i\theta}) \left( s - \alpha \right) \left[ \frac{1 - (se^{i\theta})^N}{1 - (se^{i\theta})^{\alpha}} \right]$$

$$S_2 = (se^{i\theta})^{\alpha} \left( s - \alpha \right) \sum_{k=1}^{\infty} (se^{-i\theta})^{\alpha} = (se^{-i\theta})^{\alpha} \left( s - \alpha \right) \left[ \frac{1 - (se^{-i\theta})^N}{1 - (se^{-i\theta})^{\alpha}} \right]$$

Corollary A.2:

(A.5) \[
\sum_{j=-\infty}^{\infty} S \left( s + nj \right) = (s \left( s - \alpha \right) \left[ \frac{1 - (se^{i\theta})^N}{1 - (se^{i\theta})^{\alpha}} \right]
\]

The proof follows directly from the proposition for \(\theta = 0\).
Corollary A.3:

\[(A.6) \quad \lim_{n \to \infty} \sum_{j=-\infty}^{\infty} s^{|n+j|} e^{i(z+n)j} \phi = s^{|n|} e^{i\alpha \phi}\]

The proof follows directly from (A.4) when considering that \( \alpha \to 0, N \to \infty \).

Proposition A.3:

If

\[A_{jk} = \frac{1}{N} \sum_{n=-\infty}^{\infty} s^{\pm n} e^{i\alpha (\theta_j - \theta_k)}\]

\[\theta_k = 2\pi k/N \]
\[\theta_j = 2\pi j/m \]
\[N = p m \quad p = \text{integer} \]

and

\[\theta_q = 0\]

then

\[S(m, N, s) = N \sum_{j=1}^{\infty} \sum_{k=1}^{N} A_{jk} A_{kj} - 2N \sum_{k=1}^{N} A_k^2\]

equals

\[S(m, N, s) = \left[ (2-m)(1+s^m) + 2m \frac{1-s^{2N-2}}{(1-s^2)(1-s^{2N-2})} - 4 \frac{1-s^{2N-2}}{(1-s^2)(1-s^{2N-1})} \right. \]

\[+ \frac{2Ns^N}{1-s^2} - \frac{4Ns^N}{1-s} \bigg]/ (1-s^n)\]

Proof:

For \( \theta_q = 0 \) we have:
\begin{align*}
(A.7) \quad B_j &= N \sum_{k=1}^{N} A_{jk} A_{jk} = \sum_{n=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} s^{ln + il \ell} e^{i \ell \theta_j} \sum_{k=1}^{N} e^{-i(n+\ell)\theta_k} = \\
&= N \sum_{n=-\infty}^{\infty} \sum_{r=-\infty}^{\infty} s^{ln + ir \ell} e^{i(n+\ell)\theta_j} = N \sum_{n=-\infty}^{\infty} s^{ln} e^{i \theta_j} \sum_{r=-\infty}^{\infty} s^{lr + \ell r} \\
\text{and} \\
(A.8) \quad \sum_{j=1}^{m} B_j &= N m \sum_{t=-\infty}^{\infty} s^t \sum_{r=-\infty}^{\infty} s^{rt + \ell r} \\
\text{Inserting (A.5) into (A.7) for } \theta_j = 0, \text{ we obtain:} \\
N \sum_{k=1}^{N} A_{jk}^2 &= \sum_{n=-\infty}^{\infty} \sum_{r=-\infty}^{\infty} s^{ln + \ell} = \frac{1}{1-s^l} \sum_{n=-\infty}^{\infty} (s^{ln} + s^{ln + \ell}) = \\
&= \frac{1}{1-s^l} \left[ -1 - s^n + 2 \sum_{k=0}^{n-k} \left( s^{2n-k} + s^{n+k} \right) \right] = \\
&= \frac{1}{1-s^l} \left[ -1 - s^n + \frac{2(2n-1)}{1-s^2} \frac{s^{(2n-1)l} + 2n-2}{1-s^2} + 2N \sum_{k=0}^{\infty} s^{n+k} \right] = \\
&= \frac{1}{1-s^l} \left[ -1 - s^n + \frac{2(1-s^{2n-2})}{(1-s^2)(1-s^{2N-1})} + \frac{2Ns^n}{1-s} \right] \\
\text{In the same way we obtain from (A.8):} \\
\frac{1}{N} \sum_{j=1}^{N} B_j &= \frac{m}{1-s^l} \left[ -1 - s^n + \frac{1-s^{2n-2}}{(1-s^n)(1-s^{2N-1})} + \frac{2ps^n}{1-s^2} \right]
\end{align*}
so that

\[ S(m, N, s) = \frac{1}{N} \sum_{j=1}^{N} B_j - 2N \sum_{k=1}^{N} A_{ik} \]

\[ = \left[ -m - ms^2 + 2m \frac{1-s^{2N-2}}{(1-s^2)(1-s^{2N-2})} + \frac{2Ns^N}{1-s^2} + 2 + 2s^N - \frac{4(1-s^{2N-2})}{(1-s^2)(1-s^{2N-2})} - \frac{4Ns^N}{1-s^2} \right] / (1-s^2) \]

Proposition A.4:

If

\[ \Delta g(\theta) = b \cos(\ell \theta) + c \sin(\ell \theta) , \quad \ell > 0 \]

and

\[ \Delta \tilde{g}_k(\theta) = \frac{a_k}{u_{k,s}} v_{k,s}(\theta) e^{i \ell \theta} + \frac{a_{-k}}{u_{k,s}} v_{l,s}(\theta) e^{-i \ell \theta} \]

where

\[ a_k = \frac{1}{2} (b - ic) , \quad a_{-k} = \frac{1}{2} (b + ic) \]

\[ v_{k,s}(\theta) = \sum_{j} s^{m_j + \ell j} e^{i s \theta} \]

and

\[ u_{k,s} = v_{k,s}(0) \]

then

\[ R_k = \frac{\int_{0}^{2\pi} (\Delta \tilde{g} - \Delta g)^2 \, d\theta}{\int_{0}^{2\pi} \Delta g^2 \, d\theta} = \begin{cases} S_k & \text{if } k \neq \frac{m}{2} p , \ p = 1, 2, \ldots \ \\
\frac{1+2\beta}{2(1+\beta)} + \frac{1}{2s} S_k & \text{if } k = \frac{m}{2} p \end{cases} \]

where
\[ S_\ell = 1 + \frac{1-s^2}{1+s^2} \cdot \frac{1+s^{2s-4n}}{(1+s^{s-2n})^2} - \frac{1-s^2}{1+s^{s-2n}} \cdot 2s^{\ell-n} \]

\[ n = \ell - \left\lfloor \frac{\ell}{m} \right\rfloor m \]

\[ \beta = (c/b)^2 \]

**Proof:**

\[ I_1 = \frac{1}{2\pi} \int_0^{2\pi} \Delta g^2 \, d\theta = (b^2 + c^2)/2 \]

\[ I_2 = \frac{1}{2\pi} \int_0^{2\pi} (\Delta g_\ell - \Delta g) \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} (\epsilon_\ell^2 + \epsilon^{2\ell} + 2\epsilon_\ell \epsilon^{\ell}) \, d\theta = I_{21} + I_{22} + I_{23} \]

where

\[ \epsilon_\ell = a_{\ell} \left( \frac{v_\ell}{u_\ell} - 1 \right) e^{i\ell\theta} \]

\[ \epsilon^{-\ell} = a^{-\ell} \left( \frac{v_\ell}{u_\ell} - 1 \right) e^{-i\ell\theta} \]

and

\[ I_{21} = \frac{1}{2\pi} \int \epsilon_\ell^2 \, d\theta = \begin{cases} a_{\ell}^2 \frac{u_\ell^{(2)}}{u_\ell^2} - 2s^\ell / u_\ell & \text{if } \ell = \frac{m}{2} \rho \\ 0 & \text{otherwise} \end{cases} \]

\[ I_{22} = \frac{1}{2\pi} \int \epsilon^{-\ell}^2 \, d\theta = \begin{cases} a_{-\ell}^2 \frac{u_\ell^{(2)}}{u_\ell^2} - 2s^{-\ell} / u_\ell & \text{if } \ell = \frac{m}{2} \rho \\ 0 & \text{otherwise} \end{cases} \]

\[ I_{23} = \frac{1}{2\pi} \int \epsilon_\ell \epsilon^{-\ell} \, d\theta = 2a_{\ell}a^{-\ell} \left[ 1 + u_\ell^{(2)} / u_\ell^2 - 2s^\ell / u_\ell \right] \]

\[ u_\ell^{(2)}(s) = u_\ell(s^2) \]

The proposition readily follows for:

\[ R_\ell = (I_{21} + I_{22} + I_{23})/I_1 \]
and

$$(a_2 + a_{-2})^2 = b^2/4, \quad a_2 a_{-2} = (b^2 + c^2)/4$$