CANONICAL CORRELATIONS WITH RESPECT TO A COMPLEX STRUCTURE

BY

STEEN A. ANDERSSON

TECHNICAL REPORT NO. 33
JULY 1978

PREPARED UNDER CONTRACT NO0014-75-C-0442
(NR-042-034)
OFFICE OF NAVAL RESEARCH

THEODORE W. ANDERSON, PROJECT DIRECTOR

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA
CANONICAL CORRELATIONS WITH RESPECT TO A COMPLEX STRUCTURE

by

STEEN A. ANDERSSON*
University of Copenhagen

TECHNICAL REPORT NO. 33
JULY 1978

PREPARED UNDER CONTRACT NO.0014-75-C-0442
(NR-042-034)
OFFICE OF NAVAL RESEARCH

Theodore W. Anderson, Project Director

Reproduction in Whole or in Part is Permitted for
any Purpose of the United States Government.
Approved for public release; distribution unlimited.

Also issued as Technical Report No.132 under National Science Foundation
Grant MPS 75-09450 - Department of Statistics, Stanford University.

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

*Work was finished at Stanford University with support from the
Danish Natural Science Research Council.
1. Introduction

Let \( E \) be a vector space of dimension \( 2p \) over the field of real numbers \( \mathbb{R} \). Let \( x_1, \ldots, x_N \) (\( N \geq 2p \)) be identically distributed independent observations from a normal distribution with mean value 0 and unknown covariance \( \Sigma \). That is, \( \Sigma \) is a positive definite form on the dual space \( E^* \) to \( E \). The maximum likelihood estimator \( \hat{\Sigma} \) for \( \Sigma \) is well-known to be given by

\[
\hat{\Sigma} (x_1, \ldots, x_N) = \left( (x^*, y^*) \right) \frac{1}{N} \sum_{i=1}^{N} x^*(x_i) y^*(x_i); x^*, y^* \in E^*
\]

The distribution of \( \hat{\Sigma} \) is the Wishart distribution on the set \( \rho(E^*)_\mathbb{R} \) of positive definite forms on \( E^* \) with \( N \) degrees of freedom and parameter \( \frac{1}{N} \Sigma \). Suppose now that \( E \) is also a vector space over the field \( \mathbb{C} \) of complex numbers such that the restriction to the subfield of real numbers in \( \mathbb{C} \) is the original vector space structure on \( E \).

The dimension of \( E \) as a vector space over \( \mathbb{C} \) is then \( p \). The vector space \( E^* \) is then also a vector space over the complex numbers under the definition \( zx^* = x^* \circ \bar{z} = (x \to x^*(\bar{z}x)); x \in E \), \( x^* \in E^*, z \in \mathbb{C} \). The set \( \rho_{\mathbb{C}}(E^*)_\mathbb{R} = \{ \Sigma \in \rho(E^*)_\mathbb{R} | \Sigma(zx^*, y^*) = \Sigma(x^*, \bar{z} y^*), \forall x^*, y^* \in E^*, \forall z \in \mathbb{C} \} \) defines a null hypothesis in the statistical model described above. The condition \( \Sigma(zx^*, y^*) = \Sigma(x^*, \bar{z} y^*) \), \( \forall x^*, y^* \in E^*, \forall z \in \mathbb{C} \) is in Andersson [2] called the \( \mathbb{C} \)-property and in terms of matrices it has the formulation: For every basis \( e_1^*, \ldots, e_p^* \) for the complex vector space \( E^* \) the matrix for a \( \Sigma \) with the \( \mathbb{C} \)-property with respect to the basis \( e_1^*, \ldots, e_p^*, ie_1^*, \ldots, ie_p^* \) for the real vector space \( E^* \) has the form

\[
\begin{pmatrix}
\Pi & F \\
-F & \Pi
\end{pmatrix}
\]
The statistical problem of testing $\Sigma \in \mathcal{P}_c(E^*)$ versus $\Sigma \in \mathcal{P}(E^*)$ is invariant under the action of the group $GL_c(E)$ of complex one-to-one linear mappings onto the sample and parameter space $\mathcal{P}(E^*)$ given by

$$GL_c(E) \times \mathcal{P}(E^*) \rightarrow \mathcal{P}(E^*)$$

$$(f, \Sigma) \rightarrow \Sigma \circ (f^*xf^*)$$

where $f^*$ is the dual mapping to $f \in GL_c(E)$. The restriction of the action to the subset $\mathcal{P}_c(E^*)$ is transitive. Since all tests invariant under (1.2) have a factorization through a maximal invariant function we shall find a representation of a maximal invariant function into $K_+^p$, describe the distribution as a density with respect to a restriction of the Lebesque measure and state an interpretation of this representation.

The matrix for a complex linear mapping of $E$ with respect to a basis of the form $e_1, \ldots, e_p, ie_1, \ldots, ie_p$, where $e_1, \ldots, e_p$ is a basis for the complex vector space $E$ is of the form

$$(1.3) \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$

The expression $\Sigma \circ (f^*xf^*)$ from (1.2) in matrix formulation becomes

$$(1.4) \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \begin{pmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}' & \Pi_{22} \end{pmatrix} \begin{pmatrix} A' & -B' \\ B' & A' \end{pmatrix}$$

with respect to the dual basis $e_1^*, \ldots, e_p^*, ie_1^*, \ldots, ie_p^*$ in $E^*$ to $e_1, \ldots, e_p, ie_1, \ldots, ie_p$ in $E$. 

2
2. Representation of the maximal invariant

2.1. Lemma. Let $\Pi$ be a positive definite form on the $\mathbb{R}$-space $E$. Then there exists a basis $e_1, \ldots, e_p$ for the $\mathbb{C}$-space $F$ such that the $2p \times 2p$ real matrix for $\Pi$ with respect to $e_1, \ldots, e_p$, $ie_1, \ldots, ie_p$ has the form

$$
\begin{bmatrix}
I & D_{\lambda} \\
D_{\lambda} & I
\end{bmatrix}
$$

where $I$ is the $p \times p$ identity matrix and

$$
D_{\lambda} = \text{diag}(\lambda_1, \ldots, \lambda_p) \quad \text{with} \quad 1 > \lambda_1 > \cdots > \lambda_p > 0.
$$

Furthermore, the matrix $D_{\lambda}$ is uniquely determined by $\Pi$; and if $\lambda_1 > \lambda_2 > \cdots > \lambda_p > 0$, then $\Pi$ also determines the basis $e_1, \ldots, e_p$ uniquely up to the sign of each basis vector.

Proof: Let $e_1', \ldots, e_p'$ be a basis for the $\mathbb{C}$-space $E$ and let

$$
\begin{bmatrix}
\Pi_{11} & \Pi_{12} \\
\Pi_{12}' & \Pi_{22}
\end{bmatrix}
$$

be the $2p \times 2p$ real matrix for $\Pi$ with respect to $e_1', \ldots, e_p'$, $ie_1', \ldots, ie_p'$. The assertion is then that there exists a nonsingular complex $p \times p$ matrix $Z_1 = A + iB$ such that

$$
\begin{bmatrix}
A' & B' \\
-B' & A'
\end{bmatrix}
\begin{bmatrix}
\Pi_{11} & \Pi_{12} \\
\Pi_{12}' & \Pi_{22}
\end{bmatrix}
\begin{bmatrix}
A & -B \\
B & A
\end{bmatrix} =
\begin{bmatrix}
I & D_{\lambda} \\
D_{\lambda} & I
\end{bmatrix}
$$

(2.3)
and that $D_\lambda$ is unique; and in the case where $\lambda_1 > \ldots > \lambda_p > 0$, the columns of $Z$ are unique up to multiplication with $\pm 1$.

The equation (2.3) is equivalent to the complex matrix equations

$$
\overline{Z}_1^{'\frac{1}{2}} (\frac{1}{2} (\Pi_{11} + \Pi_{22}) + i \frac{1}{2} (\Pi_{12}^{'\prime} - \Pi_{12})) Z_1 = I
$$

(2.4)

$$
Z_1^{'\frac{1}{2}} (\frac{1}{2} (\Pi_{12} + \Pi_{12}) + i \frac{1}{2} (\Pi_{11} - \Pi_{22})) Z = D_\lambda
$$

If we define $Z = Z_1^{-1}$ and

$$
\phi = \frac{1}{2} (\Pi_{11} + \Pi_{22}) + i \frac{1}{2} (\Pi_{12}^{'\prime} - \Pi_{12})
$$

(2.5)

$$
\psi = \frac{1}{2} (\Pi_{12} + \Pi_{12}) + i \frac{1}{2} (\Pi_{11} - \Pi_{22})
$$

then (2.4) becomes

$$
\phi = \overline{Z}^{'\prime} Z
$$

(2.6)

$$\psi = Z^{'\prime} D_\lambda Z$$

Since $\phi$ respectively $\psi$ is the matrix for a positive definite hermitian form respectively symmetric form on the $C$-space $E$, it follows from [3] that we can find a complex $p \times p$ diagonal matrix $D$ and a complex nonsingular $p \times p$ matrix $Y$ such that

$$
\phi = \overline{Y}^{'\prime} Y
$$

(2.7)

$$\psi = Y^{'\prime} D Y$$

By permutation we can obtain that the diagonal elements $d_1, \ldots, d_p$ of $D$ have the property $|d_1| \geq |d_2| \geq \ldots \geq |d_p|$. If we then multiply
the \(v^\text{th}\) row of \(Y\) with \(\exp[-i\theta_v/2]\), where \(d_v = |d_v|\exp[i\theta_v]\), \(v = 1, \ldots, p\), and call this new matrix for \(Z\), we obtain (2.6) with \(\lambda_v = |d_v|, v = 1, \ldots, p\). Since \(\Pi\) is positive definite, we have \(1 > \lambda_1 > \ldots > \lambda_p \geq 0\). The uniqueness follows from a rather elementary examination of the proof in [3] or from direct matrix calculation. Since every matrix of the form (2.1) with \(1 > \lambda_1 \geq \ldots > \lambda_p \geq 0\) is positive definite it follows from Lemma (2.1) that the mapping from \(\rho(E^*)_T\) onto \(\Omega = \{(\lambda_1, \ldots, \lambda_p) \in \mathbb{R}_+^p \mid 1 > \lambda_1 > \ldots > \lambda_p \geq 0\}\) determined from Lemma 2.1 is a maximal invariant function.

3. Canonical correlations with respect to a complex structure.

Interpretation.

It follows from Lemma 2.1 that there exists a basis \(e_1, \ldots, e_p\) for the \(C\)-space \(E\) such that the \(2p \times 2p\) matrix for \(\Sigma\) with respect to \(e^*_1, \ldots, e^*_p, ie^*_1, \ldots, ie^*_p\) has the form (2.1). In (2.1) \(D_\lambda\) is unique; and if \(\lambda_1 > \ldots > \lambda_p > 0\), the basis \(e^*_1, \ldots, e^*_p\) for the \(C\)-space \(E^*\) is unique up to a sign for each element. \(\lambda_1\) is called the \(j\)-th theoretical canonical correlation of \(\Sigma\) with respect to the complex structure, and \(e^*_j\) is called the \(j\)-th theoretical canonical linear form of \(\Sigma\) with respect to the complex structure \(j = 1, \ldots, p\). Let \(x \in E^*\) have coordinates \((\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_p)\) with respect to \(e^*_1, \ldots, e^*_p, ie^*_1, \ldots, ie^*_p\). Then

\[
\Sigma(x^*, x^*) = \sum_{i=1}^p \alpha_i^2 + \sum_{i=1}^p \beta_i^2 + 2 \sum_{i=1}^p \lambda_i \alpha_i \beta_i
\]

(3.1)

\[
\Sigma(ix^*, ix^*) = \sum_{i=1}^p \alpha_i^2 - \sum_{i=1}^p \beta_i^2 - 2 \sum_{i=1}^p \lambda_i \alpha_i \beta_i
\]

(3.2)
Consider the problem of maximizing $\Sigma(x^*, ix^*)$ under the conditions $\Sigma(x^*, x^*) = \Sigma(ix^*, ix^*) = 1$. This is equivalent to maximizing

$$\Sigma \sum \lambda_i (\alpha_i^2 - \beta_i^2)$$

subject to the conditions

$$\sum \alpha_i^2 + \sum \beta_i^2 = 1$$

and

$$\sum \lambda_i \alpha_i \beta_i = 0 .$$

If we suppose that $\lambda_1 > \lambda_2 > \ldots > \lambda_p > 0$, we get by using Lagrange's multipliers that the maximum point is achieved at $\alpha_1 = \pm 1$, $\alpha_2 = \ldots = \alpha_p = \beta_1 = \ldots = \beta_p = 0$, and the maximum value is $\lambda_1$. By induction it follows that $\pm e^*_j$ are the only linear forms uncorrelated with $e^*_1, \ldots, e^*_{j-1}$ for which $\Sigma(\alpha^*_j, e^*_j) = \Sigma(i\alpha^*_j, i\alpha^*_j) = 1$ and $\Sigma(e^*_j, i\alpha^*_j)$ is maximal. The maximum values are $\lambda_j$, $j = 1, \ldots, p$.

The canonical correlations $\lambda_1, \ldots, \lambda_p$ with respect to the complex structure can be found as the positive roots of the equation

$$\left| \begin{array}{cc}
\Sigma_{12} + \Sigma_{12} & \Sigma_{22} - \Sigma_{11} \\
\Sigma_{22} - \Sigma_{11} & -\Sigma_{12} - \Sigma_{12}
\end{array} \right| - \lambda \left| \begin{array}{cc}
\Sigma_{11} + \Sigma_{22} & \Sigma_{12} - \Sigma_{12} \\
\Sigma_{12} - \Sigma_{12} & \Sigma_{11} + \Sigma_{22}
\end{array} \right| = 0$$

where

$$\Sigma = \begin{pmatrix}
\Sigma_{11} & \Sigma_{12} \\
\Sigma_{12} & \Sigma_{22}
\end{pmatrix} ,$$

with respect to a basis of the form $f^*_1, \ldots, f^*_p, if^*_1, \ldots, if^*_p$.  

6
4. The distribution of the empirical canonical correlations with respect to a complex structure.

The estimator \( \hat{\Sigma}(x_1, \ldots, x_N) \) for \( \Sigma \) in the observations point \((x_1, \ldots, x_N)\) is given in the introduction. Suppose that \( \Sigma \in \mathcal{P}_c(E^*)_r \) and let \( e_1^*, \ldots, e_p^* \) be a basis for \( E^* \) such that the \( 2p \times 2p \) matrix for \( \Sigma \) with respect to the basis \( e_1^*, \ldots, e_p^*, ie_1^*, \ldots, ie_p^* \) is the \( 2p \times 2p \) identity matrix. The distribution of \( \hat{\Sigma} \) in terms of matrices is a Wishart distribution with a representation as a density with respect to the restriction of the Lebesgue measure to all positive definite \( 2p \times 2p \) matrices \( \mathcal{P}(\mathbb{R}^{2p})_r \) as follows

\[
(4.1) \quad c \cdot |\det \Theta|^{(N-2p-1)/2} \exp\{- \frac{1}{2} \text{tr}(\Theta)\} d\Theta, \quad \Theta \in \mathcal{P}(\mathbb{R}^{2p})_r.
\]

The canonical correlations and linear forms (with respect to the complex structure) of \( \hat{\Sigma}(x_1, \ldots, x_N) \) is called the empirical canonical correlations and linear forms with respect to the complex structure.

The classical theory of canonical correlations is due to Hotelling [4]. We shall find the distribution of these. If we define \( \Phi \) and \( \Psi \) from the \( 2p \times 2p \) real matrix \( \Theta \), as in formula (2.5), we have a one-to-one and onto mapping between \( \mathcal{P}(\mathbb{R}^{2p})_r \) and \( \mathcal{P}(\mathbb{C}^p)_r \times S(\mathbb{C}^p) \), where \( \mathcal{P}(\mathbb{C}^p)_r \) respectively \( S(\mathbb{C}^p) \) denotes the set of positive definite hermitian respectively symmetric \( p \times p \) complex matrices, with Jacobian 1. Furthermore, (2.6) defines a one-to-one mapping from \( \text{GL}_+(\mathbb{C}^p) \times \Omega \) into \( \mathcal{P}(\mathbb{C}^p)_r \times S(\mathbb{C}^p) \), where \( \text{GL}_+(\mathbb{C}^p) \) is the subset of all nonsingular \( p \times p \) complex matrices with a positive real part in the first row and \( \Omega = \{ (\lambda_1, \ldots, \lambda_p) \in \mathbb{R}^p | 1 > \lambda_1 > \ldots > \lambda_p > 0 \} \).
The complementary to the image (which is an open set) of this mapping has Lebesgue measure 0; and therefore from our distribution point of view, we can forget this. To find the Jacobian of this mapping defined by (2.6), we proceed as in [1]. The method is due to Hsu [5]. We have

\[ d\Phi = (d\overline{Z}')Z + \overline{Z}'(dz) \]

\[ d\Psi = (dz')\Lambda Z + Z'(d\Lambda)Z + Z'\Lambda(dz) \]

and we shall find the absolute value of the determinant of the linear mapping \((dZ,d\Lambda) \rightarrow (d\Phi,d\Psi)\) defined by (4.2). This is a composition of

(a) \[
\begin{pmatrix}
\frac{dZ}{d\Lambda} \\
\frac{d\Lambda}{d\Lambda}
\end{pmatrix}
\rightarrow
\begin{pmatrix}
(dZ)Z^{-1} \\
d\Lambda
\end{pmatrix}
= \begin{pmatrix}
d\Phi \\
d\Lambda
\end{pmatrix},
\]

(b) \[
\begin{pmatrix}
\frac{dW}{d\Lambda} \\
\frac{d\Lambda}{d\Lambda}
\end{pmatrix}
\rightarrow
\begin{pmatrix}
d\overline{W}' + dW \\
dW'\Lambda + d\Lambda + \Lambda dW
\end{pmatrix}
= \begin{pmatrix}
d\Psi \\
dX
\end{pmatrix},
\]

(c) \[
\begin{pmatrix}
\frac{dY}{d\Lambda} \\
\frac{dX}{d\Lambda}
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\overline{Z}'dYZ \\
Z'dXZ
\end{pmatrix}
= \begin{pmatrix}
d\Phi \\
d\Psi
\end{pmatrix}.
\]

The Jacobians are \(|\det Z|^{-2p}, |\det Z|^{2(2p+2)}\) respectively

\[
c_1 \prod_{i=1}^p \lambda_i \prod_{i<j} (\lambda_i^2 - \lambda_j^2) \text{ for (a), (c) respective (b). Since}
\]

\[
\text{tr}(\Theta) = 2 \text{ tr}(\Phi) = 2 \text{ tr}(\overline{Z}'Z) \text{ and } |\det 0| = |\det \overline{Z}'Z|^2 \prod_{i=1}^p (1 - \lambda_i^2)
\]

(4.1) is transformed to the distribution

\[
c_2 \cdot |\det \overline{Z}'Z|^{N-p}
\]

\[
\exp\left\{-\frac{1}{2} \text{ tr}(\overline{Z}'Z)\right\} \prod_{i=1}^p \lambda_i (1 - \lambda_i^2)^{(N-2p-1)/2} \prod_{i<j} (\lambda_i^2 - \lambda_j^2) dZ \otimes d\lambda
\]
on $GL_+ (\mathbb{C}^p) \times \Omega$. Integrating over $Z \in GL_+ (\mathbb{C}^p)$, we get the distribution of $f_1 = \lambda_1^2, \ldots, f_p = \lambda_p^2$:
\begin{equation}
(4.4) \quad c_3 = \frac{\prod_{i=1}^{p} (1 - f_i)^{(N-2p-1)/2} \prod_{i<j} (f_i - f_j) df_1 \cdots df_p}{N-2p-1/2}.
\end{equation}

On $\Omega = \{(f_1, \ldots, f_p) \in \mathbb{R}^p | 1 > f_1 > \ldots > f_p > 0\}$. Formula (13) in [1], p. 324, for $p_1 = p-1$ and $p_2 = p$ gives the normings constant $c_3$, namely,
\begin{equation}
(4.5) \quad c_3 = \frac{\prod_{i=1}^{p-1} (1 - f_i)^{(N-2p-1)/2} \Gamma(\frac{1}{2}(N-1-i))}{\Gamma(\frac{1}{2}(N-p-1)) \Gamma(\frac{1}{2}(p-1)) \Gamma(\frac{1}{2}(p+1-i))}.
\end{equation}
REFERENCES


Office of Naval Research Contract N00014-75-C-0442 (NR-042-034)


# Report Documentation Page

<table>
<thead>
<tr>
<th>Report Number</th>
<th>33</th>
</tr>
</thead>
<tbody>
<tr>
<td>Title</td>
<td>Canonical Correlations with Respect to a Complex Structure</td>
</tr>
<tr>
<td>Author(s)</td>
<td>Steen A. Andersson</td>
</tr>
<tr>
<td>Performing Organization Name and Address</td>
<td>Department of Statistics, Stanford University, Stanford, California</td>
</tr>
<tr>
<td>Controlling Office Name and Address</td>
<td>Office of Naval Research, Statistics &amp; Probability Program Code 436, Arlington, Virginia 22217</td>
</tr>
<tr>
<td>Monitoring Agency Name and Address (If different from Controlling Office)</td>
<td></td>
</tr>
<tr>
<td>Report Date</td>
<td>July 1978</td>
</tr>
<tr>
<td>Number of Pages</td>
<td>10</td>
</tr>
<tr>
<td>Distribution Statement (of this Report)</td>
<td>Approved for public release; distribution unlimited.</td>
</tr>
<tr>
<td>Distribution Statement (of the abstract entered in Block 20, if different from Report)</td>
<td></td>
</tr>
<tr>
<td>Supplementary Notes</td>
<td>Issued also as Technical Report No. 132 under National Science Foundation Grant MPS 75-09450 - Department of Statistics, Stanford University</td>
</tr>
<tr>
<td>Key Words</td>
<td>Canonical correlations, complex structure, maximal invariants, distribution of empirical canonical correlations with respect to complex structure</td>
</tr>
<tr>
<td>Abstract</td>
<td>Suppose a 2p-variate multivariate normal distribution is of the form of a p-variate complex distribution. The set of such distributions is invariant with respect to a group of linear transformations. The invariants of the set of all 2p-variate distributions with respect to this group are obtained and interpreted. The distribution of the sample invariants is found.</td>
</tr>
</tbody>
</table>