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THE GENERALIZED ZACKS MODEL
by
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ACN 23274

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ABSTRACT

This paper generalizes the Zacks model for minefield crossings. Zacks computes in his model the probability of the Nth vehicle crossing a minefield and also the distribution of the number of vehicles crossing the field. Zacks' computations are made under the assumption that all the vehicles are of the same type and only one kind of mine is present in the field. This paper removes both these restrictions.
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The purpose of this note is to generalize the Zacks minefield model* for computing the probabilities of vehicles crossing a minefield. In the Zacks model only one type of mine and one type of vehicle are considered. This paper shows how the model can be extended to different mines and different vehicles. This note follows the Zacks paper closely, and the reader is advised to have a copy of that paper at hand when reading this note.

The Zacks model begins with the discussion of the number of mines in the path of the vehicle. In the Zacks model, only the number of mines is required; here, the number of mines for each mine type is needed.

The following notation is introduced:

- \( H = \) mine type, \( H=1,2,\ldots,M \)
- \( N(H) = \) number of clusters of the \( H \)th type mine
- \( n(I,H) = \) number of the \( H \)th type mines in the \( I \)th cluster
- \( J(H) = \) random number of the \( H \)th type mines in the path of a given tank
- \( J(I,H) = \) random number of the \( H \)th type mines from the \( I \)th cluster that are in the path of the tank

Then:

\[
J(H) = \sum_{I=1}^{N(H)} J(I,H) \quad \text{(Eq 1)}
\]

\( \binom{N}{M} \) = number of combinations of N things taken M at a time

\( \Psi(I,H) \) = probability that one of the Hth type mines from the Ith cluster is in the path

Then:

\[ \Pr\{J(I,H) = j\} = \text{probability that j mines, coming from the Ith cluster of the Hth type mine, are in the path} \]

The probability that j of these mines are in the path is given by:

\[ \Pr\{J(I,H) = j\} = \left( \frac{N(I,H)}{j} \right) \cdot \Psi(I,H)^j \cdot \{1 - \Psi(I,H)\}^{N(I,H) - j} \]  

(Eq 2)

Summing over all clusters yields the probability that a total of j mines are in the path:

\[ \Pr\{J(H) = j\} = \sum_{J=J_1+J_2+\ldots+J_{N(H)}} \sum_{k=1}^{N(H)} \Pr\{J(k,H) = j_k\} \]  

(Eq 3)

Equation 3 is Zacks' equation 3.9 with a subscript H for mine type. Note that \( \Pr\{J(I), \ldots, J(M)\} \) is the product of the \( \Pr\{J(H)\} \)'s.

Now let:

\[ P_d(H,V) = \text{probability that the Vth type vehicle will detect the Hth type mine} \]

\[ P_a(H,V) = \text{probability that the Vth type vehicle will activate the Hth type mine} \]

\[ P_k(H,V) = \text{probability that the Vth type vehicle will be killed by the Hth type mine} \]

Then the probability that a vehicle V will survive an encounter with an Hth type mine is:

\[ S(H,V) = P_d(H,V) + \{1 - P_d(H,V)\} \cdot \left[ \{1 - P_a(H,V)\} + P_a(H,V) \cdot \{1 - P_k(H,V)\} \right] \]  

(Eq 4)
Hence, the probability that the first vehicle of type V will survive the Hth type mine encounter, given J(H) = j, is given by equation 5:

\[ \Pr \{ S_1(H, V_1) \mid J(H) = j \} = S(H, V_1)^j \]  

(Eq 5)

The probability that the first vehicle will survive, given the J's, is the product \( \Pr \{ S_1(V) \mid J's \} \) over all mine types. Hence, the probability of survival for the first vehicle is the sum over the J's of the product (the probability of survival given the J's and the probability of the J's); that is:

\[
\Pr \{ S_1(V_1) \} = \prod_{H=1}^{M} \sum_{j(H)=0}^{N(H)} \Pr \{ S_1(H, V_1) \mid J(H) = j(H) \} \cdot \Pr \{ J(H) = j(H) \}
\]

where \( M \) is the number of mine types. This corresponds to Zacks' equation 4.3.

Additional notation is required to calculate the probability that the second vehicle \( V_2 \) crosses the minefield:

Let:

\[ M_1(H, V) = \text{random number of Hth type mines destroyed by the Vth type vehicle on a crossing of the path} \]

\[
\Pr \{ M_1(1, V) = m_1, M_1(2, V) = m_2, \ldots, M_1(M, V) = m_M \mid J(1) = j_1, \ldots, J(M) = j_M \} = \text{probability that } m_1 \text{ of the first type mine, } m_2 \text{ of the second type mine, } \ldots, m_M \text{ of the last type mine are destroyed, given that } j_1 \text{ of the first type mine, } j_2 \text{ of the second type mine, } \ldots, j_M \text{ of the last type mine are in the path} \]

\[
\hat{w}(H, V) = \{1 - Pr_d(H, V)\} \{1 - Pr_a(H, V)\} \]  

(Eq 6)

\[
\Pr \{ S_2 \mid M_1(1, V) = m_1, \ldots, M_1(M, V) = m_M, J(1) = j_1, \ldots, J(M) = j_M \} = \text{probability that the second vehicle survives, given } M_H \text{ of } J_H \text{ mines in the path have been destroyed} \]
Thus:

\[
Pr \left\{ S_2 \mid M_1(1,V) = m_1 \ldots M_1(M,V) = m_M, J(1) = j_1 \ldots J(M) = j_M \right\} = \prod_{H=1}^{M} S(H,V)^{j_H - m_H} \tag{Eq 7}
\]

Also:

\[
Pr \left\{ M_1(1,V) = m_1 \ldots M_1(M,V) = m_M \mid J(1) = j_1 \ldots J(M) = j_M \right\} = \prod_{H=1}^{M} \binom{j_H}{m_H} \cdot \tag{Eq 8}
\]

\[
\sum_{H=1}^{M} \left( s(H,V)^{j_H - m_H} + \sum_{H=1}^{M} (1-s(H,V)) \right)^{m_H - 1} \cdot \]  

\[
\sum_{r_H = m_H - 1}^{j_H - 1} \left( \binom{r_H}{m_H - 1} w(H,V)^{r_H - m_H + 1} \prod_{i \neq H} s(i,V)^{j_i} \right) \sum_{r_i = m_i}^{j_i} \left( \binom{r_i}{m_i} w(i,j)^{r_i - m_i} \right) \]

This is Zacks' equation 4.16; hence, the probability that the second vehicle survives is given by:

\[
Pr \{ S_2 \} = \sum_{J_1=0}^{N(1)} \ldots \sum_{J_M=0}^{N(M)} \sum_{m_1=0}^{j_1} \ldots \sum_{m_M=0}^{j_M} \Pr \left\{ M_1(1,V) = m_1 \ldots M_1(M,V) = m_M \right\} \sum_{J(1)=j_1 \ldots J(M)=j_M}^{M} \prod_{H=1}^{M} \Pr \left\{ J(H) = j_H \mid S(H,V)^{j_H - m_H} \right\} \tag{Eq 9}
\]

The calculation of the probability that the nth vehicle \( V_n \) survives is obtained by considering the probability that the vehicle survives an nth type mine encounter, given that there were originally \( j(H) \) of such mines and \( m_{n-1} \) were destroyed by the \( n-1 \) vehicles that preceded the current vehicle.
Let:

\[ M_{n-1}(H,V_{n-1}) = \text{random number of } H\text{th type mines destroyed by the } \]
\[ \text{passing of the } V_{n-1} \text{ vehicles over the path} \]
\[ S_{n}(H,V_{n}) = \text{the random event of the } n\text{th vehicle surviving the} \]
\[ \text{Hth type mine} \]

Then:

\[ \Pr \left\{ S_{n}(V_{n}) \mid M_{n-1}(1,V_{n-1}) = m_{1} \ldots M_{n-1}(M,V_{n-1}) \right\} \]
\[ J(1) = j_{1} \ldots J(M) = j_{M} \]

probability that the nth vehicle survives given that \( m_{H} \) of the \( j_{H} \)

mines have been destroyed \( = \prod_{H=1}^{M} S(H,V_{n})^{j_{H}-m_{H}} \) \hfill (Eq 10)

Thus:

\[ \Pr \left\{ S_{n}(V_{n}) \mid M_{n-1}(1,V_{n}) = m_{1} \ldots M_{n-1}(H,V_{n}) = m_{H} \right\} \]
\[ J(1) = j_{1} \ldots J(M) = j_{M} \]

\[ \Pr \left\{ M_{n-1}(1,V_{n}) = m_{1} \ldots M_{n-1}(H,V_{n}) \right\} \]
\[ J(1) = j_{1} \ldots J(M) = j_{M} \]
\[ \prod_{H=1}^{M} S(H,V_{n})^{j_{H}-m_{H}} \]

(Eq 11)

Now, the probability of \( m_{n} \) losses having occurred with the \( V_{n} \) crossing
by the \( H\)th type mine, given that there had been \( j \) mines in the path,
must be calculated.

Let:

\[ M_{n}(H,V_{n}) = \text{random number of } H\text{th type mines destroyed by the first} \]
\[ \text{n vehicle crossings} \]

These probabilities will be calculated recursively as follows:
This probability depends on the order in which vehicles enter the path. Now the probability that the nth vehicles survives is given by:

\[
\Pr\{N(1, V) = m_1 \ldots M_n(V, V) = m_n | J(1) = j_1 \ldots J(H) = j_H \} =
\sum_{r_1=0}^{m_1} \ldots \sum_{r_M=0}^{m_M} \Pr\{M_1(1, V_{m-1}) = r_1 \ldots M_n(V_{n-1}) = r_M | J(1) = j_1 \ldots J(M) = j_M \}
\]

(eq 12)

This probability depends on the order in which vehicles enter the path. Now the probability that the nth vehicles survives is given by:

\[
\Pr\{S_n(V_n) | J(1) = j_1 \ldots J(H) = j_H \} = \sum_{m_1=0}^{J_1} \ldots \sum_{m_M=0}^{J_M} \Pr\{M_{n-1}(1, V_{n-1}) = m_1 \ldots m_M | J(1) = j_1 \ldots J(M) = j_M \}
\]

(eq 13)

Hence, the probability that \( V_n \) survives, given the J's, is:

\[
\Pr\{S_n(V_n) \} = \sum_{j_1=0}^{n(1)} \ldots \sum_{j_M=0}^{n(M)} \Pr\{S_n(V_n) | J(1) = j_1 \ldots J(M) = j_M \} \prod_{H=1}^{M} \Pr\{J(H) = j_H \}
\]

(eq 14)
In conclusion, the paper considers the distribution of the number of survivors of \( N \) crossings. This is determined recursively.

Let:

\[ I(K) = \text{a random variable; 1, if the Kth vehicle survives the minefield crossing; 0, otherwise} \]

\[ X(K) = \text{a random number of vehicles out of K that have survived crossing the minefield.} \]

\[
X(K) = \sum_{h=1}^{K} I(h) \quad (\text{Eq 15})
\]

Now the joint probability of \( X \) and the \( M_1(1,V_1) \ldots M_1(M,V_1) \) given \( J(1), \ldots, J(M) \) is given as follows:

\[
\text{Pr}\left\{ X(1) = 1, M_1(1,V_1) = m_1, \ldots, M_1(M,V_1) = m_M \mid J(1) = j_1, \ldots, J(M) = j_M \right\} = \prod_{H=1}^{M} \frac{m_H^{j_H} W(H,V_1)^{j_H}}{S(H,V_1)^{j_H} - W(H,V_1)^{j_H}} \quad (\text{Eq 16})
\]

Hence:

\[
\text{Pr}\left\{ X(1) = 0, M_1(1,V_1) = m_1, \ldots, M_1(M,V_1) = m_M \mid J(1) = j_1, \ldots, J(M) = j_M \right\} = \text{Pr}\left\{ M_1(1,V_1) = m_1, \ldots, M_1(M,V_1) = m_M \mid J(1) = j_1, \ldots, J(M) = j_M \right\} - \text{Pr}\left\{ X(1) = 1, M_1(1,V_1) = m_1, \ldots, M_1(M,V_1) = m_M \mid J(1) = j_1, \ldots, J(M) = j_M \right\} \quad (\text{Eq 17})
\]
Now the recursive probabilities for $X(k)$ and $M_k(1,V_k)\ldots M_k(M,V_k)$ given $J(1)\ldots J(M)$ are written as:

$$\Pr\{X(K)=i,M_k(1,V_k)=M_1,\ldots,M_k(M,V_k)\mid J(1)=j_1,\ldots,J(M)=j_M\}$$

$$= \sum_{r_1=0}^{m_1} \ldots \sum_{r_M=0}^{m_M} \left[ \Pr\{X_{k-1}=i-1,M_{k-1}(1,V_{k-1})=r_1,\ldots,M_{k-1}(M,V_{k-1})=r_m\mid J(1)=j_1,\ldots,J(M)=j_M\} \right]$$

$$+ \Pr\{X(1)=1,M_1(1,V_1)=m_1-r_1,\ldots,M_1(M,V_1)=m_1-r_1,\ldots,M_1(M,V_1)=m_1-r_1\mid J(1)=j_1,\ldots,J(M)=j_M\} + \Pr\{X(1)=0,M_1(1,V_1)=m_1-r_1,\ldots,M_1(M,V_1)=m_1-r_1,\ldots,M_1(M,V_1)=m_1-r_1\mid J(1)=j_1,\ldots,J(M)=j_M\}$$

(Eq 18)

Then:

$$\Pr\{X(K)=i\} = \sum_{j_1=0}^{N(1)} \ldots \sum_{j_H=0}^{N(H)} \sum_{M_1=0}^{j_1} \ldots \sum_{M_H=0}^{j_H} \cdot$$

$$+ \Pr\{X(K)=i,M_K(1,V_K)=M_1,\ldots,M_K(H,V_K)=M_H\mid J(1)=j_1,\ldots,J(H)=j_H\}$$

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Mines, minefields, minefield models, minefield effectiveness

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