THE ACCURACY OF GRAVIMETRIC DEFLECTIONS OF THE VERTICAL
AS DERIVED FROM THE GEM 7 POTENTIAL COEFFICIENTS AND
TERRESTRIAL GRAVITY DATA

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Qualified requestors may obtain additional copies from the Defense Documentation Center. All others should apply to the National Technical Information Service.
The deflections of the vertical can be determined with a combined method using point gravity anomalies, mean anomalies and satellite spherical harmonics in the inner-, intermediate- and remote zones, respectively. In essential, the error sources of the intermediate and distant zone are studied.

The RMS error contribution of the remote zone was found considerably...
dependent on the distance of truncation. Another dominating error source is the error due to lack of more detailed gravity data than $1^\circ \times 1^\circ$ mean anomalies in the intermediate zone. Smaller block sizes are recommended between the distances $1^\circ$ and $10^\circ$ from the point of computation.

A significant gain of accuracy will be achieved in the total error by extending the radius of truncation to at least $30^\circ$. Anticipating an inner zone error of $1^\circ$ (a solution by collocation) the total RMS error will hardly be less than $2^\circ$, while for an inner zone error of $0^\circ$, the total error might decrease to $1^\circ$. 
Foreword

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1. Introduction

Recently, combined methods have been introduced for the determination of the deflections of the vertical. Those methods may, first of all, incorporate the following data: satellite derived potential coefficients, mean gravity anomalies and point gravity observations. The idea is that the inner zone is determined in detail from the point anomalies, for instance, by using the method of least-squares collocation, while the effect of an intermediate zone is calculated with the Vening Meinesz' formula using mean gravity anomalies, and finally, the remote zone contribution is represented by a spherical harmonic expansion.

In this paper we are going to study the errors of the combined method. The investigation is mainly restricted to the errors of the intermediate and remote zones. The error contribution of the inner zone differs being much dependent on the method of computation and density and quality of the data. For this zone, we will adopt some representative values found in the literature.

2. Computational Method

We briefly summarize the combined method presented in Lachapelle (1977). The components $\xi$ and $\eta$ are considered to consist of three subcomponents $(\xi_0, \xi_1, \xi_2; \eta_0, \eta_1, \eta_2)$ such that:

\begin{align*}
\xi &= \xi_0 + \xi_1 + \xi_2 \\
\eta &= \eta_0 + \eta_1 + \eta_2
\end{align*}

Each of the subcomponents is the contribution of the deflections from a specific zone around the point of computation (see Figure 1). Exceptions are $\xi_0$ and $\eta_0$, which also serve as "reference field" in $\sigma_1$ and $\sigma_2$.

![Figure 1. Subdivision of the Surface Around the Point of Computation into Inner Zone ($\sigma_1$), Intermediate Zone ($\sigma_2$) and Remote Zone ($\sigma_0$).](image-url)
The values of $\xi_0$ and $\eta_0$ are computed from the fully normalized potential coefficients $J_{ns}^*$ and $K_{ns}$ in the following way (Lachapelle, ibid., p. 3):

\[
\left\{ \begin{array}{l}
\xi_0 \\
\eta_0
\end{array} \right\} = - \sum_{n=2}^{n_{max}} \left( \frac{r_n}{r} \right)^{n+1} \sum_{s=0}^{n} \left[ J_{ns}^* \left\{ \frac{D\phi}{D\lambda} \right\} \bar{R}_{ns} (\phi, \lambda) + K_{ns} \left\{ \frac{D\phi}{D\lambda} \right\} \bar{S}_{ns} (\phi, \lambda) \right]
\]

where $n_{max}$ is the maximum degree of expansion, and:

\[
D\phi \left\{ \frac{\bar{R}_{ns}(\phi, \lambda)}{\bar{S}_{ns}(\phi, \lambda)} \right\} = \left\{ \begin{array}{c}
\cos m\lambda \\
\sin m\lambda
\end{array} \right\} \frac{d}{d\phi} \bar{P}_{ns} (\sin \phi)
\]

\[
D\lambda \left\{ \frac{\bar{R}_{ns}(\phi, \lambda)}{\bar{S}_{ns}(\phi, \lambda)} \right\} = \left\{ \begin{array}{c}
-sin m\lambda \\
\cos m\lambda
\end{array} \right\} \bar{P}_{ns} (\sin \phi) \frac{m}{\cos \phi}
\]

\[r_n/r = \text{ratio between the radius (} r_n \text{) of the internal sphere (to which } J_{ns}^* \text{ and } K_{ns} \text{ are referred) and the radius (} r \text{) of the point of computation.}\]

Remark 1. Formula (2) is a slight generalization of Lachapelle's formula for points at an arbitrary height above the sphere to which the coefficients $J_{ns}^*$ and $K_{ns}$ refer. For low degree expansions at the surface of the earth $n/r = 1$ is a good approximation.

The values of $\xi_2$ and $\eta_2$ are obtained by applying Vening Meinesz' integral formula to gravity anomalies that are formed by subtracting from mean terrestrial free-air anomalies $(\Delta g)$ the contribution $(\Delta g_s)$ implied by the potential coefficients used in computing $\xi_0$ and $\eta_0$. This computation, which is carried out for the zone $\sigma_2$ with $\psi_1 \leq \psi \leq \psi_2$, can be expressed in the following way:

\[
\left\{ \begin{array}{l}
\xi_2 \\
\eta_2
\end{array} \right\} = \frac{1}{4\pi G} \int \int_{\sigma_2} (\Delta g - \Delta g_s) \frac{dS(\psi)}{d\psi} \left\{ \begin{array}{c}
\cos \alpha \\
\sin \alpha
\end{array} \right\} d\sigma
\]

where $S(\psi)$ is Stokes' function, $d\sigma$ is an elemental area, and $\alpha$ is the azimuth from the point of computation to a current point.
The classical Vening Meinesz' formula is valid strictly only for a sphere. However, terrestrial data $\Delta g$ may be used provided that either they are continued analytically to the internal sphere or that a terrain correction (Molodenskii term) is applied to $\Delta g$ (Heiskanen and Moritz, 1967, pp. 315 and 313, respectively).

Lachapelle (ibid.) used $1^\circ \times 1^\circ$ mean gravity anomalies for these computations, which were extended to $\psi_2 = 8^\circ$. $\psi_1$ varied between $0.7$ and $1.5$, depending on the density of the point gravity data in the inner zone.

The values of $\xi_1$ and $\eta_1$ may be obtained in different ways; Lachapelle (ibid.) used least squares collocation. Such an approach has the advantage that there is no difficulty to compute the effect of the innermost zone and we can easily incorporate heterogeneous data in the computations. As pointed out by Tscherning (1974) a considerable gain in accuracy will be achieved if we include astrogeodetic deflection of the vertical in the set of observations. On the other hand, we have to limit the number of observations to a few hundred in order to keep the computer time at a reasonable level. Another difficulty is the instability that occurs if two observations are located close to each other. The original Vening Meinesz' formula does not suffer from these limitations.

3. Error Analysis

We now study the error propagation due to the data and other sources. It is assumed that the numerical integration of Vening Meinesz' formula is performed with such an accuracy that the integration errors can be neglected. Furthermore, the model errors caused by disregarding the flattening of the earth (in the order of $3 \times 10^{-5}$) are not considered.

The gravity data are assumed to be corrected for the terrain effect in Vening Meinesz' formula. If this correction is omitted, an additional error in the order of $0.2$, is introduced (Moritz, 1966; Dimitrijevich, 1972).

The effect of the earth's atmosphere can be estimated in the following way (Moritz, 1974). A constant distribution of air is assumed above the reference ellipsoid. This homogeneous mass has no effect on the deflection. Next, the fictitious density of the atmosphere inside the topography (above the reference ellipsoid) is subtracted from the density of the topography. (This modified density of the topography can be used for a simultaneous correction for terrain and atmosphere.) We obtain the following contribution from each compartment to the correction for the atmosphere:

$$\text{atm. corr.} = - \text{terrain corr.} \times \frac{\text{density of atmosphere}}{\text{density of topography}}$$
As the terrain correction is in the order 0".2 it is obvious that the atmosphere has no practical effect on the deflections of the vertical.

Finally, we regard the earth constants as known (without errors).

3.1 Errors from the Remote Zone

The error contribution of the remote zone is dependent on the spherical distance \( \psi_2 \) (see Figure 1). A formula for the RMS influence of this zone on the total deflection of the vertical:

\[
\theta = \sqrt{\xi^2 + \eta^2}
\]

is given in Heiskanen and Moritz (1967, p. 262). However, as pointed out by deWitte (1966) and Hagiwara (1972), Molodenskii's truncation coefficients, \( Q_n \), in this formula need to be modified. A generalization is obtained by substituting \( c_n \) by \( (r_\theta/r)^2(n+2) c_n \) (cf. Remark 1). With these modifications, the mean square contribution of the remote zone becomes:

\[
\delta \theta^2 = \frac{1}{4G^2} \sum_{n=2}^{\infty} n (n+1) \bar{Q}_n^2(\psi_2) \left( \frac{r_\theta}{r} \right)^{2(n+1)} c_n
\]

where

\[
c_n = \text{anomaly degree variances at the internal (Bjerrum)}
\]

sphere

\[
\bar{Q}_n(\psi_2) = Q_n(\psi_2) + S(\psi_2) P_n(\cos \psi_2) \sin \psi_2 / n(n+1)
\]

\( S(\psi_2) \) = Stokes' function

\( \psi_2 \) = geocentric distance of truncation

As the spherical harmonic expansion for a given set of coefficients can be regarded as a reference field in the inner and intermediate zones, errors in the coefficients will influence the remote zone contributions only. Thus, the errors of the remote zones are of two kinds: the potential coefficient errors and the truncation errors. From (2) and (5) we obtain the following mean square propagation of the potential coefficient errors:
\[ \delta \theta_{cn}^2 = \frac{1}{4G^2} \sum_{n=2}^{n_{max}} n(n+1) \overline{Q_n}^2 (\psi_0) \left( \frac{r_n}{r} \right)^2 (n+2) \quad dc_n \]

where \( dc_n \) is the mean square error of \( c_n \). This error can be determined in the following way.

Let us expand the gravity anomaly \( (\Delta g^*) \) at the internal sphere \( (r_n = r) \) into a series of spherical harmonics to degree \( n_{max} \) [cf. (2)]:

\[ \Delta g^* = \sum_{n=2}^{n_{max}} \Delta g_n^* \]

where

\[ \Delta g_n^* = G(n-1) \sum_{n=0}^{n} \left[ \overline{J_n}^* \overline{R_{ns}}(\varphi, \lambda) + \overline{K_{ns}} \overline{S_{ns}}(\varphi, \lambda) \right] \]

Thus, the coefficient errors \( \delta \overline{J_{ns}} \) and \( \delta \overline{K_{ns}} \) are propagated according to:

\[ \delta \Delta g_n^* = G(n-1) \sum_{n=0}^{n} \left[ \delta \overline{J_{ns}} \overline{R_{ns}}(\varphi, \lambda) + \delta \overline{K_{ns}} \overline{S_{ns}}(\varphi, \lambda) \right] \]

and the global mean square error of \( c_n \) is finally given by:

\[ dc_n = M[\delta \Delta g_n^{*2}] = G^2(n-1)^2 \sum_{n=0}^{n} \left( \delta \overline{J_{ns}}^2 + \delta \overline{K_{ns}}^2 \right) \]

where \( M \{x\} \) is the global average of \( x \):

\[ M \{x\} = \frac{1}{4\pi} \int \int x \ d\sigma \]

In (8) we have used the orthogonality property of the spherical harmonics over a sphere. For a different estimate of \( \delta c_n \), see Rapp (1973). In Table 1 we give the mean square errors \( \delta c_n \) computed from the errors of the GEM 7 coefficients (Wagner, 1976).
Table 1. Mean Square Errors of the Degree Variances for GEM 7 from Wagner (1976, Table 27).
Fully Normalized Harmonics. \( G = 980 \) gal.

<table>
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<tr>
<th>( n )</th>
<th>( \delta \sigma_n^2 \times 10^{18} )</th>
<th>( \delta c_n ) [mgal^2]†</th>
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<tr>
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* \( \delta \sigma_n^2 = \sum_{n=0}^{\infty} (\delta \bar{J}_n^2 + \delta \bar{K}_n^2) \)

† \( \delta c_n = G^2(n-1)^2 \delta \sigma^2 \)

Finally, by inserting these values for \( \delta c_n \) into formula (6), the potential coefficient error for \( n_{\text{max}} = 16 \) was determined. The ratio \( r_b/r \) was set equal to 1 in this low degree expansion. The result is depicted in Figure 2.

The other error source of the remote zone, the error due to the truncation of the spherical harmonic expansion, is also given by formula (5). The mean square value of the truncation error is:

\[
\delta \theta_{o,z}^2 = \frac{1}{4G^2} \sum_{n_{\text{max}}+1}^{\infty} n(n+1) Q_n^2(\psi_2) \left( \frac{r_o}{r} \right)^{2(n+2)} c_n
\]
For the numerical computation of this error, we use the degree variances \( c_n \) of Tscherning and Rapp (1974):

\[
(10) \quad c_n = A \frac{(n-1)(n-2)(n+24)}{n-3}, \quad n \geq 3
\]

\[
s = \left( \frac{r_\theta}{r} \right)^2 = 0.999617
\]

\[
A = 425.28 \text{ mgal}^2
\]

The convergence of (9) is very slow for \( \psi_2 \) close to 0 while for larger angles, say \( \psi_2 \geq 5^\circ \), it converges well. Formula (9) is illustrated in Figure 2. The excellent subroutine of Paul (1973) was used for a rapid determination of \( Q \), and the series was truncated at \( n = 2000 \). A remarkable minimum (0.06) of the RMS error is obtained for \( \psi_2 = 40^\circ \).

### 3.2 Errors Due to Lack of More Detailed Data in \( \sigma_2 \)

The application of Vening Meinesz' formula (3) for the determination of \( \xi_2 \) and \( \eta_2 \) requires, theoretically, that \( \Delta g \) is known at each point in \( \sigma_2 \). In practice we limit ourselves to using mean gravity anomalies in this area. We shall now estimate the error due to the lack of more detailed gravity material in \( \sigma_2 \). The data is assumed to be located on a mean earth sphere of radius \( r \).

Let us expand \( \Delta g \) into Laplace harmonics:

\[
\Delta g = \sum_{n=0}^{\infty} \Delta g_n
\]

The mean gravity field \( \langle \Delta g \rangle \) is related to \( \Delta g \) according to:

\[
\langle \Delta g \rangle(y) = \int_{\sigma} \int B(y \cdot x) \Delta g \, d\sigma = \sum_{n=2}^{\infty} \beta_n \Delta g_n
\]

where \( B(y \cdot x) \) is the averaging operator and \( \beta_n \) are its eigen values. If we approximate each block of mean anomalies with a circular cap with equal area we obtain (see Meissl, 1971, p. 24):
Figure 2. The RMS errors of the Total Deflection of the Vertical Due to Potential Coefficient Errors and Truncation Error.

\[ \delta \theta_0^1, \text{potential coefficient error} \]
\[ \delta \theta_0^2, \text{truncation error} \]
\[ \beta_n = \frac{[P_{n-1}(\cos \psi) - P_{n+1}(\cos \psi)]}{(2n+1)(1-\cos \psi)} \]

where \( \psi \) is the spherical radius of the cap. If \( \nu \) is the block size, then \( \psi \) is given by:

\[ \psi = \sqrt{\nu \sin \nu / \pi} \]

Now the error at each point when representing \( \Delta g \) by \( \bar{\Delta g} \) is:

\[ \delta g = \Delta g - \bar{\Delta g} = \sum_{n=2}^{\infty} (1 - \beta_n) \Delta g_n \]

Following the derivation of formula (5) in Heiskanen and Moritz (1967, pp. 261-262), we obtain the following error propagation of \( \delta g \) outside the spherical distance \( \psi \).

The error of each component \( \xi \) and \( \eta \) becomes:

\[ \left\{ \frac{\delta \xi}{\delta \eta} \right\} = -\frac{1}{2G} \sum_{n=2}^{\infty} \overline{Q}_n(\psi) \left\{ \frac{1}{\cos \phi} \frac{\partial}{\partial \lambda} \right\} \delta g \]

and the total RMS error is given by:

\[ \delta \theta^2 = M \left\{ \delta \xi^2 + \delta \eta^2 \right\} = \]

\[ = \frac{1}{4G^2} \sum_{n=2}^{\infty} \sum_{n=2}^{\infty} \overline{Q}_n(\psi) \overline{Q}_n'(\psi) M \left\{ \frac{\partial \delta g_n}{\partial \phi} \frac{\lambda}{\cos^2 \phi} + \frac{1}{\cos \phi} \frac{\partial \delta g_n'}{\partial \lambda} \right\} = \]

\[ = \frac{1}{4G^2} \sum_{n=2}^{\infty} \overline{Q}_n^2(\psi) n (n+1) M \left\{ \delta g_n^2 \right\} \]

-9-
where

\[ \delta g_n = (1 - \beta_n) \Delta g_n \]

and

\[ M \{ \delta g_n^2 \} = (1 - \beta_n^2) M \{ \Delta g_n^2 \} = (1 - \beta_n^2) \left( \frac{r_n}{r} \right)^{2(n+2)} c_n \]

Hence, by using Vening Meinesz' formula outside the spherical distance \( \psi \), the following mean square error (due to neglecting more detailed data) is committed:

\[ \delta \theta^2 = \frac{1}{4G^2} \sum_{n=2}^\infty \bar{Q}_n^2(\psi) n(n+1)(1-\beta_n^2) \left( \frac{r_n}{r} \right)^{2(n+2)} c_n \]

Finally, the corresponding contribution from the zone \( \sigma_a \) is given by:

\[ (12) \quad \delta \theta_{a,1}^2 = \frac{1}{4G^3} \sum_{n=2}^\infty \left[ \bar{Q}_n^2(\psi_1) - \bar{Q}_n^2(\psi_2) \right] n(n+1)(1-\beta_n^2) \left( \frac{r_n}{r} \right)^{2(n+2)} c_n \]

As \( \beta_n \) approaches 0 for large \( n \) this formula suffers from the same slow convergence as (9) for small angles \( \psi_1 \). For \( 1^\circ < \psi_1 < 4^\circ \) (12) was expanded to \( n = 5000 \). For larger \( \psi_1 \), \( n = 2000 \) was found to be a sufficient degree of truncation (\( \nu = 1^\circ \)). The results are shown in Figure 3.

### 3.3 Errors Due to Inaccurate Gravity Material in \( \sigma_a \)

A constant error in \( \Delta g \) will not affect the deflections of the vertical, because there is no zero-order term present in \( \xi \) and \( \eta \). Hence, the only error source that has to be considered is the random errors due to insufficient gravity data within each block to assure an accurate determination of the mean value. The propagation of this error is given by (3). We obtain approximately:

\[ (13) \quad \delta \theta_{a,2}^2 = \frac{1}{(4\pi G)^2} \sum_1 \left[ \int \int_{\sigma_1} \frac{dS(\psi)}{d\psi} d\sigma \right]^2 \]
Figure 3. The RMS Error of the Total Deflection of the Vertical Due to Lack of More Detailed Gravity Material than $1^\circ \times 1^\circ$ Mean Anomalies.
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**TETM 2.1 FROM GLOB. AND MORITZ**

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<td>0.96 2.13</td>
</tr>
<tr>
<td>0.67</td>
<td>0.36 0.17</td>
</tr>
<tr>
<td>0.41</td>
<td>0.14 0.04</td>
</tr>
</tbody>
</table>

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where \( m \) is the standard error of \( \Delta g_i \) and \( \sigma_i \) is the area of block \( i \). The evaluation of (13) requires that \( m \) is known for all blocks used in the determination of \( \xi_2 \) and \( \eta_2 \). It will therefore be very much dependent on the quality of the mean anomalies and will vary for each point of computation. In Table 2, we report the computations in four specific points. The computations are based on the 1° x 1° anomaly information described in Rapp (1977). For unknown blocks \( m \) is set to 30 mgal. In this table we also show the error estimates of Groten and Moritz (1964). By assuming uniform errors of all \( \Delta g \) blocks, they arrived at the following formula:

\[
\delta \theta^2_{\xi_2, \eta_2} = \frac{S}{8\pi (Gr)} \left[ J(\psi_1) - J(\psi_2) \right]
\]

where

\[
J(\psi) = \int_{\psi}^{\pi} \left( \frac{dS(\psi)}{d\psi} \right)^2 \sin \psi \, d\psi
\]

\( S = 0.027 \, r^2 \) (1° x 1° blocks, 1 gravity profile inside each block).

Formula (14) seems to give a reasonable approximation of \( \delta \theta^2_{\xi_2, \eta_2} \) for \( \psi_2 \geq 2^\circ \), specially in areas with a poor gravity material (points 2 and 4).

In Table 2 we also report the sum of squares of the errors considered above (the error of the inner zone is not included). From these results we conclude that the RMS errors are significantly decreasing with an increasing angle \( \psi_2 \) all the way to 30°.

3.4 An Extended View

On the basis of the previous computation results, it is also of interest to study the errors when \( \psi_2 \) is extended beyond 30°. However, such an extension would include large areas where no 1° x 1° gravity material exists today. In these computations, we will therefore assume that we have such a material with a uniform error of all blocks. Thus, we can apply formula (14) for the computation of \( \delta \theta_{\xi_2, \eta_2} \). The resulting RMS errors are shown in Figure 4.

From Figures 2 and 4 we conclude that a dominating error source is the truncation of the spherical harmonic expansion. A significant minimum of the RMS error is obtained for \( \psi_2 = 40^\circ \). It is unreasonable to extend \( \psi_2 \) beyond this minimum.
Figure 4. The Total RMS Error of the Deflection of the Vertical ($\theta$) (The Inner Zone Excluded).
3.5 The Inner Zone Error and the Total Error

The error of the inner zone depends on the quality and distribution of the observations in this area. To some extent it will also vary with the method used in the computations. By using least squares collocation with approximately 200 observations, Lachapelle (ibid.) estimated the errors of $\xi_1$ and $\eta_1$ to about $1'3$. These errors were rather independent of the cap size $\psi_1$, which varied between $0'7$ and $1'5$ in the computations. Hence, by using this technique, the total RMS error of $\theta$ will be in the order of $2'2$ for $\psi_1 = 1^\circ$ and $\psi_2 = 40^\circ$.

Kearsley (1976) applied the original Vening Meinesz' formula with the well-known subdivision into Rice Rings. The inner zone errors of $\xi_1$ and $\eta_1$ were about $0'3$ for $\psi_1 \approx 1^\circ$. Thus, a total RMS error of $\theta$ in the order of $1'3$ (for $\psi_2 = 40^\circ$) seems possible to achieve in an intensive determination of the inner zone contribution. Even though the same accuracy might be possible to obtain by using least squares collocation, there are practical limits of the number of observations in this method (Lachapelle, ibid., p. 6). Here we refer to $\delta\xi = \delta\eta = 0'3$ as the accuracy of Vening Meinesz' formula. Some total RMS errors based on the above estimates of $\delta\theta_1$ are given in Table 3.

Table 3. Total RMS Errors of the Deflections of the Vertical for $\psi_1 = 1^\circ$

<table>
<thead>
<tr>
<th>$\psi_2$</th>
<th>5$^\circ$</th>
<th>10$^\circ$</th>
<th>20$^\circ$</th>
<th>30$^\circ$</th>
<th>40$^\circ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Collocation $\delta\theta_1 = 1'84$</td>
<td>2'88</td>
<td>2'57</td>
<td>2'34</td>
<td>2'24</td>
<td>2'22</td>
</tr>
<tr>
<td>Vening Meinesz $\delta\theta_1 = 0'42$</td>
<td>2'26</td>
<td>1'85</td>
<td>1'50</td>
<td>1'35</td>
<td>1'32</td>
</tr>
</tbody>
</table>

Lachapelle (ibid.) compared predicted deflections of the vertical at some 169 astrogeodetic stations with the astrogeodetic deflection components. The RMS difference between the total deflections was $2'12$. This result agrees fairly well with Table 3.

4. Summary and Conclusions

In this paper we have studied possible error sources for the determination of the deflections of the vertical by using a combined method. The investigation deals mainly with the errors generated in the intermediate and remote zone, while for the inner zone, some error figures have been adopted from earlier studies. The numerical results are shown in Tables 2 – 3 and Figures 2 – 4.

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We assume that the only contribution to $\xi$ and $\eta$ from the remote zone is a spherical harmonic expansion to degree 16. The RMS truncation error of such a series was found considerably sensitive to changes in the distance of truncation, $\psi_2$ (see Figure 2). A local minimum ($\delta \theta_0,2 = 0.06$) is obtained for $\psi_2 = 40^\circ$.

In the intermediate zone the gravity field was assumed to be represented by $1^\circ \times 1^\circ$ mean anomalies. The error due to lack of a more detailed gravity material is depicted in Figure 3. This error source can be diminished by using a more detailed subdivision of the blocks between the distances $1^\circ$ and $10^\circ$ from the computation point.

The error propagation of the mean anomaly errors ($\delta \theta_2,2$) is shown in Table 2 for four selected points. A reasonable approximation of these errors is obtained by using formula (14) according to Moritz and Groten.

The RMS sum of the errors of the intermediate and remote zone is given in Figure 4. A significant gain in accuracy is achieved by extending $\psi_2$ to at least $30^\circ$. A minimum is obtained for $\psi_2 = 40^\circ$. Finally, by adding a representative error for the inner zone computation, the total RMS error was estimated (Table 3). If the inner zone is determined by the method of least squares collocation, the total RMS error will hardly be less than 2.2 ($\psi_1 = 1^\circ$). Substituting this method with an accurate version of Vening Meinesz' formula, the final error might decrease to 1.3. Further improvements are expected for a refined subdivision of the intermediate zone blocks. In no case should the truncation distance ($\psi_2$) exceed $40^\circ$. 

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References


Tscherning, C. C. and R. H. Rapp, Closed Covariance Expressions for Gravity Anomalies, Geoid Undulations and Deflections of the Vertical Implied by Anomaly Degree Variance Models, Department of Geodetic Science Report No. 208, The Ohio State University, Columbus, 1974.

