A FORCED QUASILINEAR WAVE EQUATION
WITH DISSIPATION

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ABSTRACT

We obtain sufficient conditions for the global existence, uniqueness, and continuous dependence of solutions of the quasilinear Cauchy problem (in one space dimension):

\[ \begin{cases} 
    y_{tt} + a y_t - (\sigma(y_x))_x = g \\
    (0 < t < \infty, x \in \mathbb{R}) \\
    y(0,x) = y_0(x), y_t(0,x) = y_1(x) (x \in \mathbb{R})
\end{cases} \] (*)

for smooth, small data \( y_0, y_1, \) and \( g. \) In (*) subscripts denote partial differentiation, \( a > 0 \) is a constant, \( \sigma : \mathbb{R} \rightarrow \mathbb{R}, g : [0,\infty) \times \mathbb{R} \rightarrow \mathbb{R}, \)
\( y_0, y_1 : \mathbb{R} \rightarrow \mathbb{R} \) are given sufficiently smooth functions, and \( \sigma \in C^2(\mathbb{R}) \) satisfies \( \sigma(0) = 0, \sigma'(\xi) \geq \varepsilon > 0 (\xi \in \mathbb{R}); \) the "genuinely nonlinear" problem \( \sigma''(\xi) \neq 0 \) is of primary interest. The results can be used to study certain nonlinear Volterra functional differential equations arising in heat flow and viscoelastic motion for "materials with memory".

AMS (MOS) Subject Classifications: 35L60, 35L05, 35A05

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This paper concerns the initial value problem for the one dimensional quasilinear wave equation with dissipation

\[
\begin{cases}
    y_{tt} + \alpha y_t - (\sigma(y_x))_x = g(t,x) & (0 < t < \infty, -\infty < x < \infty) \\
y(0,x) = y_0(x), y_t(0,x) = y_1(x) & (-\infty < x < \infty)
\end{cases}
\]

where the subscripts \(t, x\) denote partial differentiation, \(\alpha > 0\) is a fixed constant, and the given real functions \(\sigma, g, y_0, y_1\) are assumed to be sufficiently smooth. If \(\sigma(\xi) = c^2 \xi, -\infty < \xi < \infty, c\) a constant, equation (1) is the linear wave equation with dissipation which can be solved explicitly by elementary methods.

However, there are problems arising in such applications as fluid dynamics, electrodynamics, elasticity, and others in which the linear problem \(\sigma(\xi) = c^2 \xi\) does not adequately model the physical situation. If, for example, \(\sigma(\xi) = c^2 \xi + f(\xi)\), where \(f\) is a smooth function with \(f(0) = f'(0) = 0\) (e.g. \(f(\xi) = \xi^2\)), (1) can no longer be solved explicitly; yet it is important to obtain qualitative information about solutions. Before one can do this, one must know what types of solutions exist and for how long. If \(\alpha = 0, g \equiv 0\) in (1), it is known that solutions can develop singularities in the first derivatives at a finite time \(t\), even for arbitrarily smooth data \(y_0(x), y_1(x)\) (such solutions are called "shocks").

It is the purpose of this paper to obtain reasonable sufficient conditions for the global existence and uniqueness of smooth solutions of (1) for \(\alpha > 0\). Our result states that such solutions exist for \(0 < t < \infty, -\infty < x < \infty\), provided the data functions \(y_0, y_1, g\) are sufficiently smooth and small (in a suitable norm); we also show that the solutions depend continuously on the data. Consequently, shock solutions do not arise in our situation. As an application we indicate briefly how our result can be used to discuss a problem arising in nonlinear heat flow and viscoelasticity.

The method of proof is technical and involves an extension of a method of T. Nishida who studied (1) with the forcing term \(g \equiv 0\); Nishida did not consider the problem of the continuous dependence of the solution on the data \(y_0, y_1, g\).
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1. Introduction. We study the global existence, uniqueness and continuous dependence on data of smooth solutions of the initial value problem

\[ y_{tt} + ay_t - (\sigma(y_x))_x = g \quad (0 < t < \infty, x \in \mathbb{R}), \]

\[ y(0,x) = y_0(x), \quad y_t(0,x) = y_1(x) \quad (x \in \mathbb{R}), \]

where the subscripts \( t, x \) denote partial differentiation, \( a > 0 \) is a fixed constant, \( \sigma : \mathbb{R} \rightarrow \mathbb{R}, g : [0,\infty) \times \mathbb{R} \rightarrow \mathbb{R} \) and \( y_0, y_1 : \mathbb{R} \rightarrow \mathbb{R} \) are given smooth functions. We shall assume throughout that

\[ \sigma \in C^2(\mathbb{R}), \quad \sigma(0) = 0, \quad \sigma'(\xi) \geq c > 0 \quad (\xi \in \mathbb{R}; c > 0); \]

the case \( \sigma''(\xi) \neq 0 \) is of primary interest.

If \( a = 0, g \equiv 0 \) it is known [4], [7] that solutions of the Cauchy problem (1.1), (1.2) will in general develop singularities in the first derivatives even for smooth data, and smooth solutions will not exist for large \( t \). If \( a > 0, g \equiv 0 \) Nishida [10], has established the existence and uniqueness of global smooth solutions of (1.1) for smooth and sufficiently small data (1.2) by a remarkably simple method.

It is the purpose of this note to (i) extend Nishida's method to obtain the global existence and uniqueness of smooth solutions of (1.1), (1.2) with \( g \not\equiv 0 \), and (ii) study the continuous dependence of solutions of (1.1), (1.2) on the data \( y_0, y_1, g \). The result (i) is implicit in a recent paper of MacCamy [5]; however, his proof of the analogue of the important Lemma 2.3 below is not entirely complete. The result (ii) is new.

We remark that our results (i) and (ii) can be used to obtain a local existence and uniqueness result for smooth solutions of the functional differential equation

\[ y_{tt} + ay_t - (\sigma(y_x))_x = G(y) \quad (0 \leq t \leq T, x \in \mathbb{R}), \]

subject to the initial condition (1.2), for some \( T > 0 \). In (1.3) \( G \) is a given mapping defined on a suitable function space, and \( G \) satisfies a Lipschitz type condition. While limitations of space do not allow us to present this problem in detail, we point out that if \( F(g) \) denotes the solution of (1.1), (1.2) on \([0,T] \times \mathbb{R} \), then a solution of (1.3), (1.2) is a fixed point of the composition map \( K \) defined by \( K(y) = F(G(y)) \). Such a fixed point can be found with the aid of our continuous dependence result for smooth solutions of (1.1), (1.2) for sufficiently small data in a manner similar to the method we used with Crandall.

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in [1] to solve a functional differential equation in which, however, the under-
lying problem was an evolution equation of parabolic, rather than hyperbolic
type. The details will be presented in a forthcoming joint paper with C. Dafermos.

The Cauchy problem (1.3), (1.2) has arisen in certain applications in heat flow and viscoelastic motion for "materials with memory" studied by MacCamy [5], [6]; the functional $G$ has the form

$$G(y)(t,x) = G(t,x) + \beta y(t,x) + \int_0^t b(t-t)y(t,x)dt,$$

where $\beta$ is a real smooth function on $[0,\infty) \times \mathbb{R}$ such that

$$\sup_{x \in \mathbb{R}} |G(t,x)| \in L^1(0,\infty) \cap L^\infty(0,\infty), \quad \sup_{x \in \mathbb{R}} |xG(t,x)| \in L^1(0,\infty),$$

$\beta > 0$ is a constant, and $b \in L^1(0,\infty;\mathbb{R})$, the value of $G$ at $(t,x)$ depends on the restriction of $y(\cdot,x)$ to $[0,t)$. In [5], [6] the interest is in the existence of global smooth solutions of the Cauchy problem (1.3), (1.4), (1.2); this is carried out by combining Nishida's method with certain delicate a priori estimates obtained by energy methods. However, the proof in [5], [6] appears to us to be incomplete, because the local existence problem for (1.3), (1.4), (1.2), which can be handled by the method outlined above, is essentially ignored.

In Section 2 we obtain the desired results for a "diagonal" strictly hyperbolic system of first order equations equivalent to (1.1), (1.2); the results for (1.1), (1.2) follow as an easy corollary and these are stated in Section 3. We acknowledge useful discussions with M. C. Crandall, C. Dafermos, and R. J. DiPerna during the preparation of this paper.

Finally, we mention related work of Matsumuma [8], [9] received after the completion of this paper; the author generalizes Nishida's results for (1.1), (1.2) with $g \equiv 0$ from one space dimension to quasilinear hyperbolic equations in several space dimensions, and he obtains global existence of weak solutions and results concerning their decay (Nishida's method does not apply in this case).

2. Equivalent Systems and Preliminary Results. We assume that $\sigma$ in (1.1) satisfies assumptions (a). In addition, assume that $g$, and the initial functions $y_0, y_1$ in (1.1), (1.2) satisfy:

$$(g) \quad g, g_x, g_{xx} \in C([0,\infty) \times \mathbb{R}), \quad g(t) = \sup_{x \in \mathbb{R}} |g(t,x)| \in L^\infty(0,\infty) \cap L^1(0,\infty),$$

$$g_{1}(t) = \sup_{x \in \mathbb{R}} |g_{1}(t,x)| \in L^\infty(0,\infty),$$

$$(I) \quad g_0, g_1 \in B^2(\mathbb{R}), \quad y_0, y_1 \in B^2(\mathbb{R}),$$

where $B^m$ denotes the set of real functions with continuous and bounded derivatives up to and including order $m$.

Following Nishida [10] we reduce the Cauchy problem (1.1), (1.2) to the equivalent system (2.3) below. Putting $y_x = v$ and $y_t = w$ in (1.1), (1.2)
yields the equivalent Cauchy problem

\begin{align*}
\begin{cases}
\tau_t - \tau_x = 0, \quad \tau_t - \sigma'(v)\tau_x + aw = g \\
v(0,x) = y_0'(x), \quad w(0,x) = y_1(x)
\end{cases}
(0 < t \leq \tau, \quad x \in \mathbb{R})
\end{align*}

The eigenvalues of the matrix of (2.1)

\[
\begin{pmatrix}
0 & 1 \\
-\sigma'(v) & 0
\end{pmatrix}
\]

are \(\lambda = -\sqrt{\sigma'(v)}, \mu = \sqrt{\sigma'(v)}\); by assumptions (c), \(\lambda\) and \(\mu\) are real and distinct so that (2.1) is a strictly hyperbolic problem in the region \((v,w) : v \in \mathbb{R}, w \in \mathbb{R}\). To diagonalize (2.1) introduce the Riemann invariants

\begin{align*}
\rho &= w + \phi(v), \\
\sigma &= w - \phi(v), \\
\phi(v) &= \int_0^v \sqrt{\sigma'(|\xi|)} \, d\xi
\end{align*}

by (c) the mapping \((v,w) \rightarrow (\rho,\sigma)\) defined by (2.2) is one to one from \(\mathbb{R} \times \mathbb{R}\) onto \(\mathbb{R} \times \mathbb{R}\). A simple calculation shows that (2.1) is equivalent to the Cauchy problem for the diagonal, strictly hyperbolic system

\begin{align*}
\begin{cases}
\tau_t + \rho \tau_x + \frac{a}{2} (\rho + \sigma) = g \\
\sigma_t + \mu \sigma_x + \frac{a}{2} (\rho + \sigma) = g \\
\tau(0,x) = \tau(0,x), \quad \sigma(0,x) = \sigma_0(x)
\end{cases}
(0 < t \leq \tau, \quad x \in \mathbb{R}),
\end{align*}

where by (2.2) \(\lambda = \lambda(r-s), \mu = \mu(r-s) \in C'(\mathbb{R})\), and where by (2.1)

\begin{align*}
\tau_0(x) = y_1(x) + \phi(y_0'(x)), \\
\sigma_0(x) = y_1(x) - \phi(y_0'(x))
\end{align*}

by assumptions (a) and I, the initial data \(r_0, s_0 \in \beta^1(\mathbb{R})\). It is also seen that if \(r, s \) is a smooth \((\beta^1)\) solution of the problem (2.3) for \((t,x) \in \Omega \subseteq ([0,\tau) \times \mathbb{R})\), then \(y\), determined by the relations \(y_t = w(r,s), \quad y_x = v(r,s)\) (where \(v, w\) are uniquely determined by (2.2)), will be a smooth \((\beta^2)\) solution of the Cauchy problem (1.1), (1.2) and conversely; we shall therefore deduce our results for (1.1), (1.2) from (2.3).

The following local result for (2.3) is known [2; Sec. 8], [3, Theorem VI]:

**Lemma 2.1.** Let \(r_0, s_0 \in \beta^1(\mathbb{R})\), let assumptions (c) hold, and assume that \(g, g_x \in \beta^0\) for \((t,x) \in [0,T] \times \mathbb{R}\), where \(T > 0\). Then there exists a number \(0 < T_1 < T\) such that the Cauchy problem (2.3) has a unique smooth solution \(r, s \in \beta^1([0,T_1] \times \mathbb{R})\).

The objective of the next two lemmas is to obtain apriori estimates on \(r, s, r_x, s_x\) (and hence by (2.3) on \(r_t, s_t\)), independent of \(T\), which enable us to continue the local \(\beta^1\)-solution in \(t\) by a standard method.

**Lemma 2.2.** Let the assumptions of Lemma 2.1 hold. In addition, assume that \(g(t) = \sup_{x \in \mathbb{R}} |g(t,x)| \leq L^1(0,\infty)\). Define the a priori constant \(M_0 > 0\) by

\[
M_0 = \tau_0 + s_0 + 2 \int_0^\tau g(t) \, dt, \quad \tau_0 = \sup_{x \in \mathbb{R}} |\tau_0(x)|, \quad s_0 = \sup_{x \in \mathbb{R}} |s_0(x)|.
\]
For as long as the \( s \)-solution \( r, s \) of (2.3) exists one has

\[
\sup_{\tau \in \mathbb{R}} |r(\tau, x)| \leq M_0, \quad \sup_{\tau \in \mathbb{R}} |s(\tau, x)| \leq M_0.
\]

Sketch of Proof. The proof is similar to that of [10, Lemma 1], [5, Lemma 6.2].

Define the \( \lambda \) and \( \mu \) characteristics of (2.3) respectively by

\[
\begin{align*}
\lambda &= \lambda(r(t, x_1(t, \beta)) - s(t, x_1(t, \beta))), \\
\mu &= \mu[r(t, x_2(t, \gamma)) - s(t, x_2(t, \gamma))].
\end{align*}
\]

Let \( ^\prime = \frac{3}{\beta t} + \frac{3}{\beta x}, \quad ^\prime = \frac{3}{\beta t} + \mu \frac{3}{\beta x} \) denote differentiation along the \( \lambda \) and \( \mu \) characteristics respectively, thus \( r(\tau, x_1) = \frac{d}{dt} r(t, x_1(t, \beta)) \), \( s(\tau, x_2) = \frac{d}{dt} s(t, x_2(t, \gamma)) \). Equations (2.3) become the ordinary differential equations

\[
\begin{align*}
\frac{d}{dt} [r(t, x_1(t, \beta)) + \frac{a}{2} [r(t, x_1(t, \beta)) - s(t, x_1(t, \beta))] = g(t, x_1(t, \beta)) \\
\frac{d}{dt} [s(t, x_2(t, \gamma)) + \frac{a}{2} [r(t, x_2(t, \gamma)) - s(t, x_2(t, \gamma))] = g(t, x_2(t, \gamma)).
\end{align*}
\]

Note that solutions of (2.7) will exist for as long as the slopes \( \lambda, \mu \) of the characteristics \( x_1(t, \beta) \) and \( x_2(t, \gamma) \) remain bounded. Put

\[
R(t) = \sup_{0 \leq \tau \leq t} \sup_{x \in \mathbb{R}} [e^{\frac{t}{2}} |r(\tau, x)| + |s(\tau, x)|],
\]

\[
x_0 = \sup_{x \in \mathbb{R}} |x_0(x)|, \quad s_0 = \sup_{x \in \mathbb{R}} |s_0(x)|.
\]

Integrate each of the equations (2.9) using \( r(0, x_1(0, \beta)) = r_0(\beta) \), \( s(0, x_2(0, \gamma)) = s_0(\gamma) \) (see (2.3), (2.6)), add the resulting equations and take absolute values; a standard argument yields the inequality

\[
R(t) \leq x_0 + s_0 + \frac{a}{2} \int_0^t R(\xi) d\xi + 2 \int_0^t e^{\frac{t}{2}} g(\xi) d\xi.
\]

Gronwall's inequality applied to (2.8) gives

\[
R(t) \leq (x_0 + s_0) e^{\frac{at}{2}} + 2e^{\frac{at}{2}} \int_0^t g(\xi) d\xi,
\]

and thus finally

\[
\sup_{0 \leq \tau \leq t} (|r(\tau, x)| + |s(\tau, x)|) \leq M_0,
\]

and the proof is complete.

Lemma 2.3. Let the assumptions of Lemma 2.1 and (g) be satisfied. Define the constant \( D_1 > 0 \) by
For as long as the $\delta^1$-solution $r, s$ of (2.3) exists and if $D_1$ is sufficiently small, there exists a constant $M_1 = M_1(D_1) > 0$ where $M_1(D_1) \to 0$ as $D_1 \to 0$, such that

\[
\sup_{x \in \mathbb{R}} |r(t,x)| \leq M_1, \quad \sup_{x \in \mathbb{R}} |s(t,x)| \leq M_1.
\]

**Sketch of Proof.** (Compare [10, Lemma 2], [5, Lemma 6.3].) Differentiate the first equation in (2.3) obtaining (recall $\lambda = \lambda(r,s)$)

\[
\frac{\partial r}{\partial t} + \lambda \frac{\partial r}{\partial x} = -\lambda \frac{\partial^2 r}{\partial x^2} - \lambda \frac{\partial s}{\partial x} - \frac{a}{2} \left( \frac{\partial^2 r}{\partial x^2} \right) + g_x.
\]

We remark that although Lemma 2.1 does not assert the existence of $r_{xx}$ and $r_{xt}$, note that the left side of (2.12) is $r_x$ and this does exist for as long as the $\delta^1$-solution $r, s$ of (2.3) exists. This observation also justifies the validity of equations (2.12)-(2.18) which follow. Since $\mu = -\lambda$ the second equation in (2.3) gives

\[
s_x = \frac{s_r}{2\lambda} + \frac{a}{4\lambda} (r+s) - \frac{g}{2\lambda} \quad \left( \frac{s}{r} = \frac{3}{2} + \lambda \frac{3}{2\lambda} \right).
\]

Define

\[
h = \frac{1}{2} \log(-\lambda(r,s)).
\]

Differentiating $h$ along the $\lambda$-characteristic and using $\lambda_g = -\frac{\lambda}{x}$ gives

\[
h' = \frac{r}{2\lambda} \left( -\frac{a}{2} (r+s) + g - s' \right).
\]

Substitution of (2.13), (2.15) into (2.12) yields

\[
r_x + \frac{a}{2} + \lambda r_x + h'r_x = -\frac{a}{4\lambda} s' - \frac{2}{8\lambda} (r+s) + \frac{a}{4\lambda} g + g_x,
\]

or equivalently

\[
(e^{h} r_x)' + \frac{a}{2} + \lambda e^{h} r_x = \left( -\frac{a}{4\lambda} s' - \frac{2}{8\lambda} (r+s) + \frac{a}{4\lambda} g + g_x \right) e^{h}.
\]

Define the function $z$ by

\[
z(r,s) = \int_{0}^{r} \frac{a}{4\lambda(\xi)} e^{h(\xi)} d\xi,
\]

then $z' = -\frac{a}{2\lambda} e^{h(r,s)} + \frac{a}{4\lambda} e^{h} g - \frac{a}{4\lambda} e^{h} s'$ and (2.16) becomes

\[
(e^{h} r_x)' + \frac{a}{2} + \lambda e^{h} r_x = z' + e^{h} g_x.
\]

To integrate (2.18) along the $\lambda$-characteristic put

\[
k(t) = \frac{a}{2} + \frac{a}{4\lambda} \left( r(t,x_1(t,\beta)) - s(t,x_1(t,\beta)) \right) r_x(t,x_1(t,\beta))
\]

\[
p(t) = r_x(t,x_1(t,\beta)) \exp\left[ h(t,x_1(t,\beta)) \right]
\]

\[
p(t) = z'(t,x_1(t,\beta)) + g_x(t,x_1(t,\beta)) \exp[h(t,x_1(t,\beta))].
\]
Then

\begin{equation}
(2.20) \quad R(t) = R(0) \exp \left( - \int_0^t k(\tau) \, d\tau \right) + \int_0^t p(\xi) \exp \left( - \int_\xi^t k(\tau) \, d\tau \right) \, d\xi.
\end{equation}

Suppose we can show that for any solution \( r, s \) of (2.3)

\begin{equation}
(2.21) \quad |\lambda_x r_x| \leq \frac{a}{4}.
\end{equation}

Then \( \lambda(t) = \frac{a}{2} + \lambda_r (v) \lambda_x (v) \geq \frac{a}{4} \) and by an easy calculation

\begin{equation}
(2.22) \quad |R(t)| \leq |R(0)| + 3 \sup_{0 \leq \xi \leq t} |z(\tau, x_1(\tau, \beta))|
\end{equation}

\[ + \frac{a}{4} \sup_{0 \leq \xi \leq t} |g_x (\tau, x_1(\tau, \beta))) \exp h(x, x_1(\tau, \beta))|.
\]

We next show that (2.21) holds for any solution \( r, s \) of (2.3), provided the constant \( D_1 > 0 \) is sufficiently small. Indeed, at \( t = 0 \) and for any \( \beta \in \mathbb{R} \)

\begin{equation}
(2.23) \quad |x_0(\beta) x_1(\beta) - s_0(\beta))| \leq \sup_{x \in \mathbb{R}} |x_0(\beta) x_1(\beta) - s_0(\beta))| < \frac{a}{4}
\end{equation}

provided \( D_1 > 0 \) is sufficiently small. By (2.19) \( \lambda_x = \lambda_{x, \alpha} \exp \) and by (3), (2.14) and Lemma 2.2 \( |\lambda_x \exp | \) is uniformly bounded in \( (t, x) \) by a constant \( K_1(D_1) > 0 \), where \( K_1(D_1) \to 0 \) as \( D_1 \to 0 \), for any solution \( r, s \) of (2.3). By (2.14), (2.17), (2.22) and Lemma 2.2 the quantity \( R \) is uniformly bounded for any solution \( r, s \) of (2.3) as follows:

\begin{equation}
(2.24) \quad |R(t)| \leq |R(0)| + 3 \sup_{0 \leq \xi \leq t} |z(\tau, x_1(\tau, \beta))|
\end{equation}

\[ + \frac{a}{4} \sup_{0 \leq \xi \leq t} \sqrt{A(\xi, x_1(\tau, \beta))) \exp h(x_1(\tau, \beta))}\]

\[ \leq K_2(D_1), \text{ where } K_2(D_1) \to 0 \text{ as } D_1 \to 0. \]

Therefore \( |\lambda_x r_x| \leq K_1(D_1)K_2(D_1) \) uniformly in \( (t, x) \) for any solution \( r, s \) of (2.3). The assertion (2.21) holds at \( t = 0 \), for \( D_1 \) sufficiently small (by (2.23)). Choosing \( D_1 \) smaller if necessary so that \( K_1(D_1)K_2(D_1) \leq \frac{a}{4} \), we conclude that (2.21) continues to hold for as long as the solution \( r, s \) exists.

Returning to (2.22), (2.24) and using \( x_1 = \exp \) establishes the estimate for \( x_1 \) in (2.11); a similar argument yields the estimate for \( x_2 \) and completes the proof of Lemma 2.3.

The a priori estimates of Lemmas 2.2 and 2.3, together with \( \|S\|_{L^\infty(0, a)} < \) yield uniform a priori estimates for \( r, s \), for any solution \( r, s \) of (2.3), provided \( D_1 > 0 \) is sufficiently small. Then Lemmas 2.1, 2.2, 2.3 and a standard continuation argument give the first part of the following global result for the Cauchy problem (2.3).
Theorem 2.1. Let the assumption \((c), (g)\) be satisfied, and let the initial data \(r_0, s_0 \in \mathcal{F}_1(\mathbb{R})\). If the constant \(D_1\) (see Lemma 2.3) is sufficiently small, then the Cauchy problem (2.3) has a unique \(\mathcal{F}_1\)-solution \(r, s\) for \(0 < t < \infty, x \in \mathbb{R}\) and the a priori estimates (2.5), (2.11) are satisfied for \(0 < t < \infty\).

Let the above assumptions be satisfied by initial data \(\tilde{r}_0, \tilde{s}_0\) and \(\tilde{r}_0', \tilde{s}_0'\), and forcing functions \(g, \tilde{g}\), denoted by \(\bar{r}, \bar{s}\) and \(\tilde{r}', \tilde{s}'\) the corresponding \(\mathcal{F}_1\)-solutions of (2.3) on \([0, \infty) \times \mathbb{R}\).

Define
\[
\zeta(t) = \sup_{x \in \mathbb{R}} |\bar{r}(t,x) - \bar{r}(t,x)| + \sup_{x \in \mathbb{R}} |\bar{s}(t,x) - \bar{s}(t,x)|.
\]

Then there exists a constant \(\mu_2 = \mu_2(\mu, M_0) > 0\) such that
\[
(2.25) \quad \zeta(t) \leq e^{2H_1} \left( \zeta(0) + \int_0^t e^{2H_1 s} \sup_{x \in \mathbb{R}} |g(t,x) - \tilde{g}(t,x)| \, ds \right) \quad (0 < t < \infty),
\]
where \(M_0, H_1\) are the bounds in (2.5), (2.11) respectively.

Remark. The continuous dependence result (2.25) also holds for local solutions \(r, s\) and \(\bar{r}, \bar{s}\) on \([0, T_1) \times [0, \infty)\) of the Cauchy problem (2.3) (see Lemma 2.1), but only for \(0 < t < T_1\).

Proof of Theorem 2.1. It remains only to prove (2.25). If \(r, s\) and \(\bar{r}, \bar{s}\) are \(\mathcal{F}_1\)-solutions of (2.3) for the situation in the theorem, one has
\[
(2.26) \quad \begin{cases}
(r-r)_t + \lambda r_x - \lambda \bar{r}_x - \lambda (r-s) + g - \tilde{g} = 0 < t < \infty, \\
(s-s)_t + \mu s_x - \mu \bar{s}_x - \mu (r-s) + g - \tilde{g} = 0 < t < \infty,
\end{cases}
\]
suject to the initial conditions
\[
(2.27) \quad r(0,x) - \bar{r}(0,x) = r_0(x) - \bar{r}_0(x), s(0,x) - \bar{s}(0,x) = s_0(x) - \bar{s}_0(x) \quad (x \in \mathbb{R}),
\]
where \(\lambda = \lambda(r-s), \mu = \mu(r-s), \bar{r} = \lambda(\bar{r}-\bar{s}), \bar{s} = \mu(\bar{r}-\bar{s})\). But
\[
\lambda r_x - \lambda \bar{r}_x = \lambda(r-x) + (\lambda-x) \bar{r}_x \\
\mu s_x - \mu \bar{s}_x = \mu(s-x) + (\mu-x) \bar{s}_x;
\]
Therefore (2.26) can be written as
\[
(2.28) \quad \begin{cases}
(r-r)_t + \lambda(r-r)_x = \frac{a}{2} [(r-r) + (s-s)] + g - \tilde{g} - (\lambda-x) \bar{r}_x \\
(s-s)_t + \mu(s-s)_x = \frac{a}{2} [(r-r) + (s-s)] + g - \tilde{g} - (\mu-x) \bar{s}_x,
\end{cases}
\]
Recalling the definitions of \(\lambda, \mu\) and using (0) and the mean value theorem one has
\[
\lambda - \bar{\lambda} = \lambda(r-s) - \lambda(\bar{r}-\bar{s}) = \frac{dt}{dt} (r - \bar{r} + \theta_1 (r - s - (\bar{r}-\bar{s}))(r - s - (\bar{r}-\bar{s}))
\]
\[
\mu - \bar{\mu} = \mu(r-s) - \mu(\bar{r}-\bar{s}) = \frac{dt}{dt} (r - \bar{r} + \theta_2 (r - s - (\bar{r}-\bar{s}))(r - s - (\bar{r}-\bar{s})),
\]
for some \(0 < \theta_1, \theta_2 < 1\). Therefore, (2.28) becomes
\[
\begin{align*}
(x \cdot \tau) + \frac{a}{2} (x \cdot \tau) &= - \frac{a}{2} (s \cdot \omega) - \frac{a}{2} \frac{\partial}{\partial t} (\cdot) (x \cdot \tau) - (s \cdot \omega) \tau + g - \tilde{g} \\
(s \cdot \omega) + \frac{a}{2} (s \cdot \omega) &= - \frac{a}{2} (x \cdot \tau) - \frac{a}{2} \frac{\partial}{\partial t} (\cdot) (x \cdot \tau) - (s \cdot \omega) \omega + g - \tilde{g},
\end{align*}
\]

where \( \tau = \frac{\partial}{\partial t} + \frac{1}{2} \frac{\partial}{\partial x} \cdot \omega \) and \( \omega = \frac{1}{2} \mu \frac{\partial}{\partial x} \). We next note that assumption (a) and

Lemmas 2.2 and 2.3 imply the existence of a constant \( M_2 = M_2 (0, M_0) \) such that

\[
\frac{d}{dt} (\cdot) \tau \leq M_2 M_1, \quad \frac{d}{dt} (\cdot) \omega \leq M_2 M_1,
\]

uniformly in \( (t, x) \in (0, \infty) \times \mathbb{R} \) where \( M_1 \) is the bound in Lemma 2.3. Integrating the first equation in (2.29) along any \( \lambda \)-characteristic and the second along any \( \mu \)-characteristic and making simple estimates one obtains the pair of inequalities

\[
\begin{align*}
\frac{5}{2} + \sup_{x \in \mathbb{R}} | \tau(t, x) - \tau(0, x) | &\leq \sup_{x \in \mathbb{R}} | \tau_0 (x) - \tau_0 (x) | + \int_0^t \frac{5}{2} \sup_{x \in \mathbb{R}} | g(t, x) - \tilde{g}(t, x) | dt \\
&\quad + \int_0^t \frac{5}{2} \sup_{x \in \mathbb{R}} | g(t, x) - \tilde{g}(t, x) | dt + M_1 M_2 \int_0^t \frac{5}{2} \sup_{x \in \mathbb{R}} | \tau(t, x) - \tau(t, x) | dt, \\
\frac{5}{2} + \sup_{x \in \mathbb{R}} | \tilde{\tau}(t, x) - \tilde{\tau}(0, x) | &\leq \sup_{x \in \mathbb{R}} | \tilde{\tau}_0 (x) - \tilde{\tau}_0 (x) | + \int_0^t \frac{5}{2} \sup_{x \in \mathbb{R}} | g(t, x) - \tilde{g}(t, x) | dt \\
&\quad + \int_0^t \frac{5}{2} \sup_{x \in \mathbb{R}} | g(t, x) - \tilde{g}(t, x) | dt + M_1 M_2 \int_0^t \frac{5}{2} \sup_{x \in \mathbb{R}} | \tilde{\tau}(t, x) - \tilde{\tau}(t, x) | dt.
\end{align*}
\]

Adding these inequalities one obtains on using the definition of \( \zeta \)

\[
\begin{align*}
(2.31) \quad \frac{5}{2} + \frac{5}{2} &\leq \zeta(0) + 2 \int_0^t \frac{5}{2} \sup_{x \in \mathbb{R}} | g(t, x) - \tilde{g}(t, x) | dt \\
&\quad + \int_0^t (\frac{5}{2} + 2M_1 M_2) \frac{5}{2} \zeta(t) dt \quad (0 \leq t < \infty).
\end{align*}
\]

Finally, applying Gronwall's inequality to (2.31) yields the result (2.25) completing the proof.

3. Global Existence, Uniqueness, and Continuous Dependence for the Cauchy Problem \((1.1), (1.2)\). As an immediate consequence of Theorem 2.1 and of the equivalence of the Cauchy problems (1.1), (1.3) and (3.1), (3.4) we obtain the main result of this paper.

**Theorem 3.1.** Let the assumptions \((a), (g), (I)\) be satisfied. Define the constant

\[
D = \sup_{x \in \mathbb{R}} | y_0 (x) | + \sup_{x \in \mathbb{R}} | y_1 (x) | + \sup_{x \in \mathbb{R}} | y_0^* (x) | + \| g \|_{L^1 (0, \infty)} + \| g \|_{L^1 (0, \infty)} + \| g_1 \|_{L^1 (0, \infty)}.
\]

If \( D \) is sufficiently small, then the Cauchy problem \((1.1), (1.2)\) has a unique \( \mathcal{S} \)-solution \( y \) on \((0, \infty) \times \mathbb{R} \), and the solution \( y \) satisfies the a priori estimates.
\begin{align}
(3.2) \sup_{x \in \mathbb{R}} |y_x(t,x)|, \sup_{x \in \mathbb{R}} |y^t(t,x)| & \leq \sup_{x \in \mathbb{R}} |y_0(x)| + \sup_{x \in \mathbb{R}} |y_1(x)| + 2 \int_0^t g(x) \, dt = M_0 \quad (0 \leq t \leq \infty),
\end{align}
and there exists a constant $M_1 = M_1(D) > 0$ (which $\to 0$ as $D \to 0$) such that
\begin{align}
(3.3) \sup_{x \in \mathbb{R}} |y_{xx}(t,x)|, \sup_{x \in \mathbb{R}} |y_{xt}(t,x)|, \sup_{x \in \mathbb{R}} |y^t_{tt}(t,x)| & \leq M_1 \quad (0 \leq t \leq \infty).
\end{align}

Let, in addition, the assumptions (g) and (I) be satisfied also by functions $g$ and $y_0, y_1$ and let $\bar{y}$ denote the corresponding $\delta$-solution on $(0, \infty) \times \mathbb{R}$. Define
\begin{align}
\zeta(t) = \sup_{x \in \mathbb{R}} |y_x(t,x) - \bar{y}_x(t,x)| + \sup_{x \in \mathbb{R}} |y^t(t,x) - \bar{y}^t(t,x)|.
\end{align}
then there exists a constant $M_2 = (\sigma, M_0) > 0$ such that
\begin{align}
(3.4) \zeta(t) \leq \sigma \left( t \right) + \int_0^t \frac{t - 2M_1 \zeta(t)}{M_1 M_2} \sup_{x \in \mathbb{R}} |g(x) - \bar{g}(x)| \, dt \quad (0 \leq t \leq \infty),
\end{align}
where $M_0, M_1$ are bounds in (3.2), (3.3) respectively, moreover (using
\begin{align}
y(t,x) = y_0(x) + \int_0^t y^t(t,x) \, dt
\end{align}
\begin{align}
\sup_{x \in \mathbb{R}} |y(t,x) - \bar{y}(t,x)| & \leq \sup_{x \in \mathbb{R}} |y_0(x) - \bar{y}_0(x)| + \frac{2M_1 \zeta(t)}{M_1 M_2} \left( \sup_{x \in \mathbb{R}} |y^t_0(x) - \bar{y}_0^t(x)| + \sup_{x \in \mathbb{R}} |y^t_1(x) - \bar{y}^t_1(x)| \right)
+ \int_0^t \frac{t - 2M_1 \zeta(t)}{M_1 M_2} \sup_{x \in \mathbb{R}} |g(x) - \bar{g}(x)| \, dt \quad (0 \leq t \leq \infty).
\end{align}

References


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A FORCED QUASILINEAR WAVE EQUATION WITH DISSIPATION

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Quasilinear Unique
Cauchy Problem Continuous Dependence
Wave Equation Characteristics
Smooth Solutions Riemann Invariants
Global Existence Strictly Hyperbolic

We obtain sufficient conditions for the global existence, uniqueness, and continuous dependence of solutions of the quasilinear Cauchy problem (in one space dimension):

\[
\begin{cases}
  y_{tt} + ay_t - (\sigma(y_x))_x = g \\
  y(0,x) = y_0(x), y_t(0,x) = y_1(x)
\end{cases}
\]

(*)

for smooth, small data \( y_0, y_1, \) and \( g \). In (*) subscripts denote partial
differentiation, $\alpha > 0$ is a constant, $\sigma : \mathbb{R} \to \mathbb{R}$, $g : [0, \infty) \times \mathbb{R} \to \mathbb{R}$, $Y_0, Y_1 : \mathbb{R} \to \mathbb{R}$ are given sufficiently smooth functions, and $\sigma \in C^2(\mathbb{R})$ satisfies $\sigma(0) = 0$, $\sigma'(t) \geq \varepsilon > 0$ ($t \in \mathbb{R}$); the "genuinely nonlinear" problem $\sigma''(t) \neq 0$ is of primary interest. The results can be used to study certain nonlinear Volterra functional differential equations arising in heat flow and viscoelastic motion for "materials with memory".