Fast Ions from Laser Plasmas: 
Analytic Solution and Scaling Laws 

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**Abstract:**
New analytic solutions describe ambipolar acceleration of fast ions from a steep, localized density gradient with a high electron temperature. A scaling law and estimates for the peak energy of ion species are presented.
Fast ions are a characteristic phenomenon observed in irradiation of targets by high-power lasers. Electrostatic acceleration by a steep, localized pressure gradient with a high electron temperature is a suggested mechanism for producing the fast ions. Recent work at NRL demonstrates that a multi-species, ambipolar expansion model can account for most of the high-energy ion distribution measured far from a Nd-laser target. However, the ions of highest energy can be treated more effectively with the analytic solution presented in this note.

The only previous analytic solution describing plasma expansion by this process of electrostatic acceleration is a self-similar solution. Neither the self-similar solution nor the closely related numerical solution involves a characteristic length, and neither yields scaling laws for the maximum ion energies, since both feature ion velocities that increase indefinitely.

In this note we present a new analytic solution that, unlike the self-similar solution, involves a characteristic scale height, and yields an expression for the maximum energy of ion species in terms of their charge, mass, and scale height, and the electron temperature. We also suggest a means by which the expansion described by the new solution might evolve into a self-similar expansion.

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Our assumptions and equations describing the expanding plasma are essentially those of Refs. 3 and 4. We begin by describing the plasma by a two-fluid model involving hot electrons and relatively cold ions. Later we generalize the analytic solution to more than one ion species. Each fluid satisfies a continuity equation. The quasi-neutrality condition replaces the Poisson equation, because the characteristic scale height is taken to be much greater than the electron Debye length. The electrons are assumed to have a Boltzmann distribution (electron pressure balanced by ambipolar potential) with constant temperature. The validity of assuming isothermal electrons depends upon high thermal conductivity from a heat reservoir supplied by the laser. The electron temperature is assumed high enough (~ tens of keV) to neglect electron-ion collisions. The ion temperature is assumed to be high enough to neglect viscosity, but much less than the electron temperature, so that the ion pressure can be ignored. Consequently, the ion momentum equation has only the ambipolar electric field, or equivalently the electron pressure gradient, as its source term.

The system of equations in plane geometry reduces to

\[
\begin{align*}
\frac{\partial (\rho n)}{\partial t} + \nabla \cdot (\rho n \mathbf{v}) + c \frac{\partial (\rho v / c)}{\partial x} &= 0, \\
\frac{\partial (\rho v / c)}{\partial t} + \nabla \cdot (\rho v / c \mathbf{v}) + c \frac{\partial (\rho n v \mathbf{v})}{\partial x} &= 0,
\end{align*}
\]  

(1)

in which \( n \) is the ion density and \( \mathbf{v} \) the velocity of the ion fluid. The constant sound speed is \( c = (Z T_e / M)^{1/2} \), in which \( Z \) and \( M \) are the charge state and mass of the ions, and \( T_e \) is the electron temperature in energy units. These equations can be posed as a Cauchy initial-value
problem, and solved generally by the method of characteristics, but
analytic solutions are useful.

Our initial conditions are \( n = n_0 \exp(-x/\ell) \), and \( v = 0 \), in which
\( n_0 \) and \( \ell \) are a constant density and scale height. These initial
conditions are consistent with a quick deposition of some laser energy
at the target surface (during the rise of a laser pulse for example).
With these initial conditions, our analytic solution to the eqs. (1) is
\[
\begin{align*}
  n &= n_0 \exp(-x/\ell + c^2 t^2/2 \ell^2), \\
  v &= c^2 t/\ell.
\end{align*}
\]

In a reference frame having acceleration \( c^2/\ell \), the solution be-
comes \( n = n_0 \exp(-x/\ell) \) and \( v = 0 \). Thus the analytic solution describes
an exponential density profile maintaining its shape as it accelerates
at a rate \( c^2/\ell \). In the lab frame the energy per ion is
\[
E = \frac{3}{2} m v^2 = \frac{3}{2} \left( \frac{Z^2}{M} \right) \left( \frac{T_e}{\ell} \right)^2 \ell^2
\]

Here is evident the \( Z^2/M \) scaling law found elsewhere theoretically
and experimentally. Moreover eq. (3) demonstrates that fast ions are
produced by high electron temperatures and steep density gradients. An
estimate for the acceleration time scale (and therefore the peak energy)
will be provided below.

If the initial conditions in some region of positive \( x \) are instead
\( n = n_0 \exp(-x/\ell) \) and \( v = cx/\ell \), then the solution to eqs. (1) is the
self-similar solution \( n = n_0 \exp[-(x + ct)/(\ell + ct)] \) and
\( v = c(x + ct)/(\ell + ct) \). However if the density gradient extends over
several scale heights, then these initial conditions imply initial
ion velocities of several times sound speed, or ions that are already fast. Thus, the initial conditions for the self-similar solution require some strong acceleration mechanism to have been in operation prior to \( t = 0 \), while the initial conditions leading to the solution (2) do not. We now provide a hybrid analytic-numerical example to illustrate how the solution (2) can provide the acceleration mechanism necessary to launch the self-similar plasma expansion.

Let two families of curves, \( C_1 \) and \( C_2 \), be defined in the \( x,t \) plane by \( \frac{dx}{dt} = v + c \) and \( \frac{dx}{dt} = v - c \) respectively. \( C_1 \) are the forward-running characteristics, and \( C_2 \) the backward-running characteristics. The Riemann invariants are \( v + ct\ln\left(n/n_0\right) \), which has a constant value along \( C_1 \), and \( -v + ct\ln\left(n/n_0\right) \), which has a constant value along \( C_2 \). If the values of \( n \) and \( v \) are specified along a segment of the line \( t = 0 \) in the \( x,t \) plane, then with the aid of the Riemann invariants, (1) can be solved as a Cauchy initial-value problem.

For example consider a density profile, initially at rest, given by \( n = n_0 \exp\left(-x/t\right) \) for \( x \geq 0 \), and \( n = n_0 \) for \( x \leq 0 \), as illustrated by the dashed line in Fig. 1a. The characteristics are shown in Fig. 1b in the two regions where analytic solutions are known. In region I the characteristics are straight lines, indicating that the plasma remains at rest at constant density there. In region III the equations of the \( C_1 \) and \( C_2 \) characteristics are \( x = c^2t^2/2t + ct + x_1 \) and \( x = c^2t^2/2t - ct + x_2 \) respectively, in which \( x_1 \) and \( x_2 \) are constants. In this region the fluid accelerates to the right at a rate \( c^2/t \), and the solution (2) pertains.
At $t = 0$, the signal communicating the abrupt change in density at $x = 0$ propagates to the left at sound speed $c$, and to the right at sound plus flow speed, $c + c^2t/\lambda$. Within the expanding region II, bounded by these two signal fronts, the solution can be seen in Fig. 1a to be nearly self-similar (linear velocity and logarithmic density profiles).

The solution in Fig. 1a was calculated by a 1-D hydrodynamics code from the initial profiles shown by the dashed curves. In the region to the right of the $C_1$ signal emanating from $x = 0$, the velocity plateau calculated by the code agrees exactly with the analytic value given by (2), namely $v = c^2t/\lambda$. This example therefore suggests that the solution (2) describes a mechanism whereby ions can be accelerated from rest to the velocities necessary to initiate the self-similar expansion of a plasma.

In a reference frame having acceleration $c^2/\lambda$, the fluid in region III would appear to be at rest, and the $C_1$ and $C_2$ would be straight lines with $dx/dt = \pm c$. Therefore any localized, exponential density profile of width $L$ is eroded at each end, and extinguished after a time $L/2c$. For example in Fig. 1b it can be seen that an exponential profile of width $L = 4\lambda$ is extinguished after a time $2\lambda/c$. Abrupt changes in the profiles in Fig. 1a appear at the locations of the signals that originated at $x = 0$.

If more than one ion species is present, and each is treated within the context of the fluid model described earlier, then their momentum equations decouple in any region in which each species starts
from rest with an exponential density profile of the same scale height. In such a region each ion species behaves as though the others were not present, and the solution (2) is valid for each species.

If such a region has width $L$, then it is extinguished after a time $t_{\text{final}} = L/2c_s$, in which $c_s$ is the sound speed of the multi-species fluid. Substituting $t_{\text{final}}$ in (3), we obtain a lower bound for the peak ion energy of any ion species, assuming that each species is accelerated from coexisting exponential density profiles with scale height $l$,

$$E_{\text{PEAK}} \geq \frac{1}{8} \frac{e^2}{M} \left( \frac{L}{l} \right)^2 \left( \frac{T_e}{c_s} \right)^2.$$  

Thus if the exponential density profile is a few scale heights wide, the directed energies of the ions can exceed the electron thermal energy before the self-similar expansion becomes the dominant acceleration process. However if the ion velocity becomes comparable with the electron thermal velocity, then the assumption of a Boltzmann distribution breaks down, and this model is no longer applicable.

In conclusion we have found an analytic solution for the ambipolar expansion of a cold, collisionless ion fluid from a steep pressure gradient with a hot isothermal electron background. The solution, in contrast to the self-similar solution, describes a means for accelerating ions from rest while maintaining their exponential-density scale height. However, a steep, localized, exponential gradient is eroded at both ends at sound speed, and finally extinguished, probably leaving behind
a plasma accelerating in an approximately self-similar expansion. A scaling law for the peak ion energies in a multi-species ion fluid was derived that shows a strong dependence on electron temperature and density scale height, as well as the $Z^2/M$ scaling.

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Fig. 1 — (a) Velocity (in units of sound speed $c$) and logarithmic density profiles initially (dashed) and at time $2\xi/c$ (solid), $\xi$ is scale height, and (b) characteristic curves for same initial conditions as (a). Analytic solutions of density and velocity in regions I and III are given in text.