UNIQUENESS OF SOLUTIONS TO HYPERBOLIC CONSERVATION LAWS.

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ABSTRACT

We study strictly hyperbolic systems of conservation laws. We consider the class \( K \) of solutions which lie in \( L^1 \) and satisfy the entropy admissibility criterion. We show under certain hypotheses that admissible piecewise Lipschitz solutions are unique within \( K \). We establish the \( L^2 \)-stability of classical solutions relative to perturbations in \( K \) and we show that admissible piecewise Lipschitz solutions to the quasilinear wave equation satisfy the entropy rate criterion relative to a broad subclass of solutions in \( K \).

Subject Matter Classification: 35160, 35165

Key Words: Hyperbolic conservation laws, initial value problem, entropy inequality

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SIGNIFICANCE AND EXPLANATION

It is known that conservative systems of differential equations which result from continuum mechanics (e.g. the equations of shallow water waves, fluid dynamics, magneto-fluid dynamics and certain elasticity problems) do not have unique solutions. Thus the problem arises of proving that systems of this type have only one physically meaningful solution. In this report we show that there exists at most one solution satisfying an entropy condition which generalizes the second law of thermodynamics.

The responsibility for the wording and views expressed in this descriptive summary lies with NRC, and not with the author of this report.
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1. Introduction

We consider strictly hyperbolic systems of conservation laws,

\[ \frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \quad -\infty < x < \infty. \]

Here \( u = u(x,t) \in \mathbb{R}^n \) and \( f \) is a smooth nonlinear mapping from \( \mathbb{R}^n \) to \( \mathbb{R}^n \). The condition of strict hyperbolicity requires that the Jacobian of \( \text{If} \) of \( f \) have \( n \) real and distinct eigenvalues:

\[ \lambda_1(u) < \ldots < \lambda_n(u). \]

By a weak solution we shall mean an element of \( L^\infty \cap BV(D) \), \( u \in \mathbb{R}^n \), which satisfies (1.1) in the sense of distributions. Here \( BV(D) \) denotes the space of functions which have locally bounded total variation in the sense of Cesari; i.e., the space of \( BV(\Omega) \) functions whose first order partial derivatives are locally finite Borel measures, cf. Section 2.

It is well known that the Cauchy problem for (1.1) does not have in general globally defined smooth solutions; the nonlinear structure of the eigenvalues leads to the development of discontinuities in the solution. On the other hand, uniqueness is lost within the broader class of weak solutions; it is possible for many weak solutions to share the same initial data. Thus the problem arises of identifying the class of stable weak solutions.

In this connection, we recall that a function \( \eta : D \to \mathbb{R} \) defined on an open domain \( D \subset \mathbb{R}^n \) is an entropy for (1.1) with entropy flux \( q : D \to \mathbb{R} \) if all smooth solution with range in \( D \) satisfy an additional conservation law of the form

\[ \frac{\partial \eta}{\partial t} + \frac{\partial q(u)}{\partial x} = 0. \]

Friedrichs and Lax [13] have observed that most of the conservative systems which result from continuum mechanics (in one and in several space dimensions) are endowed with a globally defined strictly convex entropy. We mention among others the equations of shallow water waves, fluid dynamics, magnetofluid dynamics, elasticity in certain special cases and the general symmetric hyperbolic system; these systems are recorded in [9] along with their corresponding entropies and constitutive assumptions. In addition Lax [26] has shown how to construct locally defined strictly convex entropies for arbitrary strictly hyperbolic systems (1.1) of two equations and globally defined strictly convex entropies for a broad class of systems (1.1) of two equations.

For the class of systems (1.1) endowed with a strictly convex entropy \( \eta \), Lax [26] and Kruskov [24] postulate the following entropy criterion: a weak solution \( u \) with range in \( D \) is admissible if

\[ \eta(u)_t + q(u)_x < 0 \]

in the sense of distributions. For the equations of fluid dynamics, minus the classical entropy density serves as \( \eta \) and minus the classical entropy flux density serves as \( q \). In the setting of fluid dynamics, equation (1.2) expresses the fact that entropy is conserved in smooth flows; the entropy of each fluid particle remains constant during its motion. The entropy inequality (1.3) expresses the second law of thermodynamics: the entropy of each fluid particle must increase upon crossing a shock front. We note that the entropy criterion has been postulated in order to characterize the stable solutions to a specific class of systems, the class of systems each of whose eigenvalues \( \lambda_j \) is either genuinely nonlinear or linearly degenerate in the sense of Lax [25], i.e. for each \( j \) either

\[ r_j \cdot V \lambda_j \geq 0 \quad \text{or} \quad r_j \cdot V \lambda_j \geq 0 \]

where \( r_j = r_j(u) \) denotes the right eigenvector of \( \text{If} \) corresponding to \( \lambda_j \).

This class of systems includes the aforementioned equations of shallow water...
waves, fluid dynamics, magneto-fluid dynamics and in certain special cases
elasticity. For systems with eigenvalues satisfying (1.4), Lax [26] has
shown that, in the case of solutions with moderate oscillation, the entropy
criterion is equivalent to the Lax shock conditions [25] which govern the
number and type of characteristics impinging on a shock wave. We note that
the Lax shock conditions are necessary for the stability of the solution in
the linearized sense and that they are designed to single out the stable weak
solutions to systems with eigenvalues satisfying (1.4) independently of the
existence of a strictly convex entropy. It is known that for arbitrary
strictly hyperbolic systems, the entropy criterion is not sufficiently power-
ful to rule out all unstable solutions and a stronger criterion is needed.

We refer the reader to the work of Oleinik [34,36], Wendroff [43,44], Dafermos
[4,5] and Liu [27,28] concerning admissibility criteria for more general sys-
tems. In this paper we shall restrict our attention to systems of equations
(1.1) with a strictly convex entropy and eigenvalues of the form (1.4) and con-
sider the uniqueness problem for weak solutions satisfying the entropy criterion.

We refer the reader to the work of Oleinik [34,36], Vol'pert [42], Keyfitz [23]
and Frisch [24] for results on the uniqueness of solutions to scalar conserva-
tion laws.

In view of the existence theory for conservation laws, it is natural to
consider the uniqueness problem in the space $L^\infty \cap BV$. We recall that the Glimm
difference scheme generates globally defined solutions to the Cauchy problem
which lie in $L^\infty \cap BV$; convergence has been established by Glimm [14] for
general strictly hyperbolic systems of $n$ equations with initial data having
small total variation and by several authors [1,7,8,20,29,30,32,33] for
special classes of systems with initial data having large total variation.

Furthermore, Lax [26] has shown that the solution constructed by the Glimm
scheme satisfies the entropy criterion.

In this framework the uniqueness problem may be formulated as follows.
Consider a system of equations (1.1) with a strictly convex entropy and
eigenvalues of the form (1.4). Let

$$Z(T) = \{(x,t) : 0 \leq t < T\}$$

and let $K = K(Z(T))$ denote the class of admissible weak solutions defined
on the strip $Z(T)$. The problem is to prove that if $u$ and $v$ are two
solutions in $K$ whose initial data coincide at almost all $x$ then $u$ and
$v$ coincide at almost all $(x,t)$ in $Z(T)$. The sense in which a weak solu-
tion assumes its initial data is discussed in Section 2. In this paper we
consider a somewhat less general problem. We consider the class $PL$ of
admissible piecewise Lipschitz solutions and the problem of proving that
each solution in $PL$ is unique within $K$. More precisely, by $PL =
PL(Z(T))$ we mean the subclass of solutions in $K$ with the following prop-
erty: for each $t$ in $[0,T]$ there exists a set of isolated points $x_j =
x_j(t)$ such that the restriction $u(\cdot,t)$ of $u$ to each interval $(x_j,x_{j+1})$
is a Lipschitz function of $x$; the dependence of the Lipschitz constant on
the interval $(x_j,x_{j+1})$ as well as the dependence of the partition points
$x_j$ on $t$ is arbitrary. We note that $PL$ forms a broad subclass of $K$;
$PL$ contains for example the classical piecewise smooth solutions, i.e.,
solutions consisting of isolated shock waves, contact discontinuities,
contected and noncentered rarefaction waves and compression waves and their
interactions. In particular, $PL$ contains the classical solution of the
Riemann problem [25]. We shall comment further on the relationship be-
tween $PL$ and $K$ below. We refer the reader to Greenberg [17, 16, 19]
for the construction and analysis of interactions in piecewise smooth

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solutions and to Oleinik [35], Kozhdeevski [37], Godunov [16], Hurd [21, 22] and Liu [27,28] for results which establish that certain types of piecewise smooth solutions are equal if their data are equal. We refer the reader to Douglas [10] and Lysgidevski [11] for stability results within certain special classes of solutions.

In this paper we first consider the uniqueness problem for genuinely nonlinear systems to two equations. We recall that (1.1) is said to be genuinely nonlinear if all of its eigenvalues are genuinely nonlinear. For such systems we prove that for every state \( \hat{u} \in K \) there exists a constant \( \delta > 0 \) depending only on \( \hat{u} \) and \( f \) with the following property. If \( w \in PL, u \in K, |w(\cdot, \cdot) - \hat{u}|_\infty < \delta, |u(\cdot, \cdot) - \hat{u}|_\infty < \delta \) and \( w(x, 0) = u(x, 0) \) for almost all \( x \) then \( w = u \) for almost all \( (x, t) \) in \( S(T) \). Here the \( L^1 \)-norm is taken over the strip \( S(T) \). We note that the restriction to solutions with small oscillation is not essential for the argument. By the same method we consider the quasilinear wave equation

\[
\begin{align*}
\frac{1}{2} u^2_x + p(u)^2_x &= 0 \\
\frac{1}{2} u^2_t - \frac{1}{2} u^2_x &= 0
\end{align*}
\]  

(1.5)

with \( p' < 0 \) and \( p'' > 0 \) and establish uniqueness of solutions with arbitrarily large oscillation: we prove that if \( w \) and \( u \) are arbitrary solutions of (1.5) which lie in \( PL \) and \( K \) respectively and whose initial data coincide at almost all \( x \) then \( w \) and \( u \) coincide at almost all \( (x, t) \). Under the hypotheses \( p' < 0 \) and \( p'' > 0 \), (1.5) represents a class of genuinely nonlinear systems having a globally defined strictly convex entropy; this class includes the equations of shallow water waves, isentropic fluid dynamics and the equations of certain thin elastic beams.

Our interest in (1.5) stems in part from the fact that it serves as the prototype for the broad class of genuinely nonlinear systems of two equations introduced by Smoller and Johnson [41], cf. Section 2. In the case of arbitrary genuinely nonlinear systems of two (or more) equations, it will be necessary to supplement the entropy inequality (1.3) with additional inequalities in order to rule out all unstable solutions. It is currently an open problem to determine what these inequalities should be.

We note that the uniqueness problem for conservation laws is a local problem in space-time. We establish the corresponding local version of the theorems above with the aid of the notion of generalized characteristic introduced by Dafermos [6], cf. Theorem 6.3. Our approach is applicable in both situations to systems of \( n \) equations but it appears that an additional a'priori estimate will be needed to treat the case \( n > 2 \).

For systems of equations with linearly degenerate eigenvalues we establish certain preliminary results. We prove, for example, that for systems of two equations and solutions with small oscillation the classical solution to the Riemann problem is unique within \( K \), cf. Theorem 6.4.

For general systems the classes \( PL \) and \( K \) can be compared in terms of the shock sets of their member solutions. We recall that with each function \( u \) in \( BV \) there is associated a set of points \( \Gamma = \Gamma(u) \) at which \( u \) experiences an approximate jump discontinuity [11,42]; in the context of weak solutions the set of jump points \( \Gamma(u) \) is referred to as the shock set of the solution \( u \). In a forthcoming paper on the regularity of solutions we prove that if \( u \) is a solution in \( K \) of a genuinely nonlinear system of two equations and if \( U \) is an open domain whose intersection with \( \Gamma(u) \) has vanishing 1-dimensional Hausdorff measure then \( u \) is Lipschitz continuous on any compact subdomain of \( U \) after a possible modification on a set vanishing 2-dimensional Lebesgue measure. We conjecture that this result

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generalizes to genuinely nonlinear systems of \( n \) equations. Thus, in the
case of genuinely nonlinear systems of two equations, \( \partial \) is essentially
the class of admissible weak solutions with isolated shock waves. It would
be of interest to establish uniqueness in the case where both \( u \) and \( w \)
contain accumulating shock waves, i.e. where both \( u \) and \( w \) are arbitrary
solutions in \( K \).

In Section 3 we establish some preliminary results on the stability of
solutions. Consider an arbitrary system of conservation laws (1) which
possesses a strictly convex entropy \( \eta \) and let \( L = L^1(\mathbb{T}) \) denote the
class of Lipschitz solutions defined on \( \mathcal{S}(\mathbb{T}) \). We prove that solutions in
\( L \) are \( L^2 \)-stable relative to perturbations in \( K \); if \( w \in L, u \in K \) and
\( 0 \leq t < \mathbb{T} \) then for every \( M > 0 \)
\[
(1.6) \quad \int_{|x| \leq M} |w(x,t) - u(x,t)|^2 dx \leq c_2 \int_{|x| \leq M} |w(x,0) - u(x,0)|^2 dx
\]
where the constant \( c_2 \) depends on \( f \) and the \( L^\infty \) norm of \( u \) and \( w \)
while the constant \( c_2 \) depends on \( f, T \), the \( L^\infty \) norms of \( u \) and \( w \)
and \( L^p \) norm of \( w \). As before the range of \( u \) and \( w \) are assumed to lie within
the domain of definition of \( \eta \). Strictly speaking, the estimate (1.6)
holds if \( u \) is replaced by its symmetric mean \( \bar{u} \); \( \bar{u} \) is obtained from \( u \)
by replicating with radially symmetric approximations of the \( \delta \)-function and
is defined \( \mathbb{H}_1 \) almost everywhere, c.f. Section 2. Throughout this paper we
shall assume that each given weak solution \( u \) is replaced by its symmetric
mean \( \bar{u} \). In particular, we conclude from the uniqueness theorems above
that \( \bar{u} \) and \( w \) coincide at \( \mathbb{H}_1 \) almost all points \((x,t)\) in \( \mathcal{S}(\mathbb{T}) \).

We emphasize that the proof of (1.6) does not require any assumptions
on system (1) beyond the smoothness of \( f \) and the existence of a strictly
convex entropy \( \eta \). We also establish the corresponding version of (1.6)
for systems of conservation laws in several space dimensions.

\[
u_t + \sum_{j=1}^{n} \frac{\partial f_j(u)}{\partial x_j} = 0
\]

which are endowed with a strictly convex entropy \( \eta \) and entropy-
fluxes \( q_j \), \( j = 1, 2 \), and for systems with the property that all smooth solu-
tions with range in \( \mathcal{D} \) satisfy an additional conservation law of the form
\[
\eta(u)_t + \sum_{j=1}^{n} q_j^2(u) \frac{\partial \eta}{\partial x_j} = 0.
\]

This class of systems includes the equations of fluid-dynamics, magneto-
fluid dynamics and the general symmetric hyperbolic systems in several
space dimensions. Returning to systems in one space dimension, we note that
in general the solution operator for (1.1) is at best \( \mathcal{H}^{1/4} \) Holder continuous in
\( L^\infty \). We conjecture that if \( u \) and \( v \) are arbitrary solutions in \( K \),
then
\[
(1.7) \quad \int_{|x| \leq M} |u(x,t) - v(x,t)|^2 dx \leq C_4 \left( \int_{|x| \leq M} |u(x,0) - v(x,0)|^2 dx \right)^{1/2}
\]
where the constant \( C_4 \) depends only on \( f \) and the \( L^\infty \) norms of \( u \) and \( v \)
while \( C_4 \) depends only on \( f, T \) and \( L^p \) norms of \( u \) and \( v \). It would
be of interest to prove (1.7) or the corresponding estimates in \( L^p, p \neq 2 \).

The role of the entropy criterion in conservation laws is illuminated
by comparison with other admissibility criteria. In [4,5] Dafermos postu-
lates an entropy rate criterion which identifies the relevant solutions as
those which dissipate entropy at the highest possible rate, c.f. Section 7.

The entropy rate criterion was introduced by Dafermos for the purpose of
dealing with systems having general eigenvalues. We note that the entropy
rate criterion is stronger than the entropy criterion: any solution in
\( L^\infty \cap \mathcal{H} \) which satisfies the entropy rate criterion necessarily satisfies
the entropy criterion. It is interesting to consider the converse, i.e. to
consider a solution in \( L^\infty \cap \mathcal{H} \) which satisfies the entropy criterion and
determine in what class of solutions it satisfies the entropy rate criterion. In this direction we first prove that for arbitrary systems of $n$ equations with a strictly convex entropy, all Lipschitz continuous solutions satisfy the entropy rate criterion relative to a broad class $G_1$ of solutions in $L^\infty_nBV$, cf. Theorem 7.1. The class $G_1$ requires a very mild form of finite propagation speed from its member solutions. In this regard we note that an arbitrary weak solution need not possess the property of finite propagation speed for waves; indeed, it is easy to construct non-constant weak solutions with identically constant initial data. Secondly, we prove that all solutions in $\mathcal{L}$ for the quasilinear wave equation with $p^* < 0$ and $p^* > 0$ satisfy the entropy rate criterion relative to a fairly large class $G_2$ of solutions in $L^\infty_nBV$, cf. Theorem 7.2. We note that Theorem 7.2 should be regarded as a preliminary result; we expect that $G_2$ can be substantially enlarged.

Next, we shall comment on certain aspects of the uniqueness proof. Consider a system of $n$ conservation laws (1.1) with a strictly convex entropy $\mathcal{E} : \mathbb{R}^n \to \mathbb{R}$ and entropy flux $q : \mathbb{R}^n \to \mathbb{R}$. Let $(u,v)$ denote an arbitrary pair of weak solutions defined on some region $\Omega$ with range contained in $\mathbb{R}$. We estimate the distance between $u$ and $v$ in $L^2$ of the space variable.

For this purpose we associate with each pair of weak solutions $(u,v)$ a Borel measure $\gamma = \gamma(u,v)$ defined on $\Omega$ as follows: $\gamma$ is the divergence a vector-field $(\alpha,\beta)$ obtained from the quadratic part of $n$,

$$\gamma = \alpha(u,v) + \beta(u,v),$$

where $u = u(x,t)$, $v = v(x,t)$ and $\alpha$ and $\beta$ are the mappings

$$\alpha(u,v) = n(u) - n(v) - \nabla n(u) \cdot \partial D + \partial R,$$

$$\beta(u,v) = q(u) - q(v) - \nabla q(u) \cdot (f(u) - f(v)) - \partial D + \partial R.$$ 

We note that $\alpha$ and $\beta$ form an $n$-parameter family of strictly convex entropies $\alpha^*(\cdot,v)$ and entropy fluxes $\beta^*(\cdot,v)$ with $v \in \mathcal{D}$ serving as the parameter; as observed by Dafermos [3] for each fixed $v \in \mathcal{D}$, the quadratic part of $n$ at $v$, i.e., $a$, is an entropy in $u$ with entropy flux $\beta$.

This observation is also useful in studying the large-time decay of solutions [9].

According to the generalized Green's theorem for measures [1,2,2], the $\gamma$-measure of the density points of $\mathcal{D}$ with finite parameter equals the flux of the vector-field

$$\{u(x,t), v(x,t), \beta(u(x,t), v(x,t))\}$$

across the essential boundary of the set. In the case where $u$ and $v$ are defined on the strip $\Omega(T)$ and have compact support in $x$,

$$\gamma(\Omega(T)) = \int_0^T \alpha(u(x,t), v(x,t)) dt - \int_0^T \beta(u(x,t), v(x,t)) dt$$

for $0 \leq t < T$. The integral of $\gamma$ at time $t$ is equivalent to the square of the special $L^2$ norm of $u - v$ by virtue of the strict convexity of $\gamma$.

Given a pair of solutions $u \in \mathcal{C}(\Omega(T))$ and $v \in \mathcal{PL}(\Omega(T))$ whose initial data coincide at almost all $x$, we prove uniqueness by applying (1.8) with $v = \tilde{w}$ where $\tilde{w}$ is an (apparent) modification of $w$ and then establishing a singular integral inequality for

$$\int_0^T \alpha(u(x,t), \tilde{w}(x,t)) dt$$

where the singularity is nonintegrable but mild relative to the rate at which (1.9) approaches zero as $t$ approaches zero; it follows that $\alpha = \beta$ at almost all $(x,t)$ in $\Omega(T)$. We note that the integral of $\alpha$ does not satisfy a standard Gronwall inequality since the solution operator is at best Hölder continuous in $L^2$. The Hölder continuity of the solution operator is also reflected in the fact that $\gamma$ is not absolutely continuous with respect to $2$-dimensional Lebesgue measure; the support of $\gamma$...
contains sets with finite 1-dimensional Hausdorff measure whose γ-measure depends on the associated pointwise values of the solutions. The corresponding local uniqueness problem is treated by applying γ to appropriate trapezoidal regions of the x-t plane bounded by characteristics in the sense of Dafermos.

In the case of arbitrary pairs of weak solutions \((u,v)\), one may regard γ as measuring the dissipation and dispersion in the solution \(u\) relative to the solution \(v\). The γ-measure of an arbitrary Borel set \(B\) is determined by the amount of entropy which \(u\) dissipates in \(B\) relative to \(v\), by the geometry of the joint shock set \(Γ(u, v) = Γ(u) \cup Γ(v)\) and by the rates of focusing and spreading of characteristics in \(u\) and \(v\). The connection between those quantities is expressed through certain decompositions of γ. The primary decomposition is obtained by restricting γ to \(Γ(u, v)\) and to the complementary region \(Γ^c(u, v)\) of approximately continuous flow:

\[
γ = γ|_{Γ(u, v)} + γ|_{Γ^c(u, v)}
\]

We shall discuss below the structure of each of these restrictions and their relationship to the stability of shock waves and rarefaction waves. We shall do so first in the general case where \((u, v)\) forms an arbitrary pair of weak solutions and then describe the application of these results to the case where \(u \in \mathcal{K}\) and \(v \in \mathcal{L}\).

Several preliminary remarks are in order concerning the measurement of dissipation in a weak solution. By a shock wave in a weak solution \(u\) we shall mean a Borel measurable subset of \(Γ(u)\). If \(E\) is a shock wave then the total entropy dissipated by \(E\) is given by \(δ_u(E)\) where

\[
δ_u = η(u) + q(u)
\]

We shall refer to a shock wave \(E\) as admissible if the restriction of the measure \(δ_u\) to \(E\) is non-positive, i.e., if \(δ_u(B) ≤ 0\) for all Borel subset \(B \subset E\). In general, the \(δ_u\)-measure of a Borel set \(B\) equals total entropy dissipated by all of those shock waves of \(u\) which lie in \(B\); it is not difficult to prove that \(δ_u\) is concentrated on \(Γ(u): \delta_u(B) = \delta_u(B \cap Γ(u))\).

The relationship between the geometry of a shock wave \(E\) and the amount of entropy which it dissipates is expressed by Green’s theorem \([11, 42]\).

\[
δ_u(E) = \int_E [n] + \nu \cdot [q] d\mathcal{H}_1
\]

Here \(\mathcal{H}_1\) denotes 1-dimensional Hausdorff measure, \(v = (v_x, v_t)\) denotes the unit normal to \(E\) in the sense of Federer \([11]\) and bracket denotes the jump in the enclosed quantity across \(E\) in the direction of \(v\):

\[
[n] = [n](P) = n_i u_i(P) - n_i u_i(P)
\]

where \(I_{νx}(P)\) denote the approximate limits of \(u\) at \(P ∈ E\) with respect to the half-planes

\[
H_{τ_ν}(P) = \{Q; (Q-P, τ_ν) > 0\}
\]

We shall normalize the direction of \(v\) by requiring that \(v_x < 0\) and refer to

\[
u_L = \frac{v}{|v|} \quad \text{and} \quad u_L = \frac{u}{|u|}
\]

as the left and right hand approximate limits of \(u\) at \(P\). With this convention, bracket denotes the jump from left to right across a shock.

We note that in an arbitrary weak solution \(v_x ≠ 0\) at \(H_{νx}\) about all points of \(Γ(u)\).

It follows from \(1.10\) that the rate of dissipation of entropy at \(P ∈ E\) is given by

\[
τ[p] = [q]
\]

where \(τ = -v_t(P)/v_x(P)\) denotes the speed of propagation of \(E\) at \(P\).
Indeed

\[ \theta_u(E) = \int \gamma(u) - [q] \, dt \]

where \( dt = -v \, du \). In the proof of uniqueness an important role is played by the function

\[ d : R \times R^{2n+1} \times R \]

\[ d(t,u,v) = \gamma(u) - [q] . \]

We shall refer to \( d \) as the dissipation function for \( u \) and to \( \theta_u \) as the dissipation measure.

More generally, one may regard the restriction of \( \gamma(u,v) \) to the shock set \( \Gamma(u) \) as measuring the dissipation in \( u \) relative to the solution \( v \), cf. Sections 3 and 4. If \( E \) is a shock wave in \( u \) then the \( v \)-measure of \( E \) is influenced by the limiting values of \( v \) along \( E \). It follows from Green's theorem that

\[ \gamma(E) = \int \Gamma(t,v)_{u,v} \, dt \]

where \( v_1 \) and \( v_2 \) denote the approximate left and right hand limits of \( v \) along \( E \) and \( \Gamma \) denotes the mapping

\[ \Gamma : R \times R^{2n+1} \times R \]

\[ \Gamma(t,v,\bar{v}) = \gamma(t,u) - [s] \]

where \( [s] = s_{\bar{v}}(v), v \) etc. In general, the restriction of \( \gamma \) to the joint shock set \( \Gamma(u,v) \) is determined by the relative dissipation of entropy and by the geometry of waves in \( \Gamma(u) \) relative to those in \( \Gamma(v) \).

We show that the restriction of \( \gamma \) to \( \Gamma(u,v) \) admits a decomposition into the relative dissipation measure \( \theta_{u,v} \) and a measure \( \mu \) which involves the relative speeds of propagation of waves in \( \Gamma(u) \) and \( \Gamma(v) \). We study the decomposition

\[ \gamma|_{\Gamma(u,v)} = \theta_{u} - \theta_{u,v} + \mu \]

in order to understand how the relative dissipation and relative speed of propagation of pairs of shock waves (one in \( u \) and the other in \( v \)) influence the time evolution of the special \( L^2 \) norm of \( u-v \). For this purpose we associate with each shock wave \( S_u \) in a weak solution \( u \) a subset \( S_v \) of the product of space-time with a state space of dimension \( 2n+1 \):

\[ S_u \equiv \{(x,t) : \tau(x,t), u_1(x,t), u_2(x,t) : (x,t) \in S_u \} \subset R^2 \times R^{2n+1} \subset E . \]

Given a pair of weak solutions \( (u,v) \) we consider sets of the form

\[ S_u \times S_v \subset E \times E \]

together with their projections

\[ P_{2n+1} S_u \times S_v \quad \text{and} \quad P_{2n+1} S_u \times S_v \]

onto \( R^2 \) and \( R^{2n+1} \times R^{2n+1} \) respectively. Here,

\[ P_{2n+1} : E = R^2 \quad \text{and} \quad P_{2n+1} E = R^{2n+1} \]

denote projection onto the first two and last \( 2n+1 \) variables respectively, e.g. \( P_{2n+1} S_u = S_u \). We first note that the \( v \)-measure of pairs of shock waves \( S_u \times S_v \) does not have a distinguished sign even if \( S_u \) and \( S_v \) are admissible. However we show that if \( S_u \) and \( S_v \) are admissible then the projections

\[ P_{2n+1} S_u \times S_v \]

lie near a special class of subsets of \( R^{2n+1} \times R^{2n+1} \) for which the time derivative \( D \) of \( \gamma \) is non-positive. In order to describe this special class it is convenient to introduce the projection

\[ P_{4n+1} : R^{2n+1} \times R^{2n+1} \to R^{4n+1} \]

\[ P_{4n+1} (t,u,v,\bar{v}) = (t,u,v,\bar{v}) \]

where \( \bar{v} \) denotes the speed of shock waves in \( v \). We associate with each point of \( P_{2n+1} S_u \times S_v \) rates of dissipation \( d(t,u,v) \) and \( d(t,v,\bar{v}) \) and we associate with each point...
For convenience we shall refer to any Borel subset of \( \Gamma(\omega) \) as a shock wave in \( \omega \), etc. Now, in the case where a pair of shock waves \( S_u \) and \( S_v \) are equal, i.e. occupy the same position in space-time, we note that
\[
y(S_u \cup S_v) = y(S_u) + \int D(t, u_t, u_{t}^*, v_{t}^*; v_{t}^*) dt
\]
and that
\[
y(S_u \cup S_v) < 0
\]
if \( P_{n-1}(L_u^*) \neq P_{n-1}(L_v^*) \) is an attracting set. In general one is presented with pairs of shock waves \( S_u \) and \( S_v \) which are not equal, but the effect of the projection \( P_{n+1} \) is to deform the wave \( S_v \), so to speak, until it occupies the same position in space-time as \( S_u \). Given a pair of solutions \( u \in K \) and \( w \in PL \), the operator \( P_{n+1} \) may be realized as a mapping which associates with the pair \( (u, \omega) \) an approximate solution \( \tilde{\omega} \). The function \( \tilde{\omega} \) is obtained by continuing the solution \( w \) across the sides of its shock waves via the solution operator of the Cauchy problem. The continuation process leads to a three-sheeted surface which contains the solution surface associated with \( w \). The function \( \tilde{\omega} \) is obtained from this three-sheeted extension by making a single-valued selection according to \( P_{n+1} \) in such a way that the dominant waves of \( u \) and \( \tilde{\omega} \) occupy the same position in space-time. Using (1.12) and the fact that the projection \( P_{n+1}(L_u^*) \neq P_{n+1}(L_v^*) \) lie near the class of attracting sets we show that
\[
y(u, \tilde{\omega}) \mid \Gamma(\tilde{\omega}) \leq 0
\]
The estimate (1.13) is the first of the two main steps in the derivation of the singular integral inequality for the integral of \( a(u, \tilde{\omega}) \).

Next, we shall discuss the connection between \( \gamma \) and the stability of rarefaction waves. As we remarked above, one may regard the restriction of \( \gamma \) to the shock set \( \Gamma(u) \) as measuring the dissipation in u.
relative to the solution v. In general, if E ∈ f(u) then γ(E)
depends upon the limiting values of v along E. However, if v is
nearly stationary on E in the sense that v is approximately continuous
on E (as opposed to having a jump discontinuity) then the γ-measure of
E reduces to the β_u-measure of E, cf. Lemma 3.1: if

\[ E \cap I(v) = \emptyset \]

then

\[ γ(E) = β_u(E). \]

This reduction property is essentially equivalent to the fact that γ is
expressed as the divergence of the vector-field obtained from the quadratic
part of u. More generally, if the solution v is approximately continuous
on an arbitrary Borel set B then the restriction of γ to B
splits into two mutually singular measures, the dissipation measure β_u
and a measure τ = τ(u,v) which couples u and v quadratically with a
weight depending upon the geometry of characteristics in v:

\[ γ|_B = β_u|_B + τ|_B \]

(1.14)

where

\[ τ(u,v) = q(u,v)v|_B \]

and q = 0(u-v)^2. The L^2-stability of Lipschitz solutions is an immediate
consequence of the decomposition (1.14): if w ∈ L^2(T) and u ∈ K(T)
then

\[ γ(u,w) ≤ τ(u,w) ≤ \text{const.} |u-w|^2 \]

since \( w \in L^2 \) and a standard Gronwall inequality follows for the integral
of \( τ(u,w) \). An analogous result holds in several space dimensions.

If the solution v is merely approximately continuous on B then
the strength of the coupling between u and v as measured by γ depends
upon the rate of focusing of characteristics in v. For simplicity we shall

comment on the situation where v is a similarity solution \( v = v(x/t) \) consist-
ing of a centered \( j \)-rarefaction wave separating two constant states; in
this case

\[ v = \begin{cases} 0 & \text{if } x/t \leq a \\ i_j(v(x/t))/t & \text{if } a < x/t \leq b \\ 0 & \text{if } b \leq x/t \end{cases} \]

We first note that in the case of a single genuinely nonlinear equation,
formal arguments using characteristics indicate that the solution operator
for admissible solutions is Lipschitz continuous at those data points which
generate centered rarefaction waves. Indeed, it follows from the results
of Section 6 that if u and v are two solutions of a single genuinely non-
linear equations with \( u \in K(T) \) and v consisting of a centered rare-
faction wave separating two constant states then

\[ \int |u(x,t) - v(x,t)|^2 dx \leq \text{const.} \int |u(x,0) - v(x,0)|^2 dx \]

(1.5)

where the constant depends only on \( \mathcal{F} \) and the L^2-norm of u and v.

Although (1.15) does not hold in general for systems of equations it leads
one to conjecture that the coupling between u and v as measured by γ
on a centered \( j \)-rarefaction wave in v depends on the orientation of the
vectors \( u(x,t) - v(x,t) \in \mathbb{R}^n \) and that favorable coupling occurs in the
direction corresponding to the eigenvector \( r_j \). This turns out to be the
case, cf. Section 6, and we deduce that

\[ γ \leq β_u + \text{const.} \int \frac{c_k^2 dx dt}{x_j} + \text{const.} \int |u-v|^2 dx dt \]

(1.16)

where the components \( c_k \) are defined by

\[ u - v = \sum c_k r_k(v) \]

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Thus, while the rate of focusing of characteristics in \( v \) produces the nonintegrable term \( 1/t \) the only coupling coefficients which enter are those which correspond to the complementary directions \( \theta, k \neq 0 \). We then show using a specially constructed family of non-convex entropies that if \( u \) and \( v \) are solutions to a system of two equations with the same initial data then the dissipation measure \( \frac{d}{dt} \Theta_u \) nearly balances the second term of (1.16) leading to a singular integral inequality with a factor of the form \( c/t \) where \( c \) is small.

2. Preliminaries

We shall begin with several remarks concerning the classes of solutions and systems which we shall treat in subsequent sections. We recall that a real valued function \( u = u(y) \) defined on a region \( \Omega \subset \mathbb{R}^n \) is an element of \( BV(\Omega) \) if it is locally integrable and if its gradient is a locally finite Borel measure:

\[
\int u \nabla \phi \, dx = -\int \phi \, du
\]

for all \( \phi \in C_c^\infty(\Omega) \) where \( u \) is a Borel measure on \( \Omega \) such that

\[
|\mu(\Omega')| < \infty
\]

for all compact subsets \( \Omega' \subset \Omega \). More generally, a function \( u \) defined on \( \Omega \subset \mathbb{R}^n \) with values in \( \mathbb{R}^m \) is an element of \( BV(\Omega) \) if each of its components satisfies (2.1) and (2.2). We note that \( u : BV(\Omega, \mathbb{R}^m) \subset \mathbb{R}^n \) if and only if the local total variation in \( x \) is locally integrable in \( t \) while the local total variation in \( t \) is locally integrable in \( x \). For example, \( u \in BV(\Omega(\bar{T})) \), if and only if

\[
\int_0^{\bar{T}} TV u(\cdot, t) \, dt < \infty \quad \text{and} \quad \int_0^{\bar{T}} TV u(x, \cdot) \, dx < \infty
\]

for all \( \bar{T} \). A similar result holds for \( BV \) functions on \( \Omega \subset \mathbb{R}^n \).

As we remarked in the introduction, it is convenient to replace a given weak solution \( u \) by its symmetric mean \( \bar{u} \):

\[
\bar{u} = \lim_{n \to \infty} u
\]

where \( \{u_n\} \) is a sequence of radially symmetric functions approximating the \( \delta \)-function [42]. We recall that \( \bar{u} \) is defined \( H^1 \) almost everywhere and that \( u \) and \( \bar{u} \) coincide with \( H^2 \) almost everywhere. If \( u \) is a weak solution defined on \( S(\bar{T}) \) then it follows from the form of the system (1.1) that for all \( \bar{T} \), \( \bar{u} \) is an absolutely continuous function of \( t \) with values in \( L^1(-\bar{T}, \mathbb{R}^n) \). In particular, the weak solution \( \bar{u} \) assumes its
initial data in $L^1_{\text{loc}}$. We shall assume throughout that we are dealing with the symmetric mean of the solution, i.e., we shall assume that $u = 0$.

When studying solutions with large oscillation, we shall restrict our attention to systems in the Smoller-Johnson class [41]. By the Smoller-Johnson class we mean the class of genuinely nonlinear systems of two equations with the following properties:

\[ A_1 \quad \frac{x f_1}{u^2} + \frac{y f_2}{u^1} < 0 \]

\[ A_2 \quad 1_j \cdot \nu (r_k, r_k) > 0 \quad \text{if} \quad k \neq j \]

\[ A_3 \quad \text{For each} \quad v, \quad \{ u : (u^2 - u^1)^2 f(u) = (u^2) \cdot f(u) \} \]

is connected. Here $u = (u^1, u^2)$, $f = (f^1, f^2)$ and $1_j$ and $f_j$ denote the left and right eigenvectors of $\nu$ corresponding to $\lambda_j$ normalized so that $r_j \cdot \nu_j > 0$, $1_j \cdot r_j > 0$.

We note that the Smoller-Johnson class contains the quasilinear wave equation (1.6) if $p^+ > 0$ and $p^- < 0$.

For systems in the Smoller-Johnson class, it is not difficult to show that the Lax shock conditions are equivalent to the entropy inequality: if the triple $(\tau, u_k, u_e)$ satisfies the Rankine-Hugoniot relations,

\[ R(\tau, u_k, u_e) = |\nu| = [f] = 0 \]

then

\[ \tau(n) - [q] < 0 \]

if and only if

\[ \lambda_j(u_k) < \tau < \lambda_j(u_e) \]

for some $j$. Here $[u] = u_k - u_e$, etc. We note that under the assumptions $A_1^r A_3^*$, condition (2.4) is equivalent to the full Lax shock conditions, i.e., if $j = 1$ then (2.4) implies that $\tau < \lambda_2(u_k)$ and if $j = 2$, then (2.4) implies that $\tau > \lambda_1(u_k)$.

The assumptions $A_1^r A_3^*$ guarantee certain basic properties for the system (1.1). Assumption $A_1^*$ implies that (1.1) is strictly hyperbolic. Assumption $A_2^*$ is equivalent to the Glimm-Lax condition that the interaction of two shocks of the same kind produces a shock of that kind and a centered rarefaction wave of the opposite kind. Assumption $A_3^*$ together with $A_1^*$ and $A_2^*$ implies that for each $v$ the projection of the set of all solutions $(\tau, u, v)$ of the Rankine-Hugoniot relations onto the $u_k - u_e$ plane consists of four globally defined smooth curves emanating from $v$ cf. [41]. Two of the curves (usually referred to as shock wave curves) consist of the projection of states $(\tau, u_k, u_k)$ which satisfy (2.4); the other two consist of the projection of states $(\tau, u_k, u_e)$ which satisfy the reverse inequality

\[ \lambda_j(u_k) < \tau < \lambda_j(u_e) \]

For additional background we refer the reader to the work of Smoller [38, 39, 40] concerning the Riemann problem for systems of the above type.

In this paper we employ the concept of a generalized characteristic introduced by Dafermos [6]. A (generalized) j-characteristic is a weak solution $u$ defined as the trajectory of the equation

\[ \dot{x} = \lambda_j(u(x, t)) \]

where (2.5) is interpreted in the sense of Fillipov [12]. Thus, a j-characteristic is a Lipschitz continuous curve $(x(t), t)$ whose speed of propagation $\dot{x}(t)$ lies for almost all $t$ between the essential minimum and the essential maximum of $|\lambda_j(u(t), t))$ at the point $(x(t), t)$. We also employ the notions of the minimal and maximal forward j-characteristic.
through a point \((x_0, t_0)\) which are defined as the lower and upper envelopes of the set of all solutions to (2.5) in the sense of Fillipov which pass through \((x_0, t_0)\).

3. Lipschitz Solutions

In this section, we shall establish the \(L^2\) stability of Lipschitz solutions and a form of finite propagation speed for waves. In this connection, we recall that if the initial data \(w_0(x)\) are Lipschitz continuous, then the Cauchy problem for (1.1) has a solution \(w\) which is defined and Lipschitz continuous on a strip \(S(T)\) where \(T\) depends only on \(f\), \(|w_0|_{L^\infty}\) and Lip \(w_0\). We shall assume throughout that system (1.1) is endowed with a smooth entropy-entropy flux pair \((\eta, q)\) where \(\eta\) is strictly convex and that the range of each solution considered lies within the domain of definition of \(\eta\) and \(q\) unless otherwise stated.

Theorem 3.1. Suppose that \(w \in L^2(S(T))\) and \(u \in K(S(T))\). If \(0 \leq t < T\) then

\[
\int_{|x| < M} |u(x, t) - w(x, t)|^2 \, dx \leq c_1^{1/2} \left( \int_{|x| < M} |u(x, 0) - w(x, 0)|^2 \, dx \right)^{1/2}
\]

where the constant \(c_1\) depends on \(f\) and the \(L^\infty\)-norms of \(u\) and \(w\) while the constant \(c_2\) depends on \(f, T\), the \(L^\infty\)-norms of \(u\) and \(w\) and Lip \(w(x, 0)\).

After proving Theorem 3.1, we shall establish the following more refined version of finite propagation speed for waves. Suppose that \(u \in K\) and that the restriction of its initial data \(u(x, 0)\) to the interval \((a, b)\) is equal almost everywhere to a Lipschitz function \(\hat{w}_0\). Let \(x^0_m(t)\) denote the maximal forward \(n\)-characteristic through \((a, 0)\) and let \(x^1_m(t)\) denote the minimal forward \(1\)-characteristic through \((b, 0)\).

Theorem 3.2. There exists a constant \(T > 0\) and a solution \(\hat{w}\) which is defined and Lipschitz continuous on

\[
\{(x, t) : x^0_m(t) \leq x \leq x^1_m(t), 0 \leq t < T\}
\]
such that \( \tilde{u}(x,0) = \tilde{u}_0(x) \) for all \( x \) and \( \tilde{u}(x,t) = u(x,t) \) almost everywhere in \( x \) for each \( t \) in \( [0,T] \). Here, the constant \( T \) depends only on \( f, \tilde{u}_0 \) and \( \text{Lip} \tilde{u}_0 \).

We shall begin by studying the restriction of \( \gamma(u,v) \) to the set \( \gamma^E(v) \) in the general setting where \( u \) and \( v \) are arbitrary weak solutions. For this purpose we shall consider the mappings

\[
D : R \times R^n \rightarrow R \quad \text{and} \quad D : R \times R^{2n} \times R^{2n}
\]
given by

\[
d(t;u_t,u_x,v) = \gamma(n) - [\gamma] - \gamma(n(u_t) - n(u_x)) - \gamma(q(u_t)) + \gamma(q(u_x))
\]

\[
D(t;u_t,u_x,v_x,v) = \gamma(n) - [\gamma] + \gamma(n(u_t,v_x)) - 2\gamma(u_t,v_x) + \gamma(u_x,v_x).
\]

We shall say that two states \( u_t \) and \( u_x \) in \( \mathbb{R}^n \) are connected by a shock wave with speed \( \tau \) if the Rankine-Hugoniot relations are satisfied, i.e., if

\[
R(t;u_t,u_x) = \gamma(n(u_t) - n(u_x)) - \gamma(q(u_t)) + \gamma(q(u_x)) = 0.
\]

The following lemma and corollary establish the reduction property for \( \gamma \).

**Lemma 3.1.** If \( u_t \) and \( u_x \) are connected by a shock wave with speed \( \tau \) then for all \( v \)

\[
D(t;u_t,u_x,v,v,v) = d(t;u_t,u_x).
\]

**Corollary 3.1.** If \( u \) and \( v \) are arbitrary weak solutions and \( \mathcal{E} \in \gamma(u) \cap \gamma^E(v) \) then

\[
\gamma(\mathcal{E}) = \delta_{u,v}(\mathcal{E}).
\]

**Proof of Lemma 3.1 and Corollary 3.1.** It follows from the definitions that

\[
D(t;u_t,u_x,v,v,v) = d(t;u_t,u_x) - \gamma(n(v)R(t;u_t,u_x)).
\]

Therefore

\[
\gamma(\mathcal{E}) = \int_E D(t;u_t,u_x,v,v,v)dt = \int_E d(t;u_t,u_x)dt = \delta_{u,v}(\mathcal{E})
\]

if \( \mathcal{E} \in \gamma(u) \cap \gamma^E(v) \).

The restriction of \( \gamma \) to domains of approximate continuity for both \( u \) and \( v \) is quadratic in the difference \( u-v \) with a weight depending on the geometry of characteristics in \( v \).

**Lemma 3.2.** Suppose the \( u \) and \( v \) are arbitrary weak solutions. If \( E \in \gamma^E(u) \cap \gamma^E(v) \) then

\[
\gamma(\mathcal{E}) = -\int_E \gamma^2(v) Q_f(u,v) v_x^2 dt
\]

where \( Q_f \) denotes the quadratic part of \( f \) at \( v \).

**Proof:** We note that \((n,q)\) is an entropy-entropy flux pair if and only if

\[
\gamma(u) = \gamma(v) = \gamma(u,v)
\]

and the right hand side vanishes for all smooth \( u \) if and only if (3.4) holds. Now, restricting \( \gamma \) to the set \( \gamma^E(u) \cap \gamma^E(v) \) we obtain by the chain rule [42]

\[
\gamma = \gamma_t + \gamma_x = \gamma_u + \gamma_v v_t + \gamma_u u_t + \gamma_v v_x
\]

\[
= \gamma_t + \gamma_u(v_t) |u_x| + |v_x| v_x.
\]

The coefficient of the measure \( u_x \) vanishes since \( u \) and \( v \) form an entropy-entropy flux pair in \( u \) and a short calculation shows that the coefficient of \( v_x \) equals

\[
-\frac{1}{2} (f(u) - f(v)) v_x^2 + (u-v)^2 n(v) v_x.
\]

Since the matrix \( \gamma^2(v) \) is symmetric [13], the coefficient of \( v_x \) may be rewritten in the desired form

\[
-\frac{1}{2} (f(u) - f(v)) v_x^2 + (u-v)^2 \gamma^2(v) v_x.
\]

This completes the proof of Lemma 3.2.

We conclude from Corollary 3.1 and Lemma 3.2 that the restriction of \( \gamma \) to shock-free domains in \( v \) may be decomposed into the mutually singular
sum of the dissipation measure \( \theta \) and the measure \( -\sqrt{2} \eta \partial_v v_x \), i.e., if \( E : \mathbb{R}^n \) then

\[
\gamma(E) = \theta(E) - \left\{ \int_{\mathbb{R}^n} \eta(v) Q(u,v)v_x \right\}.
\]

Proof of Theorem 3.1. Fix \( M > 0 \) and consider domains of the form

\[
\mathcal{C}(t) = \left\{ (x,s) : |x| < t(a), \begin{array}{l} 0 \leq s \leq t, \\ 0 \leq t \leq T, \end{array} \right\},
\]

where \( i(s) = M + C_1 (T-s) \) and the constant \( C_1 \) will be chosen below. It follows from Green's theorem for measures [11,42] that

\[
\gamma(\mathcal{C}(t)) = \int_{\mathbb{R}^n} v_x^2 + v_x \delta \, d x
\]

\[
= \int_{\mathbb{R}^n} v_x^2 + v_x \delta \, d x + \int_{\mathbb{R}^n} a(t) dx - \int_{|x| \leq t} a(0) dx.
\]

Since \( \eta \) is strictly convex, the ratio \( \eta/\delta \) is bounded if the arguments lie in a compact set and we may choose the constant \( C_1 \) (depending only on \( f \) and the \( L^2 \)-norm of \( u \) and \( w \)) so large that

\[
v_x^2 + v_x \delta = a(v_x + v_x \delta/\delta) \geq 0
\]

on the lateral boundary \( \partial \mathcal{C}(t) \). Now, it follows from (3.4) that

\[
\gamma \leq \sqrt{2} \eta \partial_v v_x \leq C \, |u-w|^2 dx dt
\]

and we conclude that the function

\[
s(t) = \int_{|x| \leq t} a(t) dx
\]

satisfies an integral inequality of the form

\[
s(t) \leq s(0) + C \int_0^t \sqrt{\gamma(s)} \, dt.
\]

This completes the proof of Theorem 3.1.

The proof of Theorem 3.2 is somewhat technical and may be read after reading the remaining sections of the paper. In order to prove Theorem 3.2 we shall establish a preliminary lemma. Suppose that \( u \in K(\mathcal{C}(t)) \) and that its initial data \( u(x,0) \) coincides almost everywhere on \((-\infty,0)\) with a Lipschitz continuous function \( \tilde{u}_0 \). Consider the Cauchy problem with initial data of the form

\[
u_0(x) = \begin{cases} 
\tilde{u}_0(x) & \text{if } x \leq 0, \\
\tilde{u}_0(1^+) & \text{if } x \geq 0.
\end{cases}
\]

where

\[
\tilde{u}_0(1^+) = \lim_{x \to 0} \tilde{u}_0(x).
\]

This problem has a solution \( w \) which is defined and Lipschitz continuous in some strip \( S(T) \). We shall show that for each \( \delta > 0 \) the solutions \( u \) and \( w \) coincide almost everywhere to the left of the ray

\[
x(t) = (\lambda - \delta) t
\]

where \( \lambda \geq \lambda_1(\tilde{u}_0) \) provided that \( t \) is sufficiently small and that \( u \) has the appropriate limiting behavior at the origin. This is made precise as follows. Let

\[
u_\delta(t) = u((\lambda - \delta)t - 0,t).
\]

We note that \( u_\delta \) exists for almost all \( t \) since the restriction \( u(\cdot,t) \) of \( u \) to almost all lines \( t = \text{const.} \) is a classical function of bounded variation having limits \( u(x_0,0,t) \) at each point \( x_0 \).

Lemma 3.3. If

\[
es(x) = \lim_{t \to 0} u_\delta(t) = \tilde{u}_0(1^+)
\]

then there exists a constant \( T > 0 \) depending only on \( u \) and \( \delta \) such that \( u \) and \( w \) coincide almost at all points \((x,t)\) satisfying

\[
x < (\lambda - \delta) t, \quad 0 \leq t \leq T.
\]

Remark: If the initial data \( u(x,0) \) coincides at almost all \( x \) with a Lipschitz function \( \tilde{u}_0 \), defined on \((0,\infty)\) then an analogous result holds for rays propagating at speeds slightly faster than \( \lambda_n(\tilde{u}_0) \) where

\[
\tilde{u}_0(1^+) = \lim_{x \to 0} \tilde{u}_0(x).
\]
We recall that
\[
\lim_{t \to 0} u_0(t) = u_0^-
\]
if and only if for every \( \epsilon > 0 \) there is \( t_\epsilon \) such that
\[
(t : |t| < t_\epsilon \text{ and } |u_0(t) - u_0^-| > \epsilon)
\]
has Lebesgue measure zero.

**Proof of Lemma 3.1.** Applying Green's theorem to the regions

\[
\Omega(t) = \{(x,s) : x < s(x), 0 < s < t\}
\]
yields the identity
\[
x(t) = \int_{x_0}^x a(t)dx + g(t) = \gamma(\Omega(t))
\]
where
\[
g(t) = -\int_0^t (x-s) \delta(u_0(t),w(x,t),t) - \delta(u_0(t),w(x,t),t) dt.
\]
If we prove that \( g(t) \) is non-negative for small \( t \) then it follows from
(3.4) that the integral of \( a \) satisfies a Gronwall inequality. Thus, we
need only show that
\[
p(u,v) \leq (k^2 - \delta) a(u,v) - \delta(u,v)
\]
is non-positive if the states \( u \) and \( v \) are sufficiently close to \( u_0^- \).

Now it follows from Taylor's theorem that
\[
u(u,v) = \frac{1}{2} \varphi_u^2(v)(u-v)^2 + 0(u-v)^3
\]
and therefore
\[
g = \frac{1}{2} \varphi_u^2(v)(u-v)^2 + 0(u-v)^3.
\]
If \( u \) and \( v \) are sufficiently close to \( u_0^- \) then \( \lambda_{-1}^-(v) \) is small and
\[
g \leq -\delta/2a + \frac{1}{2} \varphi_u^2(v)(1 - \lambda_{-1}^-(v)) = \varphi_u^2(v)(u-v)^2.
\]
We observe that the matrix \( \varphi_u^2(v)(1 - \lambda_{-1}^-(v)) \) is non-positive. Indeed, it follows
from the symmetry of \( \varphi_u^2(v) \) that
\[
\lambda_{-1}^-(v) \varphi_u^2(v) \varphi_u^2(v) \lambda_{-1}^-(v) = \lambda_{-1}^-(v) \varphi_u^2(v) \varphi_u^2(v) \lambda_{-1}^-(v) = 0
\]
if \( k \neq j \). This completes the proof of the lemma.

**Proof of Theorem 3.2.** We shall first recall a known estimate on Lipschitz
continuous solutions. Let \( s(u) \) denote the matrix whose \( j \)th column is
the normalized eigenvector \( \begin{pmatrix} \lambda_j(u) \\

r_j(u)
\end{pmatrix} \)

\[
\forall j \quad r_j = \lambda_j r_j, \quad |r_j| = 1
\]
Consider a solution \( u \) which is defined and Lipschitz continuous on some
strip \( S(t) \). Let
\[
L u(t) = |s^{-1}(u(t),t) | u_x(t) |
\]
It is known that there exists a constant \( c \) depending only on \( f,T \) and
\( |u|_\infty \) such that
\[
(3.5) \quad L u(t) \leq e^{ct} L u(0)/(1 - ct L u(0))
\]
if \( t \leq 1/c L u(0) \). We recall that Lipschitz norm of a function is equal
to the \( L^\infty \)-norm of its distributional gradient; it is somewhat more convenient here to work with \( Lu \) rather than with Lip \( u \). The estimate
(3.5) is proved by considering the diagonalized system satisfied by
\( z(x,t) = s^{-1}(u) u_x \) and deriving an integral inequality for the quantity
\[
|z(t)| = \frac{1}{j} \int |y_j(t)| = L u(t)
\]
by integrating along characteristics. One may first consider \( C^2 \) solutions
and then obtain the general case by passing to the limit.

In the paragraphs below we shall assume for simplicity that all
solutions are defined on domains contained in the strip \( S(t) \). Then for a given
system, the constant \( c \) appearing in (3.5) depends only on the \( L^\infty \)-norm
of the solution. Thus, one obtains the following uniform estimate on the growth
rate of the Lipschitz norm of the solution: fix \( k > 0 \) and suppose that \( u \)
is an admissible weak solution which is defined on the strip
\[
\{(x,t) : t_1 < t < t_2\}
\]
and satisfies there $|u|_s \leq N$. If $t_2 - t_1 < 1/c(N)k$ and

$$L u(t) \leq h(t, c, k),$$

then

$$L u(t) \leq h(t, c, k)$$

provided that $t_1 \leq t \leq t_2$ where

$$h(t, c, k) = e^{c t k} h_{1-c(k)}.$$

The basic idea of the proof of Theorem 3.2 is to consider the largest domain on which the solution is Lipschitz continuous and show that it necessarily contains the region bounded by the extreme characteristics. For simplicity in terms of, say we shall call a measurable function Lipschitz continuous of it is equal almost everywhere to an everywhere defined function which is Lipschitz continuous in the standard sense. Let $u$ be a solution in $K(\mathbb{C}^1)$ and suppose that its initial data $u_0(x)$ are Lipschitz continuous on the interval $I_0 = (-\infty, 0)$. We shall prove that $u$ is Lipschitz continuous to the left of the minimal $1$-characteristic $\gamma(x)$ passing through the origin. Let $|u|_s(I)$ denote the $L^s$-norm over the set $I$ and put

$$N = 2 |u_0|_{s(I_0)}$$

and $k = \|u_0 |_{s(I_0)}$. Let $I(t) = (-\infty, \gamma(t))$ denote the largest open interval with the property that the restriction of $u(\cdot, t)$ to $I(t)$ is Lipschitz continuous and satisfies

$$L(u(t), I(t)) = \|s^{-1}(u_0)\|_{s(I(t))} = h(t, c, k).$$

We shall show that if $t \leq 1/c(N)k$ then $I(t)$ is nonempty and $\gamma(t) \geq x(t)$. To this end, we shall first prove that

$$\gamma(t) \geq \text{const.} \ t,$$

for small $t$. Consider the Cauchy problem with initial data

$$v_0(x) = \begin{cases} u_0(x) & \text{if } x < 0 \\ u_0(0-) & \text{if } x \geq 0 \end{cases}$$

There exists a solution $v$ to the problem (3.7) which is defined and Lipschitz continuous on $C(I_v)$ where $I_v$ depends only on $|v_0|_s$ and $\text{Lip} v_0$. If $I_v$ is chosen so small that

$$|v_0|_{C(I_v)} \leq N,$$

then Theorem 3.1 guarantees that for each $t$ in $[0, T_v)$, the solutions $u$ and $v$ coincide at almost all $x$ satisfying $x \geq \text{const.} \ t$ where the constant depends only on $N$. Since

$$|s^{-1}(v_0)v_0|_s \leq |s^{-1}(u_0)u_0|_s = k$$

it follows that

$$L v(t) = \|s^{-1}(v_0)v_0\|_{s(I_v)} = h(t, c, k)$$

and therefore (3.6) holds for $0 \leq t < T_v$ with a constant depending only on $N$.

In order to prove that $\gamma(t) \geq x(t)$, we shall assume for simplicity that $\sup_{v}(v) < 0$ and show that

$$y(t) = \inf_{v}(y(s) : 0 \leq s \leq t)$$

is defined and Lipschitz continuous on the interval $[0, 1/c(N)k)$ and satisfies there $y(t) \geq x(t)$. In view of the monotonicity of $y(t)$ it is sufficient to show that

$$y(t_2) - y(t_1) \geq \text{const.} (t_2 - t_1), \ t_2 > t_1$$

if $t_2 - t_1$ is sufficiently small. The lower bound (3.8) may be proved in exactly the same way as (3.6): one need only replace the Cauchy problem (3.7) by the Cauchy problem at time level $t = t_1$ with data

$$v(x, t_1) = \begin{cases} u(x, t_1) & \text{if } x \leq y(t_1) \\ u(y(t_1) - 0, t_1) & \text{if } x > y(t_1) \end{cases}$$

The same constant will serve in both (3.6) and (3.8).
In order to prove that \( y(t) \geq x(t) \) we shall show that for each \( \delta > 0 \)
\[(3.9) \quad \dot{y}(t) \geq \min_{\gamma \in \Gamma_1} \left\{ u(y(t), y(t)) - \delta \right\} \]
for almost all \( t \) in \([0, T_0] \). Here, we denote the essential minimum of a
function \( g(x, t) \) at the point \( y \) by
\[
\min g(y, t) = \inf \left\{ \min_{p \in [y]} g(p, t) \right\}.
\]
It follows from (3.9) that \( y(t) \geq x(t) \) if \( x \) denotes the minimal
solution \([12]\) of the forward initial value problem
\[
\dot{x} = \lambda_1(u(x(t), y(t))) - \delta, \quad x(0) = 0.
\]
After passing to a subsequence we may conclude that \( x_\delta \) converges uniformly
to the minimal 1-characteristic \( x(t) \) and that \( y(t) \geq x(t) \) for \( 0 \leq t \)
\( < T_0 \).

In order to prove (3.9) we observe that the condition
\[
\dot{y}(t) \geq [u] = 0 \quad \text{and} \quad \dot{y}(t) \leq [q] \leq 0
\]
hold for almost all \( t \) where \([u] = u(y(t) - 0, t) = u(y(t) + 0, t) \) etc.

Let
\[
J = \{ t : 0 \leq t < T_0, \quad [u] \neq 0 \}.
\]
It follows from the entropy inequality that
\[
\dot{y}(t) \geq \lambda_1(u(y(t) + 0, t))
\]
almost everywhere on \( J \). Therefore, (3.9) holds almost everywhere on \( J \)
and we need only prove (3.9) almost everywhere on \( S^2 \). To this end, let us
consider the set
\[
J_1 = \{ t \in S^2 : \dot{y}(t) \leq \lambda_1(u(y(t), y(t))) - \delta \}.
\]
If \( J_1 \) has measure zero the proof is finished. If not there exists a
point \( t_1 \) in \( J_1 \) which has non-zero density with respect to 1-dimensional
Hausdorff measure. We shall reach a contradiction as follows. Let
\[
\lambda = \lambda_1(u(y(t), y(t)))
\]
for almost all \( c_1 \) and \( c_2 \) the restrictions of \( u \) to the lines
\[
\{(x, t) : t = c_1 \} \quad \text{and} \quad \{(x, t) : x = (s - t/3) t + c_2 \}
\]
are classical functions of bounded variation whose points of continuity and
 discontinuity as a function of one variable correspond to points of approxi-
mate continuity and approximate jump discontinuity in \( (x, t) \). We conclude
that there exist points \( (y_2, t_2) \) with \( y_2 = y(t_2) \), \( t_2 < T_0 \)
almost arbitrarily close to the point \( (y(t_1), t_1) \) and at which \( u \) has the following
limiting behavior:
\[
u(y_2 - 0, t_2) = \lim_{t \to T_0} u((s - t/3) t + c_2) = \lambda.
\]
Since we may assume without loss of generality that \( u \) is Lipschitz con-
tinuous in the standard sense to the left of the curve \( y(t), t \), we may
choose one such point with the additional property that
\[(3.10) \quad \left| \lambda_1(u(y_2, t_2)) - \lambda_1(u(y_1, t_1)) \right| < \delta/2.
\]
At such a point the speed of propagation of \( y(t) \) is strictly less than
that of the ray
\[
x = (s - t/3)(t - t_2) + y_2, \quad t < t_2.
\]
We may now apply Lemma 3.3 to obtain the desired contradiction. Consider
the Cauchy problem at time level \( t = t_2 \) with data
\[
v(x, t_2) = \begin{cases} u(x, t_2) & \text{if } x \leq y_2 \\ u(y_2 - 0, t_2) & \text{if } x > y_2 \end{cases}.
\]
This problem has a solution \( v(x, t) \) which is defined and Lipschitz contin-
uous on some band \( [t_2 - \delta, t_2 + \delta] \). By Lemma 3.3 there exists a constant \( \varepsilon > 0 \)
such that \( u \) and \( v \) coincide at almost all points \( (x, t) \) which satisfy
\[
x < (1_{t_2} - 6/3)(t - t_2), \quad 0 \leq t - t_2 < \varepsilon.
\]
Thus it follows from the definition of \( y(t) \) that
\[
y(t) \geq (1_{t_2} - 6/3)(t - t_2)
\]
for \( 0 \leq t - t_2 < \varepsilon \). This completes the proof of Theorem 3.2.
4. Shock Waves

In this section we study the restriction of γ to the joint shock set \( \Gamma(u,v) \) and then establish the uniqueness of admissible shock waves in piecewise Lipschitz solutions to genuinely nonlinear systems of two equations.

Consider a genuinely nonlinear system (1.1) of two equations and suppose that \( w \) is a solution on \( S(T) \), which is Lipschitz continuous on either side of an admissible j-shock wave passing through the origin. More precisely, assume that \( w \) is Lipschitz continuous in regions of the form

\[
(x,t) : x < y(t), 0 \leq t < T \quad \text{and} \quad (x,t) : x > y(t), 0 \leq t < T
\]

where \( \gamma \in C^1(0,T), \gamma(0) = 0 \) and

\[
R(y' \, u^+, u^+) = 0
\]

\[
l_1(u^+) < y' < l_2(u^+)
\]

where \( u^+ = u(y(t) - 0, t) \). (We note that the assumptions above imply that the speed of propagation \( y(t) \) is a Lipschitz function of \( t \).) It is not difficult to prove that there exists a solution \( w \) of the above form if one is given initial data \( w_0(x) \), which are Lipschitz continuous on each of the intervals \((-\infty, 0)\) and \((0, \infty)\) and if the limiting values at the origin

\[
w_0^- = w_0(0 + 0)
\]

are connected by a sufficiently weak admissible j-shock wave, i.e., if there exists a number \( \tau \) such that

\[
R(\tau, w_0^-, w_0^+) = 0
\]

\[
l_1(w_0^-) < \tau < l_2(w_0^+).
\]

In the case where (1.1) belongs to the Smoller-Johnson class one need not require that \( w_0^+ \) and \( w_0^- \) are close.

Theorem 4.1. For every \( \tilde{u} \in R^2 \) there exists a constant \( \delta > 0 \) depending only on \( \tilde{u} \) and \( \tilde{u} \) with the following property. If \( u \in K(S(T)) \),

\[
u - \tilde{u} \leq \delta, \quad |w - \tilde{w}| \leq \delta \quad \text{and} \quad u(x,0) = w(x,0) \quad \text{for almost all} \quad x \quad \text{then}
\]

\[
u = w \quad \text{for almost all} \quad (x,t) \quad \text{in} \quad S(T).
\]

Theorem 4.2. Suppose that (1.1) is a system in the Smoller-Johnson class. If \( u \in K(S(T)) \) and \( u(x,0) = w(x,0) \) for almost all \( x \) then \( u = w \) for almost all \( (x,t) \) in \( S(T) \).

Remarks. If (1.1) is an arbitrary strictly hyperbolic system of two equations then there exists in a neighborhood of each point \( \tilde{u} \in R^2 \) a smooth strictly convex entropy [26]. If (1.1) lies in the Smoller-Johnson class then there exists in a neighborhood of each compact set in \( R^2 \) a smooth strictly convex entropy [26]. In proving Theorem 4.2 one chooses an entropy whose domain of definition contains the range of both \( u \) and \( w \).

We shall begin with several lemmas which describes the structure of \( \gamma(u,v) \) on the joint shock set \( \Gamma(u,v) \) in the case where \( u \) and \( v \) are arbitrary weak solutions to a system of \( n \) equations. As always we assume that (1.1) has a smooth entropy-entropy flux pair \( (n,q) \) with \( n \) strictly convex and that the range of solutions considered lies within the domain of definition of \( n \) and \( q \).

Lemma 4.1. Suppose that \( \psi_k \) and \( \psi_k \) are connected by a shock wave with speed \( s \). Then, for all \( u \),

\[
D(n(u,v_k,\psi_k) - n(u,v_k)) = -D(n_v(u,v_k) - n_v(\psi_k)) \quad \text{where} \quad [n_v] = n_v(\psi_k) - n_v(\psi_k).
\]

Proof. By definition

\[
D = \sigma(\alpha_k - \alpha_k) - \beta_k + \beta_k
\]

where \( \alpha_k = n(u) - n(v_k) - n(\psi_k)(v_k - v_k) \) and \( \beta_k = q(u) - q(v_k) - n(\psi_k)(f(u) - f(v_k)) \).
Thus,

\[ u_i - u_i = -\eta_i + \eta_i - \psi(u_i) + \psi(u_i) \]
\[ = \psi(u_i) - \psi(u_i) + \psi(u_i) - \psi(u_i) \]

where \( \eta_i = \eta_i(v_i) \) etc. Hence,

\[ D(u, u_i, v_i, v_i') = -D(v_i, v_i') - \psi(u_i, v_i') + \psi(u, v_i) + \psi(u, v_i) \]
\[ + \psi(v_i) - \psi(v_i) - \psi(v_i) - \psi(v_i) \]

The identity (4.1) follows from (4.2) and the Rankine-Hugoniot relations

\[ \sigma v_i - f(v_i) = \sigma v_i - f(v_i) \]

We conclude from Lemma 4.1 that the restriction of \( \gamma \) to the set \( \mathcal{S}^C(u) \cap \mathcal{S}(v) \) splits into two measures, the negative dissipation for \( v \),

\[ -\theta_v \]

and a measure which couples the limiting values of \( u \) and \( v \) on \( \mathcal{S}^C(u) \cap \mathcal{S}(v) \).

**Corollary 4.1.** Suppose that \( u \) and \( v \) are arbitrary weak solutions. If \( E \in \mathcal{S}^C(u) \cap \mathcal{S}(v) \) then

\[ \gamma(E) = -\theta_v(E) - \int \psi(v)R(u, u_i, v_i)\mathcal{H}_1 \]

**Remark:** In general the coupling term \( |\psi|\mathcal{H} \) does not have a distinguished sign and is only first order in the difference between \( u \) and \( v \):

\[ |\psi|\mathcal{H} = O(|u_i - v_i|) \min(|u_i - v_i|, |u_i - v_i|) \]

since \( R(u, u_i, v_i) = R(u, u_i, v_i) \).

**Corollary 4.1** is useful in estimating the \( \gamma \)-measure of a pair of nearby shock waves. Consider a shock wave \( S_u \) which propagates through points of approximate continuity in \( v \) and a shock wave \( S_v \) in \( v \) which propagates through points of approximate continuity in \( u \), i.e. consider sets \( S_u \) and \( S_v \) satisfying

\[ S_u \in \mathcal{S}(u) \cap \mathcal{S}(v), \quad S_v \in \mathcal{S}(u) \cap \mathcal{S}(v) \]

The restriction of \( \gamma \) to \( S_u \cup S_v \) may be decomposed into the relative dissipation measure \( \theta_u - \theta_v \) and a measure which couples the values of \( u \) and \( v \) along \( S_u \cup S_v \); if \( E \in S_u \cup S_v \) then

\[ \gamma(E) = \theta_u(E) - \theta_v(E) - \int \psi(v)R(0, u_i, v_i)\mathcal{H}_1 \text{ d}E_v \]

Special interest attaches to the case where the restriction of \( u \) to \( S_v \) coincides with one of the limiting values of \( v \) on \( S_u \), assume that for each \( t_0 \) in the interval \( (t, t_0) \) the sets \( S_u \) and \( S_v \) intersect the line \( t=t_0 \) at points \( P \) and \( Q \) respectively and that either

\[ u(P) = u(Q) \text{ or } u(P) = u(Q) \]

In this situation the coefficient of the measure \( H_1 \) in (4.3) may be expressed in terms of the relative speed of propagation of \( S_u \) and \( S_v \).

**Lemma 4.2.** Suppose that \( u \) and \( u_i \) are connected by a shock wave with speed \( v \) while \( v_i \) and \( v_i \) are connected by a shock wave with speed \( u \).

Then

\[ D(u, u_i, v_i, v_i') = -D(v_i, v_i') - \psi(u_i, v_i') + \psi(u, v_i) + \psi(u, v_i) \]

\[ + \psi(v_i) - \psi(v_i) - \psi(v_i) - \psi(v_i) \]

**Proof:** Substituting \( u = u_i \) in (4.1) yields

\[ D(u_i, u_i, v_i, v_i') = -D(v_i, v_i') - \psi(u_i, v_i') + \psi(u, v_i) + \psi(u, v_i) \]

we obtain (4.5b) using the identity

\[ R(u_i, u_i, v_i, v_i) = R(u_i, u_i, v_i) - (u_i - u) \]

(4.5a) is proved in a similar way.

Therefore, if (4.4) holds the \( \gamma \)-measure of a pair of shock waves \( S_u \cup S_v \) may be expressed in terms of the relative dissipation \( (u, v) \) and the relative speed of propagation of \( S_u \cup S_v \) modulo an error term: if \( E \in S_u \cup S_v \) and (4.4) holds then

\[ \gamma(E) = \theta_v(E) - \theta_v(E) + \int \psi(v)\text{ d}E_v \]

where
error = $\int_{E \cap S_v} |Vn| \, dx.$

We note that in general the state $u(r)$ does not lie close to either $u_1(r)$ or $u_2(r)$ and the $\gamma$-measure of $E \cap S_v$ is not bounded by the Euclidean distance between $E$ and $S_v$ in physical space. However by associating with each shock wave $S_u$ in a weak solution $u$ the set

$$S_u = \{(x,t,r,v_1,v_2) : (x,t) \in S_u \subset \mathbb{R}^2 \times \mathbb{R}^{2n+1} \cap E,$$

described in the introduction, one may regard the second term in (4.5a,b) as measuring the distances between the sets $F_{2n+1}(S_u)$ and $F_{2n+1}(S_v)$ in the state space $\mathbb{R}^{2n+1}$. In this connection, we establish the following lemma.

**Lemma 4.3.** Suppose that $u_1$ and $u_2$ are connected by a shock wave with speed $\gamma$ while $v_1$ and $v_2$ are connected by a shock wave with speed $\alpha$.

Then

$$\begin{align*}
(4.6a) \quad D(r,v_1,v_2) &= d(r,v_1,v_2) - \delta(v_1,v_2) - \delta(v_1,v_2) \gamma + \gamma d(r,v_1,v_2) \\
(4.6b) \quad D(r,v_1,v_2) &= d(r,v_1,v_2) - \alpha(v_1,v_2) \gamma + \gamma d(r,v_1,v_2).
\end{align*}$$

In order to establish the decompositions (4.6a) and (4.6b) we write the function $D$ as the sum of two transition functions, one in which $v$ is stationary and the other in which $u$ is stationary.

**Proof.** Let $q_{s,1}(x,v_1) = a(u_1,v_1)$, $q_{s,2}(x,v_2) = a(u_2,v_2)$. Then

$$D = \{(r,v_1,v_2) : \sigma(r,v_1,v_2) - \sigma(r,v_1,v_2) + \delta(r,v_1,v_2) - \delta(r,v_1,v_2) \gamma + \gamma d(r,v_1,v_2) \\
= \delta(r,v_1,v_2) \gamma + \gamma d(r,v_1,v_2) \\
= \{(r,v_1,v_2) : \sigma(r,v_1,v_2) = \sigma(r,v_1,v_2) + \delta(r,v_1,v_2) - \delta(r,v_1,v_2) \gamma + \gamma d(r,v_1,v_2)
\}
$$

Substituting the expression (4.1) for $D(r,v_1,v_2)$ we obtain

$$D = \{(r,v_1,v_2) : \sigma(r,v_1,v_2) \gamma + \gamma d(r,v_1,v_2) \\
= \{(r,v_1,v_2) : \sigma(r,v_1,v_2) - \sigma(r,v_1,v_2) - \delta(r,v_1,v_2) - \delta(r,v_1,v_2) \gamma + \gamma d(r,v_1,v_2)
\}
$$

A short calculation shows that

$$\begin{align*}
\sigma(r,v_1,v_2) &= a(u_1,v_1) - [Vn](u_1,v_1) = -\delta(v_1,v_2) - [Vn](u_1,v_1).
\end{align*}$$

Thus,

$$\begin{align*}
(4.7) \quad D &= d(r,u_1,v_1) - d(0,v_1,v_2) \\
&= \delta(v_1,v_2) \gamma + \gamma d(r,v_1,v_2).
\end{align*}$$

The identity (4.6a) follows by factoring out $[Vn](u_1,v_1)$ from (4.7) and making use of the relation

$$\delta(r,v_1,v_2) \gamma + \gamma d(r,v_1,v_2) = \delta(r,v_1,v_2) \gamma + \gamma d(r,v_1,v_2).$$

The identity (4.6b) is proved in a similar way.

We may reformulate Lemma 4.3 as follows. Let $u$ be an arbitrary weak solution defined on $S$ and $v$ an arbitrary function in $L^\infty \cap BV(S)$. If $E \subset T(u,v)$ and if almost all limiting states $v_1(x)$ and $v_2(x)$, $x \in E$, are connected by a shock wave with speed, say, $\sigma(x)$ then

$$\gamma(E) = \int_E D(r,v_1,v_2) \, dx$$

where $\gamma(E) \delta(r,v_1,v_2)$ denotes the speed of propagation of $E$, $dt = -\gamma \, dx$, and $E$ is given by either (4.6a) or (4.6b) with $\sigma$ replaced by $\gamma(x)$. As always $v$ is normalized by the requirement that $u_1 < 0$. Later we shall employ the representation (4.8) with $v$ replaced by an approximate solution. Although $D$ does not have a distinguished sign in general we show that $D$ is negative definite on a special class of points. We shall refer to a point

$$A = (r,v_1,v_2) \epsilon E \cap S_v$$

as attracting if the following three properties hold:

$$\begin{align*}
R(r,v_1,v_2) &= 0 = R(v_1,v_2) \\
\lambda_j(u_1) &< \gamma_i(v_2) \quad \text{and} \quad \lambda_j(v_1) < \gamma_i(v_2), \quad \text{for some} \quad j.
\end{align*}$$

Either $u_1 = v_1$ or $u_2 = v_2$. 

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We shall refer to a subset of $\mathbb{R}^{2n+1}$ as attracting if each of its points is attracting.

Lemma 4.4. For every $\hat{u}$ in the domain of definition $D$ of $\eta$ and $q$ there exists a neighborhood $N(\hat{u}) \in D$ with the following property. Suppose that $A$ is an attracting point such that $u_t, \dot{u}_t, \delta_t$ and $v_x$ lie in $N(\hat{u})$.

If $u_x = v_x$ then

\begin{equation}
D(P^{2n+1}(A)) \leq \text{const.} \cdot |u_x - v_x| (1 - o)^2.
\end{equation}

If $u_x = v_x$ then

\begin{equation}
D(P^{2n+1}(A)) \leq \text{const.} \cdot |u_x - v_x| (1 - o)^2.
\end{equation}

where the constants are positive and depend only on $\hat{u}$ and $f$.

Proof: Suppose that $u_x = v_x$. Fixing $u_x$ and the wave index $j$ we may regard $u_x$ as a function of the wave speed $v$ via the Rankine-Hugoniot relations if $u_x$ is close to $u_x$.

\[ R(t, u_x, u_x(t)) = 0. \]

Consider the function

\[ \hat{v}(t) = d(t, u_x, u_x(t)). \]

A straightforward computation shows that

\[ \hat{v}(t) = n(u_x, u_x(t)) \cdot \]

Now, if $u_x = v_x$ it follows from (4.6a) that

\[ D(P^{2n+1}(A), v_x, v_x) = \hat{v}(t) - \hat{v}(o) = \hat{v}(o)(1 - o). \]

A short calculation shows that

\[ \hat{v}(t) = \nabla_u n(u_x, u_x). \]

Since $\eta$ is strictly convex, we conclude that $\hat{v}$ is a concave function of the wave speed $v$. This completes the proof of (4.9). The case $u_x = v_x$ is treated in a similar fashion.

Remark: For systems (1.1) in the Smoller-Johnson the corresponding result holds if we merely assume that $u_t, u_x, v_x$ and $v_x$ lie in a compact subset of $\mathbb{R}^2$.

One may regard (4.9) and (4.10) as expressing the stability of shock waves relative to a special class of perturbations, namely those which share a common end state. The stability of shock waves relative to general perturbations is discussed below. We shall establish Theorems 4.1 and 4.2 with the aid of an approximate solution $\hat{w}$. Suppose that $u$ is a solution in $K(S(T))$ whose initial data coincide with $w(x, 0)$ at almost all $x$. The approximate solution $\hat{w}$ is constructed from $w$ and $u$ by a two step procedure. The first step involves the extension of $w$ across the shock wave

\[ \hat{w} = \{(y(t), t) : 0 \leq t < T\}. \]

Consider the restrictions $\hat{w}$ of $w$ to the right and left of $S_u$, i.e. to the regions

\[ \hat{w} = \{(x, t) : x \in (0, T) \} \quad \text{and} \quad \hat{w} = \{(x, t) : x \in (T, 0) \}. \]

Since $S_u$ is non-characteristic we may employ the classical existence theory to extend $\hat{w}$ across $S_u$ to the left: there exists $\hat{\tau} > 0$ and a solution $\hat{\hat{w}}$ which is defined and Lipschitz continuous in the region

\[ \hat{\hat{w}} = \{(x, t) : x \in (0, T) \}, \]

and which coincides on $S_u$ with the restriction of $\hat{w}$ to the right side of $S_u$:

\[ \hat{\hat{w}}(P) = \hat{w}(P) \text{ if } P \in S_u. \]

Similarly, there exists $\hat{\tau} > 0$ and a solution $\hat{\hat{w}}$ which is defined and Lipschitz continuous in the region

\[ \hat{\hat{w}} = \{(x, t) : x \in (T, 0) \}, \]

and which coincides on $S_u$ with the restriction of $\hat{w}$ to the left side of $S_u$:

\[ \hat{\hat{w}}(P) = \hat{w}(P) \text{ if } P \in S_u. \]
Both \( \hat{w} \) and \( \check{w} \) may be constructed by extending \( S_\omega \) slightly below the x-axis and putting, say constant data along the extended part.

The approximate solution \( \hat{w} \) is defined in terms of \( \hat{w}^1 \) and \( \hat{w}^2 \) in the following way. There are two cases to consider accordingly as \( S_\omega \) is a shock of the first or second kind. If \( S_\omega \) is a 1-shock let

\[
\hat{u}_1^1 = \{(x_1(t), t) : 0 \leq t < T_0\}
\]

denote the minimal forward 1-characteristic in \( u \) and choose \( T_0 \) so small that restriction of \( \hat{u}_1^1 \) to the interval \( 0 < t < T_0 \) lies within the domains of definition of \( \hat{w} \). We define

\[
\hat{w}(x,t) = \begin{cases} u(x,t) & \text{if } x < x_1^1(t) \\ \hat{w}(x,t) & \text{if } x > x_1^1(t) \end{cases}
\]

for \( (x,t) \in \hat{S}(T_0) \). If \( S_\omega \) is a 2-shock let

\[
\hat{u}_2^2 = \{(x_2(t), t) : 0 \leq t < T_0\}
\]

denote the maximal forward 2-characteristic in \( u \) and choose \( T_0 \) so small that the restriction of \( \hat{u}_2^2 \) to the interval \( 0 < t < T_0 \) lies within the domains of definition of \( \hat{w} \). In this case, we define

\[
\hat{w}(x,t) = \begin{cases} \hat{w}(x,t) & \text{if } x < x_2^1(t) \\ u(x,t) & \text{if } x > x_2^1(t) \end{cases}
\]

for \( (x,t) \in \hat{S}(T_0) \).

Proof of Theorem 4.1. We shall first show that \( u \equiv \hat{w} \) at almost all \( (x,t) \) in \( \hat{S}(T_0) \). By the generalized Green's theorem

\[
y(S(t)) = \int_{\hat{S}_\omega} g(u(x,t), \hat{w}(x,t)) \, dx
\]

where \( y = y(u, \hat{w}) \). It follows from (3.4) that

\[
y(S(t) \cap \hat{S}_\omega^1) \leq \text{const.} \int_0^T |\hat{u} - \hat{w}|^2 \, dx \, dt,
\]

where \( S_\omega = \hat{S}_\omega^1 \). In order to prove that \( u \) and \( \hat{w} \) coincide almost everywhere, we need only show that

\[
y(S(t) \cap \hat{S}_\omega^1) \leq 0
\]

for \( 0 < t < T_0 \). Indeed, the approximate solution \( \hat{w} \) was constructed in order that the \( y(u, \hat{w}) \)-measure of the dominant wave \( S_\omega \) would be non-positive.

For concreteness let us assume that \( S_\omega \) is a 1-shock. Since \( u \) and \( \hat{w} \) coincide by definition to the left of \( S_\omega \), we have \( u \leq \hat{w} \) and

\[
y(S(t) \cap \hat{S}_\omega^1) = \int_0^T D(y, u, \hat{w}) \, dx \leq \int_0^T \beta_y \, dx dt
\]

where \( \beta_y = \alpha(u, \hat{w}) \) and \( \beta_y \) is \( \hat{y}(u, \hat{w}) \). We first note that \( u(T_0) \) and \( \hat{w}(T_0) \) coincide at \( \hat{H}_1 \) almost all points \( P \) of \( S_\omega \cap \hat{S}_\omega^1 \). Indeed, it follows from Theorem 3.2 that \( u \) and \( \hat{w} \) coincide on

\[
\{(x,t) : x < \min\{x(t), y(t)\}\}
\]

where \( x(t) \) denotes the minimal l-characteristic in \( u \). Since the state on the right side of a shock wave is uniquely determined by the state on the left together with the speed of propagation it follows that \( u_\omega(P) \) and \( \hat{w}_\omega(P) \) coincide at \( \hat{H}_1 \) almost all points \( P \) of \( S_\omega \cap \hat{S}_\omega^1 \). At such points \( u_\omega(P) = \hat{w}_\omega(P) \) by construction. Thus the quantity

\[
-\int_{\hat{H}_1} \beta_y \, d\hat{S}_\omega^1
\]

vanishes at \( \hat{H}_1 \) almost all points of \( S_\omega \cap \hat{S}_\omega^1 \). Therefore, in order to prove (4.11) we need only show that

\[
y(S(t) \cap \hat{S}_\omega^1) \leq 0
\]

for \( 0 < t < T_0 \). Since \( S_\omega \cap \hat{S}_\omega^1 \) is a relatively open subset of \( S_\omega \) we may restrict our attention to the open components; for concreteness we shall assume that

\[
\Delta(t) = y(t) - x(t) > 0
\]

if \( t \in (t_1, t_2) \), \( t_2 < T_0 \) and \( \Delta(t_1) = 0 \) and show that

\[
y(S(t_1) \cap \hat{S}_\omega^1) = 0
\]
for \( t_1 < t < t_2 \) where

\[ \mathcal{T}(t_1, t) = \{(x, t) : t_1 < t < t_2 \} \]

To this end we proceed as follows. Let

\[
\begin{align*}
\tilde{w}_x &= \tilde{u}_x(P) & \tilde{w}_r &= \tilde{w}_r(P) & w_x &= w_x(Q) \\
\tilde{w}_x &= \tilde{u}_x(P) & \tilde{w}_r &= \tilde{w}_r(P) & w_x &= w_x(Q)
\end{align*}
\]

where \( P = (x(t), t), \ Q = (y(t), t) \) and \( t_1 < t < t_2 \). We recall that \( u_x = \tilde{u}_x \).

**Proposition.** If the parameter \( \delta \) appearing in the statement of Theorem 4.1 is sufficiently small then

\[ (4.13) \quad \left| \frac{\partial}{\partial t} u_x, u_r \left( \tilde{w}_r \right) \right| \leq \text{const.} \left| \tilde{w}_r - u_x \right|^2 + \text{const.} \left| w_x - u_x \right|^2 + \text{const.} \left| \tilde{w}_r - w_x \right|^2 \]

where the constants are positive.

**Remark:** The estimate (4.13) provides a sense in which the projections \( P_{2n+1} E_n \mapsto P_{2n+1} E_n \) associated with pairs of shock waves \( S_u \) and \( S_w \) lie near the class of attracting sets.

**Proof of Proposition:** We shall establish a result which is slightly more general than the one stated in the proposition. Fix the state \( \tilde{u} \) and let \( B_\delta(\tilde{u}) \) denote the ball of radius \( \delta \) centered at \( \tilde{u} \). We claim that there exists a constant \( \delta \) depending only on \( \tilde{u} \) and \( f \) with the following property. Suppose that \( (\tilde{w}_r, \tilde{w}_x), (\tilde{w}_r, \tilde{w}_x) \), and \( (u_x, u_x) \) are three pairs of states in \( B_\delta(\tilde{u}) \) such that \( \tilde{w}_r = u_x \) and such that \( u_x \) and \( u_x, \tilde{w}_x \) as well as \( \tilde{w}_x \), and \( \tilde{w}_x \) are connected by admissible 1-shock waves. If \( \left| \tilde{w}_r - u_x \right| < \epsilon > 0 \) then

\[ (4.14) \quad \left| \frac{\partial}{\partial t} u_x, u_r \left( \tilde{w}_r \right) \right| \leq c_1 \left| \tilde{w}_r - u_x \right|^2 + c_2 \left| \tilde{w}_r - u_x \right|^2 + c_3 \left| \tilde{w}_r - \tilde{w}_x \right|^2 \]

where the constants \( c_1 \) and \( c_2 \) depend only on \( \epsilon \). cf. Figure 1. As usual, \( \tau \) denotes the wave speed corresponding to \( u_x \) and \( u_x \), i.e.

\[ R(t, u_x, u_x) = 0 \]

If the system (1.1) lies in the Smoller-Johnson class then the corresponding global result holds: if the three pairs lie in a compact subset of the domain of definition of \( n \) and \( q \) and have the stated properties then (4.14) holds with constants \( c_1 \) and \( c_2 \) depending only on \( \epsilon \).

The estimate (4.13) follows immediately from (4.14) by noting that for a solution \( w \) of the specified form

\[ \left| w_x - u_x \right| \geq \epsilon \]

for some \( \epsilon > 0 \).

Since \( D \) is smooth we may assume in proving (4.14) that without loss of generality

\[ \left| w_x - u_x \right| \quad \text{and} \quad \left| \tilde{w}_x - u_x \right| \]

are small. In this situation there exists a state \( w_x \) which lies on the 1-shock wave curve through \( u_x \) and whose wave speed \( \tau \) satisfies

\[ \tau = \frac{1}{\alpha} \quad \text{satisfies} \]

where \( \alpha \) denotes the wave speed of the pair \( (w_x, w_x) \), i.e.

\[ R(w_x, w_x) = 0 \quad \text{and} \quad R(w_x, w_x) = 0 \]

The idea is to regard \( D \mapsto \tilde{u}_x, e, \tilde{e}_x, \tilde{e}_x \) as a perturbation of

\[ D' \mapsto \alpha(u_x, w_x) + \beta(u_x, w_x) \]

To this end, we observe that

\[ \alpha(u_x, w_x) = \alpha(u_x, w_x) + \alpha(u_x, w_x) (\tilde{w}_r - w_x) \]

where \( \beta \) lies on the line segment joining \( \tilde{w}_r \) and \( w_x \). Since \( \alpha(u_x, u_x) = 0 \), we have

\[ \left| \alpha(u_x, \beta) \right| \leq e \left| \beta - u_x \right| \]

This fact together with a similar estimate on \( \beta \) yields

\[ (4.15) \quad \left| D - D' \right| \leq e \left| \beta - u_x \right| \left| \tilde{w}_r - w_x \right| \]

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A simple application of the triangle inequality shows that

\[ |\hat{w}_t - w_t| \leq |\hat{w}_x - w_x| + |w_t - w_x| + c|\hat{w}_x - w_x| \]

\[ |\hat{w}_x - w_x| \leq |\hat{w}_x - w_x| + c|\hat{w}_x - w_x| \]

Therefore,

\[ |\hat{w}_t - w_t| \leq \frac{1}{2} |\hat{w}_x - w_x|^2 + c|\hat{w}_x - w_x|^2 + |\hat{w}_x - w_x|^2. \]

Since, \[ D' \leq c|\hat{w}_x - w_x| |\hat{w}_t - w_t| \] it follows from (4.15) and (4.16) that

\[ D \leq c|\hat{w}_t - w_t| |\hat{w}_x - w_x| + c|\hat{w}_x - w_x|^2 + c|\hat{w}_x - w_x|^2 + \frac{c}{2}(\hat{w}_x - w_x)^2 + |\hat{w}_x - w_x|^2. \]

Since the coefficient of the leading term satisfies

\[ |\hat{w}_t - w_t| \leq c|\hat{w}_x - w_x| - c|\hat{w}_x - w_x| \]

we have for an appropriate choice of \( u \)

\[ D \leq c|\hat{w}_x - w_x| |\hat{w}_x - w_x|^2 + c|\hat{w}_x - w_x|^2 + |\hat{w}_x - w_x|^2. \]

The desired estimate (4.14) follows by observing that

\[ |\hat{w}_x - w_x| \leq |\hat{w}_x - w_x| + |\hat{w}_x - w_x| + c|\hat{w}_x - w_x| \]

and hence

\[ -|\hat{w}_x - w_x|^2 \leq -\frac{1}{2} |\hat{w}_x - w_x|^2 + c|\hat{w}_x - w_x|^2. \]

This completes the proof of (4.14).

Continuing with the proof of (4.12) we observe

\[ D \leq c_1 |\hat{w}_x - w_x|^2 + c_2 \Delta^2(t) \]

since

\[ |\hat{w}_x (p) - \hat{w}_x (q)| \leq \text{const.} \Delta(t) \] and \[ |u_x (p) - \hat{w}_x (q)| \leq \text{const.} \Delta(t). \]

We may relate the separation distance \( \Delta(t) \) to the deviation of the limiting states on the sides of the \( S_u \) and \( S_w \) with the aid of the Rankine-Hugoniot relations: using the fact that

\[ \Delta(t) \leq \int_{t_1}^{t} |y'(t) - x'(t)| dt \]

together with the smooth dependence of the shock speeds on the corresponding limiting states we may deduce in a straightforward way that

\[ (4.18) \quad A(t) \leq \text{const.} \int_{t_1}^{t} |x - \hat{x}| dt. \]

Combining (4.17) and (4.18) yields

\[ D \leq \text{const.} |u - \hat{u}| \phi(t) \int_{t_1}^{t} g(t) dt \]

where \( g(t) = c_1 |u - \hat{u}|^2 (t) \). Dropping the factor of \( (t-t_1) \) and integrating \( D \) with respect to \( t \) yields

\[ \int_{t_1}^{t} D dt \leq -\text{const.} \int_{t_1}^{t} g(t) dt \]

for \( t \) near \( t_1 \) since

\[ h(t) \geq \text{const.} \int_{t_1}^{t} g(t) dt \]

is non-decreasing. Thus, if we restrict our attention to a strip \( S(T_0) \) where \( T_0 \) is sufficiently small we conclude that (4.12) holds for \( 0 \leq t < T_0 \) and therefore that \( u \) and \( \hat{u} \) coincide almost everywhere on \( S(T_0) \). Since \( T_0 \) depends only on the magnitude of the shock wave \( S_u \) we may continue the argument to deduce that (4.12) holds on \( (0, T_0) \) and that \( u \) and \( \hat{u} \) coincide almost everywhere on \( S(T_0) \).

Lastly, we shall show that \( u \) and \( \hat{u} \) coincide almost everywhere on \( S(T) \). It follows from the analysis above that both \( u \) and \( \hat{u} \) are Lipschitz continuous on the complement of \( S_u \) and \( S_w \). Thus the symmetric part \( \gamma \) of \( \gamma_u \) of \( \gamma_u \) vanishes on \( S_u \) and \( S_w \) and is bounded on \( (S_u \cup S_w)^C \) by a tame measure:

\[ |\gamma_u| \leq \int_{E} \text{const.} |u-w|^2 d\sigma dt \]

if \( E \subset (S_u \cup S_w)^C \). Theorem 4.1 follows by an application of Gronwall's inequality to the function

\[ \int_{E} a(u(x,t), v(x,t)) dx. \]

This completes the proof of Theorem 4.1. The proof of Theorem 4.2 is virtually identical.
Remark: If one is interested in working with a measure which is symmetric in $u$ and $v$, then it will be necessary to employ a measure which does not reduce to the dissipation measure when one solution is constant. Such measures are more singular than the divergence of the vector-field obtained from the quadratic part of entropy.

**Figure 1.**

### 5. Rarefaction Waves

In this section we shall establish the uniqueness of centered rarefaction waves. Suppose that $w$ is a solution in $K_i(S(T))$. Consider the set

$$\Omega_j = \{(x,t) : x^j(t) \leq x \leq x^j(t), 0 \leq t < T\}$$

where $x^j_M$ and $x^j_m$ denote the minimal and maximal $j$-characteristics emanating from the origin. We shall say that $\Omega_j$ is a centered Lipschitz rarefaction wave if the following two conditions hold: $x^j_M(t) < x^j_m(t)$ for $0 < t < T$ and the restriction of the measure $\mu_k$ to $\Omega$ has the form

$$\mu_k = \frac{1}{\tau_k} \int_{\Omega_j} r_k(u(x,t)) \cdot dxdt$$

where $dxdt$ denotes 2-dimensional Lebesgue measure and $\tau_k = \tau_k(x,t)$

$k = 1, 2, \ldots, n$, are measurable functions satisfying

$$\text{const} \leq \tau_k \leq \text{const}/t$$

and $|\tau_k| \leq \text{const}$, if $k \neq j$.

For the purposes of this paper it would suffice to impose the weaker condition that coefficient of $\tau_k$ be bounded from below by an integrable function of $t$. We note that a classical centered $j$-rarefaction is a smooth similarity solution $u = u(x/t)$ which is defined in a domain of the form $a < x/t < b$ and satisfies there

$$u_x = r_j(u(x/t))/t$$

The classical centered $j$-rarefaction wave can be generated by solving the Riemann problem with initial states having the same $j$-Riemann invariants [25].

Consider a solution $w$ of a genuinely nonlinear system of two equations which is defined in a strip $S(T)$ and has the following properties: $w$ is Lipschitz continuous in each of the regions

$$\{(x,t) : x \leq x^j_M(t), 0 \leq t < T\} \quad \text{and} \quad \{(x,t) : x \geq x^j_m(t), 0 \leq t < T\}$$

and the set $\Omega_j$ defined by (5.1) is a Lipschitz centered $j$-rarefaction wave.

It is not difficult to prove that a solution $w$ with the above structure exists in a small strip $S(T)$ provided one is given initial data of the form
\[ w_0(x) = \begin{cases} w_1(x) & \text{if } x < 0 \\ w_2(x) & \text{if } x > 0 \end{cases} \]

where \( w_1 \) and \( w_2 \) are Lipschitz continuous functions whose limiting values at the origin, \( w_1(0^-) \) and \( w_2(0^+) \), have the same \( J \)-Riemann invariants.

**Theorem 5.1.** For every \( \tilde{u} \in \mathbb{R}^n \) there exists a constant \( \epsilon > 0 \) depending only on \( \tilde{u} \) and \( f \) with the following property. If \( u \in \mathcal{K}(\tilde{u}^+) \) with \( \|u - \tilde{u}\| < \epsilon \), \( \|\nabla u\| < \epsilon \), and \( u(x, 0) = w(x, 0) \) for almost all \( x \) then \( u = w \) for almost all \((x, t)\) in \( S(T) \).

**Theorem 5.2.** If \( w \) is a solution to the quasilinear wave equation (1.5) with \( p' < 0 \) and \( p^* < 0 \) and if \( u \in \mathcal{K}(w_0) \) with \( u(x, 0) = w(x, 0) \) for almost all \( x \) then \( u = w \) for almost all \((x, t)\) in \( S(T) \).

We shall begin by studying the structure of the measure \( \gamma(u, v) \) on centered rarefaction waves in the base solution \( v \) for general systems of equations.

**Lemma 5.1.** Let \( u \) and \( v \) be arbitrary weak solutions. Suppose that \( \tilde{u}_j \) is a centered Lipschitz \( J \)-rarefaction wave in \( v \). If \( E \in \tilde{u}_j \) then

\[ \gamma(E) = \gamma(E) = \int_0^1 \sum_{k=1}^n \phi_k(v) \mathcal{Q}f(u, v) \, dx \, dt \]

where \( \phi_k(v) = \varphi_k(v)/\varphi_n(v) \) and \( \mathcal{Q}f \) is the quadratic part of \( f \) at \( v \),

\[ \mathcal{Q}f = f(u) - f(v) - \mathcal{V}f(v)(u - v) \]

**Proof:** As we observed in Lemma 3.2 the right eigenvectors of \( \mathcal{V}f \) are bi-orthogonal with respect to \( \varphi_n^2 \). Thus \( \varphi_k \) is a left eigenvector of \( \mathcal{V}f \) and the lemma follows from the decomposition (3.4) and the representation (5.2).

The dominant coefficient in (5.3), \( a_{j,j} \), reflects the increased coupling between \( u \) and \( v \) near the center \((0,0)\) of the wave. The remaining coefficients are tame since \( a_k \) lies in \( L^\infty \) if \( k \neq j \).

The stability of a \( j \)-rarefaction wave is a consequence of the geometry of \( j \)-characteristics; the influence of waves crossing transversely, i.e. \( k \)-waves, \( k \neq j \), is subordinate to the spreading of \( j \)-characteristics. We shall presently show that the only unfavorable coupling occurs in the directions complementary to \( \tilde{u}_j \). For the purpose of formulating this result, let us restrict our attention to a small neighborhood \( N(\tilde{u}) \) of a fixed state \( \tilde{u} \) in \( \mathbb{R}^n \) and choose a coordinate system of functions \( \tilde{e}_j = \tilde{e}_j(u) \), \( j = 1, 2, \ldots, n \) which are defined in \( N(\tilde{u}) \) and satisfy

\[ \tilde{r}_j(\tilde{u}) \cdot \tilde{V}_{\tilde{e}_j}(\tilde{u}) = 1 \]

The functions \( \tilde{e}_j \) provide a convenient way to estimate the coordinates of a given vector in the basis of eigenvectors: if \( u \) and \( v \) lie in \( N(\tilde{u}) \) then

\[ u - v = \sum_{k=1}^n c_k \tilde{r}_k(v) \]

where

\[ c_k = \tilde{e}_k(u) - \tilde{e}_k(v) + O(|u - v|^2) \]

**Lemma 5.2.** There exist positive constants \( c_1, c_2 \) and \( \delta \) with the following property. If \( u \) and \( v \) lie in \( N(\tilde{u}) \) and \( |u - v| < \delta \) then

\[ \tilde{l}_{j}(v) \mathcal{Q}f(u, v) \geq c_1 (\tilde{e}_j(u) - \tilde{e}_j(v))^2 - c_2 \sum_{k \neq j} (\tilde{e}_k(u) - \tilde{e}_k(v))^2 \]

**Proof:** For the moment let \( \tilde{r}_k \) and \( \tilde{l}_k \) be arbitrary right and left eigenvectors of \( \mathcal{V}f \). Differentiating the eigenvalue equation \( \mathcal{V}f \tilde{r}_k = \lambda \tilde{r}_k \) in the direction \( \tilde{r}_k \) yields

\[ \mathcal{V}f \tilde{V}_{\tilde{r}_k} \cdot \tilde{r}_k + \tilde{V}^2 f \tilde{r}_k = \lambda \tilde{r}_k \cdot \tilde{V}_{\tilde{r}_k} - \tilde{r}_k \cdot \tilde{V}_{\tilde{r}_k} \]

Taking the inner product with \( \tilde{l}_k \) yields the identity

\[ \tilde{l}_k \mathcal{V}^2 f \tilde{r}_k = (\tilde{l}_k \cdot \tilde{r}_k) \tilde{V}_{\tilde{r}_k} \]

Thus, under the normalization \( \tilde{r}_j \cdot \tilde{V}_{\tilde{j}} = 1 \), the vector \( \tilde{l}_j \tilde{V}_{\tilde{j}} \varphi_n \) satisfies

\[ \tilde{l}_j \varphi_n \tilde{V}^2 f \tilde{r}_j \tilde{r}_j^* \varphi_n \tilde{r}_j > 0 \]
Letting \( u-v = \sum \chi_{k} \hat{g}_{k}(v) \) we obtain

\[
\begin{align*}
\ell_{j}(u) \mathcal{Q}(u,v) &= \sum \chi_{k} \hat{g}_{k}(v) + \mathcal{O}(u-v)^{3} \\Rightarrow \\sum \chi_{k} \hat{g}_{k}(v) \\text{is bounded}.
\end{align*}
\]

This completes the proof of the lemma.

**Corollary 5.1.** Suppose that \( \eta_{j} \) is a centered Lipschitz \( j \)-rarefaction wave in a solution \( v \) and suppose that \( u \) is an arbitrary weak solution.

If \( |u-v|_{\infty} \leq \delta \) and \( E \subset \mathbb{R} \) then

\[
|u|_{\infty} + |w|_{\infty} \leq M
\]

\[
osc \; u + osc \; w \leq \delta
\]

\[
(\text{5.5}) \quad \int_{0}^{t} \int_{\mathbb{R}^{2}} (\hat{g}_{j}(u)-\hat{g}_{j}(v))^{2} dx \leq \text{const.} \cdot |\hat{g}_{j}(w(t))|_{\infty}^{2}
\]

where \( \delta \) depends only on \( |u-v|_{\infty} \) and \( f \).

**Remark:** The constant which appear in (5.4) depend only on the bounds for the coefficients \( \hat{g}_{k} \). The estimate (5.4) may be sharpened by taking advantage of the favorable sign of the coefficient \( \{ e_{j}(u) - e_{j}(v) \} \). We shall not make use of this refinement.

Henceforth, we shall restrict our attention to genuinely nonlinear systems of two equations and solutions with small oscillation. An extension to solutions with large oscillation will be discussed below. The following lemma shows that the coupling between \( u \) and \( w \) on a \( j \)-rarefaction wave in \( w \) and in \( L^{1} \) of the complementary direction \( r_{k} \), \( k \neq j \) is bounded by a small fraction of the dissipation in \( u \) plus a quantity on the order of the \( L^{1} \)-deviation between \( u \) and \( w \) in space-time. Suppose that \( j = 1 \) and let

\[
\xi_{n}(t) = ((x,s) ; x < x_{n}(s) ; 0 \leq s \leq t)
\]

where \( x_{n}(s) \) is the maximal \( l \)-characteristic in \( w \) passing through the origin.

**Lemma 5.1.** For every \( \epsilon > 0 \) and \( M > 0 \) there exists a constant \( d(\epsilon,M) \) with the following property. If \( u \) is a solution in \( K_{2}(\mathbb{R}) \) whose initial data coincide with \( w(x,0) \) at almost all \( x \) and if

\[
|u|_{\infty} + |w|_{\infty} \leq M
\]

\[
osc \; u + osc \; w \leq \delta
\]

then for \( 0 \leq t < T \)

\[
(\text{5.6}) \quad \int_{0}^{t} \int_{\mathbb{R}^{2}} (\hat{g}_{j}(u(s,t)) - \hat{g}_{j}(w(s,t)))^{2} dx \leq \text{const.} \cdot |\hat{g}_{j}(w(t))|_{\infty}^{2}
\]

\[
+ \text{const.} \int_{0}^{t} \int_{\mathbb{R}^{2}} |u(s,t) - w(s,t)|^{2} dx dt
\]

**Remarks:** The constants appearing in (5.5) depend only on \( f \) and \( M \). A similar result holds in the case \( j = 2 \) for the region \( s_{m}^{2} \) to the right of the minimal characteristic \( s_{m}^{2} \) passing through the origin:

\[
(\text{5.6}) \quad \int_{0}^{t} \int_{\mathbb{R}^{2}} (\hat{g}_{j}(u(s,t)) - \hat{g}_{j}(w(s,t)))^{2} dx \leq \text{const.} \cdot |\hat{g}_{j}(w(t))|_{\infty}^{2}
\]

\[
+ \text{const.} \int_{0}^{t} \int_{\mathbb{R}^{2}} |u(s,t) - w(s,t)|^{2} dx dt
\]

if \( u(x,0) \) and \( w(x,0) \) coincide for almost all \( x > 0 \).

In order to prove Lemma 5.3 we shall construct two one-parameter families of non-convex entropies. We may assume without loss of generality that \( \{ \hat{g}_{j} \} \) forms a coordinate system of Riemann invariants, i.e.,

\[
\hat{g}_{j}(u) \cdot V_{A}(u) = \delta_{jk}
\]

For each \( \hat{u} \in \mathbb{R}^{2} \) we shall construct two families of entropies \( \eta_{j} = \eta_{j}(\hat{u},.) \) and corresponding fluxes \( q_{j} = q_{j}(\hat{u},.) \), \( j = 1,2 \), which are defined and twice continuously differential on a set of the (convenient) form

\[
B(\hat{u}) \times I_{j}(\hat{u})
\]

where

\[
B(\hat{u}) = \{ \hat{u} \in \mathbb{R}^{2} ; |\hat{g}_{j}(\hat{u}) - \hat{g}_{j}(\hat{u})| \leq \epsilon, j = 1,2 \}
\]

\[
I_{j}(\hat{u}) = \{ \hat{u} \in \mathbb{R}^{2} ; |\hat{u} - \hat{g}_{k}(\hat{u})| \leq \epsilon, k \neq j \}
\]

We regard \( \eta_{j} \) and \( q_{j} \) as parametrized by \( \phi \). In addition \( \eta_{j} \) and \( q_{j} \) will have the following three properties. If \( u \) and \( v \) lie in \( B(\hat{u}) \) and \( \hat{g}_{k}(v) \) lies in \( I_{j}(\hat{u}) \) then
\[ P_1 : \quad \text{const} \cdot (\mathbf{e}_k(u) - \mathbf{e}_h(v))^2 \leq \eta_j(u_{k, h}(v)) \leq \text{const} \cdot (\mathbf{e}_k(u) - \mathbf{e}_h(v))^2, \quad k \neq j. \]

\[ P_2 : \quad \int_{\mathbf{e}_h(\mathbf{q})}^2 \eta_j(u_{k, h}(v), \mathbf{q}) \geq 0. \]

\[ P_3 : \quad \mathbf{q}_1(u_{k, h}(v), \mathbf{q}) - \mathbf{q}_1(u_{k, h}(v)) \leq 0 \quad \text{and} \quad \mathbf{q}_1(u_{k, h}(v), \mathbf{q}) - \mathbf{q}_1(u_{k, h}(v)) \geq 0. \]

where the constants in \( P_1 \) are positive. Roughly speaking, \( \eta_j \) is nearly convex in the minor characteristic direction \( r_k \) and nearly flat in the major characteristic direction \( r_j \). In order to construct \( \eta_j \) and \( \mathbf{q}_j \) let us take the curl of the compatibility equation \( \nabla \times \mathbf{f} = \mathbf{q} \) to obtain a smooth strictly hyperbolic second order linear equation for \( \eta : \)

\[ \text{curl} (\nabla \eta(u) \times \mathbf{f}(u)) = 0. \]

The characteristic directions for (5.7) are given by the right eigenvectors \( r_j \) of \( \mathbf{f} \): taking the inner product of the compatibility equation with \( r_j \) yields

\[ \mathbf{e}_j \cdot \mathbf{q}_j - \mathbf{q}_j \cdot r_j = 0. \]

Thus, the level curves of the Riemann invariants \( \mathbf{e}_j \) are precisely the characteristics of (5.7) and one may deal equally well with the canonical form of (5.4):

\[ \mathbf{e}_1 = \alpha \mathbf{e}_1 + \beta \mathbf{e}_2 + \gamma \mathbf{e}_3 + \mathbf{e}_4 = 0. \]

In constructing \( \eta_j \) and \( \mathbf{q}_j \) it is convenient to employ the rarefaction wave curves and the compression wave curves through a specified point \( u_0 \):

\[ \mathbf{R}_j(u_0) = \{ u : \mathbf{e}_j(u) = \mathbf{e}_j(u_0), \mathbf{e}_k(u) = \mathbf{e}_k(u_0), k \neq j \} \]

\[ \mathbf{C}_j(u_0) = \{ u : \mathbf{e}_j(u) = \mathbf{e}_j(u_0), \mathbf{e}_k(u) = \mathbf{e}_k(u_0), k \neq j \}. \]

First, we shall construct the family of entropies \( \eta_j(u_0) \). Fix a point \( \mathbf{u} \in \mathbb{R}^3 \) and consider a nearby point \( \mathbf{v} \neq \mathbf{u} \) which lies on \( \mathbf{R}_j(u_0) \). Let \( \mathbf{v} \) denote a typical point on the wave curve \( \mathbf{W}_j(\mathbf{v}) = \mathbf{W}_j(\mathbf{v}) \cup \mathbf{C}_j(\mathbf{v}) \). We shall associate with such \( \mathbf{v} \) the following two local Goursat problems for the equation (5.7), cf. Figure 2.

\[ \eta_j = 0 \text{ on } \mathbf{C}_j(\mathbf{v}) \text{ and } \eta_j = (\mathbf{e}_j(u) - \mathbf{e}_j(v))^2 \text{ on } \mathbf{R}_j(\mathbf{v}). \]

\[ \eta_j = 0 \text{ on } \mathbf{C}_j(\mathbf{v}) \text{ and } \eta_j = (\mathbf{e}_j(u) - \mathbf{e}_j(v))^2 \text{ on } \mathbf{R}_j(\mathbf{v}). \]

The entropy \( \eta_j = \eta_j(u_0) \) is defined as follows. Let \( \mathbf{v}(\mathbf{e}) \) denote the point of \( \mathbf{W}_j(\mathbf{v}) \) such that

\[ \mathbf{e}_j(\mathbf{v}(\mathbf{e})) = \mathbf{e}. \]

In the quadrant \( \mathbf{Q}_j(\mathbf{v}(\mathbf{e})) \) define \( \eta_j(u_0, \mathbf{e}) \) to be the solution of problem (I) and in \( \mathbf{Q}_2(\mathbf{v}(\mathbf{e})) \) define \( \eta_j(u_0, \mathbf{e}) \) to be the solution of problem (II). It follows directly from the classical representation formula for the solution of the Goursat problem that \( \eta_j \) is defined and twice continuously differentiable in a neighborhood \( \mathbf{N}(\mathbf{v}) \) of \((\mathbf{v}, \mathbf{r}_j(\mathbf{v}))\) of the form

\[ \mathbf{N}(\mathbf{v}) = \{ u \in \mathbb{R}^3 : |\mathbf{e}_j(u) - \mathbf{e}_j(\mathbf{v})| < \mathbf{v}, \mathbf{r}_j(\mathbf{v}) \times (\mathbf{v} - \mathbf{r}_j(\mathbf{v})) < \mathbf{v}, \text{ and satisfies there property } P_1. \}

Furthermore, we may guarantee that \( \mathbf{v} \) is uniformly bounded away from zero if \( \mathbf{v} \) is confined to a compact set. Thus, one may associate with each vector \( \mathbf{v} \) in a given compact subset of \( \mathbb{R}^3 \) an auxiliary vector \( \hat{\mathbf{v}} \) such that

\[ \mathbf{N}(\mathbf{v}) = \mathbf{N}(\hat{\mathbf{v}}) \times \mathbf{I}_j(\mathbf{v}) \]

and \( \mathbf{v} \) is bounded away from zero. We observe that property \( P_2 \) is a corollary of property \( P_1 : P_1 \) implies that

\[ \mathbf{r}_2(\hat{\mathbf{v}}) \mathbf{v} \mathbf{e}_j(\mathbf{v} + \mathbf{r}_2(\hat{\mathbf{v}})) \mathbf{r}_2(\hat{\mathbf{v}}) > 0 \]

and

\[ \mathbf{r}_2(\hat{\mathbf{v}}) \mathbf{v} \mathbf{e}_j(\mathbf{v} + \mathbf{r}_2(\hat{\mathbf{v}})) \mathbf{r}_2(\hat{\mathbf{v}}) = 0 \]

since \( \eta_j \) vanishes along \( \mathbf{C}_j(\mathbf{v}). \)

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Lastly, we shall show that property \( P_3 \) holds in \( B(\hat{u}) \times I(\hat{u}) \) with a possibly smaller choice of \( \Gamma \). We normalize the flux \( q_1(u, s) \) by the requirement that it vanish at \( \nu(s) \):

\[
q_1(u, s) = \int_{\nu(s)}^{1} \eta_1(u, s, v) \nu(v) \, dv.
\]

Suppose now that \( u \) and \( v \) lie in \( B(\hat{u}) \) and that \( \delta_2(v) \) lies in \( I(\hat{u}) \).

Let us fix \( v \) and examine the function

\[
u(u) = \lambda_1(v) \eta_1(u, \delta_2(v)) - q_1(u, \delta_2(v)).
\]

Our goal is to show that \( \nu \) is non-positive. We first observe that \( \nu \) vanishes on that portion of the wave curve \( W_1(v) = \eta_1(v) \cup C_1(v) \) which lies in \( B(\hat{u}) \) : \( \eta_1 \) vanishes by definition and \( q_1 \) vanishes since

\[
r_1(u) - \nu \eta_1(u, \delta_2(v)) = \lambda_1(v) r_1(u) - \nu \eta_1(u, \delta_2(v)) = 0
\]

if \( u \) lies on \( W_1(v) \). Similarly it can be seen that the derivative of \( \nu \) in the direction \( r_2 \) vanishes on \( W_1 \) : if \( u \in W_1 \) then

\[
r_2(u) = \nu \eta_1(u, \delta_2(v)) = \lambda_2(v) r_2(u) - \nu \eta_1(u, \delta_2(v)) = 0
\]

by property \( P_2 \). Next, we shall expand \( \nu \) in a finite Taylor series. Fix a point \( u \) near \( v \) and let \( z \) denote the point of intersection of \( W_1(v) \) and \( W_2(u) \). Then

\[
\eta_1(z, \delta_2(v)) = \frac{1}{2} r_2^2(z) \left[ u_0 \eta_1(z, \delta_2(v)) r_2(z) (\delta_2(u) - \delta_2(v))^2 + o(\delta_2(u) - \delta_2(v))^2 \right] (5.9)
\]

\[
q_1(z, \delta_2(v)) = \frac{1}{2} r_2^2(z) \left[ u_0 q_1(z, \delta_2(v)) r_2(z) (\delta_2(u) - \delta_2(v))^2 + o(\delta_2(u) - \delta_2(v))^2 \right].
\]

Differentiating the compatibility equation twice in the direction \( r_2 \) shows that the leading terms in (5.9) differ by a factor of \( \lambda_2^2 \). Therefore

\[
\lambda_2(z) \eta_1(z, \delta_2(v)) - q_1(z, \delta_2(v)) = o(\delta_2(u) - \delta_2(v))^2.
\]

We conclude that

\[
\nu = (\lambda_1(v) - \lambda_2(v)) \eta_1(u, \delta_2(v)) + \lambda_2(v) \eta_1(u, \delta_2(v)) - q_1(u, \delta_2(v))
\]

\[
= (\lambda_1(v) - \lambda_2(v)) \eta_1(u, \delta_2(v)) + o(\delta_2(u) - \delta_2(v))^2 \leq 0,
\]

if \( u \) and \( v \) are sufficiently close. This completes the verifications of properties \( P_1 - P_3 \) for \( \eta_1 \) and \( q_1 \). The construction of \( \eta_2 \) and \( q_2 \) is similar.

Proof of Lemma 5.3. For concreteness we shall consider the case where \( w(x, t) \) contains a centered 1-rarefaction wave. Applying Green's theorem to the measure

\[
u \eta_1(u, \delta_2(w)) \, dx + q_1(u, \delta_2(w)) \, dx,
\]

we obtain

\[
\nu \eta_1^1(t) = \int_{-\infty}^{\infty} \nu \eta_1(u(x, t), \delta_2(w(x, t))) \, dx + \int_{-\infty}^{\infty} \nu \eta_1 + \nu \, q_1 \, ds
\]

where the second integral is taken over \( \{x(s) : x = \eta_1^1(t) ; 0 \leq s < t\} \).

Since the second integral is non-negative by virtue of property \( P_2 \) we need only estimate the \( \nu \)-measure of \( \eta_1^1(t) \). To this end, let us partition the domain into jump points of \( \nu \) and points of approximate continuity of \( \nu \):

\[
\eta_1^1(t) = J \cup A
\]

where \( J = J(t) \equiv \eta_1^1(t) \cap \Gamma(u) \) and \( A = A(t) = \eta_1^1(t) \cap \Gamma(u) \). We claim that

\[
\nu(J) \leq \text{const.} \, \epsilon \|\nu(J)\|
\]

if the range of \( u \) and \( w \) is contained in a sufficiently small set. It follows from a theorem of Lax [26] that if \( \eta_1^1(t) \) is an arbitrary \( C^2 \) entropy and \( \tilde{q}(u) \) its associated flux and if \( u_L \) and \( u_R \) are connected by a \( j \)-shock with speed \( \tau \) and magnitude \( \omega \) then the associated rate of dissipation is third order in \( \omega \):

\[
\tau[\eta_1^1] - [\tilde{q}] = \frac{1}{2} r_j^2(u) \nu \tilde{q}^2(u) r_j(u) \nu \tilde{q}(u) \nu^3 + o(\omega^3).
\]

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Therefore, if \( u \) and \( v \) are two nearby vectors in \( K \), property \( P_2 \) implies that
\[
\varepsilon^2_{\varepsilon} = (u, u)^2 (v) \geq -\varepsilon
\]
and we conclude that
\[
\tau((u, u)^2 (v)) = \eta_{u}(u, u)^2 (v)) - q_{2}(u, u)^2 (v)) + q_{1}(u, u)^2 (v)) \leq \text{const.} \cdot |v|^{3} \leq \text{const.} \cdot |\tau(n) - \varepsilon|
\]
if \( u \) and \( u \) are connected on a weak admissible \( j \)-shock. This completes the proof of (5.10). Next, we shall estimate the \( \mu \)-measure of \( A \). We claim that if \( E \subset A \) then
\[
|\omega(\xi)| \leq \int_{E} |u-w|^{2} \frac{3}{\bar{\nu}_{2}} \, dx \, dt
\]
where \( \bar{\nu}_{2} = \bar{\nu}_{2}(w(x,t)) \) and the constant depends only on \( E \). Now, by the chain rule we have on the set \( A \)
\[
u = \frac{n_{u}(u,t) + q_{u}(u,t)}{x}
\]
where we have suppressed the subscripts for simplicity in printing. We observe that the coefficient of \( u_{x} \) vanishes and that \( \phi = \phi(w(x,t)) \) satisfies the characteristic equation
\[
\varepsilon_{t} + \lambda_{2}(w) \phi_{x} = 0.
\]
Thus on \( A \), \( \nu = p(u(x,t), w(x,t), \phi(x,t)) \phi_{x}(x,t) \) where
\[
p(u,w,t) = \frac{q_{2}(u,t) - \lambda_{2}(w) n_{u}(u,t)}{x}.
\]
In order to prove (5.11) it is sufficient to show that
\[
|p(u,w,t)| \leq \text{const.} \cdot |u-w|^{2}
\]
If we write \( n(u, \phi) \) in the form \( n(\phi) = a(\phi, \phi)(\phi(u) - \phi)^2 \) and then differentiate with respect to \( \phi \) we obtain
\[
\eta_{\phi}(u, \phi) = -2a(\phi(u) - \phi)^2 + a_{2}(\phi(u) - \phi)^2
\]
\[
= -2(\phi(u) - \phi)^2 + O((\phi(u) - \phi)^2).
\]

The boundedness of \( a_{2} \) follows from the integral representation of \( \nu(u, \phi) \).
In a similar way we obtain
\[
q_{2}(u, \phi) = -2q_{2}(u, \phi)/(\phi(u) - \phi) + O((\phi(u) - \phi)^2).
\]
Therefore,
\[
p(u,w,t)(\phi(u, \phi) = \frac{p(u,w,t)(\phi(u) - \phi)^2 + O((\phi(u) - \phi)^2}{\text{const.}}.
\]
We shall complete the proof of (5.11) by showing that
\[
|p(u,w,t)(\phi(u, \phi) = \text{const.} \cdot |u-w|^{2}(\phi(u) - \phi)^2.
\]
Fix the states \( u \) and \( v \) and let \( u' \) denote the point of intersection of \( W_{1}(u) \) and \( W_{2}(v) \). We observe that
\[
\lambda_{2}(v) = n(u', \phi(v)) - q(u', \phi(v)) = O((\phi(u') - \phi(v)^2)
\]
\[
\eta_{u}(u,v) = O((\phi(u') - \phi(v)^2) + O((\phi(u') - \phi(v))^{2})
\]
These facts together with corresponding estimates for \( q \) establish (3.12).
This completes the proof of Lemma 5.3.

**Proof of Theorem 5.1.** Suppose \( u \) lies in \( K \) and \( u_{0}(x) = \omega_{0}(x) \) for almost all \( x \). Then
\[
\int_{K} a(u(x,t), w(x,t)) dx = \gamma(t)
\]
Let us assume for concreteness that \( w \) contains a centered Lipschitz 1-

rarefaction wave \( \gamma(u) \). Since \( w \) is Lipschitz on \( 0_{1} \),
\[
\gamma(t) \subset \gamma(u) \subset \gamma(t) + \text{const.} \cdot |u-w|^{2} \, dx \, dt,
\]
while Corollary 5.1 implies that
\[
\gamma(t) \subset \gamma(u) \subset \gamma(t) + \text{const.} \cdot (\phi(u) - \phi(u))^{2}
\]
\[
\gamma(t) \subset \gamma(u) \subset \gamma(t) + \text{const.} \cdot |u-w|^{2} \, dx \, dt.
\]
Therefore,
\[
\int_{K} a dx = \gamma_{u}(t) + \text{const.} \int_{0}^{t} (\phi_{2}(u) - \phi_{2}(u))^{2} \, dx \, dt
\]
\[
+ \text{const.} \cdot |u-w|^{2} \, dx \, dt.
\]
It follows from Lemma 5.3 that

\[ \phi_u (g^1(t)) \leq \int_0^t \frac{\text{const.}}{c} \left( \dot{\phi}_2 (u) - \dot{\phi}_2 (w) \right)^2 \, dx + \int_0^t \frac{1}{\text{const.}} |u-w|^2 \, dt \]

if the oscillation of \( u \) and \( w \) is sufficiently small. Therefore,

\[ \frac{1}{k_N(t)} \int_0^t \text{const.} \frac{1}{1+c} \left( \dot{\phi}_2 (u) - \dot{\phi}_2 (w) \right)^2 \, dx + \int_0^t \text{const.} \left( \phi_2 (u) - \phi_2 (w) \right) \frac{1}{k_N(t)} \int_0^t \text{const.} \frac{1}{1+c} \left( \dot{\phi}_2 (u) - \dot{\phi}_2 (w) \right)^2 \, dx + \int_0^t \frac{\text{const.}}{c} |u-w|^2 \, dt. \]

If we let \( q(t) \) denote the right hand side of (5.13) then

\[ 0 \leq q(t) \leq \int_0^t \left( \frac{\text{const.}}{c} + \frac{\text{const.}}{t} \right) g(s) \, ds \]

which in general implies that \( g \leq 0 \) if \( g = o(t^c) \) as \( t \) approaches zero.

However the latter condition is satisfied since we may take \( c \) small and

\[ \int_0^t |u(s,t) - v(s,t)|^2 \, ds \leq \text{const.} \, t \]

by virtue of finite propagation speed. This completes the proof of Theorem 5.1.

Proof of Theorem 5.2. A straightforward calculation shows that

\[ \tau_j (v) \Omega(v) \geq 0 \]

for both \( j=1 \) and \( j=2 \) if \( \rho' < 0 \) and \( \rho'' > 0 \). We conclude from Lemma 5.1 that in the case of quasilinear wave equation (1.5)

\[ \gamma(u,w) = \int E \text{const.} \, |u-w|^2 \, dx \]

if \( E \) is contained in a centered Lipschitz \( j \)-rarefaction wave, \( j = 1 \) or \( j = 2 \). Theorem 5.2 follows immediately by applying \( \gamma \) to the strip \( \mathcal{S}(T) \).

We may now combine the results of the previous sections to obtain the uniqueness of piecewise Lipschitz solutions.

Theorem 6.1. Consider a genuinely nonlinear system of two conservation laws of the form (1.1). For every \( \tilde{u} \in \mathbb{R}^2 \) there exists a constant \( \delta > 0 \) depending only on \( \tilde{u} \) and \( f \) with the following property. If \( u \in K(S(T)) \neq \emptyset \), then \( \|u - \tilde{u}\|_{\infty} < \delta \) and \( u(x,0) = w(x,0) \) for almost all \( x \) then \( u = w \) for almost all \( (x,t) \) in \( S(T) \).

Theorem 6.2. If \( u \in K(S(T)) \) and \( w \in PL(S(T)) \), \( u \neq w \), are solutions to the quasilinear wave equation (1.5) with \( p' < 0 \) and \( p'' > 0 \) and if \( u(x,0) = w(x,0) \) for almost all \( x \) then \( u = w \) for almost all \( (x,t) \) in \( S(T) \).

Suppose that \( u \in K(S(T)) \) and \( w \in PL(S(T)) \). Let \( \chi_1^1(t) \) and \( \chi_2^1(t) \) denote the minimal 1-characteristics in \( u \) and \( w \) passing through the point \( (a,0) \). Let \( \chi_1^2(t) \) and \( \chi_2^2(t) \) denote the maximal 2-characteristics in \( u \) and \( w \) passing through the point \( (a,0) \). Let \( \chi_1^2(t) = \max(\chi_1^2(t),\chi_2^2(t)) \) and \( \chi_2^2(t) = \min(\chi_1^2(t),\chi_2^2(t)) \).

Theorem 6.3. Suppose that \( u(x,0) = w(x,0) \) for almost all \( x \in (a,b) \). Under the hypotheses of Theorem 6.1 and Theorem 6.2 respectively, \( u(x,t) = w(x,t) \) for almost all \( (x,t) \) satisfying \( \chi_1^1(t) < x < \chi_2^2(t), \ 0 \leq t < T \).

We also obtain uniqueness of classical solutions to the Riemann problem for systems with degenerate eigenvalues. We recall that the classical solution of the Riemann problem is a similarity solution \( w = w(x/t) \) which consists of constant states separated by admissible shock waves, contact discontinuities and centered rarefaction waves.

Theorem 6.4. Consider a system of two conservation laws whose eigenvalues \( \lambda_j \) are either genuinely nonlinear or linearly degenerate in the sense of Lax. For every \( \tilde{u} \in \mathbb{R}^2 \) there exists a constant \( \delta > 0 \) depending only on \( \tilde{u} \) and \( f \) with the following property. If \( w \) is a classical solution to the Riemann problem and if \( u \in K \) with \( \|w - \tilde{u}\|_{\infty} < \delta, \|u - \tilde{u}\|_{\infty} < \delta \), then \( u(x,0) = w(x,0) \) almost everywhere then \( u = w \) almost everywhere.

Sketch of proof: We observe that if \( \lambda_j \) is linearly degenerate, i.e., \( x_j - 1 \eta_j = 0 \), then the dissipation function

\[
d(t, w_t, w_{tt}) = \tau(n(w_t) - n(w)) - q(w) + q(w)
\]

vanishes if \( w = w_{\tau} \) and \( w_{\tau} \) are connected by a \( j \)-contact with speed \( \tau \). Therefore, the function \( D(t, u, u_t, v, v_t) \) vanishes if both pairs \( (u, v) \) and \( (v, v_t) \) are connected by \( j \)-contacts such that either \( u = v \) or \( v = v_t \). Let us assume for simplicity that \( w = w(x/t) \) consists of a 1-contact separating two constant states:

\[
w = \begin{cases} w_t & \text{if } x/t < \tau \\ w_{\tau} & \text{if } x/t > \tau \end{cases}
\]

where \( v_{\tau}, w_{\tau} \) and \( \tau \) satisfy (6.1) with \( j = 1 \). Let \( S^1_m \) denote the minimal 1-characteristic in \( u \) passing through the origin and consider the comparison function \( \tilde{w} \) defined by

\[
\tilde{w} = \begin{cases} w_{\tau} & \text{if } x < x(t) \\ w_t & \text{if } x > x(t) \end{cases}
\]

We observe that

\[
\gamma(u, \tilde{w})(S^1_m) = 0
\]

since both of the limiting states of \( u \) and \( \tilde{w} \) on the right side of \( S^1_m \) lie on the 1-contact curve through \( w_{\tau} \). In addition, by (3.4) \( \gamma(u, \tilde{w}) \) is
non-positive on the complement of \( \frac{1}{w^2} \) since \( \tilde{w} \) is identically constant there. We conclude that \( u \) and \( \tilde{w} \) coincide almost everywhere. Hence 
\( \frac{1}{w} \) is a straight line propagating with speed \( \frac{1}{2} (w_t) \) and \( w = \tilde{w} \). This completes the proof of Theorem 6.3.

In a similar way one may obtain uniqueness theorems in the large for the Riemann problem for systems of two equations such that one characteristic field is linearly degenerate while the other satisfies \( \frac{1}{2} QF \geq 0 \).

7. The Entropy Rate Criterion

In this section we shall show that the entropy rate criterion of Dafermos [4] is satisfied by Lipschitz continuous solutions to general systems of \( n \) equations and by PL solutions to the quasilinear wave equation. For simplicity we shall formulate the entropy rate criterion for weak solutions defined on the strip \( S(T) \). A weak solution \( u \) defined on \( S(T) \) is said to satisfy the entropy rate criterion with respect to a class \( \mathcal{G} \) of weak solutions if for every \( a \) and \( b \) there exists a nullset \( \mathcal{K}(a,b) \) of \( (0,T) \) with the following property: Fix \( \tau \in \mathcal{K}(a,b) \) and suppose that \( u \) is a weak solution whose domain of definition \( \mathcal{U} \) is open and contains

\( \{(x,t) : a \leq x \leq b, 0 \leq t < \tau \} \).

If

\( u(x,t) = w(x,t) \)

for almost all \( x \) in \( \mathcal{U} \cap \{(x,t) : t = \tau \} \), then

\[
D \delta_u (R(\tau)) \leq D^+ \delta_u (R(\tau))
\]

where \( D^+ \) denotes the lower right hand derivative and

\( R(\tau) = \{(x,t) : a < x < b, 0 \leq t < \tau \} \).

We note that the derivative

\( D \delta_u (R(\tau)) \leq \frac{d}{dt} \delta_u (R(\tau)) \)

electronically exists for almost all \( \tau \) since \( \delta_u (R(\tau)) \) is an absolutely continuous function of \( \tau \). (As always we assume that \( u = \tilde{w} \)). Indeed, if \( u \) is an arbitrary weak solution then

\[
\theta_u (R(\tau)) = \int_a^b \eta \rho (u(x,t)) + n (u(x,t)) dx + \int_a^\tau q (u(b,0,s) - q (u(a+0,s)) ds
\]

where

\( R(\tau) = \{(x,s) : a < x < b, t \leq s < \tau \} \).

The terms on the right hand side of (7.2) are absolutely continuous functions. Using (7.2) the entropy rate criterion can be stated in terms of
the rate of decay of local entropy: for almost all \( x \), \( D\theta_q \) equals the time rate of change of total entropy in \((a,b)\) modulo the flux of \( q \) at the boundary points \( a \) and \( b \), i.e.

\[
D\theta_q R(t) = D \int_a^b \eta(u(x,t)) dx + q(u(b-0,t)) - q(u(a+0,t))
\]

One may, of course, formulate slight variants of (7.1).

We shall first show that Lipschitz continuous solutions satisfy (7.1) with respect to the class \( G_1 \) of weak solutions \( u \) with the following properties: \( u \) is defined on an open domain \( U \subset \mathbb{R}^T \) and if

\[
\text{esslim}(u(x,t_0)) = x \ast x_0
\]

exists where \( (x_0,t_0) \in \mathbb{R} \) then

\[
\text{esslim}(u(x,t)) = (x,t \ast (x_0,t_0), t > t_0)
\]

exists and they are equal. We recall that the existence of the essential limit means that the function can be modified on a set of zero Lebesgue measure in such a way that the corresponding pointwise limit exists. The above condition may be regarded as a very mild form of finite propagation speed for waves.

**Theorem 7.1.** Suppose that (1.1) is a system of conservation laws with a smooth entropy-entropy flux pair \( (\eta,q) \) where \( \eta \) is strictly convex.

If \( \psi \) is a Lipschitz continuous solution defined on \( \mathbb{R}^T \) then \( \psi \) satisfies (7.1) with respect to the class \( G_1 \).

**Proof:** Fix \( a \) and \( b \) with \( 0 < a < b \) and \( t > t_0 \). Consider the trapezoidal regions of the form

\[
\zeta(x,t) = ((x,s) : x < x \ast y(x), x \ast y(x) < s < t)
\]

where

\[
x(s) = a + c(s-t), \ y(s) = b - c(s-t)
\]

and \( t - t_0 \) is small. Suppose that \( \psi \) is a solution in \( G_1 \) where domain

of definition contains

\[
(x,t) : a < x < b, \ t = t_0
\]

and such that \( u(x,t) = w(x,t) \) for almost all \( x \). It follows from the proof of Theorem 3.1 that, for a sufficiently large constant \( c \),

\[
g(t) \leq \theta_t[G(t,t)] + \text{const.} \int_0^t \int_x \eta(s) ds
\]

where

\[
g(t) = y(t)
\]

Thus,

\[
g(t) \leq \theta_t[G(t,t)] + \text{const.} \int_0^t \int_x \eta(s) ds
\]

and we conclude that

\[
0 \leq \theta_t[G(t,t)] + o(t-1)
\]

since \( \theta_t[G(t,t)] \) is continuous and vanishes at \( t = 1 \). We shall complete the proof by showing that

\[
\theta_t[R(t,t)] = \theta_t[G(t,t)] + o(t-1)
\]

Combining (7.3) and (7.4) yields

\[
D\theta_q(R(t)) = 0 \leq D^\theta_q[R(t)]
\]

Now, a straightforward calculation shows that

\[
\theta_t[R(t,t)] = \int_0^t \int_x \eta(s) ds
\]

where, for almost all \( s \), the summation is taken over all points of discontinuity \( x_j \) in the restriction \( u(s,t) \) of \( u \) to the plane \( t = s \):

\[
\sum_{i=1}^{k} \int_a^b \frac{c_i[u(x_j,t_0) - q_i(u(x_j,t_0))]}{c_i[u(x_j,t_0)]}
\]

Relation (7.4) follows from (7.5) and the fact that \( u(x,t) \) coincides with \( w(x,t) \) for almost all \( x \). Indeed, since

\[
\sum_{i=1}^{k} \int_a^b \frac{c_i[u(x_j,t_0) - q_i(u(x_j,t_0))]}{c_i[u(x_j,t_0)]}
\]

The difference between \( \theta_t[R(t,t)] \) and \( \theta_t[G(t,t)] \) is \( o(t-1) \) by virtue of the fact that \( u(s,t) \) is approximately continuous at \( x = a \) and at \( x = b \).
This completes the proof of Theorem 7.1.

Next, we shall show that PL solutions to the quasilinear wave equation (1.5) with $p' < 0$ and $p'' > 0$ satisfy the entropy rate criterion with respect to a large class $G_2$ of solutions. For simplicity in the formulation, we shall call a measurable function Lipschitz continuous if it is equal almost everywhere to a function which is Lipschitz continuous in the standard sense. By $G_2$ we shall mean the class of weak solutions $u$ with the following three properties. First, $u$ is defined on an open domain $\mathcal{D} \subset \mathcal{E}(T)$. Second, if $u(x,t_0)$ is Lipschitz continuous on $(a,b)$ then $u(x,t)$ is Lipschitz continuous in the region between the maximal forward 2-characteristic through $(a,t_0)$ and the minimal forward 1-characteristic through $(b,t_0)$ intersect a small strip of the form

$$(x,t) : t_0 \leq t < t_0 + \varepsilon.$$

Third, if $u(x,t_0)$ is piecewise Lipschitz on $(a,b)$ with a jump at $c$, $a < c < b$, i.e. if

$$u(x,t_0) = \begin{cases} u_1(x) & a < x < c \\ u_2(x) & c < x < b \end{cases}$$

where $u_1(x)$ and $u_2(x)$ are Lipschitz, then the speeds of propagation of the minimal forward 1-characteristic through $(c,t_0)$ and the maximal forward 2-characteristic through $(c,t_0)$ approach the speeds of propagation, of the corresponding extreme characteristics in the classical solution to the Riemann problem with initial data $u_1(c-0)$ and $u_2(c+0)$ as $t$ approaches $t_0$, $t > t_0$.

Theorem 7.2. Consider the quasilinear wave equation (1.5) with $p' < 0$ and $p'' > 0$ and suppose that $(n,q)$ is a smooth entropy-entropy flux pair with $n$ strictly convex. If $w \in \text{PL}(S(T))$ then $w$ satisfies (7.1) with respect to the class $G_2$.

Remark: The purpose of presenting Theorem 7.2 is simply to indicate that the entropy rate criterion is satisfied by PL solutions relative to a fairly broad class of comparison solution $G_2$. We expect that $G_2$ can be substantially enlarged.

Proof of Theorem 7.2. For simplicity, let us assume that $w$ is Lipschitz continuous on either side of an admissible 1-shock

$$S_w = \{(y(t),t) : 0 < t < T\}$$

passing through the origin. Let $u$ be a solution in $G_2$ whose initial data coincides with $w_0$ and consider the approximate solution $\hat{w}$ defined in terms of $u$ and $w$ by (4.11). It follows from Green's theorem that

$$\int_0^T \sum a(u(x,t),\hat{w}(x,t))dt = \gamma(S(t)).$$

Let $S_u(t)$ denote the restriction of the minimal forward 1-characteristic in $u$ to the interval $[0,t]$:

$$S_u(t) = \{(x(s),s) : 0 \leq s < t\}.$$

Then

$$\gamma(S_u(t)) = \int_0^t D(t,u_x,u_t,x_\hat{w},\hat{w}_t)dt$$

where $\gamma = x'(t)$ and the subscripts denote the limiting values of $u$ and $\hat{w}$ along the edges of $S_u$.

Lemma 4.3 implies that

$$D(t,u_x,u_t,x_\hat{w},\hat{w}_t) = D(t,u_x,u_t,w_x,w_t) + O(|w_x-r_x| + |w_t-r_t|)$$

$$= d(t,u_x,u_t) - d(o,w_x,w_t) - a(w_x-w_t)(t-o) + O(|u_x-r_x| + |u_t-r_t| + |w_x-r_x|).$$

where $o = \gamma(t)$. It follows from the definitions of $G_2$ and $\hat{w}$ that $\gamma = o$, $x_\hat{w} = w_x$, and $\hat{w}_t = w_t$ as $t$ approaches zero where

$$w_x = w(y(t)-0,t) \quad \text{and} \quad w_t = w(y(t)+0,t).$$

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Then

\[ \gamma(\{j(t)\}) = \theta \left[ \{j(t)\} - \theta \right] \}

\[ \text{where } \{j(t)\} = \{j(t)\} \cap [0, t]. \] Since

\[ \gamma(\{j(t)\}) = \theta \left[ \{j(t)\} - \theta \right] \}

\[ \int_0^T |u(x, t) - \tilde{w}(x, t)|^2 \, dt, \]

we deduce from (7.6) and (7.7) that

\[ g(t) \leq \theta \left[ \{j(t)\} - \theta \right] \}

\[ + o(t) + \int_0^t \text{const. } g(s) |ds| \]

where \( g(t) = \int_0^t |u(x, t) - \tilde{w}(x, t)|^2 \, dx. \) Therefore,

\[ 0 \leq \theta \left[ \{j(t)\} - \theta \right] \]

and the theorem follows in the case of PL solutions \( w \) with the indicated structure. We note that the proof did not appeal to the special structure of the quasilinear wave equation; indeed, the result above is valued for such solutions \( w \) to systems in the Smoller-Johnson class. Unfortunately, in order to treat the general solution in PL we must appeal to the following special property of the quasilinear wave equation:

\[ \int_0^T \langle u, v \rangle g(u, v) \geq 0 \]

for both \( j = 1 \) and \( j = 2 \), cf. proof of Theorem 5.2. If (7.8) holds then the restriction of \( \gamma \) to Lipschitz centered rarefaction waves in \( w \) is bounded from above by a measure which is absolutely continuous with respect to 2-dimensional Lebesgue measure and we may use virtually the same proof given above for the case of general solutions in PL. We omit the details.

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