Some Qualitative Considerations on
the Numerical Determination of Minimum Mass Structures
with Specified Natural Frequencies

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Some Qualitative Considerations on
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Abstract. The problem of the axial vibration of a cantilever
beam is investigated analytically. The range of values of
the frequency parameter having technical interest is de-
termined.

Key Words. Structural optimization, cantilever beams,
axial vibrations, fundamental frequency constraint.

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### Notation

- **E**: Modulus of elasticity, lb ft$^{-2}$
- **L**: Length of the beam, ft
- **m**: Normalized mass per unit length, $m = ML/M_0$
- **M**: Mass per unit length, lb ft$^{-2}$ sec$^2$
- **M_0**: Reference mass, lb ft$^{-1}$ sec$^2$ (Sections 2-3)
- **M_t**: Tip mass, lb ft$^{-1}$ sec$^2$ (Sections 4-5)
- **M_*$**: Total mass of the beam, lb ft$^{-1}$ sec$^2$
- **x**: Normalized axial coordinate, $x = X/L$
- **X**: Axial coordinate, ft
- **u**: Normalized axial displacement, $u = Y(X)/Y(L)$
- **Y**: Axial displacement, ft
- **β**: Frequency parameter, $β = ωL/(ρ/E)$
- **ρ**: Density, lb ft$^{-4}$ sec$^2$
- **ω**: Natural frequency, sec$^{-1}$

### Superscript

- ' Derivative with respect to the normalized axial coordinate $x$ (for example, $u' = du/dx$)
1. **Introduction**

In this memorandum, we consider the problem of the axial vibration of a cantilever beam. With reference to a constant-section beam, we determine the range of values of the frequency parameter $\beta$ having technical interest. This range of values of the frequency parameter is important in the solution of a subsequent problem: the determination of the mass distribution that minimizes the total mass of a beam for a given fundamental frequency constraint.
2. Nonoptimal Beam without a Concentrated Mass

Let $m$ denote the normalized mass per unit length, $u$ the normalized axial displacement, and $\beta$ the frequency parameter. Let $x$ denote the axial coordinate, normalized so that $x=0$ at the base of the beam and $x=1$ at the tip of the beam. Let the prime denote total derivative with respect to the axial coordinate $x$. With this understanding, the fundamental equation to be solved is the following:

$$\left(\mu u'\right)' + \beta^2 \mu u = 0.$$ (1)

In this equation, the frequency parameter $\beta$ is related to the natural frequency $\omega$, the length $L$, the density $\rho$, and the modulus of elasticity $E$ by the relation

$$\beta = \omega L \sqrt{\rho / E}.$$ (2)

In the absence of a concentrated mass attached at the tip of the beam, the boundary conditions for Eq. (1) are as follows:\(^5\)

$$u(0) = 0, \quad m(1)u'(1) = 0.$$ (3)

If the mass distribution

$$m = m(x)$$ (4)

is prescribed a priori, then (1) is a second-order differential

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\(^5\)Equations (3) must be completed by the normalization condition $u(1) = 1$. 
equation, to be solved in conjunction with the boundary conditions (3).

**Constant Section.** Next, we consider the particular case of a constant-section structure, that is, a structure with a constant mass per unit length:

\[ m = \text{const}. \]  

For this particular case, the differential equation (1) and the boundary conditions (3) simplify as follows:

\[ u'' + \beta^2 u = 0, \]  
\[ u(0) = 0, \quad u'(1) = 0. \]

The solution of (6) consistent with the initial condition (7-1) is the following: \(^6\)

\[ u = A \sin(\beta x), \]

with the implication that

\[ u' = A\beta \cos(\beta x). \]

From (9) and the final condition (7-2), we conclude that

\[ \cos \beta = 0, \]  

so that

\[ \beta = (2n+1)\pi/2, \quad n = 0, 1, 2, \ldots \]

\(^6\)The constant \( A \) has the value \( A = 1/\sin \beta \).
Therefore, for this problem, the smallest nontrivial value of the frequency parameter is

\[ \beta = \pi/2 . \] (12)
3. Optimal Beam without a Concentrated Mass

Now, suppose that a constant-section structure has been studied in accordance with Section 2. Suppose that the frequency parameter $\beta$ which allows satisfaction of the boundary conditions (7) has been determined, namely, $\beta=\pi/2$. The total mass of the structure studied in Section 2 is given by

$$M_*/M_0 = \int_0^1 m dx, \quad m = \text{const.} \tag{13}$$

Therefore, it is natural to pose the following question: for the same value of the frequency parameter $\beta=\pi/2$, is there a better beam, that is, one having a smaller total mass? In particular, is there a beam which yields the smallest total mass for the given value of $\beta$? This question leads to the following variational problem: Minimize the total mass

$$M_*/M_0 = \int_0^1 m dx, \quad m = m(x), \tag{14}$$

with the understanding that the following constraints must be satisfied:

$$\beta^2 m u + (mu')' = 0, \quad m(0) = 0, \quad m(1)u'(1) = 0, \tag{15}$$

and with the further understanding that $\beta=\pi/2$. Owing to
the fact that the problem (15)-(16) is homogenous, the obvious solution under the physical constraint

\[ m(x) \geq 0 \]  

is

\[ m(x) = 0 , \]  

with the implication that

\[ \frac{M^*}{M_0} = 0 . \]  

In order to avoid the occurrence of the above trivial solution, Ineq. (17) could be changed as follows:

\[ m(x) > m_0 . \]  

Then, the solution would become

\[ m = m_0 . \]  

To arrive at solutions other than constant mass solutions, it is necessary to postulate some different physical situation (e.g., a concentrated mass attached at the end of the beam). In turn, this results in a change in the boundary condition (16-2), and this change makes it unnecessary to employ inequality constraints of the form (17) or (20).

7The symbol \( M_0 \) denotes a reference mass.

8Equations (16) must be completed by the normalization condition \( u(1) = 1 \).
4. Nonoptimal Beam with a Concentrated Mass

In this section, we assume that a concentrated mass $M_o$ is attached at the tip of the beam. Using the same terminology as in Section 2, we see that the governing differential equation (1) still holds:

$$ (mu')' + \beta^2 mu = 0. \quad (22) $$

On the other hand, the boundary conditions (3) are modified as follows:

$$ u(0) = 0, \quad m(l)u'(l) = \beta^2. \quad (23) $$

**Constant Section.** Again, we consider the particular case of a constant-section structure. Under condition (5) and after observing that

$$ \frac{M_s}{M_o} = m, \quad (24) $$

then problem (22)-(23) becomes

$$ u'' + \beta^2 u = 0, \quad (25) $$

$$ u(0) = 0, \quad u'(l) = \left( \frac{M_o}{M_s} \right) \beta^2. \quad (26) $$

The solution of (25) consistent with the initial condition (26-1) is the following:

Equations (23) must be completed by the normalization condition $u(l) = 1$. 

\[ u = A \sin(\beta x), \] (27)

with the implication that
\[ u' = A \beta \cos(\beta x). \] (28)

From (28) and the final condition (26-2), we conclude that
\[ A \cos \beta = \left( \frac{M_0}{M_*} \right) \beta. \] (29)

Owing to the fact that
\[ u(1) = A \sin \beta, \] (30)

elimination of A from (29)-(30) leads to the following transcendental equation:
\[ \beta \tan \beta = (M_*/M_0)u(1), \] (31)

which, for \( u(1) = 1 \), reduces to
\[ \beta \tan \beta = M_*/M_0. \] (32)

This equation supplies the frequency parameter \( \beta \) in terms of the mass ratio (ratio of beam mass \( M_* \) to tip mass \( M_0 \)).

In order to understand the significance of (32), let us consider two limiting cases: (i) negligible mass ratio and (ii) infinite mass ratio. If \( M_*/M_0 = 0 \), then the solution of (32) is
\[ \beta = n\pi, \quad n = 0, 1, 2, \ldots. \] (33)
On the other hand, if $M_\ast/M_0 = \infty$, then the solution of (32) is

$$\beta = (2n+1)\pi/2, \quad n = 0, 1, 2, \ldots$$

which is identical with (11). Since the first natural frequency corresponds to $n = 0$, we conclude that, for mass ratios in the range

$$0 \leq M_\ast/M_0 \leq \infty,$$  \hspace{1cm} (35)

the smallest frequency parameter $\beta$ consistent with the transcendental equation (32) lies in the range

$$0 \leq \beta \leq \pi/2.$$  \hspace{1cm} (36)
5. Optimal Beam with a Concentrated Mass

As in Section 3, we can formulate the problem of finding the optimal mass distribution. The problem is as follows:

Minimize the total mass

\[ \frac{M_\ast}{M_0} = \int_0^1 m dx, \quad m = m(x), \quad (37) \]

with the understanding that the following constraints must be satisfied:

\[ (mu')' + \beta^2mu = 0, \quad (38) \]

\[ u(0) = 0, \quad m(1)u'(1) = \beta^2, \quad (39) \]

and with the further understanding that the frequency parameter \( \beta \) has some fixed value in the range

\[ 0 \leq \beta \leq \pi/2. \quad (40) \]

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10 Equations (39) must be completed by the normalization condition \( u(1) = 1 \).
References


SOME QUALITATIVE CONSIDERATIONS ON THE NUMERICAL DETERMINATION OF MINIMUM MASS STRUCTURES WITH SPECIFIED NATURAL FREQUENCIES.

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