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ANALYSIS OF THREE-DIMENSIONAL OPTIMAL Evasion WITH LINEARIZED K-ETC(U)
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J SHINAR, Y ROTSZTEIN, E BEZNER
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by

J. Shinar, Y. Rotsztein and E. Bezner

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WITH LINEARIZED KINEMATICS

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ANALYSIS OF THREE-DIMENSIONAL OPTIMAL EVASION WITH LINEARIZED KINEMATICS

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Technion - Israel Institute of Technology
Haifa, Israel

Abstract
Three-dimensional optimal missile avoidance is analyzed with a linearized kinematic model. The solution requires maximum load factor and the problem is reduced to optimal roll position control having two phases: (1) orientation of the lift vector into the optimal evasion plane, (2) rapid 180° roll maneuvers governed by a switch function. For circular missile vectorgram the plane of optimal evasion is perpendicular to the line of sight. Evading from roll stabilized missiles of rectangular vectorgram, further advantage can be taken maximizing the target-missile maneuver ratio. Bounded roll-rate reduces the miss distance but does not affect the optimal evasive maneuver structure.

Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
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<tr>
<td>a</td>
<td>lateral acceleration.</td>
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<td>A</td>
<td>system matrix (14).</td>
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<td>A1</td>
<td>single channel matrix (15).</td>
</tr>
<tr>
<td>B</td>
<td>control matrix (18).</td>
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<tr>
<td>H</td>
<td>variational Hamiltonian.</td>
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<td>J</td>
<td>pay-off function (48).</td>
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<td>K</td>
<td>constant of true proportional navigation (9).</td>
</tr>
<tr>
<td>m</td>
<td>miss distance (46).</td>
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<td>N</td>
<td>effective prop. nav. ratio (9).</td>
</tr>
<tr>
<td>PR</td>
<td>roll rate (control variable).</td>
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<tr>
<td>R</td>
<td>relative distance</td>
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<tr>
<td>S1, S2</td>
<td>switch functions (55), (68).</td>
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<tr>
<td>t</td>
<td>time.</td>
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<tr>
<td>ti</td>
<td>nominal time of flight (8).</td>
</tr>
<tr>
<td>u</td>
<td>control vector.</td>
</tr>
<tr>
<td>V</td>
<td>velocity.</td>
</tr>
<tr>
<td>x1</td>
<td>state vector components.</td>
</tr>
<tr>
<td>y1,z</td>
<td>components of R, perpendicular to the initial line of sight (12).</td>
</tr>
<tr>
<td>(\tau)</td>
<td>dynamic similarity parameter (92).</td>
</tr>
<tr>
<td>(\delta_{1,2})</td>
<td>costate dependent coefficients (52), (53).</td>
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<td>(\zeta_{1,2})</td>
<td>normalized time-to-go (65).</td>
</tr>
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<td>(\lambda_1)</td>
<td>costate vector components.</td>
</tr>
<tr>
<td>(\mu)</td>
<td>missile-target maneuver ratio (30).</td>
</tr>
<tr>
<td>(\mu_1)</td>
<td>&quot; &quot; &quot; &quot; &quot; of a single channel (33).</td>
</tr>
<tr>
<td>(t)</td>
<td>missile time constant.</td>
</tr>
<tr>
<td>(\Phi)</td>
<td>roll angle.</td>
</tr>
<tr>
<td>(\Phi_T)</td>
<td>normalized target roll rate limit.</td>
</tr>
<tr>
<td>(\gamma)</td>
<td>azimuth angle.</td>
</tr>
<tr>
<td>(\Omega)</td>
<td>angular velocity.</td>
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</table>

Subscripts

<table>
<thead>
<tr>
<th>Symbol</th>
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<tbody>
<tr>
<td>c</td>
<td>commanded value.</td>
</tr>
<tr>
<td>f</td>
<td>final value.</td>
</tr>
<tr>
<td>i</td>
<td>index.</td>
</tr>
<tr>
<td>M</td>
<td>missile.</td>
</tr>
<tr>
<td>r</td>
<td>required value.</td>
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<tr>
<td>S</td>
<td>line of sight.</td>
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<tr>
<td>T</td>
<td>target.</td>
</tr>
<tr>
<td>o</td>
<td>initial value.</td>
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<tr>
<td>(*)</td>
<td>column vector.</td>
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1 Introduction

The missile-aircraft pursuit-evasion problem can be either formulated as a zero-sum differential game, or decomposed to two reciprocal optimal control problems whose respective objectives are to determine: (a) optimal guidance laws against maneuvering targets; (b) optimal evasive maneuvers from guided missiles.

Regardless of the formulation, the problem is of an inherent complexity. The relative pursuer-evader kinematics are expressed by a nonlinear three-dimensional vector equation. Both vehicles' dynamics are expressed by sets of nonlinear differential equations. Moreover the guidance law of the pursuer is implemented by a rather complicated transfer function. The exact solution for each of the alternative formulations requires the solution of a nonlinear two-point boundary value problem of very high dimension. The computation of such a solution, although feasible, is so time-consuming that it makes this approach impractical for systematic studies.

For a systematic analysis which is necessary to create an insight into this complex problem, simplified analytic solutions are required. Important simplification can be achieved by: (a) neglecting guidance dynamics. (b) restricting the motion in a plane. (c) trajectory linearization.

It turns out that the attractive assumption, made by neglecting the dynamics of the pursuer, yields seriously misleading results1.2,3. As a consequence of this assumption, the direction of the optimal evasive maneuver is constant and is determined by the initial or terminal conditions. Moreover, if the pursuer's maneuverability is sufficient, the final miss distance is always zero3.

Most analytic studies in the past used two-dimensional models4-10. Whenever guidance dynamics were considered (even if by an approximation of a first order time constant or a pure time delay) an oscillating or "bang-bang" structure of the optimal evasive maneuver became apparent. It was also shown that optimal evasion can guarantee non-zero miss distance even from a pursuer of unlimited maneuverability7 or from one of an optimal guidance strategy9. It has been indicated however10 that the validity of 2-D analysis is limited to near "head-on" or "tail-chase" engagements. For other initial conditions three-dimensional analysis is required. The same study also demonstrated that, due to the

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"bang-bang" structure of the optimal evasive maneuvers, trajectory linearization is a good approximation for a wide range of parameters.

The objective of this paper, motivated by the above mentioned results, is to analyze the problem of optimal missile avoidance using a three-dimensional linearized kinematic model. Analysis is based on the following set of assumptions:

1. Pursuer and evader are both considered as constant speed mass points.
2. The pursuer is a homing missile launched against an initially non-maneuvering evader (target) in a collision course.
3. Relative pursuer-evader trajectory can be linearized around the initial line of sight.
4. Pursuer and evader both have perfect information on the relative state.
5. Gravity can be neglected for both vehicles (not effecting relative trajectory).
6. The pursuing missile has two identical and independent guidance channels to execute proportional navigation in two perpendicular directions in a plane normal to the line of sight (true proportional navigation).
7. The dynamics of each guidance channel is assumed to be (for sake of simplicity only) of first order.
8. The first five assumptions and the effects of more complex pursuer dynamics are discussed in detail in Ref. 10.

Based on the above listed hypotheses the 3-D missile avoidance is formulated as a fixed duration optimal control problem maximizing a terminal payoff (the square of the miss distance). The control variable is the lateral acceleration vector of the evading airplane. This acceleration is perpendicular to the relative velocity vector, its magnitude is bounded by the maximum lift factor (or maximum lift) and its direction is controlled by the airplane's roll orientation.

First a mathematical model of unbounded missile maneuverability and infinite airplane roll-rate is used. This linear formulation leads to a closed form solution and provides the basic insight into the problem. In consecutive steps saturation of missile acceleration and realistic roll dynamics of the evading airplane are introduced.

The solutions obtained by the linearized 3-D model are compared both to the prediction of a 2-D linearized analysis and to results of complete non-linear (6 degrees of freedom) simulation.

II Mathematical Modelling

Three-dimensional vector formulation

A three-dimensional pursuit-evasion is described by the vector equations

\[ \dot{X}_P = \dot{V}_P - \dot{V}_R \]
\[ \dot{X}_M = \frac{\dot{V}_M \times \dot{V}_P}{|\dot{V}_P|^2} \]

The acceleration command of the pursuing missile is given by Assumption 6 as

\[ \dot{V}_M = \frac{(\dot{V}_P \times \dot{X}_M)}{|\dot{V}_P|} \]

while the actual acceleration is determined (see Ass. 7) by

\[ \dot{V}_M + \dot{V}_R = \dot{V}_N \]

The acceleration of the constant speed evader (the target) is normal to its vector velocity

\[ \dot{V}_T = (\dot{V}_T \times \dot{V}_T) \]

Trajectory linearization around the initial collision course (Ass. 2 & 3) yields

\[ \dot{X}(t) = \dot{V}_T - \dot{V}_R = \text{const} \]

and as a consequence

\[ \dot{X}(t) \cdot \dot{X}_0 - V_{RT} = \frac{V_R(t_f-t)}{V_R} \]

determining the final time of the pursuit by

\[ t_f = \frac{|\dot{X}_0|}{V_R} \]

Substituting (6) and (7) into (3) and defining

\[ K_N \triangleq \frac{\dot{X}_N}{V_R} \]

yields

\[ \dot{V}_M = \frac{-N v}{(t_f-t)^2} [\dot{X}(t)+n(t_f-t)\dot{X}(t)] \]

The system of differential equations (1), (4), (5), (8) and the linearized feedback relation (10) determine the 9 components of the vectors \( \dot{X}(t) \), \( \dot{V}_T(t) \) and \( \dot{V}_R(t) \), if initial conditions and the target angular velocity vector \( \dot{V}_T(t) \) are given. In the problem of optimal missile avoidance this last quantity is the control variable.

Non-dimensional scalar equations - linear case

The initial collision plane (Ass. 2) is taken as plane of reference for the direction of the vectors \( \dot{X}_0, \dot{V}_M \) and \( \dot{V}_R \) (see Fig. 1).

![Fig. 1. Initial collision geometry](image)

By choosing the X axis of the coordinate system to coincide with the initial line of sight, only those components of the relative motion which are normal to this direction have to be considered. The linearized equation of motion along this axis is already solved by (7). The state vector is reduced to be of six components

\[ \dot{x}^T = (x_1 \ldots x_6) \hat{a} (y, z, \dot{y}, \dot{z}, \dot{y}, \dot{z}) \]

"y" and "z" being the relative displacements.
perpendicular to the initial line of sight (see Fig. 2)

\[
\begin{align*}
  y &= y_T - y_N \\
  z &= z_T - z_N
\end{align*}
\]  
(12)

The decoupled structure of the state matrix is preserved with

\[
\begin{bmatrix}
  0 & 1 & 0 \\
  0 & 0 & -1 \\
  \frac{N'}{(\xi f'-\xi)} & \frac{N'}{(\xi f'-\xi)} & -1 \\
\end{bmatrix}
\]  
(24)

Nonlinear effects

The state equation (13) or (21), describing system dynamics, is linear due to the implicit assumptions of unlimited missile maneuverability and infinite target roll-rate. A more realistic mathematical model has to consider the constraints on these variables. The state equation including such effects will no longer be linear.

Limited missile maneuverability

When missile maneuverability constraints are taken into account it is necessary to redefine the state vector and the state equation. The components of the lateral acceleration, \( \dot{y}_N \) and \( \dot{z}_N \), are to be replaced by their required value \((\dot{y}_N)_{\text{req}}\) and \((\dot{z}_N)_{\text{req}}\) in nondimensional form

\[
\begin{align*}
  \dot{x}_3 &= \frac{(\dot{y}_N)_{\text{req}}}{(\dot{y}_p)_{\text{max}}} \\
  \dot{x}_6 &= \frac{(\dot{z}_N)_{\text{req}}}{(\dot{z}_p)_{\text{max}}}
\end{align*}
\]  
(25-26)

As these variables are not affected by the constraint, the differential equations for...
\[ \frac{dx_3}{dt} \text{ and } \frac{d^2x_3}{dt^2} \text{ remain unchanged.} \]

The relation between the components of the acceleration, which are subject to constraints and their required value can be expressed by the nonlinear saturation function defined as

\[
\text{sat}\left( \frac{a}{b} \right) = \begin{cases} \frac{a}{b} & \text{if } |a| < |b| \\ \text{sign}(a) \frac{b}{a} & \text{if } |a| > |b| \end{cases}
\]

(27)

Consequently the state equation will not have the linear form of (21) but has to be written as

\[
\frac{dx}{dt} = F(x, t) + Bu
\]

(28)

For such saturation two alternative formulations exist expressed by two different vectograms:

a. Circular (isotropic) vectogram (see Fig. 4) showing that the constraint of maneuverability applies to the resultant lateral acceleration

\[
a_M^2 = y_M^2 + z_M^2 = \left(\frac{a_{\text{max}}}{a_{\text{max}}} \right)^2 \text{sat}\left( \frac{\gamma_M}{a_{\text{max}}} \right) + \left(\frac{\gamma_M}{a_{\text{max}}} \right)^2
\]

(29)

\[\text{Fig. 4. Circular vectogram of missile acceleration.}\]

Such vectogram represents the maneuverability of a thrust vector controlled (T.V.C.) missile or of a cruciform configuration with unknown roll orientation. In this model it is assumed that saturation of both guidance channels takes place simultaneously.

By introducing the missile-target maneuver ratio which is one of the similarity parameters of the problem\(^1\), as

\[\mu = \frac{\gamma_T}{\gamma_M} \text{max}\]

(30)

(29) can be written using (25) and (26) in a non-dimensional form

\[
\gamma_T^2 + \gamma_M^2 = \frac{d^2y}{dt^2} + \frac{d^2z}{dt^2} = \mu^2 \text{sat}\left( \frac{\gamma_T}{\gamma_M} \right)
\]

(31)

b. Rectangular (square) vectogram (see Fig. 5) indicating that saturation may occur in each guidance channel separately:

\[
\gamma_T = \text{sat}\left( \frac{\gamma_T}{\gamma_M} \right)
\]

(32)

with similar relation for \(z_M\). Vectogram of this type represents a roll stabilized cruciform missile with known roll orientation.

\[\text{Fig. 5. Vectogram of cruciform missile.}\]

Defining the relative maneuverability of the guidance channels as

\[\nu_1 = \frac{\gamma_T}{\gamma_M} \text{max} = \frac{\gamma_T}{\gamma_M} \text{max}\]

(33)

enables to write (32) in nondimensional form

\[
\frac{d^2y}{dt^2} = \nu_1 \text{ sat}\left( \frac{\gamma_T}{\gamma_M} \right)
\]

(34)

A similar expression holds for the second channel.

For rectangular vectogram the actual missile target maneuver ratio \(\mu\) depends on the orientation of the target acceleration. It can be easily seen that

\[\nu_1 \leq \mu \leq \sqrt{2} \nu_1\]

(35)

**Limited target roll-rate**

Realistic description of the evader's roll dynamics requires that its roll orientation \(\phi_T\) be considered not as a control but an additional state variable.

\[x_T = \phi_T\]

(36)

The roll dynamics can be expressed by several alternative formulations:

1) The control variable is the target's roll-rate

\[x_T = P_T (\phi_T)_{\text{max}}\]

subject to the constraint

\[|P_T| < 1\]

(38)

2) Closed loop roll control based on the required roll orientation

\[x_T = k_l (x_T - x_T)\]

subject to saturation

\[x_T = \text{sat}\left( \frac{\gamma_T}{\gamma_M} \right)\]

(40)
Complete description of the evading airplane's roll dynamics controlled by an aerodynamic rolling moment

\[ \dot{\alpha}_r + e_1 \dot{\alpha}_r = \eta \]

(41)

assuming both

\[ |\dot{\alpha}_r| \leq (\dot{\alpha})_{max} \]

(42)

and

\[ |\alpha_r| \leq (\alpha)_{max} \]

(43)

The common feature in all these formulations is the limited roll-rate expressed by \((\dot{\alpha})_{max}\). In non-dimensional form this constraint is expressed by

\[ \dot{\alpha}_{\text{r}}' / \dot{\alpha} = \dot{\alpha}_{\text{r}}' = \dot{\alpha}_{\text{r}} / \dot{\alpha} = \alpha_r / \alpha \]

(44)

defining \(\dot{\alpha} = \) another similarity parameter of the problem.

Considering target roll orientation \(\hat{\alpha}_r\) as a state variable makes the overall dynamic system nonlinear even in the absence of missile saturation. In this case the state equation has the coupled form of

\[ \frac{d^2 \alpha'}{dt^2} + \dot{\alpha}' = 0 \]

(45)

with control vector \(\dot{\alpha}'\) defined according to one of the alternative formulations.

Formulation of the Optimal Control Problem

The objective of the missile avoidance is to maximize the survivability of the evading aircraft, assuming uniformly performing warhead and proximity fuse allows to determine the payoff as the absolute value of the square of the miss distance. For linearized kinematics the last one is expressed as

\[ \alpha' = \alpha' (\hat{\alpha}_r) + \alpha (\hat{\alpha}_r) \]

(46)

with \(\alpha_r\) given by (6).

The optimal missile avoidance with a linearized kinematic model can therefore be formulated as a fixed duration optimal control problem maximizing a terminal payoff (problem of Mayer).

Using nondimensional variables the formulation (for unlimited missile maneuverability and target roll-rate) in the following:

\[ \dot{\alpha}' = \dot{\alpha}' (\hat{\alpha}_r) + \alpha (\hat{\alpha}_r) \]

(47)

which maximizes the payoff

\[ \dot{\alpha}' (\hat{\alpha}_r) + \alpha (\hat{\alpha}_r) \]

(48)

for the fixed value of \(\hat{\alpha}_r\) given by

\[ \dot{\alpha}' = \frac{\dot{\alpha}_r}{\dot{\alpha}_r} = \frac{\dot{\alpha}_r}{\dot{\alpha}_r} \]

(49)

If missile saturation or and target's roll dynamic are considered a similar formulation holds with (21) replaced by (21b) or (45) and stating the appropriate control structure with the additional constraints.

Formal Solutions

Linear Case

For the assumptions of unlimited missile maneuverability and infinite target roll rate, stated in the previous section, the variational Hamiltonian

\[ H(x, \dot{x}, \dot{\alpha}_r, \hat{\alpha}_r) = \dot{x}^T (\dot{\alpha}_r, \hat{\alpha}_r, \hat{\alpha}_r) \]

(50)

can be written, separating the part independent of the control variables, as

\[ H = H_0 (x, \dot{x}, \dot{\alpha}_r, \hat{\alpha}_r) + \eta_1 (\dot{\alpha}_r, \hat{\alpha}_r, \hat{\alpha}_r) \]

(51)

with

\[ \eta_1 = \cos \gamma (\dot{\alpha}_r, \hat{\alpha}_r, \hat{\alpha}_r) \]

(52)

\[ \eta_2 = \cos \gamma (\dot{\alpha}_r, \hat{\alpha}_r, \hat{\alpha}_r) \]

(53)

The optimal control variables \(\dot{\alpha}_{\text{r}}'\) and \(\dot{\alpha}_r'\) have to maximize the Hamiltonian, yielding

\[ (\dot{\alpha}_r)' = \frac{1}{2} (|\text{sign} (\dot{\alpha}_r) + 1) \]

(54)

with

\[ (\dot{\alpha}_r)' = \frac{1}{2} \text{sign} (\dot{\alpha}_r) \]

(55)

and

\[ (\dot{\alpha}_r)' = \dot{\alpha}_r - \dot{\alpha}_r \]

(56)

Substituting (56) into (55) leads to

\[ \dot{\alpha}_r = k (\dot{\alpha}_r^2 + \dot{\alpha}_r^2) \geq 0 \]

(57)

\[ k \text{ being a positive constant of proportionality determined by} \]

\[ k^2 = (\dot{\alpha}_r^2) (\dot{\alpha}_r^2) \]

(58)

Equations (54) and (57) indicate directly that for optimal missile avoidance maximum lateral load factor has to be always used.

The components of the costate vector \(\lambda\) involved in \(S_1\), \(S_2\) and \(S_3\) are determined by the adjoint equation

\[ \frac{\alpha'}{\eta'} = - \frac{\dot{\alpha}_r}{\dot{\alpha}_r} (\dot{\alpha}_r) \]

(59)

with the terminal conditions

\[ \lambda (T) = \frac{\frac{\dot{\alpha}_r}{\dot{\alpha}_r} (\dot{\alpha}_r)}{\frac{\dot{\alpha}_r}{\dot{\alpha}_r} (\dot{\alpha}_r)} \]

(60)

resulting in

\[ \lambda (T) = - 2(\dot{\alpha}_r) \dot{\alpha}_r - 2(\dot{\alpha}_r) \dot{\alpha}_r \]

(61)

\[ \lambda (T) = 0 \]

(62)

(63)

(64)

(65)
The system of equations (64) can be reduced to two identical scalar differential equations of the form
\[ \ddot{\theta} + \frac{d^2\theta}{d\tau^2} + N^i \frac{d\theta}{d\tau} = 0 \quad i = 3, 6 \] (66)
which were solved in a closed form in Ref. 10. This solution combined with (65) yields
\[ \lambda_i(0) = \lambda_{i0} e^{-b \theta} \quad i = 1, 4 \] (67)
\[ \lambda_{i+1}(0) = \lambda_{i+10} e^{-b \theta} \quad i = 1, 4 \] (67)
\[ \lambda_{i+2}(0) = \lambda_{i+20} e^{-b \theta} \quad i = 1, 4 \] (67)
\[ P_i(0) \] are functions of \( \theta \) depending on \( N^i \) only.

For integer values of \( N^i \) these functions are polynomials of the order \((N^i-2)\) (see Table A-1 in Ref. 10).

As a consequence of (67) and (61), the time dependent part of \( \lambda_2 \) and \( \lambda_5 \) are identical:
\[ \lambda_2(\tau) = -2 y \tilde{S}_2(\tau' - \tau) \] (68)
\[ \lambda_5(\tau) = -2 y \tilde{S}_2(\tau' - \tau) \] (68)

Substituting (68) into (52) and (53) yields
\[ \delta_1 = 2 \cos \theta \tilde{y} \cos \Phi - \tilde{z} \sin \Phi \tilde{y}_2(\tau' - \tau) \] (69)
\[ \delta_2 = -2 \tilde{y} \tilde{y}_2 \tilde{z} \sin \Phi \tilde{y}_2(\tau' - \tau) \] (70)

The expressions in the brackets are the components of the miss distance vector: the first in the plane of the initial collision and the second in the direction perpendicular to this plane.

As by (54) and (57)
\[ \dot{\theta}(\tau)^* = 1 \] (71)
maximizing the Hamiltonian is equivalent to maximizing \( S_1 \). Inspection of eq (57), (69) and (70) makes it obvious that for any given miss distance, maximization is achieved only by
\[ \delta_1 = 0 \] (72)
and as a consequence
\[ t g(\theta(\tau))^* = 0 \] (73)
This clearly indicates that the direction of the optimal evasive maneuver has to be perpendicular to plane of initial collision. The optimal roll orientation, either \( \phi_0 \) or \( \phi_\alpha \), is determined by
\[ \cos(\phi_0(\tau)) = -\text{sign}(S_2(\tau' - \tau)) \] (74)

is determined by the sign of the switch function \( S_2 \), which is identical to the one obtained in 2-D analysis.10

Case of circular missile vectogram

If missile maneuver constraints are taken in consideration the linear state equation (21) is replaced with one of a nonlinear form of (28). Nevertheless, this nonlinearity does not alter the structure of the optimal solution. As the controls appear separately in (28), the Hamiltonian preserves its separated form of (51). The control dependent part is not affected by the saturation type nonlinearity and the optimal control functions are given in this case also by (52) and (53). Moreover, as saturation takes place in both guidance channels simultaneously, the time dependent parts of \( \lambda_2 \) and \( \lambda_5 \) remain identical (although they are different from the costate variables of the linear case). As a consequence both (71) and (74) hold, yielding the same type of "bang-bang" maneuver perpendicular to the initial collision plane as for unlimited missile maneuverability. The switch function governing this maneuver is however different from the one obtained in closed form for the linear problem.

When missile saturation takes place in the terminal phase of the pursuit (it was shown12 that it always occurs in this phase), the state equation of the system, and as a consequence the adjoint equation, both change. As both components of missile acceleration are constants in the saturated phase, the submatrix \( A_2(\tau) \) in (24) is modified. For \( \tau \rightarrow T \), its second line will contain only zeroes. The adjoint equations are given in this case as a function of normalized time-to-go \( \theta \), as follows:
\[ \frac{d\lambda_i}{d\theta} = N^i \gamma^2 \lambda_{i+2} \] (75)
\[ \frac{d\lambda_{i+1}}{d\theta} + N^i \gamma \lambda_{i+2} \] (76)
\[ \frac{d\lambda_{i+2}}{d\theta} = \lambda_{i+2} \] (77)
with the initial conditions (61).

From (77) and (61) it is obvious that for \( \theta \in \theta_0 \), \( \lambda_2 \) and \( \lambda_5 \) are both zero. This fact confirms that in the saturated phase the required accelerations have no influence on the solution. Consequently
\[ \lambda_1(\theta \in \theta_0) = (\lambda_1(0) - \text{const}) \] (78)

It is easy to see that, due to the monotonicity of \( \lambda_2 \) and \( \lambda_5 \), no switch can occur when the missile is saturated. It is also important to note that, as \( \tilde{y} \tilde{y}_2(\tau) \) in (28) has no discontinuity, an optimal maneuver is however different from the one obtained in closed form for the linear problem.

The time of saturation is one of the unknowns and has to be determined with the complete solution of the costate variables, by solving a two-point boundary value problem. Fortunately, due to the "bang-bang" type solution a simple and efficient search technique developed in Ref. 10 can be used as an alternative. The results obtained are identical with those of a previous 2-D analysis.10 They show that the optimal switch function and the resulting miss distance both depend on the missile-target maneuver ratio \( \mu \). The dependence of the normalized miss distance can be expressed approximately as
\[ N^i(\tilde{\eta}, \tilde{\tau}, \mu) \approx N^i(\tilde{\eta}, \tilde{\tau}, \mu) + b(N^i, \tilde{\tau}, \mu)/\mu^2 \] (80)

Case of rectangular missile vectogram

In this case, representing a roll stabilized cruciform missile with known roll orientation, saturation occurs in each guidance channel separately. The optimal control functions are determined, as for the previous case, by (52) and (53). As (57) is not influenced by the saturation, (71) remains valid. However, the time dependent parts of \( \lambda_2 \) and \( \lambda_5 \) are no more identical and the optimal target roll
orientation given by (56)

$$\tg(\phi_T)^* = \frac{\delta_1}{\delta_2} \cos \chi_0 \frac{\lambda_1 \sin \chi - \lambda_2 \cos \chi}{\lambda_1 \sin \chi - \lambda_2 \cos \chi}$$

cannot be determined by simple inspection.

It is, however, intuitively obvious, in view of (80), that for maximum miss distance the effective missile-target maneuver ratio has to be minimum. This problem has a straightforward geometric solution shown in Fig. 6. It yields, for $0 < \psi_T / \psi_A$, which is the relevant one for cruciform missiles,

$$\tg(\phi_T)^* = \cos \chi_0 \tg_0$$

This is a suboptimal solution, which maximizes the projection of target acceleration on the more susceptible guidance channel, but it can be easily implemented. Moreover it takes a definite advantage of the known missile roll orientation by providing always

$$v_{\text{eff}} \leq \sqrt{2} v_1$$

$$\text{Fig. 6. Minimization of the effective missile-target maneuver ratio.}$$

Case of limited target roll-rate

The "bang-bang" solution obtained in the previously discussed cases assumes an infinite roll-rate of the evading aircraft. Whenever the real limitation on the target roll-rate is taken in account the roll orientation $\phi_T$ becomes an additional state variable on the target roll-rate is taken into account the roll orientation is.

The system is controlled by

$$\dot{\eta} = A\eta + Bp$$

The initial conditions $\eta_0$ are given and the terminal state is not specified.

Find the optimal control $\eta$ subject to the constraints (47) and (38) that maximizes the terminal payoff (48) for the fixed $\psi_T$ given by (49).

The variational Hamiltonian of the problem is

$$H = \frac{1}{2} \lambda_1^T T_0 = H_0(\chi, \psi_T) + \frac{1}{2} S_1(\chi, \psi_T) + \frac{1}{2} S_2(\chi, \psi_T) + \psi_T + \gamma$$

with $S_1$ written explicitly in (55).

The first optimal control variable is $\psi_T^*$ given as previously, by (71). The second control component maximizing the Hamiltonian has to be ($f \lambda^m$)

$$\left(\psi_T^*\right) = \text{sign} \lambda_T$$

The time derivative of the costate variable is

$$\lambda_T^* = -\frac{3h^2}{2x_0} - \frac{1}{4} S_1 h x_0$$

yielding (as a function of the normalized time-to-go)

$$\lambda_T^* = \frac{1}{2} \delta_1(1, \lambda_2, \lambda_3) \cos \delta_2(1, \lambda_2, \lambda_3) \sin \delta_3$$

with the initial condition ($\lambda_T^* = 0$).

A singular control is possible if $\lambda_T \lambda_T^* / d \lambda_T > 0$, requiring by (80)

$$\tg(\phi_T)^* = \frac{\delta_1(1, \lambda_2, \lambda_3)}{\delta_2(1, \lambda_2, \lambda_3)}$$

Assuming that (91) holds, the singular value of $\psi_T^*$ can be obtained from the second derivative. This value turns out to be zero. Comparing (91) and (56) indicates that the required roll orientation, predicted by (56) under the assumption of an infinite roll-rate, does not change by the introduction of the roll rate constraint.

Such steady state ($\psi_T = 0, \lambda_T > 0, \tg(\phi_T)^* = \delta_1 / 2$) is however very unlikely. The mutual relations (85), (88) and (89), shown in the block diagram of Fig. 7, predict limit-cycle type oscillations around the equilibrium value of (91). These oscillations may damp out if higher order roll dynamics are introduced in the model. Roll oscillation of small amplitude have no appreciable effect on the solution. The major effect is the reduction in the optimal miss distance as the value of the normalized maximum roll-rate decreases. This phenomenon was already predicted by the 2-D analysis.

Fig. 7. Roll rate as control variable.

V Concluding Remarks

In this paper the problem of the 3-D optimal missile avoidance is analyzed in nondimensional form for realistic missile and aircraft models using linearized kinematics. The solution, derived by rigorous mathematical treatment, is presented in simple geometric terms, providing a clear insight
into this inherently complex problem.

First, it is shown that the optimal evasion does not take place in the initial collision plane. Thus the effort in 3-D analysis is justified. Nevertheless, the optimal evasive maneuver is confined to a plane which, for circular missile vectorgram, is perpendicular to the initial plane of collision. Eviding from a roll stabilized cruciform missile, represented by a rectangular vectorgram, further advantage can be taken by choosing a maneuver plane which minimizes the missile-target maneuver ratio.

The solution of the optimal control problem, maximizing the miss distance, is a "bang-bang" type maneuver with the continuous use of maximum load factor of the evading airplane. It can be therefore reduced to an optimal roll-position control problem of two consecutive phases: (1) Orienting the airplane lateral acceleration vector into the plane of optimal evasion. (2) Changing the direction of this acceleration, which has to be maximal, by rapid roll maneuvers of 180° in accordance with an optimal switch function.

\[
\begin{array}{c|cccccc}
M^* & 3-D \text{ Linearized} & 3-D \text{ non-linear} \\
15 & N = 4.0 & 10 & \dot{\gamma} = 0.13 & 1 & \dot{\gamma} = 12.5 & 0.5 \\
& & & \mu = 0.4 & & \phi = 4.2 & \\
0 & 15 & 30 & 45 & 60 & 75 & 90 \\
\end{array}
\]

\[ X(0) [\text{deg}] \]

Fig. 8. Comparison of 3-D and 2-D tactics.

In Fig. 8 the optimal miss distances, obtained by a 2-D analysis with exact (nonlinear) kinematics and the present 3-D study based on a linearized model, are compared. The comparison was carried out by a 6 degrees of freedom simulation which used the optimal control functions derived by the respective studies. The comparison shows that, for the set of nondimensional parameters chosen, the 3-D tactics have a definite advantage if the initial target azimuth angle \( \gamma_0 > 45° \).

The existence of an optimal maneuver plane enables to use some results of the 2-D analysis which avoids the solution of two-point boundary value problems, which seems a priori necessary if missile saturation and limited airplane roll-rate are considered. By the way, it can be noted that the optimal miss distances predicted by the three-dimensional linearized kinematic model compare very well with results of complete (nonlinear) 6 degrees. Such a good agreement is not unexpected.

It has been shown previously\(^{10}\), that the "bang-bang" nature of the optimal maneuver seems to justify the assumption of linearized kinematics. However it is proposed to distinguish between two phases when defending the choice of a linearized kinematic model: a) a priori justification can be based on examination of the value of the "dynamic similarity parameter" introduced for nonlinear kinematics in a recent report\(^{12}\) and defined as

\[
\dot{v}_T = \frac{(v_T)_{\text{max}}}{v_T} \tag{92}
\]

If this parameter is sufficiently small use of linearized kinematics can be attempted for the analysis. b) a posteriori, it has to be verified that the solution of the linearized kinematic model does not predict excessively long maneuvers in one direction. If it does happen, trajectory linearization is not appropriate for the specific problem.

References


References (cont'd)

