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SINGULAR SEMI-LINEAR EQUATIONS
IN $L^1(\mathbb{R})$

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ABSTRACT

Let $g$ be a positive continuous function on $\mathbb{R}$ which tends to zero at $-\infty$ and which is not integrable over $\mathbb{R}$. The boundary-value problem $-u'' + g(u) = f$, $u'(\pm \infty) = 0$, is considered for $f \in L^1(\mathbb{R})$. We show that this problem can have a solution if and only if $g$ is integrable at $-\infty$ and if this is so then the problem is solvable precisely when

$$\int_{-\infty}^{\infty} f(t)dt > 0.$$ 

Some extensions of this result are also given.

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SINGULAR SEMI-LINEAR EQUATIONS IN $L^1(\mathbb{R})$

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\[
\begin{cases}
-u''(x) + \beta(u(x)) = f(x), & -\infty < x < \infty \\
u'(\pm \infty) = 0 \\
u'' \in L^1(\mathbb{R})
\end{cases}
\]  

has a solution for each $f \in L^1(\mathbb{R})$ with $\int_{\mathbb{R}} f > 0$ if (and only if) $\beta$ is integrable at $-\infty$. Here $\beta$ is a given positive monotone increasing continuous function on $\mathbb{R}$. In fact, they discuss the more general situation when $\beta$ is a maximal monotone graph. In this paper we consider several extensions of the problem (\star) and provide another technique for proving that these equations have a solution. In particular, we recover the result of Crandall and Evans by different means.

Theorem 1. Let $g$ be a positive continuous function on $\mathbb{R}$ with

\[
\lim_{t \to \infty} g(t) = 0, \quad \int_{-\infty}^{\infty} g(s)ds \text{ divergent}.
\]

Let $L^1_+ = \{f \in L^1(\mathbb{R}) : \int_{\mathbb{R}} f > 0\}$; for $f \in L^1_+$ consider the problem

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\[ -u''(x) + g(u(x)) = f(x), \quad -\infty < x < \infty \]

\[
\begin{cases}
  u'' \in L^1(\mathbb{R}) \\
  u'(\pm \infty) = 0.
\end{cases}
\]

The following are equivalent:

(a) (1) has a solution for all \( f \in L^1_+ \)

(b) (1) has a solution for some \( f \in L^1_+ \)

(c) \( g \) is integrable at \(-\infty\).

Proof. (a) implies (b) is trivial. To see that (b) implies (c) suppose there is a function \( u \) with \( u'' \in L^1, \, u'(\pm \infty) = 0, \) and

\[
-u'' + g(u) = f
\]

for some \( f \in L^1_+ \). Then \( u' \in L^\infty \) and \( u \) tends to \(-\infty\) at both \( \pm \infty \) for the following reason. Suppose there is a sequence \( x_n \to \infty \) with \( \lim u(x_n) = L > -\infty \). Let \( \{y_n\} \) be any other sequence of real numbers \( n \to +\infty \). Then from (2) we get

\[
-\frac{1}{2} (u'(y_n))^2 + \frac{1}{2} (u'(x_n))^2 + H(u(y_n)) - H(u(x_n)) = \int_{x_n}^{y_n} f u' \]

where

\[
H(t) = \int_0^t g(s)ds.
\]

Hence, \( \lim H(u(y_n)) \) exists and equals \( H(u(L)) \). Thus, \( H(u(t)) \) has a
limit at $\infty$ which implies that $u$ has limit $L$ at $\infty$ since $H$ is strictly monotone. But then $g(u(t))$ tends to $g(L) > 0$ as $t \to \infty$ which contradicts the fact that $g(u(t))$ is in $L^1(\mathbb{R})$. An identical argument shows $u$ tends to $-\infty$ at $-\infty$. With $H$ as above we also have

$$\frac{1}{2} [(u'(y))^2 - (u'(0))^2] + H(u(0)) - H(u(y)) = \int_y^0 f u'$$

for each $y, y < 0$. Thus, $H(u(y))$ has a finite limit as $y \to -\infty$. Since $u(y) \to -\infty$ as $y \to -\infty$ we find that $H(s)$ has a finite limit as $s \to -\infty$ implying that $g$ is integrable at $-\infty$. The proof that (c) implies (a) is the most difficult. The first step is to show that the set of those $f \in L_+^1$ for which (1) is solvable is closed in $L_+^1$; the second step is then obviously to show that the set of those $f \in L_+^1$ for which (1) is solvable is dense in $L^1$. To prove the first assertion, let $f_n \to f$ in $L^1(\mathbb{R})$, with $f_n \in L_+^1$. Let $u_n$ satisfy

(3a) $-u_n'' + g(u_n) = f_n$

(3b) $u_n'(\pm \infty) = 0$

(3c) $u_n'' \in L^1(\mathbb{R})$.

Integrate both sides of (3a) from $-\infty$ to $x$ and then from $x$ to $+\infty$ and use the fact that $g \geq 0$. This gives $\|u_n'(x)\| \leq \|f_n\|_1 + 1$ for all large $n$ and hence

(4) $\|u_n'\|_{L^\infty(\mathbb{R})} \leq A_n$, $n = 1, 2, \ldots$. 

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This in turn implies that \( \{u_n\} \) is equicontinuous. We may assume, therefore, that \( \{u_n\} \) converges uniformly on compact subsets of \( \mathbb{R} \) to either \( +\infty \), or \( -\infty \), or to a continuous function \( u \). Set

\[
G(x) = \int_{-\infty}^{\infty} g(s)ds.
\]

For any \( x \in \mathbb{R} \) and any \( n \) we have

\[
G(u_n(x)) = \int_{-\infty}^{\infty} g(u_n(t))u_n'(t)dt
= \frac{1}{2} \left( u_n'(x) \right)^2 + \int_{-\infty}^{\infty} f u_n'
\leq A_1.
\]

Hence, \( u_n(x) \leq C \) for all \( n \) and all \( x \). Thus, it is obviously impossible that \( \{u_n\} \) tends to \( +\infty \). Suppose that \( \{u_n\} \) tends to \( -\infty \) uniformly on compact subsets of \( \mathbb{R} \). Again we have

\[
-\frac{1}{2} \left( u_n'(x) \right)^2 + G(u_n(x)) = \int_{-\infty}^{\infty} f u_n'(t)dt
\]

and hence

\[
0 = \int_{-\infty}^{\infty} f u_n'.
\]

We may assume that \( \{u_n'\} \) converges weak-* in \( L^\infty(\mathbb{R}) \) to a function \( p \) and also that \( \{u_n'(0)\} \) converges. Integrating (3a) from 0 to \( x \) we see that \( u_n'(x) \) converges pointwise to \( p(x) \) on \( \mathbb{R} \). Hence, (5) and (6) yield

\[
-\frac{1}{2} (p(x))^2 = \int_{-\infty}^{x} fp
\]

and

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Hence, \( p \) has a limit of 0 at both \(+\infty\) and \(-\infty\). Again from (3a) we obtain
\[
u'_n(y) - u'_n(x) + \int_x^y g(u_n(t))\,dt = \int_y^x f(t)\,dt,
\]
so that
\[
p(y) - p(x) = \int_y^x f(t)\,dt.
\]
Now let \( y \to -\infty \) and \( x \to +\infty \); we find
\[
0 < \int_{-\infty}^{\infty} f = p(-\infty) - p(+\infty) = 0,
\]
a contradiction. Note that this argument is dependent on \( g \) in only a minor way. In particular, if \( \{g_n\} \) is a sequence of positive continuous functions converging uniformly on compact subsets to a positive continuous function \( g \) which tends to 0 at \(-\infty\) and which lies in \( L^1(-\infty, 0) \) but not in \( L^1(\mathbb{R}) \) and if, say, \( \{g_n\} \) increases to \( g \) on \((-\infty, \infty)\), then the functions \( v_n \) which satisfy
\[
-v''_n + g_n(v_n) = f, \quad v_n(\pm \infty) = 0, \quad f \in L^1_+\mathbb{R}
\]
are equicontinuous and uniformly bounded on compact subsets of \( \mathbb{R} \). We shall make use of this later on.

Returning to the functions \( \{f_n\} \) and \( \{u_n\} \) we see that \( \{u_n\} \)
converges uniformly on compact subsets of \( \mathbb{R} \) to a continuous function \( u \). We clearly have \( u''_n \to u'' \) in \( L^1_{\text{loc}} \) so that \( u \) satisfies

\[
0 = \int_{-\infty}^{\infty} fp.
\]
(7) 
\[-u'' + g(u) = f \text{ on } \mathbb{R}.
\]
Fatou's lemma implies \( g(u) \) is in \( L^1(\mathbb{R}) \) and hence \( u'' \in L^1(\mathbb{R}) \); thus \( u' \) has limits at both \( \pm \infty \) and \( u \) tends to \(-\infty\) at both \( \pm \infty \) as in the implication (b) implies (c). From (5) and (6) we get
\[
-\frac{1}{2}(u'(x))^2 + G(u(x)) = \int_x^\infty f u' 
\]
and
\[
0 = \int_{-\infty}^\infty f u'.
\]
Hence, \( u' \) tends to 0 at both \( \pm \infty \), so that \( u \) is a solution of (1).

Note also that
\[
\int_{-\infty}^\infty |u'' + f| = \int_{-\infty}^\infty (u'' + f) = \int_{-\infty}^\infty f 
\leq \|f\|_1
\]
and hence
(8) \[\|u''\|_1 \leq 2\|f\|_1.\]

The second assertion, that there is a dense set of \( f \in L^1_+ \) for which (1) is solvable, will be proved in the following way. Let \( f \) be a continuous function on \( \mathbb{R} \) in \( L^1_+ \) with support in the interval \( I = [a, b] \). We shall show (1) is solvable for this \( f \). We assume temporarily that \( g \) is \( C^1 \) on \( \mathbb{R} \).

We shall need the following Proposition.

Proposition. Let \( a < b \) and let \( g \) be a positive \( C^1 \) function on \( \mathbb{R} \) which is integrable at \(-\infty\) and bounded at \( +\infty \); set
Then for each \( \alpha, \beta \) the initial value problem

\[
\begin{cases}
-\varphi''(x) + g(\varphi(x)) = f(x), & a < x < b, \quad f \in L^2(a, b) \\
\varphi(a) = \alpha, & \varphi'(a) = \beta
\end{cases}
\]  

(9)

has a unique solution. If \( \alpha_n \to \alpha \) and \( \beta_n \to \beta \) and if \( \varphi_n \) is the solution of (9) for \( (\alpha_n, \beta_n) \), then \( \varphi_n \) converges uniformly to the solution \( \varphi \) of (9) for \( (\alpha, \beta) \). Finally, the family \( \{\varphi_{\alpha \beta}\} \) of solutions of (9) corresponding to the initial values \( \{(\alpha, \beta) : -\infty < \alpha \leq \alpha_0, \quad |\beta| \leq M\} \) is equicontinuous on \([a, b]\).

Proof. Once the equicontinuity is established the existence and uniqueness follow from standard results; see [1], Chapter I. To obtain the equicontinuity assertion (from which the second assertion also follows), we multiply the top equation in (9) by \( \varphi' \) and integrate to obtain

\[
-\frac{1}{2} (\varphi'(x))^2 + G(\varphi(x)) + \frac{1}{2} \beta^2 - G(\alpha) = \int_a^x f\varphi' dx
\]

so that if \( x_0 \) is chosen with \( |\varphi'(x_0)| = \|\varphi'\|_{\infty} \) we have

\[
\|\varphi'\|_{\infty}^2 \leq \beta^2 + 2G(\alpha) + 2G(\varphi(x_0)) + A\|\varphi'\|_{\infty}
\]

\[
\leq \beta^2 + 2G(\alpha) + 2G(\alpha + (b - a)\|\varphi'\|_{\infty}) + A\|\varphi'\|_{\infty}
\]

\[
\leq \beta^2 + 2G(\alpha) + A_0 + A_1(\alpha + (b - a)\|\varphi'\|_{\infty}) + A\|\varphi'\|_{\infty}
\]

for some constants \( A_0, A_1 \) depending only on \( g \). Hence, \( \|\varphi'\|_{\infty} \) is bounded for \( |\beta| \leq M \) and \( -\infty < \alpha \leq \alpha_0 \).
Conclusion of proof of Theorem 1. Let $f$ be a continuous function in $L^1_+$ with support in the interval $(a, b)$. We shall show that (1) is solvable for this $f$. First, on $(-\infty, a]$ we show that the equation

$$g(u(x)) = u''(x)$$

$$u(a) = c_1, \ u'(-\infty) = 0$$

has a solution. Let $v$ be the function with

$$v'(t) = (2G(t))^{-1/2}, \ -\infty < t < c_1$$

$$v(c_1) = a$$

where

$$G(x) = \int_{-\infty}^{x} g(s)ds.$$  

Then $v$ is increasing and has range $(-\infty, a]$. Let $u$ be the inverse of $v$ on $(-\infty, a]$, $u(v(t)) = t$. Thus

$$u(a) = c_1$$

and

$$u'(x) = 1/v'(t) = (2G(t))^{1/2}$$

or

$$(II) \quad u'(x) = (2G(u(x)))^{1/2}.$$  

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If we differentiate both sides of (11) we see that \( u \) satisfies (10).

Similarly, there is a solution of
\[
\begin{align*}
  u''(x) &= g(u(x)) \quad b < x < \infty \\
  u(b) &= c_2, \quad u(\infty) = 0
\end{align*}
\]

which satisfies
\[
  u'(x) = -(2G(u(x)))^{1/2}, \quad b < x < \infty.
\]

Hence, to finish the proof of the theorem we need only show that there is a solution \( v \) of the equation
\[
- v'' + g(v) = f \quad \text{on} \quad (a, b)
\]

with
\[
\begin{align*}
  (a) \quad v'(a) &= (2G(v(a)))^{1/2} \\
  (b) \quad v'(b) &= -(2G(v(b)))^{1/2}.
\end{align*}
\]

Let \( v_t \) be the solution of (12) with \( v(a) = t \) and \( v'(a) = (2G(t))^{1/2} \) assured by the Proposition. (We temporarily assume that \( g \) is bounded at \( +\infty \) if, in fact, it is not.) Then
\[
  v'_t(b) = v'_t(a) + \int_a^b v''_t(s)ds \\
  = (2G(t))^{1/2} + \int_a^b g(v_t(s))ds - \rho
\]
where \( \rho = \int_a^b f(t) dt > 0 \). To show that \( \tau \) may be chosen with

\[ v'_t(b) = - (2G(v_t(b)))^{1/2} \]

we consider

\[ I(t) = (2G(t))^{1/2} + (2G(v_t(b)))^{1/2} + \int_a^b g(v_t(s)) ds - \rho. \]

The Proposition implies \( I \) is continuous. We have

\[ I(t) \geq -\rho + (2G(t))^{1/2}. \]

Since \( G \) is unbounded, there are values of \( t \) with \( I(t) > 0 \). Next let \( t \downarrow -\infty \); by the equicontinuity of the functions \( \{v_t\} \) we must have

\[ v_t \rightarrow -\infty \]

uniformly on \([a, b]\) so that \( I(t) \rightarrow -\rho < 0 \); hence, there is a \( t_0 \) at which \( I(t_0) = 0 \), and thus (12) is solvable with the boundary conditions (13).

We have now shown that (1) is solvable for all \( f \in L_+^1 \) under the assumption

\[ g \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \quad g \notin L^1(\mathbb{R}). \]  

If \( g \) is merely positive and continuous on \( \mathbb{R} \) with \( g \in L^1(-\infty, 0) \), \( g \notin L^1(\mathbb{R}) \), then there is a sequence \( \{g_n\} \) of positive functions satisfying (14) which converge uniformly on compact subsets of \( \mathbb{R} \) to \( g \) and which also increase to \( g \) on \((-\infty, \infty)\). The comments made earlier show that the solutions \( \{u_n\} \) of (1) with \( g_n \) in place of \( g \) converge to a solution of (1) for \( g \). This completes the proof of Theorem 1.

Remark. The condition \( g \notin L^1(\mathbb{R}) \) is necessary as well as sufficient in order that Theorem 1 be valid. For suppose \( g \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}) \); then the
function $G$ is bounded. If $f$ is supported on $[-1,1]$ and if (1) has a solution for $f$, then (13) must hold with $u$ in place of $v$ so that
\[
0 = \sqrt{2}G(u(-1)) + \sqrt{2}G(u(1)) + \int_{-1}^{1} g(u(s))ds - \int_{-1}^{1} f(s)ds.
\]
The first three terms of this expression are bounded, independent of $u$, and hence the integral of $f$ over $\mathbb{R}$ can not exceed some fixed number depending only on $g$.

Theorem 2. Let $g$ be a positive continuous function on $\mathbb{R}$ with
\[
\lim_{t \to -\infty} g(t) = 0, \quad g \in L^1(\mathbb{R}).
\]
Let $B(x)$ be a positive absolutely continuous function on $\mathbb{R}$ with $B' \in L^1(\mathbb{R})$ and $B$ bounded away from zero. For $f \in L^1$ consider the equation
\[
\begin{cases}
-u''(x) + B(x)g(u(x)) = f(x), & -\infty < x < \infty \\
u'' \in L^1(\mathbb{R}) \\
u'(\pm \infty) = 0.
\end{cases}
\]
Then (16) has a solution for each $f \in L^1_+$ if and only if $g$ is integrable at $-\infty$.

Proof. If (16) is solvable for some $f \in L^1_+$ with support in $[-1,1]$ then $u' > 0$ on $(-\infty,-1]$ and $u' < 0$ on $[1,\infty)$. It now follows very much as in Theorem 1 that $u$ tends to $-\infty$ at $\pm \infty$ and that $g$ is integrable at $-\infty$. 

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To show the sufficiency of the condition that \( g \) be integrable at \(-\infty\), we first show that the equations
\[
\begin{align*}
u''(x) &= B(x)g(u(x)), \quad |x| \geq a > 0 \\
u(-a) &= c_1, \quad u(a) = c_2 \\
u'(\pm \infty) &= 0
\end{align*}
\] have a solution. As in the proof of Theorem 1, the solution \( u \) must be monotone increasing for \(-\infty < x < -a\) and monotone decreasing on \((a, \infty)\); we shall only consider the details for the case \(-\infty < x < -a\), the other case being entirely similar. We wish to find a continuous function \( v \) with
\[
v'(t) = \left(2 \int_{-\infty}^{t} B(v(s))g(s)ds\right)^{-1/2}, \quad -\infty < t < c_1 \\
v(c_1) = -a.
\] If such a \( v \) exists, then the inverse function \( u \) of \( v \) will satisfy
\[
u'(x) = \left(2 \int_{-\infty}^{x} B(r)g(u(r))u'(r)dr\right)^{1/2} \\
u(-a) = c_1
\] and hence \( u \) will satisfy (17). To see that (18) has a solution let \( b_1 \) and \( b_2 \) be positive numbers with \( b_1 \leq B(s) \leq b_2 \) for all \( s \) and let \( \xi_N \) be the function defined by
\[
\xi_N(t) = \left(2 \int_{-N}^{t} g(s)ds\right)^{-1/2}, \quad -N \leq t \leq c_1.
\]
Let $\Omega_N = \{w \in C([-N, c_1]) : (2b_2)^{-1/2} \xi_N(t) \leq w(t) \leq (2b_1)^{-1/2} \xi_N(t)\}$ for all $t \in [-N, c_1]$ and let $T$ map $\Omega_N$ into $\Omega_N$ by

$$(Tw)(x) = \left(2 \int_{-N}^{x} B(\tilde{w}(s))g(s)ds\right)^{-1/2}$$

where

$$\tilde{w}'(t) = w(t), \quad \tilde{w}(c_1) = -a.$$ 

Clearly $Tw \in \Omega_N$; if $\{w_n\}$ is a bounded sequence in $\Omega_N$, then $\{\tilde{w}_N\}$ is equicontinuous and uniformly bounded. Thus, $T$ is a compact mapping and so has a fixed point $w_N$ which must satisfy

$$w_N(x) = \left(2 \int_{-N}^{x} B(\tilde{w}_N(s))g(s)ds\right)^{-1/2}, \quad -N \leq x \leq c_1.$$ 

The functions $\{\tilde{w}_N\}$ are equicontinuous and uniformly bounded on compact subsets of $(-\infty, c_1]$ and so a subsequence, again denoted by $\{\tilde{w}_N\}$, converges uniformly on compact subsets of $(-\infty, c_1]$ to a function $\tilde{w}_0$. But we also see that

$$\int_{-N}^{x} B(\tilde{w}_N(s))g(s)ds \to \int_{-\infty}^{x} B(\tilde{w}_0(s))g(s)ds$$

uniformly on compact subsets of $(-\infty, c_1]$. Hence, $w_N \to \tilde{w}_0$ uniformly on compacta; setting $v = \tilde{w}_0$ we see that $v$ satisfies (18).

The remainder of the proof of Theorem 2 is like that of Theorem 1; the condition that $B' \in L^1(\mathbb{R})$ is used to prove that the sequence $\{u_n\}$ cannot go to $-\infty$.

Corollary 3. Let $a(x) \in L^1(\mathbb{R})$, $f \in L^1(\mathbb{R})$, and let $g$ be a positive continuous function satisfying (15). Consider the equation
(19) \[
\begin{cases}
(\text{i}) & -u''(x) + a(x)u'(x) + g(u(x)) = f(x), \quad -\infty < x < \infty \\
(\text{ii}) & u' \in L^1(\mathbb{R}) \\
(\text{iii}) & u'(\pm \infty) = 0
\end{cases}
\]

Let \( g \) be integrable at \(-\infty \) and set \( w(x) = \exp\left[ -\int_0^x a(s) \, ds \right] \). A necessary and sufficient condition that (19) be solvable is that

(20) \[
\int_{\mathbb{R}} f(x)w(x) \, dx > 0.
\]

If (21) is solvable for all \( f \in L^1 \) satisfying (20), then \( g \) is integrable at \(-\infty \).

Proof. Let \( x = H(y) \) where \( H \) is the inverse of the function \( I \) defined by

\[
I'(x) = 1/w(x), \quad I(0) = 0.
\]

Then both \( H \) and \( I \) are 1-1 monotone increasing functions mapping \( \mathbb{R} \) onto \( \mathbb{R} \) and the substitution \( v(y) = u(H(y)) \) reduces (19) to

(21) \[
\begin{cases}
-v''(y) + (H'(y))^2g(v(y)) = (H'(y))^2f(H(y)) \\
v' \in L^1, \quad v'(\pm \infty) = 0
\end{cases}
\]

which has a solution according to Theorem 2 precisely when

\[
0 < \int_{-\infty}^{\infty} (H'(y))^2f(H(y)) \, dy = \int_{-\infty}^{\infty} f(x)w(x) \, dx.
\]
Remark. Let $\beta$ be a maximal monotone graph lying in the upper half-plane; that is, $\beta(x)$ is a subset of $\{y > 0\}$ for each $x \in \mathbb{R}$. Let $\beta^0(x) = \text{min}\{y : y \in \beta(x)\}$. The result of Crandall and Evans is that if
\[
\int_{-\infty}^{a} \beta^0(x)dx < \infty
\]
for some $a \in D(\beta)$, then the equation
\[(22) \quad -u''(x) + \beta(u(x)) \geq f(x), \quad u'(\pm \infty) = 0, \quad f \in \mathbb{L}_1^1
\]
is solvable. This result also follows from Theorem 1 in the following way.

Let $\{\beta_n\}$ be a sequence of positive continuous monotone increasing functions which increase to $\beta^0$ on $D(\beta)$ and which increase to $+\infty$ off $D(\beta)$. The solutions $\{u_n\}$ of (I) with $\beta_n$ in place of $g$ then decrease on $\mathbb{R}$ to a solution $u$ of (22).

A final result related to Theorem 1 is presented below.

Theorem 4. Let $g$ be a positive continuous function on $\mathbb{R}$ satisfying
\[(15). \text{ For } f \in \mathbb{L}_1^1(\mathbb{R}) \text{ consider the equation}
\begin{align*}
\begin{cases}
u''(x) + g(u(x)) = f(x), & -\infty < x < \infty \\
u'' \in \mathbb{L}_1^1(\mathbb{R}) \\
u'(-\infty) \leq \xi_1, \quad u'(+\infty) = \xi_2
\end{cases}
\end{align*}
(23)
where
\[
(24) \quad \int_{-\infty}^{\infty} f(x)dx = p > \xi_2 - \xi_1.
\]

(a) Suppose $g$ is integrable at $-\infty$. If (23) has a solution for some $f$ with compact support (which necessarily satisfies (24)) then $\xi_1 > 0 > \xi_2$.

If (23) has a solution for $f = 0$, then $\xi_1 = -\xi_2$. 

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(b) If \( g \) is integrable at \(-\infty\) and if \( \xi_1 > 0 > \xi_2 \), then (23) has a solution for all \( f \) with

\[
\xi_2 - \xi_1 < \rho = \int_{-\infty}^{\infty} f(x) dx \leq \min\{\xi_2, -\xi_1\}.
\]

(c) If (23) has a solution for some \( f \) satisfying (24), then \( g \) is integrable at \(-\infty\).

Proof. (a). If \( f \) has support in \([a, b]\), then \( u''(x) < 0 \) for \( x < a \) and \( x > b \). If \( u'(-\infty) \leq 0 \), then \( u' < 0 \) on \((-\infty, a)\) and hence \( u \) is decreasing on \((-\infty, a)\). However, \( u \) must tend to \(-\infty\) at both \(-\infty\) and \(+\infty\) if \( u \) is a solution of (23) and thus \( u \) can not decrease on \((-\infty, a)\).

Likewise, \( u'(+\infty) \) must be negative. Further, if \( u'' + g(u) = 0 \), then

\[
(u'(x))^2 + 2G(u(x)) = \text{const. on } (-\infty, \infty)
\]

which clearly implies that \( \xi_1 = -\xi_2 \).

(c) is proved exactly as in Theorem 1.

(b) is the most difficult of the assertions. First, exactly as in Theorem 1, it can be shown that the set of those \( f \) satisfying (24) for which (23) is solvable is closed in \( L^1(\mathbb{R}) \). Next, we show that if \( f \) has compact support, say in \((a, b)\), and if \( f \) satisfies

\[
\xi_2 - \xi_1 < \rho = \int_{-\infty}^{\infty} f(x) dx < \min\{-\xi_1, \xi_2\}
\]

then (23) has a solution. The key to this, as in Theorem 1, is to show two things: first that the equations

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\[
\begin{aligned}
\{ & u''(x) + g(u(x)) = 0, \quad x \notin [a, b] \\
& u'(a) = \xi_1, \quad u'(b) = \xi_2
\end{aligned}
\]

have a solution which necessarily satisfies

\(\tag{27}
\begin{aligned}
u'(a) &= (\xi_1^2 - 2G(u(a)))^{1/2} \\
u'(b) &= -(\xi_2^2 - 2G(u(b)))^{1/2}
\end{aligned}\)

and second that the equation

\(\tag{28}
u''(x) + g(u(x)) = f(x), \quad a \leq x \leq b\)

is solvable subject to the non-linear boundary conditions (27). Both these assertions are proved as the similar statements are in the proof of Theorem 1.

Remark. The upper bound in (25) is not completely satisfactory; however, the situation for (23) is more involved than that of (1) as (a) shows.

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REFERENCES


Let \( g \) be a positive continuous function on \( \mathbb{R} \) which tends to zero at \(-\infty\) and which is not integrable over \( \mathbb{R} \). The boundary-value problem
\[
-u'' + g(u) = f, \quad u'(\infty) = 0,
\]
is considered for \( f \in L(\mathbb{R}) \). We show that this problem can have a solution if and only if \( g \) is integrable at \(-\infty\) and if this is so then the problem is solvable precisely when \( \int_{-\infty}^{\infty} f(t) dt > 0 \). Some extensions of this result are also given.