Analytic Theory of the Effects of Atmospheric Scattering on the Current and Ionization Produced by the Compton Electrons from High-Altitude Nuclear Explosions

William Solifrey

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Previous studies of scattering effects have involved either ad hoc physical assumptions or Monte Carlo techniques requiring very costly computer programs. This new theory of scattering is carried as far as possible analytically. A transport equation is derived that governs the evolution in time of the electron velocity distribution function from the initial Compton production distribution function under the influence of the earth's magnetic field, energy loss processes, and atmospheric scattering. This differential equation is solved analytically by an expansion to third order in the ratio of spatial spreading to velocity spreading, thus providing sufficient accuracy to cover the significant duration of the pulse. The current and ionization are then calculated as integrals over the velocity distribution function. After the integrals are carried out as far as possible, rapid computer programs are employed. Results show that scattering produces major reductions in the current. (JDD)
Analytic Theory of the Effects of Atmospheric Scattering on the Current and Ionization Produced by the Compton Electrons from High-Altitude Nuclear Explosions.

A Project AIR FORCE report prepared for the United States Air Force

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In recent years, considerable interest has developed within the Air Force in the electromagnetic pulse generated by a high-altitude nuclear explosion. Possible effects of this pulse on military equipment have been discussed, and there has been a rekindling of investigations in the basic physics and mathematics of the phenomenon. Large and expensive computer codes have been developed to determine the fields that may result. Many organizations have been involved in these operations.

The Rand Corporation has continuing efforts in this area, conducted under the Project RAND research project "System Vulnerabilities." The work has been communicated to the Air Force in several briefings at the Air Force Weapons Laboratory, Kirtland Air Force Base, and the findings are being documented in a series of reports.

This report presents an analytic treatment of the effects of atmospheric scattering on the current and ionization produced by the burst. Studies of the problem at other locations have involved either ad hoc physical assumptions or Monte Carlo techniques requiring very costly computer programs. The methods described here use analytical procedures as far as practicable, and then employ very rapid computer programs. Since this work is all of a fundamental character, it is presented in unclassified form. Applications to the electromagnetic pulses produced by actual nuclear explosions will be developed in a classified report.

The analysis and the computer program contained in this report should be useful to persons working with the theory or applications of electromagnetic pulses from high-altitude nuclear explosions. The report presents the most comprehensive investigation to date of the basic phenomena of current and conductivity.
SUMMARY

A high-altitude nuclear explosion generates gamma rays which are absorbed in the atmosphere. The Compton electrons produced by the gamma rays form a current, which then radiates an electromagnetic pulse, and also induce secondary ionization which determines the propagation of the electromagnetic field. The electrons move under the influence of their deflection by the earth's magnetic field, lose energy by the ionization process, and suffer Coulomb scattering by the atoms of the atmosphere. This report is devoted to an analytic investigation of the current and ionization produced by these combined effects.

Although there have been satisfactory previous treatments of magnetic deflection and of energy loss, the investigations of scattering have been fraught with difficulties. Methods to handle scattering have included (1) the "obliquity" theory, which contains ad hoc assumptions and uncertain physical meaning, and (2) the Monte Carlo technique. Although the latter method is theoretically sound, it involves lengthy and expensive computer programs. Accordingly, we have been impelled to develop a theory of scattering which is carried as far as possible analytically.

The theory is based on a distribution function which represents the number of electrons per unit volume in a unit velocity interval at a point in space and time. We then derive a transport equation which governs the evolution in time of this distribution under the indicated physical processes from the initial Compton production distribution. This differential equation is then solved analytically by an expansion to third order in the ratio of spatial spreading to velocity spreading, thus providing sufficient accuracy to cover the significant duration of the pulse (10 shakes), as is shown by detailed numerical investigation. The current and ionization are then calculated as integrals over the velocity distribution function.

These integrals are carried out analytically as far as we have found possible. A computer program has been written to evaluate the
integrals which remain. This program, included as an appendix, is rapid compared to the Monte Carlo technique.

Some typical results are presented. For a gamma-ray source taken to be a delta function in time, the current per gamma ray at high altitudes, which is essentially independent of scattering and energy loss, displays a peak at about 2 shakes for the selected choice of magnetic field and gamma-ray energy, and a slow decay thereafter. As the altitude decreases, the peak becomes lower, earlier, and narrower, all effects which would be expected by the interpretation that scattering reduces the coherence of the electron beam. The magnitude of reduction is considerable, thus at an altitude of 20 km and times later than 2 shakes, the current is reduced by more than a factor of four relative to the high-altitude current per gamma ray. The effects on ionization are much less striking, and the conductivity per gamma ray associated with the ionization is fairly insensitive to altitude.

The theoretical investigation and computer program of this report should prove valuable to all persons interested in the high-altitude electromagnetic pulse.
ACKNOWLEDGMENTS

It is a pleasure to thank C. M. Cram for numerous helpful suggestions and sage advice. The careful review of the manuscript by R. W. Huff is likewise much appreciated. The computer program could not have been written without the skilled cooperation of M. D. Lakatos.

Many of the ideas developed in this report have benefited from discussions with investigators at other institutions. The author wishes to thank all those who have cooperated unselfishly with him.
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\( \hat{B} \)  
magnetic field (2.7)

\( B \)  
magnitude of earth's magnetic field (2.11)

\( C(\beta) \)  
forward component of displacement (3.12)

\( \overline{C}(\beta) \)  
forward component of displacement relative to position without scattering (3.13)

\( \overline{C}(\beta') \)  
the function \( \overline{C} \) regarded as a function of its lower limit \( \beta' \) with the upper limit set equal to \( h(\cos \theta) \) (4.12)

\( C_1 \)  
\( \overline{C}(\beta') - \overline{C}(\beta_1) \) (4.12)

\( D(\zeta', \theta, \psi) \)  
ratio of retarded time to electron reference time (4.21)

\( D_1^j(\beta) \)  
Legendre expansion coefficient (3.21)

\( E \)  
electron energy (2.9)

\( E_\gamma \)  
gamma-ray energy (before 2.29)

\( \bar{E} \)  
electric field (2.37)

\( E_1 \)  
mean ionization energy (2.41)

\( E_j^1 \)  
Legendre expansion coefficient (3.21)

\( F(r, v, t) \)  
electron velocity distribution function (Introduction)

\( G(r, \beta, \theta, \varphi, t, \beta', \theta', \varphi', t') \)  
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\( H \)  
atmospheric scale height (before 2.1)
\(H(r, \theta, \phi, r', \beta', \theta', \phi')\) reduced Green's function (3.8)

\(I\) ionization potential of air (2.12)

\(I(r_0, \beta_0, \theta_0, \phi_0, r', \beta', \theta', \phi')\) reduced Green's function expressed in initial variables (after 3.13)

\(\dot{J}\) total current (2.37)

\(\dot{J}_C\) Compton current (after 2.37)

\(\dot{J}_S\) conduction current (after 2.37)

\(K\) energy attenuation per unit angle of turn (4.16)

\(K(\cos \theta)\) Klein-Nishina number distribution (2.29)

\(L\) zero-order differential operator (3.17)

\(L(t-t')\) ionization delay function (2.44)

\(M\) first-order differential operator (after 3.16)

\(M_1\) reduced first-order differential operator (after 3.26)

\(N\) number of atoms per unit volume (2.6)

\(N_e(r, \nu, t)\) Compton electron production rate (2.4)

\(N_s(r, t)\) number of ionization electrons (2.38)

\(P_1(\cos \theta)\) associated Legendre polynomial

\(Q\) second-order differential operator (after 3.16)

\(Q_1\) reduced second-order differential operator (after 3.26)

\(Q(\beta, \theta, \phi, \delta, \gamma)\) differential cross section for elastic scattering (2.6)

\(R(\beta, \theta_0, \phi_0)\) inhomogeneous term in zero-order differential equation (3.20)

\(S(\beta)\) sideward component of displacement (3.12)

\(\overline{S}(\beta)\) sideward component of displacement relative to position without scattering (3.13)

\(\overline{S}(\beta')\) the function \(\overline{S}\) regarded as a function of its lower limit \(\beta'\) with the upper limit set equal to \(h(\cos \theta)\) (4.12)
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\( \sigma_c \) total Compton scattering cross section (2.30)
\( \tau = (t-r/c) \) retarded time (before 4.20)
\( \tau_m \) pulse duration (before 2.1)
\( \varphi \) azimuthal angle of velocity (2.1)
\( \varphi_0 \) initial azimuthal angle variable (3.12)
\( \chi \) polar angle between initial and final directions of velocity after scattering (2.17)
\( \psi \) space polar angle (Fig. 1)
\( \psi_B \) angle between line of sight and vertical (Fig. 1)
\( \psi(r, v, \Delta v) \) reduced transition probability (2.3)

\( \Delta \mathbf{r} \) position increment in \( \Delta t \) (2.2)
\( \Delta t \) time increment including many scatterings (2.2)
\( \Delta \mathbf{v} \) total velocity increment in \( \Delta t \) (2.2)
\( \partial \) partial derivative (2.6)
\( \nabla \mathbf{v} \) gradient operator in velocity space (2.6)
\( \nabla^2 \) Laplacian in angles of the velocity (2.22)
\( d\omega \) element of solid angle (after 2.5)
I. INTRODUCTION

The theory of the generation of electromagnetic (EM) pulses by high-altitude nuclear explosions was first put on a firm footing in investigations by Karzas and Latter. In their theory, the prompt gamma rays from the burst interact with the atmosphere via the Compton effect to produce high-velocity electrons. These Compton electrons are accelerated by the earth's magnetic field to produce a radiating current and lose energy by ionization to cause atmospheric conductivity. The propagation of the fields through the conducting medium is treated by the "high-frequency" approximation, which neglects certain spatial derivatives relative to time derivatives and is regarded as sufficiently accurate for the effective duration of the pulse.

Following this initial theoretical investigation, considerable effort was expended in developing computer codes to apply the theory to practical cases. Among these codes were LEMP I, developed at the Mission Research Corporation, and HEMP, HEMP-B, and CHEMP, developed at the Air Force Weapons Laboratory. The codes were generally of such character that additional theoretical developments could be incorporated without changing the basic structure of the code. As the physics of the calculations have been improved, the codes steadily have become more complex and costly to operate.

There have been numerous theoretical improvements to the basic studies. The dependence of atmospheric conductivity on field strength has been treated by Baum and by Higgins et al. The CHEMP code was designed to include self-consistent electron motions, allowing the electromagnetic field to interact with the electrons which produce it. The lag in the secondary ionization process was investigated by Longmire and Longley. There have been several studies of how the current is affected by atmospheric scattering.

When atmospheric scattering is not included in the theory, the motion of the electrons is determined by their deflection by the magnetic field (electromagnetic field for self-consistent calculations) and by the energy loss to ionization. Thus, there is a unique and
invertible relation between the position and velocity of an electron at a given time and its position and velocity at its birth. In the presence of scattering, an electron suffers many random deflections in the course of its motion. Consequently, neither its position nor its velocity can be explicitly deduced from initial conditions. At any space-time location there will be a distribution of electrons with respect to velocity, and this enormously complicates the situation.

The treatments of scattering to date have proceeded along two essentially independent lines. First, there is the "obliquity" theory, developed in Ref. 8 and later improved. In this model, the electron velocity distribution is replaced by a moving mean position whose velocity depends on the angular speed of the motion through an "obliquity" factor $\eta = \langle 1/\cos \theta \rangle$, where $\theta$ is the angle of the velocity from its initial direction. The assumptions involved in this theory have remained obscure, and it has generally been justified by showing its agreement, if any, with the Monte Carlo theory.

Since the scattering phenomenon is a sequence of random processes, it is amenable to the Monte Carlo technique. In this method, the motion of each electron is followed individually. The scattering is included by selecting random numbers from distributions which represent the number of scatterings per unit path length and the angular deflection suffered at each scattering. The investigation via computer code was developed by Knutson (9) and improved by Wittwer et al. (10). This procedure allows a fully satisfactory treatment of scattering effects. However, it is extremely costly in computer time, and the change of any parameter requires a complete Monte Carlo investigation.

The difficulties associated with these treatments of scattering have impelled us to develop a new theory. Atmospheric scattering is basically a statistical process. The obliquity model employs "lumped" statistics, in that the characteristics of the velocity distribution are lumped into essentially a single parameter, whereas the Monte Carlo

---

* C. L. Longmire, description of new obliquity given at a meeting at Air Force Weapons Laboratory, November 1973.
technique uses individual statistics, treating the electrons as independent. We shall treat the problem as a general stochastic process, employing the methods reviewed and summarized many years ago by Chandrasekhar. (11)

The body of this report is devoted to an exposition of this method of treating atmospheric scattering effects. The theory is based on a distribution function \( F(r, v, t) \) which represents the number of Compton electrons per unit volume per unit velocity interval at the position \( r \) at the time \( t \) with velocity vector \( v \). In Section II, we derive the partial differential equation which governs the evolution in time of the distribution function \( F \) as affected by the various physical effects. This equation is referred to variously as the Boltzmann, Fokker-Planck, or transport equation. We shall use the last term. Effects included are production of electrons by gamma rays, with the appropriate Klein-Nishina distributions of number and energy versus initial direction, electron transport, deflection by the earth's magnetic field, energy loss by ionization, and atmospheric scattering. Self-consistent field effects are not included, since they make the transport equation nonlinear and essentially intractable. The current and the density of secondary (ionization) electrons will be represented in terms of the distribution function \( F \).

Our purpose is to carry out the solution of the transport equation by analytic procedures as far as possible. These procedures are developed in Section III. We introduce a Green's function, which represents the distribution of electrons from an instantaneous uni-velocity point source. The function \( F \) is then found as a suitable convolution of the Green's function and the Compton electron production function. The transport equation for the Green's function is transformed by introducing new variables which eliminate all first derivatives from the equation. The resulting equation is then treated by successive approximations in which the expansion parameter is essentially the ratio of the effect of positional derivatives to the effect of velocity derivatives. This procedure is carried to third order. Explicit post hoc calculations in Section V show that the expansion is sufficiently
accurate for times up to 10 shakes (1 shake = $10^{-8}$ sec), which covers the significant duration of the pulse.

In Section IV, the current and the ionization density are calculated from these representations of the distribution function. As many integrals as possible are calculated analytically. The remaining integrations, which only involve the direction of the initial velocity, are brought into a form suitable for computer programming.

A computer program written to evaluate these integrals proved to be very rapid compared with the Monte Carlo technique. The program is described in Section V and some typical results given and discussed. The program itself is included as an appendix.

In Section VI, results are shown for a typical magnetic field (0.6 gauss, perpendicular to the direction of the gamma rays) and a selected gamma-ray energy (1.6 MeV). At high altitudes, the effects of scattering and energy loss are small, and the current reduces to the Karzas-Latter result. It may be expected that the effect of scattering will be to reduce the coherence of the electron distribution. Accordingly, as the altitude decreases, the peak value of the current should decrease, and the peak should occur at earlier times, since the electron beam cannot hold together. The results show precisely such behavior. For the particular case shown in Fig. 12, with a delta function time dependance for the gamma-ray source, the current per gamma ray at high altitudes peaks at 2.2 shakes with a peak value of 1.27 in the units of the figure. At 30 km, the peak occurs at 1.6 shakes with a value of 1.06; at 24 km the current peaks at 1.2 shakes with a value of 0.69; and at 18 km it peaks at 0.6 shake and is down to 0.31. Since the main electron deposition by the gamma rays takes place at these lower altitudes, we see that the effective radiating source is very considerably reduced by scattering, which will cause an attendant reduction in the electromagnetic field produced by this source. Although the secondary electron density is also affected by scattering, the results are much less striking than those for the current. The steady reduction of ionization with increasing altitude is primarily because the energy loss is proportional to the atmospheric density. The conductivity per gamma ray, which is
proportional to the number of secondary electrons divided by the atmospheric density, is relatively insensitive to altitude.

The calculations of the radiated field from the current and ionization are considerably complicated by the nonlinear dependence of the effective electron collision frequency on the electric field. At low altitudes, the effect is manifested by making the conductivity field-dependent, whereas at high altitudes (> 50 km), the effective conduction current must be treated by swarm theory. Since our main purpose in this report is to present the effects of atmospheric scattering on the collective Compton electron distribution, we have avoided this thicket by not calculating the fields produced by the current. This makes it very difficult to compare our results directly with other investigations, which usually give only the radiated fields.

This report presents our unclassified contribution to the theory of the effects of atmospheric scattering on the current and ionization density associated with high-altitude nuclear explosions. We have investigated other phenomena and will report on them separately. Applications of the theory to the pulses generated by specific nuclear weapons will be presented in a classified report.
II. DERIVATION OF THE TRANSPORT EQUATION

GENERAL CONSIDERATIONS

The geometrical configuration and coordinate system are shown in Fig. 1. A nuclear explosion occurs at height $h_B$, which will be assumed low enough that the curvature of the earth can be safely neglected, but high enough that it is above the principal altitudes at which gamma rays are absorbed. Thus, burst altitudes of 100 to perhaps 500 km are considered. This neglect of the earth's curvature simplifies the mathematics without losing any of the essential physics. An observer is on the ground, with the line of sight from burst to observer making an angle $\Psi_B$ from the vertical. This observer could also be in an airplane.

![Diagram of geometry and coordinate system](image)

Fig. 1—Geometry and coordinate system

A rectangular coordinate system is shown, with the $z$-axis along the line of sight directed from the burst to the observer. This will also be the radial direction of a spherical coordinate system. Since the effect of scattering will be to reduce the electromagnetic pulse, we shall consider circumstances which tend to both simplify the analysis and maximize the pulse in the absence of scattering. For this purpose,
we shall assume that the earth's magnetic field is perpendicular to the line of sight and will define its direction as the \( y \)-direction. Since the gamma rays produce electrons which are predominantly directed forward, this assumption maximizes the magnetic deflection. Self-consistent field effects are neglected, and the earth's magnetic field is assumed uniform over the region in which significant interaction occurs between gamma rays and the atmosphere. This region has only a 30 to 40 km height range, so the variation of magnetic field is legitimately small. The methods of this report can be used for an arbitrary direction of the magnetic field, but the analysis becomes very much more complicated in detail. The coordinate system is completed by an \( x \)-coordinate.

Consider an electron which is going to contribute to the electromagnetic pulse within the first 10 shakes. This electron must be sufficiently close to the line of sight that the sum of the travel time of the gamma ray to the birth point of the electron, the age of the electron when it emits radiation, and the travel time of the radiation to the observer does not exceed by more than 10 shakes the travel time of light along the line of sight. As a typical example, for a burst at 100 km and an angle from the vertical of 60 deg, an electron born at a height of 30 km must be within 1.6 km of the radial axis. Introduce spherical coordinates \( r, \Psi, \lambda \), with the origin at the burst and the radial direction along the line of sight. Then the polar angle \( \Psi \) will be less than 1 deg. The differential equation we shall derive later will involve derivatives with respect to the spherical angles. Since the solution is nonsingular with respect to angles and the total angular variation is small, we shall neglect these spherical angle derivatives of the electron number density compared to the radial derivative. We have worked out the extremely tedious calculations required by not making this approximation and find that the current and conductivity are only modified by a few percent over the time interval. Furthermore, the distance traveled by an electron in the 10 shake retarded time interval is small compared to the atmospheric scale height, so we shall neglect the variation of the atmospheric density over this distance. We thus neglect effects on the order of \( c \tau_m / H \), where \( c \) is the velocity of light, \( \tau_m \) the pulse duration, and \( H \) the scale height.
The gamma rays produce electrons in all directions. Hence, the velocity vector $\vec{v}$ shown in Fig. 1 can have all possible directions and magnitudes up to the maximum permitted by the Compton effect. The rectangular components of the velocity ($v_x, v_y, v_z$) are given in terms of its magnitude $v = \beta c$ and direction angles $\theta, \phi$, by

$$v_x = \beta c \sin \theta \cos \phi$$  \hspace{1cm} (2.1a)
$$v_y = \beta c \sin \theta \sin \phi$$  \hspace{1cm} (2.1b)
$$v_z = \beta c \cos \theta$$  \hspace{1cm} (2.1c)

The polar angle $\theta$ is indicated in Fig. 1.

We shall now introduce stochastic procedures, following the method described on p. 42 of Ref. 11. An electron moves under the action of the coherent acceleration processes (i.e., energy loss and magnetic deflection) and the incoherent scattering process. We assume a time interval $\Delta t$ exists such that the electron undergoes many infinitesimal scatterings, but that its position and velocity do not change appreciably. Then the increments in velocity and position in this time interval are given by:

$$\Delta \vec{v} = \vec{a} \Delta t + \delta \vec{v}$$  \hspace{1cm} (2.2a)
$$\Delta \vec{r} = \vec{v} \Delta t$$  \hspace{1cm} (2.2b)

where $\vec{a}$ is the acceleration produced by the coherent processes, and $\delta \vec{v}$ is a fluctuating quantity with a definite law of distribution. Thus, the transition probability that an electron with velocity $\vec{v}$ at position $\vec{r}$ will undergo a velocity change $\Delta \vec{v}$ and position change $\Delta \vec{r}$ is given by the function:

$$\Psi(\vec{r}, \vec{v}, \Delta \vec{r}, \Delta \vec{v}) = \Psi(\vec{r}, \vec{v}, \Delta \vec{v}) \delta(\Delta \vec{r} - \vec{v} \Delta t)$$  \hspace{1cm} (2.3)

Here, $\delta$ denotes the Dirac delta function and is not to be confused with the fluctuation velocity $\delta \vec{v}$. 
Let us now define the distribution function $F$ such that

$$F(r, v, t) \, dr \, dv$$

is the expected number of electrons at time $t$ in an element of spatial volume $dr$ in the vicinity of $r$ and in an element of velocity volume $dv$ in the vicinity of $v$. We ask how many electrons are in this phase space volume at time $t + \Delta t$. Clearly, this is the sum of those produced by the gamma rays in the volume during the time $\Delta t$ and the net change produced by inflow/outflow from the volume as a result of the acceleration forces. Thus (see Ref. 11, Eq. (3.24)), we have the integral equation:

$$F(r, v, t + \Delta t) \, dr \, dv = N_e (r, v, t) \Delta t \, dr \, dv$$

Here, $N_e$ is the rate of production of electrons by gamma rays, and the integral term depicts transitions in and out of the phase space volume. The representation (2.3) permits integration over $d\Delta r$. With a slight change of variables, Eq. (2.4) takes the form:

$$F(r + \Delta r, v + \Delta v, t + \Delta t) \, dr \, dv = N_e (r, v, t) \Delta t \, dr \, dv$$

Under the previous assumptions, the spatial dependence of $F$ involves the radial coordinate only. The functions on the left and right sides of Eq. (2.5) can be expanded in Taylor series, and the limit $\Delta t \to 0$ taken. The transport equation results, which governs the evolution in time of the distribution function $F$: 
Here, \( \vec{V}_v \) is the gradient operator in velocity space, \( N \) is the number of atoms per unit volume, and \( Q \) is the differential cross section for elastic scattering from a velocity described by the direction angles \( \theta', \varphi' \) to the angles \( \theta, \varphi \). The transition probability \( \Psi \) is the product \( N \cdot Q \). The two terms in brackets in the integral indicate that scattering takes place both into and out of the element of volume in velocity space. The space divergence operator on the left represents the net flow through the spatial element of volume, taking into account the change in the element of surface area \( r^2 dw \), where \( dw \) is an element of solid angle.

Equation (2.6) is the fundamental transport equation. We shall now consider the explicit representation of the coherent acceleration, scattering, and production terms.

### COHERENT ACCELERATION

The coherent acceleration is given by the magnetic deflection and the energy loss expressions. The equation of motion of a relativistic electron under these forces is:

\[
\frac{d}{dt} \left( \frac{mv}{\sqrt{1 - \beta^2}} \right) = - e \vec{v} \times \vec{B} - \vec{W}(\beta) \frac{\vec{v}}{v}
\]

Here, \( \beta \) is the velocity of the electron divided by \( c \), the velocity of light, \( \vec{B} \) is the earth's magnetic field, and \( \vec{W}(\beta) \) is related to the energy loss in a manner to be demonstrated. Self-consistent field effects have been omitted in Eq. (2.7). If we take the scalar product of \( \vec{v} \) and Eq. (2.7) and simplify, there results:
The left side is the rate of change of the electron energy. Consequently, the function $W(\beta)$ is given by

$$W(\beta) = -\frac{dE}{dS}$$

Thus, $W(\beta)$ is the energy loss per unit path length $dS$. This is the quantity calculated and tabulated in the literature.

We have assumed here that all the inelastic scattering processes can be represented as an energy loss term which is parallel to the electron velocity. Actually, these inelastic processes also involve change in direction, but such effects are quite small. The dominant energy loss process at Compton energies is ionization, with small momentum change per collision. The change in the direction of the velocity of the primary for such ionizing collisions has been shown experimentally to be small. We neglect the very rare large-angle inelastic scattering events. With these assumptions, the use of the term "coherent" for energy loss seems satisfactory. This treatment is consistent with all other investigators.

When Eq. (2.8) is used in (2.7), we obtain for the coherent acceleration $\dot{a}$ the form:

$$\dot{a} = -\frac{e}{m} (1-\beta^2)^{1/2} \vec{v} \times \vec{B} - \frac{W(\beta)(1-\beta^2)^{3/2}}{mc^2\beta} \frac{\vec{v}}{v}$$

This expression is to be substituted into Eq. (2.6). The first term is perpendicular to the velocity, and the divergence will only involve derivatives with respect to the angles $\theta, \phi$. The second term is parallel to the velocity, and only derivatives with respect to the velocity magnitude will appear. We have assumed the magnetic field to be in the $y$-direction. When the derivatives are simplified, we have:
\[
\nu \cdot \frac{\nabla}{\nu} \cdot \frac{\nabla F}{m} = \frac{eB}{m} (1-B^2)^{1/2} \left[ -\cos \phi \frac{\partial F}{\partial \theta} + \frac{\cos \theta \sin \varphi}{\sin \theta} \frac{\partial F}{\partial \varphi} \right] \]

\[
- \frac{1}{\beta^2} \frac{\partial}{\partial \beta} \left[ \beta^2 (1-B^2)^{3/2} \frac{W(\beta)}{mc} F \right]
\]

(2.11)

where \( B \) is the magnitude of the earth's field and we have replaced the derivative with respect to \( \nu \) by that with respect to \( \beta \).

The function \( W(\beta) \), the energy loss per unit path length, has been calculated by Bethe;\(^{(12)}\) it is given by:

\[
\frac{W(\beta)}{mc} = \frac{2\pi N Z r_e^2 c}{\beta^2} \left[ \log \left\{ \left(\frac{mc^2}{\gamma-1}\right) \left(\gamma^2-1\right)/2L^2 \right\} - \left(\frac{2}{\gamma} \right) \log 2 + \frac{1}{\gamma^2} \right]
\]

(2.12)

\[
\gamma = (1-B^2)^{-1/2}
\]

(2.13)

Here, \( N \) is the number of atoms per unit volume, \( Z \) is the number of electrons per atom \( (Z = 7.24 \text{ for air}) \), and \( r_e = e^2/mc^2 \). The quantity \( I \) is an effective ionization potential which is 80.5 eV for air. The total energy of the electron is \( \gamma mc^2 \) and its kinetic energy is \( (\gamma-1) mc^2 \).

The expression (2.12) has dimensions of inverse time. We shall measure time in shakes, so the multiplying factor becomes

\[
2\pi N Z r_e^2 c = 0.05522 \rho \ [\text{shakes}^{-1}]
\]

(2.14)

The quantity \( \rho = \rho/\rho_0 \) is the ratio of the air density at the altitude at which the electrons are located to the density at sea level. The factor in brackets in Eq. (2.12) will be designated \( V(\beta) \) and is plotted against electron kinetic energy in Fig. 2, where we also plot \( V(\beta)/\beta \). It may be seen from Fig. 2 that \( V(\beta)/\beta \) only varies slightly over most of the range in energy. If it is replaced by the constant value 21, the error will not exceed 3.5 percent for energies between 250 KeV and 2 MeV, and will only be 20 percent at 100 MeV.
Fig. 2—Energy loss factor versus energy

Thus, we shall use the approximate form:

$$W(\beta) = \frac{1.16 \beta}{\beta} \text{[shakes}^{-1}]$$

(2.15)

The coefficient of the magnetic field term is

$$\frac{eB}{m} = 0.17588 \text{ B (gauss) [shakes}^{-1}]$$

(2.16)

The form (2.15) for the energy loss per unit path length is equivalent to a constant loss per unit time, so the energy decreases linearly with time in the reference frame of the electron, down to relatively low
energies. At these low energies, the electron loses energy much
to much faster and essentially stops. We shall use the form (2.15) for all
ergies and will not expect to affect the current or ionization
significantly.

ATMOSPHERIC SCATTERING

The atmospheric scattering term on the right hand side of Eq. (2.6)
involves the differential cross section for elastic scattering. This
will generally be a function of the change in the momentum of the elec-
tron. As discussed above, we shall neglect all inelastic scattering
processes, which will be regarded as represented by the energy loss.
The change in momentum involves the invariant magnitude of the velocity,
and the cosine of the angle between the initial and final direction of
the velocity. We thus have:

\[ Q(\beta, \theta, \varphi, \theta', \varphi') = Q[\beta, \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\varphi - \varphi')] \]  \hspace{1cm} (2.17)

We shall call the second argument \( \cos \chi \). The function \( Q \) is strongly
peaked at small values of \( \chi \). We therefore expand the function
\( F(r, \beta, \theta', \varphi', t) \) in a Taylor series in \( \theta' - \theta \), \( \varphi' - \varphi \). To second order, we
have:

\[ F(r, \beta, \theta', \varphi', t) = F(r, \beta, \theta, \varphi, t) + \frac{\partial F}{\partial \theta} (\theta' - \theta) + \frac{\partial F}{\partial \varphi} (\varphi' - \varphi) \]
\[ + \frac{1}{2} \frac{\partial^2 F}{\partial \theta^2} (\theta' - \theta)^2 + \frac{\partial^2 F}{\partial \theta \partial \varphi} (\theta' - \theta)(\varphi' - \varphi) + \frac{1}{2} \frac{\partial^2 F}{\partial \varphi^2} (\varphi' - \varphi)^2 \]  \hspace{1cm} (2.18)

Since the integration is over the direction cosines associated
with \( \theta', \varphi' \), the angle differences should be represented in terms of
differences of direction cosines. Again to second order:

\[ \theta' - \theta = \frac{(\cos \theta - \cos \theta')}{\sin \theta} - \frac{\cos \theta (\cos \theta' - \cos \theta)^2}{2 \sin^3 \theta} \]  \hspace{1cm} (2.19a)

\[ \varphi' - \varphi = \frac{\sin \theta' \sin (\varphi' - \varphi)}{\sin \theta} \left[ 1 + \frac{\sin \theta - \sin \theta' \cos (\varphi' - \varphi)}{\sin \theta} \right] \]  \hspace{1cm} (2.19b)
These expressions are to be inserted into Eq. (2.18), then (2.18) itself is to be put into the integral of Eq. (2.6). With \( \varphi' - \varphi \) as a new azimuthal angle, the terms in \( \partial F/\partial \varphi \) and \( \partial^2 F/\partial \varphi \partial \varphi \) are odd functions and integrate to zero. The integration over the direction of \( \hat{v} \) can be simplified by rotating the coordinate system so the direction \( \theta, \varphi \) represents the polar direction. In this coordinate system, the angle \( \chi \), defined following Eq. (2.17), is the polar angle. We will call the corresponding azimuthal angle \( \alpha \). The relation between the quantities required in Eq. (2.19) for the \( \theta', \varphi' \) and \( \chi, \alpha \) systems is:

\[
\begin{align*}
\cos \theta - \cos \theta' &= \cos \theta (1 - \cos \chi) + \sin \theta \sin \chi \cos \alpha \\
\sin \theta - \sin \theta' \cos (\varphi' - \varphi) &= \sin \theta (1 - \cos \chi) - \cos \theta \sin \chi \cos \alpha \\
\sin \theta' \sin (\varphi' - \varphi) &= \sin \chi \sin \alpha
\end{align*}
\]

Since \( Q \) does not involve \( \alpha \), the integration over \( \alpha \) is immediate. To this point, the scattering integral of Eq. (2.6) becomes:

\[
2\pi N_v \int_0^\pi \sin \chi \, d\chi Q(\beta, \cos \chi) \left[ \left\{ 1 - \cos \chi - \frac{1}{4} \sin^2 \chi \right\} \frac{\cos \theta \partial F}{\sin \theta \partial \theta} + \frac{1}{4} \sin^2 \chi \left( \frac{\partial^2 F}{\partial \theta^2} + \frac{1}{\sin^2 \theta \partial \varphi^2} \right) \right]
\]

(2.21)

Again, keeping only second-order terms, this reduces to:

\[
2\pi N_v \cdot \frac{1}{4} S_2 \cdot \nabla^2 F
\]

(2.22a)

\[
S_2 = \int_0^\pi \sin^3 \chi \, Q(\beta, \cos \chi) \, d\chi
\]

(2.22b)

\[
\nabla^2 F = \frac{\partial^2 F}{\partial \theta^2} + \frac{\cos \theta \partial F}{\sin \theta \partial \theta} + \frac{1}{\sin^2 \theta \partial \varphi^2}
\]

(2.22c)
We carried this procedure to sixth order; the result is:

\[
\begin{align*}
2\pi Nv & \left[ \left( \frac{1}{4} S_2 + \frac{3}{32} S_4 + \frac{5}{96} S_6 \right) V_F^2 + \left( \frac{1}{64} S_4 + \frac{13}{1152} S_6 \right) V_F^2 V_2^2 + \frac{1}{2304} S_6 V_2^2 V_2^2 F \right] \\
S_{2m} &= \int_0^\pi \sin^{2m+1} \chi Q(\beta, \cos \chi) d\chi
\end{align*}
\] (2.23a)

The moments \( S_{2m} \) can be expected to fall rapidly with \( m \). We now require an expression for the elastic scattering cross section \( Q \). We are concerned with electron energies such that the momentum transfer is small and the Born approximation to scattering is valid. From the review article by Motz et al.\(^{(13)} \), we have selected the formula of Molière,\(^{(14)} \) which represents an improved version of the Fermi–Thomas theory. This formula is:

\[
Q(\beta, \cos \chi) = \frac{4Z^2 r_e^2}{1-\beta^2} \left[ \sum_{i=1}^{3} \frac{a_i}{\Lambda_i^2 + q^2} \right]^2
\] (2.24a)

\[
q = 2\beta \gamma \sin \frac{\chi}{2}
\] (2.24b)

\[
\Lambda_i = Z^{1/3} b_i / 121
\] (2.24c)

\[
a_1 = 0.10 \quad a_2 = 0.55 \quad a_3 = 0.35
\] (2.24d)

\[
b_1 = 6 \quad b_2 = 1.2 \quad b_3 = 0.3
\] (2.24e)

This expression may be substituted into Eq. (2.23b) for the scattering moments. All integrals may be performed analytically. The result is an expansion in \( \Lambda_1^2 / 2\beta \gamma \), beginning with a logarithmic term (in \( S_2 \) only), then constants, then positive even powers. This expansion parameter will be small for all significant values of \( \beta \), so only
the logarithm and constants will be kept. To this order:

\[ S_2 = 4Z^2r_2 \frac{1-\beta^2}{\beta^4} \left[ \log \frac{242 \beta Y}{Z^{1/3}} - 1.0793 \right] \]  
(2.25a)

\[ S_4 = 4Z^2r_2 \frac{1-\beta^2}{\beta^4} \frac{2}{3} \]  
(2.25b)

\[ S_6 = 4Z^2r_2 \frac{1-\beta^2}{\beta^4} \frac{2}{5} \]  
(2.25c)

We shall modify these formulas by replacing \( Z^2 \) by \( Z(Z+1) \). This is the "standard" procedure* for including scattering by the orbital electrons, whereas Eq. (2.24) corresponds to scattering by the screened nucleus. When Eqs. (2.25) are substituted into (2.23a), the coefficient of \( \nabla^2 F \) becomes

\[ 2\pi Z(Z+1)r_2^2 \frac{1-\beta^2}{\beta^3} \log (114.78 \beta Y Z^{-1/3}) \]  
(2.26)

If the higher order terms had not been included, the coefficient in the argument of the logarithm would have been 82.24. The logarithm is between 3 and 5 over the energy range. The coefficients of the higher derivatives in Eq. (2.23a), corresponding to the logarithm in Eq. (2.26), are respectively 0.05972 and 0.00069, so it is fully legitimate to neglect the fourth and sixth derivative terms. Substituting numbers into Eq. (2.26) yields:

\[ 0.455 \rho \frac{(1-\beta^2)}{\beta^3} \log (59.33 \beta Y) \text{ [shakes}^{-1}] \]  
(2.27)

*See Ref. 12, pp. 261 and 284 for many other references.
The logarithm and log/β are plotted against energy in Fig. 3. Again, log/β varies only slightly over the entire range. We shall replace it by the constant value 5.52, which is in error by only 3 percent between 250 KeV and 2 MeV, and only 20 percent at 100 KeV. The ratio between the scattering coefficient and the energy loss coefficient is more nearly constant than either of them. This was initially pointed out in Ref. 8. (The argument of the log was 65.36 by in Ref. 8, representing a slightly different choice of lower limit in the integration.) With this constant value, the scattering integral reduces to:

\[
2.51 \frac{\rho}{\beta^2} \frac{(1-\beta^2)}{\beta^2} V^2 F \text{ [shakes}^{-1}\text{]} \tag{2.28}
\]

This will be used in Eq. (2.6) at all energies.
COMPTON ELECTRON PRODUCTION

The term \( \dot{N}_e \) in Eq. (2.6) represents the production of Compton electrons by gamma rays and acts as a driving function for the differential equation. The gamma rays will interact with the atmosphere via the Compton effect, and the number of electrons produced per unit time will be proportional to the number of gamma rays which undergo interaction and to the differential cross section for production of an electron with a specified momentum vector. At each scattering of a primary gamma ray, a secondary gamma ray will also be produced, which will in turn be scattered at a later time, leading to a general cascade process. The retarded time until a second scattering takes place will usually exceed the 10-shake retarded time scale of this report, so we will limit our considerations here to the production of electrons by the first scatterings of gamma rays from the burst. We have investigated the cascade process on a longer time scale and will report the results in a separate publication.

We shall consider a source of gamma rays of a single energy, \( E_\gamma = \hbar c^2 \). If the burst is represented by an energy distribution, our results can be integrated over \( k \) to obtain the total current and ionization. The nature of the Compton scattering process is such that there is a unique relation between the angle at which an electron is emitted and its velocity. The function \( \dot{N}_e \) can then be written as a product of four factors:

\[
\dot{N}_e(r, \vec{v}, t) = g(r) \frac{f(t - r/c) K(\cos \theta) \delta(\beta - h(\cos \theta))}{2\pi c^2} \tag{2.29}
\]

These functions have the following interpretations. First, \( g(r) \) is the total number of gamma rays which are absorbed per unit volume at the distance \( r \) from the burst. Let \( \sigma_c \) be the total Compton scattering cross section, given by:

\[
\sigma_c = 2\pi e^2 \left[ \frac{2 + 8k + 9k^2 + k^3}{k^2(1 + 2k)^2} \right. \\
- \left. \frac{(2 + 2k - k^2) \log (1 + 2k)}{2k^3} \right] \tag{2.30}
\]
Let the yield of the burst in MeV of gamma rays be \( Y \). Then the function \( g(r) \) is given by

\[
g(r) = \frac{Y \bar{\rho}}{4\pi E_{\gamma} r^2} \exp \left( -\mu \int_0^r \bar{\rho}(r') \, dr' \right)
\]

(2.31a)

\[
\mu = 2\pi r^2 N_o s(k) = 18.42 \, s(k) \, [\text{km}^{-1}]
\]

(2.31b)

where \( s(k) \) is the function in brackets in Eq. (2.30), \( N_o \) is the atomic density at sea level, and \( \bar{\rho} \) is the ratio of the density at \( r \) to the sea level density. Equation (2.31a) involves a mixture of units. With some further changes of units, we will write

\[
g(r) \, [\text{cm}^{-3}] = \frac{3.66 \times 10^{10} \, Y_{\text{KT}} \, s(k) \, \bar{\rho} \exp \left( -18.42 \, s(k) \int_0^r \bar{\rho}(r') \, dr' \right)}{E_{\gamma} \, [\text{MeV}] \, r_{\text{KM}}^2}
\]

(2.32)

where now \( Y_{\text{KT}} \) is the gamma yield in kilotons, the distance \( r \) from the burst is measured in kilometers, the gamma energy is in MeV, and the total deposition \( g(r) \) is per cubic centimeter. The integral in the exponent is proportional to the total air mass traversed by the gamma rays to reach the distance \( r \). We have plotted \( g(r) \) versus altitude in Fig. 4 for a burst at 100 km altitude, a yield of 1 kT, a gamma energy of 1.6 MeV \(( k = 3.1311 \)\), and an angle from the vertical of 60 deg. At this energy, \( s(k) = 0.3321 \). The production for other yields, energies, and geometries will have the same general character, a peak at some altitude in the 20 to 40 km range with a sharp drop-off at lower altitudes, and an exponential decline at higher altitudes essentially proportional to density. We have used the Coopair International Reference Atmosphere 1961. The peak is actually quite broad, becoming half the value at the height of maximum (33 km) at 25.5 and at 44.5 km. The density varies by a factor of 37 over this altitude range.
Fig. 4—Total electron production versus altitude

The function $f(t - r/c)$ represents the time history of the gamma source. Rather than use specific functions, we shall calculate with a delta function for $f$. The actual time dependence of the source can then be brought in by convolution. The computer program in the appendix can be augmented by a simple integration process to obtain the current and ionization produced by an arbitrary gamma-ray source time dependence. No further differential equations need be solved. We anticipate that any actual time dependence of the gamma source will cause the current to display a slower initial rise rate than will be obtained with the delta function.

The function $K(\cos \theta)$ corresponds to the angular distribution of the Compton production process. The literature generally gives this Klein-Nishina distribution in terms of the differential cross section for producing a gamma ray of momentum $\mathbf{k}'$ from a gamma ray of momentum $\mathbf{k}$. The cross section for electron production is obtained from this expression by replacing $\mathbf{k}'$ by $\mathbf{k} - \mathbf{p}$, where $\mathbf{p}$ is the electron momentum,
and then multiplying by the ratio of the unit solid angle for the photon $k'$ to that of the electron $p$. We normalize so the integral of $K(\cos \theta) \sin \theta$ is unity. Omitting the detailed algebraic manipulations, there results:

$$K(x) = \frac{2(1+k)^2 x}{s(k)[(1+k)^2 - k^2 x^2]^2} \left[ \frac{(1+k)^2 - k(2+k)x^2}{(1+k)^2 - k^2 x^2} + \frac{(1+k)^2 - k^2 x^2}{(1+k)^2 - k(2+k)x^2} \right]$$

$$- \frac{4(1+k)^2 x^2(1-x^2)}{[(1+k)^2 - k(2+k)x^2]^2}$$

(2.33)

where $x = \cos \theta$. We have plotted $K(\cos \theta)$ and $K(\cos \theta) \sin \theta$ in Figs. 5 and 6 for $E_Y = 1.6$ MeV ($k = 3.1311$). These are respectively the number per unit solid angle and the number per unit angle. Note the factor of 10 change in scale between Figs. 5 and 6. The sharp peak of $K(\cos \theta)$ at small angles is evident in Fig. 5. However, the curve of Fig. 6, the number versus angle distribution, which will appear in the calculation of current and ionization, begins at zero, peaks at 10 deg, and falls off essentially linearly. This much broader distribution indicates that the often-made assumption that the Klein-Nishina distribution can be replaced by forward-directed electrons is fraught with considerable danger. Only half the number of electrons are produced at angles below 27 deg.

The final delta function factor in Eq. (2.29) relates the electron velocity to angle. The normalization is such that the integral over $\beta$ will be unity. The electron velocity and kinetic energy are found from the Compton formulas, and are:

$$h(\cos \theta) = \frac{2k(1+k) \cos \theta}{(1+k)^2 + k^2 \cos^2 \theta}$$

(2.34)

$$E(\cos \theta) = \frac{m c^2 2k^2 \cos^2 \theta}{(1+k)^2 - k^2 \cos^2 \theta}$$

(2.35)
Fig. 5—Electron production per unit solid angle; $E_{\gamma} = 1.6$ MeV

Fig. 6—Electron production per unit angle; $E_{\gamma} = 1.6$ MeV
These functions are plotted against angle in Fig. 7 for $E_\gamma = 1.6$ MeV. Although the velocity drops slowly with angle, the energy drops rather steeply. The average energy per electron can be calculated using Figs. 6 and 7; it is almost exactly 0.8 MeV. Thus, half the energy of the original gamma rays has gone into the production of these first-generation electrons and half remains in the photon cascade for later deposition. Half the total energy of the distribution is associated with electrons produced at angles less than 18 deg, so the effective energy distribution $E(\cos \theta) K(\cos \theta)$, not shown, is narrower than the electron number distribution.

![Fig. 7—Electron velocity and kinetic energy; $E_\gamma = 1.6$ MeV](image)

**CONNECTION WITH CURRENT AND SECONDARY IONIZATION**

At this point, we can write the complete equation for the evolution of the distribution function $F$. Using the results of the previous subsections, we have:
\[
\frac{\partial F}{\partial t} + \beta c \cos \theta \left( \frac{1}{r^2} \frac{\partial}{\partial r} r^2 F + \frac{eB}{m} (1 - \beta^2)^{1/2} \left( \cos \varphi \frac{\partial F}{\partial \varphi} - \frac{\cos \theta \sin \varphi}{\sin \theta} \frac{\partial F}{\partial \varphi} \right) \right)
\]

\[
- \frac{1.16 \bar{\rho}(r)}{\beta^2} \frac{\partial}{\partial \beta} \beta (1 - \beta^2)^{3/2} F - 2.51 \bar{\rho}(r) \frac{(1 - \beta^2)}{\beta^2} \left\{ \frac{\partial^2 F}{\partial \varphi^2} \right\}
\]

\[
+ \frac{\cos \theta}{\sin \theta} \frac{\partial F}{\partial \vartheta} + \frac{1}{\sin^2 \theta} \frac{\partial^2 F}{\partial \varphi^2}
\]

\[
= \frac{g(r) f(t - r/c) K(\cos \theta) \delta(\beta - h(\cos \theta))}{2\pi \beta^2}
\]

(2.36)

This is the complete transport equation. The time is in shakes and, if Eq. (2.32) is to be used for the total production, the units of $F$ are electrons per cubic centimeter. The reduced density $\bar{\rho}(r)$ is a function of altitude or, equivalently, of range to the burst.

The physical quantities to be determined from $F$ are found from Maxwell's equations. In MKS units, these are:

\[
\nabla \times \vec{E} = - \frac{\partial \vec{B}}{\partial t} \quad (2.37a)
\]

\[
\nabla \times \vec{B} = \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} + \mu \vec{J} \quad (2.37b)
\]

where $\vec{J}$ denotes the total current, including the Compton current $\vec{J}_C$ and the conduction current $\vec{J}_S$. At altitudes below 50 km, the secondary electrons experience so many collisions with the neutral molecules of the atmosphere in a time during which the electromagnetic field changes appreciably that the conduction current is a true conduction term $\sigma \vec{E}$.

The conductivity is then given by:

\[
\sigma = \frac{e^2 N_S(r,t)}{m \nu(r)} \quad (2.38)
\]
where \( v(r) \) is the collision frequency which is proportional to the density and \( N_s(r,t) \) is the total number of secondary electrons generated in the atmosphere by the ionization process.

The Maxwell equations are reduced and simplified by dropping the angular derivatives and employing the high-frequency approximation of Ref. 1. This enables us to write a single equation

\[
\frac{\partial}{\partial r} r E + \frac{\mu c \sigma}{2} r E = \frac{\mu c}{2} r J_c
\]

(2.39)

where the arguments of \( E \) and \( J_c \) are \( r, t - r/c \). The derivative is with respect to \( r \), with \( t - r/c \) held constant. Only the transverse components of \( E \) or \( J_c \) appear in Eq. (2.39). Thus, the quantities we shall calculate from \( F \) are \( \mu c J_c / 2 \) and \( N_s \). We note that, although \( v(r) \) has been written as a function of position only, it actually is strongly dependent on the field, so Eq. (2.39) is a nonlinear equation for the field which must be integrated step-wise. Since we shall not calculate fields in this report, we shall not treat Eq. (2.39) further.

The Compton current is obtained from the velocity distribution function by multiplying by \( ev \) and integrating over all values of the velocity. This yields:

\[
\frac{\mu c}{2} J_c = \frac{e}{2\epsilon} \int_0^1 \beta^3 d\beta \int_0^\pi \int_0^{2\pi} \sin^2 \theta \cos \phi d\phi d\theta d\rho F(r, \beta, \theta, \phi, t)
\]

(2.40)

The numerical factor \( e/2\epsilon = 0.90467 \times 10^{-8} \). The other component will integrate to zero.

For the secondary electrons, we have the rate relation

\[
E_i \delta N_s = \frac{dE}{dt} F
\]

(2.41)

which asserts that the number of secondaries produced per unit time multiplied by the mean energy per secondary equals the number of primaries multiplied by the energy loss per unit time of the primaries. Using a mean energy per secondary of 34 eV and with the help of Eq. (2.15), we have
The total rate of production of secondaries is thus:

\[ \delta \dot{N}_S = 17435 \, \bar{\rho} F \]  

(2.42)

\[ \dot{N}_S = 17435 \, \bar{\rho} \int_0^1 \beta^2 d\beta \int_0^\pi \int_0^{2\pi} \sin \theta \ d\theta d\varphi \ F(r, \theta, \varphi, t) \]  

(2.43)

The integral on the right is also the total number of primaries \( N_0 \).

To obtain the actual number of secondaries from their rate of production, we shall use the model of Ref. 8, which takes into account the manner in which the electrons produced by Compton ionization in turn produce additional ionization. With this model, we have:

\[ N_S = \int_{-\infty}^t dt' \dot{N}_S(t') \ L(t-t') \]  

(2.44a)

\[ L(t-t') = \frac{34}{86} \left[ 1 + 0.16 \ \log \left( 1 + 33100 \ \bar{\rho}(t-t') \right) \right] \]  

(2.44b)

\[ = 1 \text{ if Eq. (2.44b) exceeds 1} \]  

(2.44c)

We have modified the actual formula of Ref. 8 by introducing \( \bar{\rho} \) instead of the atomic density \( N \) and have expressed time in shakes. Electron attachment has been neglected, which should be satisfactory at these altitudes on this time scale. We thus see that the quantities to be calculated from \( F \) are the integrals of Eqs. (2.40) and (2.43).

With Eqs. (2.36), (2.40), and (2.43), we have completed the derivation and shall now proceed to the solution. Lengthy and complicated mathematical treatment follows. We considered delegating this material to appendices, but feel the report would be impossible to follow if we leaped from the formulation of the transport equation to (1) its solution, which in abbreviated form in Eq. (3.27) occupies ten lines of solid mathematical symbols, or (2) the representations for current and ionization rate, which are of comparable length. We have attempted to make the presentation clear without becoming either too concise or hopelessly bogged down in detail.
III. SOLUTION OF THE TRANSPORT EQUATION

THE GREEN'S FUNCTION

Equation (2.36) represents the evolution in time of the electron distribution function in response to the Compton production function. We shall introduce a Green's function, which is the response to an instantaneous point source with a specified velocity vector. The true distribution function can then be determined by a convolution between the Green's function and the Compton production function. The equation for the Green's function is simpler to solve than is Eq. (2.36).

First, we shall consider the distance scale of the electron motion. In the absence of scattering, the fastest electrons of the initial distribution will spiral around the field lines with a helical radius of about 100 m. We do not expect scattering to change this scale significantly. This scale is small compared to the 6 km atmospheric scale height, so we shall neglect the variation of the density $\rho(r)$ and the deposition $g(r)$ over the electron path. We extract a velocity-dependent factor, and write:

$$F = \frac{g(r) W(r, \beta, \theta, \varphi, t)}{\beta(1-\beta^2)^{3/2}}$$  \hspace{1cm} (3.1)

$$\frac{\partial W}{\partial t} + c \cos \varphi \frac{\partial W}{\partial r} + \frac{eB}{m} (1-\beta^2)^{1/2} \left( \cos \varphi \frac{\partial W}{\partial \theta} - \frac{\cos \theta \sin \varphi}{\sin \theta} \frac{\partial W}{\partial \varphi} \right)$$

$$- 1.16 \frac{1}{\beta^2} \frac{\partial W}{\partial \beta} - 2.51 \frac{1}{\beta^2} \frac{\partial^2 W}{\partial \beta^2} - 2.51 \frac{1}{\beta^2} \frac{\partial^2 W}{\partial \beta \partial \varphi}$$

$$= (1-\beta^2)^{3/2} \frac{f(t - r/c) K(\cos \theta) \delta (\beta - h(\cos \theta))}{2\pi \beta}$$  \hspace{1cm} (3.2)

$$\frac{\gamma c}{2} J_e(r', t') = \frac{e g(r')}{2} \int_0^1 \frac{\beta^2 d\beta'}{(1-\beta^2)^{3/2}} \int_0^\pi \int_0^{2\pi} \sin^2 \theta' \cos \varphi' \, d\theta' \, d\varphi'$$

$$W(r', \beta', \theta', \varphi', t')$$  \hspace{1cm} (3.3)
\[
N_S(r', t') = 17435 \bar{\rho} g(r') \int_0^1 \frac{\beta' d\beta'}{(1-\beta'^2)^{3/2}} \int_0^{\pi} \int_0^{2\pi} \sin \theta' d\theta' d\varphi'
\]

\[
W(r', \beta', \theta', \varphi', t')
\]

(3.4)

where the primes have been introduced to facilitate later work.

The point of the transformation (3.1) is, of course, to eliminate
the velocity factor under the \(\beta\) derivative. The initial condition on
the distribution \(W\) is that \(W\) vanish for \(t < r/c\).

The Green's function \(G\), a function of the 10 variables, \(r, \beta, \theta, \varphi, t, r', \beta', \theta', \varphi', t'\), is taken to satisfy the differential equation

\[
\frac{\partial G}{\partial t} + \beta \cos \theta \frac{\partial G}{\partial r} + \frac{eB}{m} (1-\beta'^2)^{1/2} \left( \cos \varphi \frac{\partial G}{\partial \theta} - \cos \theta \sin \varphi \frac{\partial G}{\partial \varphi} \right)
\]

\[
- 1.16 \rho \frac{(1-\beta^2)^{3/2}}{\beta} \frac{\partial G}{\partial \beta} + 2.51 \rho \frac{(1-\beta^2)}{\beta^2} V^2 G
\]

\[
= (1-\beta^2)^{3/2} \delta(r-r') \delta(\beta-\beta') \delta(\theta-\theta') \delta(\varphi-\varphi') \delta(t-t') \sin \theta
\]

(3.5)

We choose the initial condition \(G = 0\) for \(t > t'\).

Multiply Eq. (3.2) by \(BG \sin \theta/(1-\beta^2)^{3/2}\); Eq. (3.5) by \(BW \sin \theta/(1-\beta^2)^{3/2}\); add; and integrate over all values of \(r, \beta, \theta, \varphi, t\). The several terms of the integration on the left lead, after integration by
parts, to the following boundary terms:

\[
\begin{align*}
GW & \bigg|_{t=\infty}^{t=-\infty}, \quad GW & \bigg|_{r=\infty}^{r=0}, \quad GW & \bigg|_{\beta=1}^{\beta=0}, \quad \sin \theta \ GW & \bigg|_{\theta=\pi}^{\theta=0}, \\
\sin \varphi & GW \bigg|_{\varphi=2\pi}^{\varphi=0}, \quad \sin \theta & \left( \frac{\partial G}{\partial \theta} - \frac{\partial G}{\partial \theta} \right) \bigg|_{\theta=\pi}^{\theta=0}, \quad \left( \frac{\partial G}{\partial \varphi} - \frac{\partial G}{\partial \varphi} \right) \bigg|_{\varphi=2\pi}^{\varphi=0}
\end{align*}
\]

(3.6)
The first term vanishes at $t = \infty$ from the condition on $G$, and at $t = -\infty$ from the condition on $W$. If $t$ is finite, $W$ vanishes for $r + \infty$. We take $G(r = 0) = 0$. Since the function $K(\cos \theta)$, Eq. (2.34), has an upper limit, there will be no electrons at $\beta = 1$, so the third boundary term vanishes at $\beta = 1$. To make the lower limit contribution vanish, we take $G = 0$ for $\beta < \beta'$. We assume $G$ and $W$ are bounded functions of $\theta$, which makes the $\theta$ boundary terms vanish, and that they are periodic functions of the velocity azimuthal angle $\varphi$ with period $2\pi$, as is required by the physics. As a result the $\varphi$ boundary terms vanish and the entire left side integration is zero. On the right side, the second term can be evaluated completely because of the delta functions, and is simply $-W(r',\beta',\theta',\varphi',t')$. Thus, $W$ can be expressed in terms of the Green's function $G$ by:

$$W(r',\beta',\theta',\varphi',t') =$$

$$\int_{-\infty}^{\infty} dt \int_{0}^{\infty} dr \int_{0}^{\beta} d\beta \int_{0}^{\pi} \int_{0}^{2\pi} \sin \theta d\theta d\varphi f(t - r/c) K(\cos \theta) \delta(\beta - h(\cos \theta))$$

$$\frac{G(r,\beta,\theta,\varphi,t,r',\beta',\theta',\varphi',t')}{2\pi}$$

(3.7)

The time may be eliminated from Eq. (3.5). It has the following solution, which meets the initial and boundary conditions:

$$G(r,\beta,\theta,\varphi,t,r',\beta',\theta',\varphi',t') = \delta \left( t - t' + \int_{\beta'}^{\beta} \frac{d\beta' \beta'}{1.16 \rho} (1 - \beta'^2)^{3/2} \right)$$

$$H(r,\beta,\theta,\varphi,t,r',\beta',\theta',\varphi')$$

$$\frac{1.16 \rho}{1.16 \rho}$$

(3.8)

When this form is substituted into Eq. (3.5), the terms in $\delta'$ cancel, and the terms in $\delta$ factor out on both sides. The integral in Eq. (3.8) can be evaluated, and the argument of the delta function becomes $t - t' + (\gamma - \gamma')/1.16 \rho$. Since this must be zero, we see that the energy
of each electron decreases linearly with time, exactly as it must for the purely coherent energy loss process. Substitute Eq. (3.8) into Eq. (3.7) and evaluate the $t$ integration, and we have:

$$W(r', \beta', \theta', \phi', t') =$$

$$\int_0^\infty dr \int_0^1 d\beta' \int_0^\pi d\theta' \int_0^{2\pi} \sin \theta' d\phi' f \left[ t' - \frac{r'}{c} + \frac{(y' - y)}{1.16 \rho} \right]$$

$$K(\cos \theta) \frac{\delta(\beta' - h(\cos \theta)) H(r, \beta, \theta, \phi, r', \beta', \theta', \phi')}{(2\pi 1.16 \rho)} \quad (3.9)$$

The differential equation satisfied by $H$ will be:

$$\frac{\partial H}{\partial \beta} - \frac{\beta^2}{1.16 \rho(1-\beta^2)^{3/2}} \frac{\partial H}{\partial r} - \frac{eB}{m} \frac{\beta}{1.16 \rho(1-\beta^2)^{3/2}} \left( \cos \phi \frac{\partial H}{\partial \phi} - \cos \theta \sin \phi \frac{\partial H}{\partial \rho} \right)$$

$$- \frac{2.164}{\beta(1-\beta^2)^{1/2}} q^2 H = \frac{\delta(r-r') \delta(\beta-\beta') \delta(\theta-\theta') \delta(\phi-\phi')}{\sin \theta}; \quad H = 0, \beta < \beta'$$

$$(3.10)$$

We shall define the three functions $q(\beta)$, $\epsilon(\beta)$, and $a(\beta)$ as the coefficients in this equation:

$$q(\beta) = \frac{\beta^2}{1.16 \rho(1-\beta^2)^{3/2}} \quad (3.11a)$$

$$\epsilon(\beta) = \frac{eB \beta}{m 1.16 \rho(1-\beta^2)^{3/2}} \quad (3.11b)$$

$$a(\beta) = \frac{2.164}{\beta(1-\beta^2)^{1/2}} \quad (3.11c)$$

We now tackle the solution of Eq. (3.10).
ELIMINATION OF THE FIRST DERIVATIVES

The procedure for simplification of Eq. (3.10) is indicated on p. 39 of Ref. 11. The procedure is to introduce as new independent variables a set of first integrals of the motion in the absence of scattering. When the differential equation is expressed in these variables, the first derivatives will disappear from Eq. (3.10), leaving only $\partial H/\partial \beta$ and second derivatives from the scattering term. The new variables are essentially the integration constants of the motion of an electron under magnetic deflection with energy loss, with the velocity magnitude $\beta$ regarded as the independent variable.

In the absence of scattering, the $y$-direction cosine of velocity will remain constant, whereas the $x$- and $z$-direction cosines will correspond to a steady rotation. Expressing the direction cosines in terms of their initial values, we take as the trial representation for the new variables

$$
\sin \theta \cos \varphi = \sin \theta_o \cos \varphi_o \cos \zeta(\beta) - \cos \theta_o \sin \zeta(\beta) \tag{3.12a}
$$

$$
\sin \theta \sin \varphi = \sin \theta_o \sin \varphi_o \tag{3.12b}
$$

$$
\cos \theta = \sin \theta_o \cos \varphi_o \sin \zeta(\beta) + \cos \theta_o \cos \zeta(\beta) \tag{3.12c}
$$

$$
r = r_0 - c \left[ \sin \theta_o \cos \varphi_o \sin(\beta) + \cos \theta_o \cos(\beta) \right] \tag{3.12d}
$$

This expresses the old variables $\theta, \varphi, r$ in terms of the new variables $\theta_o, \varphi_o, r_0$. The functions $\zeta(\beta), C(\beta), S(\beta)$ are to be determined to eliminate the first derivative terms in Eq. (3.10). Since the initial value of the magnitude of the velocity is $\beta^\prime$, we must choose $\zeta(\beta^\prime) = C(\beta^\prime) = S(\beta^\prime) = 0$. The inversion of Eq. (3.12) to express the new variables in terms of the old yields:

$$
\sin \theta_o \cos \varphi_o = \sin \theta \cos \varphi \cos \zeta(\beta) + \cos \theta \sin \zeta(\beta) \tag{3.13a}
$$

$$
\sin \theta_o \sin \varphi_o = \sin \theta \sin \varphi \tag{3.13b}
$$
\[ \cos \theta_0 = -\sin \theta \cos \phi \sin \zeta(\beta) + \cos \theta \cos \zeta(\beta) \quad (3.13c) \]
\[ r_0 = r + c \left( -\sin \theta \cos \phi \bar{s}(\beta) + \cos \theta \bar{c}(\beta) \right) \quad (3.13d) \]
\[ \bar{c}(\beta) = c(\beta) \cos \zeta(\beta) + s(\beta) \sin \zeta(\beta) \quad (3.13e) \]
\[ \bar{s}(\beta) = c(\beta) \sin \zeta(\beta) - s(\beta) \cos \zeta(\beta) \quad (3.13f) \]

The reduced Green's function \( H \) will be denoted by \( I \) when expressed in terms of the new variables. The derivatives of \( H \) may be expanded by the chain rule in terms of derivatives of \( I \) and derivatives of the new variables with respect to the old. The latter set is found by differentiating Eq. (3.13). The manipulation is tedious and will not be presented. To save writing, drop the arguments from \( \zeta, C, S \), and let \( d/d\beta \) be denoted by a prime.

When the expressions for the derivatives of \( r_0, \theta_0, \phi_0 \) are substituted into the chain rule relations to obtain the derivatives of \( H \), and the resulting forms are substituted into Eq. (3.10), an enormous amount of algebraic simplification takes place. The following formulas are obtained for the coefficients of the first derivative terms:

\[
\left( \frac{\partial I}{\partial r_0} \right) = c \left[ -\sin \theta \cos \phi (\bar{s}' - 2a\bar{s} - \varepsilon\bar{c}) + \cos \theta (\bar{c}' + 2a\bar{c} + \varepsilon\bar{s} - q) \right]
\quad (3.14a)
\]
\[
\left( \frac{\partial I}{\partial \theta_0} \right) = \frac{\zeta'' - \varepsilon}{\sin \theta_0} \cos \phi_0 - a \cos \theta_0 \sin \theta_0
\quad (3.14b)
\]
\[
\left( \frac{\partial I}{\partial \phi_0} \right) = \frac{\zeta'' - \varepsilon}{\sin \theta_0} \cos \phi_0 \sin \phi_0 \sin \theta_0
\quad (3.14c)
\]

The coefficients of \( \sin \theta \cos \phi \) in Eq. (3.14a) may be set equal to zero, which gives a pair of differential equations for \( \bar{c} \) and \( \bar{s} \) which may be solved easily. Setting \( \zeta'' = \varepsilon \) eliminates Eq. (3.14c) and the first term of Eq. (3.14b). The second term is part of the transformed Laplacian.
The solutions for the functions $\zeta, C, S$ satisfying the proper initial conditions are:

$$\zeta = \int_{B^-}^{B^+} d\beta_1 \epsilon(\beta_1) \quad (3.15a)$$

$$C = \int_{B^-}^{B^+} d\beta_1 q(\beta_1)e^{-2\int_{B_1}^{B} d\beta_2 a(\beta_2)} \cos \int_{B_1}^{B} d\beta_3 \epsilon(\beta_3) \quad (3.15b)$$

$$S = \int_{B^-}^{B^+} d\beta_1 q(\beta_1)e^{-2\int_{B_1}^{B} d\beta_2 a(\beta_2)} \sin \int_{B_1}^{B} d\beta_3 \epsilon(\beta_3) \quad (3.15c)$$

$$\bar{C} = \int_{B^-}^{B^+} d\beta_1 q(\beta_1)e^{-2\int_{B_1}^{B} d\beta_2 a(\beta_2)} \cos \int_{B_1}^{B} d\beta_3 \epsilon(\beta_3) \quad (3.15d)$$

$$\bar{S} = \int_{B^-}^{B^+} d\beta_1 q(\beta_1)e^{-2\int_{B_1}^{B} d\beta_2 a(\beta_2)} \sin \int_{B_1}^{B} d\beta_3 \epsilon(\beta_3) \quad (3.15e)$$

It is clear from these equations that $\zeta$ represents the angle through which the velocity of the electron would turn in the magnetic field while changing velocity magnitude from $B^-$ to $B$ in the absence of scattering. The functions $C$ and $S$ ($\bar{C}$ and $\bar{S}$) describe the forward and sideward motion of the electron during this turn, and the exponential is associated with the effects of scattering.

The coefficients of the second derivatives must now be computed. These require straightforward but lengthy manipulations. The result is the transformed differential equation:
\[
\frac{\partial I}{\partial \beta} = a(\beta) + 2c \left\{ (C \sin \theta_o - S \cos \theta_o \cos \phi_o) \frac{\partial^2 I}{\partial r_o \partial \theta_o} + S \frac{\sin \phi_o}{\sin \theta_o} \frac{\partial^2 I}{\partial r_o \partial \phi_o} \right\} \\
+ c^2 \left\{ C^2 + S^2 - (C \cos \theta_o + S \sin \theta_o \cos \phi_o)^2 \right\} \frac{\partial^2 I}{\partial r_o^2}
\]

\[
\frac{\delta(r_o - r') \delta(\beta - \beta') \delta(\theta_o - \theta') \delta(\phi_o - \phi')}{\sin \theta_o}
\]

\[
I = 0 \quad \beta < \beta'
\]

**APPARENT SOLUTION OF THE DIFFERENTIAL EQUATION**

Define the differential operators on the second and third lines of Eq. (3.16a) as M and Q respectively. We see that the first line differential operator includes only derivatives with respect to the initial velocity angles \(\theta_o, \phi_o\). The second line differential operator M involves mixed derivatives with respect to initial position \(r_o\) and initial velocity angles, and the third line differential operator Q contains only position derivatives.

We would expect that the primary effect of scattering for small changes in velocity magnitude (early times) would be to change the velocity direction. As the velocity change increases, the effect on position becomes significant. We would therefore anticipate that an expansion in terms of the "ratio" of the operators M and Q to the first line operator would in some sense be rapidly convergent for small velocity changes, and hope that this convergence persists for the significant duration of the current. We shall perform this expansion to third order and obtain after the fact that the convergence has remained satisfactory.
We see from Eqs. (3.15d,e) that, for small changes in velocity from the initial velocity $\beta'$, $C$ is proportional to $(\beta - \beta')$ and $S$ to $(\beta - \beta')^2$. The operator $M$ is thus a first-order and $Q$ a second-order operator, both in the velocity change and in the order of the derivative with respect to $r_0$. Let us introduce the operator:

$$L = \frac{\partial}{\partial \beta} - a(\beta) \left[ \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + \csc^2 \theta \frac{\partial^2}{\partial \phi^2} \right]$$

(3.17)

There should be no confusion between the differential operator $L$ and the ionization lag function $L$ of Eq. (2.44b). We now make the symbolic expansion:

$$I = I_o + I_1 + I_2 + I_3$$

(3.18)

substitute into Eq. (3.16a), and match terms of corresponding orders. There results:

$$LI_0 = \frac{\delta (r_o' - r') \delta (\beta - \beta') \delta (\theta - \theta') \delta (\phi_o - \phi')}{\sin \theta}$$

(3.19a)

$$LI_1 = MI_0$$

(3.19b)

$$LI_2 = MI_1 + QI_0$$

(3.19c)

$$LI_3 = MI_2 + QI_1$$

(3.19d)

The expansion can be continued indefinitely, but the calculation of the higher terms becomes so complex that we have not attempted to proceed beyond third order, and have in some parts of the calculation stopped at second order.

From the structure of Eqs. (3.19), we see that we wish to solve the general equation:
\[ Lu = \frac{\partial^2 u}{\partial \beta^2} - a(\beta) \left[ \frac{\partial^2 u}{\partial \theta_o^2} + \frac{\cos \theta_o}{\sin \theta_o} \frac{\partial u}{\partial \theta_o} + \frac{1}{\sin^2 \theta_o} \frac{\partial^2 u}{\partial \varphi_o^2} \right] = R(\beta, \theta_o, \varphi_o) \quad (3.20) \]

Since the operator \( L \) does not involve \( r_o \), the dependence of the right side on \( r_o \) need not be indicated, and \( u \) is to be regular in \( \theta_o, \varphi_o \). This equation is obviously suitable for an expansion in Legendre functions in \( \theta_o, \varphi_o \). Thus, we assume the following solution form, in which the orthogonality properties of the Legendre functions yield the coefficient representation:

\[ u = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{2\pi} (i+\frac{1}{2})! (i-j)! p^j_i(\cos \theta_o) \left[ D^j_i(\beta) \cos j\varphi_o + E^j_i(\beta) \sin j\varphi_o \right] \quad (3.21a) \]

\[ D^j_i = \int_0^\pi \int_0^{2\pi} \sin \theta_1 d\theta_1 d\varphi_1 p^j_i(\cos \theta_1) u(\beta, \theta_1, \varphi_1) \cos j\varphi_1 \quad (3.21b) \]

\[ E^j_i = \int_0^\pi \int_0^{2\pi} \sin \theta_1 d\theta_1 d\varphi_1 p^j_i(\cos \theta_1) u(\beta, \theta_1, \varphi_1) \sin j\varphi_1 \quad (3.21c) \]

Here we have written \( c_j = 1 \) if \( j = 0 \), \( 2 \) if \( j \neq 0 \), and have, in anticipation of later results, changed the letter for the dummy integration variable. We assume a corresponding expansion for \( R \), with coefficients \( T^j_i, U^j_i \). Substitute these expansions into Eq. (3.20), multiply by the factors of Eqs. (3.21b,c), and integrate. The differential operator in \( \theta_o, \varphi_o \) in Eq. (3.20) reduces to \(-i(i+1)\) when applied to the appropriate term in the expansion. The orthogonality of the Legendre functions then completely separates the terms of the expansion, leaving the differential equations

\[ \frac{\partial}{\partial \beta} D^j_i + i(i+1) a(\beta) D^j_i = T^j_i \quad (3.22a) \]

\[ \frac{\partial}{\partial \beta} E^j_i + i(i+1) a(\beta) E^j_i = U^j_i \quad (3.22b) \]
Assuming that everything vanishes at $\beta = \beta_-$, the solution of these equations is:

$$D_1^j = \int_{\beta_-}^{\beta} d\beta_1 T_1^j (\beta_1) e^{-i(i+1) \int_{\beta_1}^{\beta} d\beta_2 a(\beta_2)} \tag{3.23}$$

and correspondingly for $E_1^j$. Put these solutions into the expansion (3.21a), obtaining

$$u(\beta, \theta_0, \varphi_0) =$$

$$\int_{\beta_-}^{\beta} d\beta_1 \int_0^{2\pi} \sin \theta_1 \, d\theta_1 \, d\varphi_1 \sum_{i=0}^{\infty} \sum_{j=0}^{i} \frac{e_i}{2\pi} \frac{(i+j)!}{(i+1)!} e^{-i(i+1) \int_{\beta_1}^{\beta} d\beta_2 a(\beta_2)}$$

$$p^j_i (\cos \theta_0) p^j_i (\cos \theta_1) \cos j(\varphi_0 - \varphi_1) R(\beta_1, \theta_1, \varphi_1) \tag{3.24}$$

Using the addition theorem for Legendre functions, this becomes symbolically:

$$u(\beta, \theta_0, \varphi_0) =$$

$$\int_{\beta_-}^{\beta} d\beta_1 \int_0^{2\pi} \sin \theta_1 \, d\theta_1 \, d\varphi_1 \sum_{i=0}^{\infty} \sum_{j=0}^{i} \frac{e_i}{2\pi} \frac{(i+j)!}{(i+1)!} e^{-i(i+1) \int_{\beta_1}^{\beta} d\beta_2 a(\beta_2)}$$

$$R(\beta_1, \theta_1, \varphi_1) X(\beta, \theta_0, \varphi_0, \beta_1, \theta_1, \varphi_1) \tag{3.25a}$$

$$X(\beta, \theta_0, \varphi_0, \beta_1, \theta_1, \varphi_1) =$$

$$\frac{1}{2\pi} \sum_{i=0}^{\infty} (i+1) e^{-i(i+1) \int_{\beta_1}^{\beta} d\beta_2 a(\beta_2)}$$

$$p^j_i (\cos \theta_0 \cos \theta_1 + \sin \theta_0 \sin \theta_1 \cos (\varphi_0 - \varphi_1)) \tag{3.25b}$$
The function $X$ is the Green's function for the differential equation (3.20). We shall write the arguments of $X$ symbolically as $(\xi_0, \xi'_1)$, where the letter $\xi_1$ stands for the triad $\beta_1, \theta_1, \varphi_1$, and will write $d\omega_1$ for $\sin \theta_1 d\theta_1 d\varphi_1$. In all these equations, $\beta'$ stands for a value of $\beta$ immediately below $\beta'$. We now apply this general solution to the sequence of Eqs. (3.19a–d). We see immediately that:

$$I_o = \delta(r_0 - r') X(\xi_0, \xi'_1) \quad (3.26)$$

The differential operators $M, Q$ may be split into products of $\partial/\partial r_0$ and operators in $\beta, \theta_0, \varphi_0$, which we denote $M_1, Q_1$. The $\partial/\partial r_0$ will act to produce derivatives of $\delta(r_0 - r')$, whereas the other operators will appear in integrals of the type in Eq. (3.25a). We shall write down the results of the successive solution of Eqs. (3.19a–d) without the intermediate details:

$$1 + \delta(r_0 - r') X(\xi_0, \xi'_1) - 2c \delta'(r_0 - r') \int_{\beta'}^{\beta} d\beta' \int_{\omega_1}^{\omega} d\omega_1 X(\xi_0, \xi'_1) M_1(\xi'_1) X(\xi'_1, \xi'_1)$$

$$+ 4c^2 \delta''(r_0 - r') \int_{\beta'}^{\beta} d\beta' \int_{\omega_1}^{\omega} d\omega_1 X(\xi_0, \xi'_1) M_1(\xi'_1) \int_{\beta'}^{\beta} d\beta'' \int_{\omega_1}^{\omega} d\omega_2$$

$$X(\xi_1, \xi_2) M_1(\xi_2) X(\xi_2, \xi'_1)$$

$$+ c^2 \delta''(r_0 - r') \int_{\beta'}^{\beta} d\beta' \int_{\omega_1}^{\omega} d\omega_1 X(\xi_0, \xi'_1) Q_1(\xi'_1) X(\xi'_1, \xi'_1)$$

$$- 8c^3 \delta'''(r_0 - r') \int_{\beta'}^{\beta} d\beta' \int_{\omega_1}^{\omega} d\omega_1 X(\xi_0, \xi'_1) M_1(\xi'_1) \int_{\beta'}^{\beta} d\beta'' \int_{\omega_1}^{\omega} d\omega_2$$

$$X(\xi_1, \xi_2) M_1(\xi_2) \int_{\beta'}^{\beta} d\beta' \int_{\omega_1}^{\omega} d\omega_3 X(\xi_2, \xi'_1) M_1(\xi'_1) X(\xi'_1, \xi'_1)$$

$$- 2c^3 \delta'''(r_0 - r') \int_{\beta'}^{\beta} d\beta' \int_{\omega_1}^{\omega} d\omega_1 X(\xi_0, \xi'_1) M_1(\xi'_1) \int_{\beta'}^{\beta} d\beta'' \int_{\omega_1}^{\omega} d\omega_2$$

$$X(\xi_1, \xi_2) Q_1(\xi_2) X(\xi_2, \xi'_1)$$

$$- 2c^3 \delta'''(r_0 - r') \int_{\beta'}^{\beta} d\beta' \int_{\omega_1}^{\omega} d\omega_1 X(\xi_0, \xi'_1) Q_1(\xi'_1) \int_{\beta'}^{\beta} d\beta'' \int_{\omega_1}^{\omega} d\omega_2$$

$$X(\xi_1, \xi_2) M_1(\xi_2) X(\xi_2, \xi'_1)$$

(3.27)
This representation for \( I \) holds for \( \beta > \beta' \). For \( \beta < \beta' \), \( I = 0 \). At \( \beta = \beta' \), \( I \) reduces to the first term. If we let \( \beta \) approach \( \beta' \) from above, \( X \) tends to \( \delta(\theta_o - \theta') \delta(\varphi_o - \varphi')/\sin \theta \), so \( I \) satisfies the proper discontinuity condition.

The reduced Green's function \( H \) is found from \( I \) by re-expressing the variables \( r_o, \theta_o, \varphi_o \) in terms of \( r, \theta, \varphi \), using Eq. (3.13). Then \( H \) is to be substituted in Eq. (3.9) to obtain \( W \), the response to the Compton production function, as represented in the variables \( r', \xi', t' \). This representation is then to be inserted into Eqs. (3.3) and (3.4) and, after a change of lettering, integrated over \( \xi' \). The result will be the current and the ionization production rate. We shall carry out these procedures in the next section.
IV. CALCULATION OF THE CURRENT AND IONIZATION

The highest order term in the expansion (3.27) involves integration over nine variables. There are four integrations in Eq. (3.9) and three in Eq. (3.3) or (3.4). Thus, we have to deal with 16-fold integrals. We shall now perform analytically as many as we have found possible. We shall interchange orders freely but carefully and will make great use of the orthogonality properties of the Legendre functions.

The first integration is the outermost, the integration over $\theta', \varphi$, in Eqs. (3.3) and (3.4). We observe that in Eq. (3.27) these variables occur only in the extreme right factors of each term in the expansion. In $X(\xi, \xi')$, the angular variables appear only as arguments of Legendre functions. We thus have to evaluate the integrals:

\[ \int d\omega' P_i \left[ \cos \theta_k \cos \theta' + \sin \theta_k \sin \theta' \cos (\varphi_k - \varphi') \right] \] (4.1a)

\[ \int d\omega' \sin \theta' \cos \varphi' P_i \] (4.1b)

where $\theta_k, \varphi_k$ refer to the first arguments of $X$ in the appropriate term of Eq. (3.27). Rotate the coordinate system so that the pole is in the direction $\theta_k, \varphi_k$, and Eq. (4.1) becomes:

\[ \int d\omega' P_i (\cos \theta') \] (4.2a)

\[ \int d\omega' \left[ \sin \theta_k \cos \varphi_k \cos \theta' + \cos \theta_k \cos \varphi_k \sin \theta' \cos \varphi' 
- \sin \varphi_k \sin \theta' \sin \varphi' \right] P_i (\cos \theta') \] (4.2b)

The first integral is zero unless $i = 0$, in which case it equals $4\pi$. The second and third terms in the bracket of Eq. (4.2b) vanish when integrated over $\varphi'$. The first integrates to zero unless $i = 1$, when
it becomes $4\pi \sin \theta_k \cos \phi_k / 3$. Using these results in the series representation then yields:

$$\int d\omega X(\xi_k, \xi) = 1$$  \hspace{1cm} (4.3a)

$$\int d\omega \sin \theta \cos \phi X(\xi_k, \xi) = \sin \theta_k \cos \phi_k e^{-2 \int_{\beta_k}^{\beta_k} d\beta'' a(\beta'')}$$  \hspace{1cm} (4.3b)

Since $M_1$ is a differential operator, when it is applied to a constant the result is zero. Hence, there is no contribution to the ionization density from the second, third, fifth, and seventh terms of the expansion (3.27). We shall make a further simplification by dropping the third-order terms in the current. This will be justified by later numerical work.

Next are the angular integrations in Eq. (3.27). We shall demonstrate the procedure by calculating the first-order term in the current and then writing out the result for the complete expression. In this term, the integral to be evaluated is:

$$\int d\omega_1 X(\xi_0, \xi_1) M_1(\xi_1) \sin \theta_1 \cos \phi_1$$  \hspace{1cm} (4.4)

The differential operator $M_1$ is given in Eq. (3.16a), with the variables to be given the subscript 1. There results:

$$M_1(\xi_1) \sin \theta_1 \cos \phi_1 = C_1 \sin \theta_1 \cos \theta_1 \cos \phi_1$$

$$- S_1 (1 - \sin^2 \theta_1 \cos^2 \phi_1)$$  \hspace{1cm} (4.5)

Insert this into the integral in Eq. (4.4) corresponding to the $i$th term in the series for $X$, and rotate the coordinate system so that the pole is in the $\theta_0, \phi_0$ direction. We then have, after simplification:
\[
\int d\omega_1 P_1(\cos \theta_1) \left[ - S_1 + (\sin \theta_o \cos \varphi_o \cos \theta_1 + \cos \theta_o \cos \varphi_o \sin \theta_1 \cos \varphi_1 \\
- \sin \varphi_o \sin \theta_1 \sin \varphi_1) \left\{ C_1(\cos \theta_o \cos \theta_1 \\
- \sin \theta_o \sin \theta_1 \cos \varphi_1) + S_1(\sin \theta_o \cos \varphi_o \cos \theta_1 \\
+ \cos \theta_o \cos \varphi_o \sin \theta_1 \cos \varphi_1 - \sin \varphi_o \sin \theta_1 \sin \varphi_1) \right\} \right]\]
(4.6)

Integrating over \( \varphi_1 \) yields:

\[
2\pi \int_0^\pi \sin \theta_1 d\theta_1 P_1(\cos \theta_1) \left[ C_1 \cos \theta_o \sin \theta_o \cos \varphi_o \left( \frac{3}{2} \cos^2 \theta_1 - \frac{1}{2} \right) \\
+ S_1 \left\{ - 1 + \frac{1}{2} \sin^2 \theta_1 \\
+ \sin^2 \theta_o \cos^2 \varphi_o \left( \frac{3}{2} \cos^2 \theta_1 - \frac{1}{2} \right) \right\} \right]
(4.7)

From the orthogonality properties of the Legendre functions, the integral is zero unless \( i = 0 \) or \( 2 \). Evaluating these cases and restoring the series representation gives the result:

\[
\int d\omega_1 X(\xi_o, \xi_1) M_1(\xi_1) \sin \theta_1 \cos \varphi_1 =
\begin{align*}
&-6 \int_{\beta_1}^\beta d\beta_2 a(\beta_2) \\
&+ \frac{2}{3} + \left( \frac{1}{3} - \sin^2 \theta_o \cos^2 \varphi_o \right) e^{-6 \int_{\beta_1}^\beta d\beta_2 a(\beta_2)} \\
&- S_1 \left[ \right. \\
&\left. -6 \int_{\beta_1}^\beta d\beta_2 a(\beta_2) \right]
(4.8)
\end{align*}
\]
In this expression we have:

\[ C_1 = \int_{\beta^*}^{\beta_1} d\beta_2 q(\beta_2) e^{-2 \int_{\beta_2}^{\beta_1} d\beta_3 a(\beta_3)} \cos \int_{\beta^*}^{\beta_2} d\beta_4 e(\beta_4) \]  

(4.9)

and a corresponding expression for \( S_1 \). All the angular integrals of Eq. (3.27) can be evaluated in exactly the same manner, but the manipulations become very tedious for the higher order terms. We write out the result to third order in the integrand for the ionization and to second order in the current:

\[
\int g\, dz' = 5(r_0 - r') + c^2 \delta'(r_0 - r') \int_{\beta^*}^{\beta} d\beta a(\beta) \left[ \left( c^2 + s^2 \right) \left( \frac{2}{3} + \frac{1}{3} e^{-6 \int_{\beta}^{\beta^*} d\beta e(\beta)} \right) - \left( c_1 \cos \theta_o + s_1 \sin \theta_o \cos \varphi_o \right)^2 e^{-6 \int_{\beta}^{\beta^*} d\beta e(\beta)} \right] 
\]

\[ + \frac{4}{5} e^3 \delta''(r_0 - r') \int_{\beta^*}^{\beta} d\beta_1 d\beta_2 a(\beta_1) \int_{\beta^*}^{\beta_2} d\beta_3 d\beta_4 a(\beta_3) \left( -2 \int_{\beta}^{\beta^*} d\beta d\beta' e(\beta') + 2 \int_{\beta}^{\beta^*} d\beta d\beta' e(\beta') \right) 
\]

\[ \left[ c_1 c_2 + s_1 s_2 \right] \left( c_1 \cos \theta_o + s_1 \sin \theta_o \cos \varphi_o \right) \left( 2 e^{-6 \int_{\beta}^{\beta^*} d\beta e(\beta)} + 1 \right) 
\]

\[ + \left( c_2 + s_2 \right) \left( c_1 \cos \theta_o + s_1 \sin \theta_o \cos \varphi_o \right) \left( 2 e^{-6 \int_{\beta}^{\beta^*} d\beta e(\beta)} + 1 \right) 
\]

\[ - 5 \left( c_1 \cos \theta_o + s_1 \sin \theta_o \cos \varphi_o \right) \left( c_2 \cos \theta_o + s_2 \sin \theta_o \cos \varphi_o \right)^2 e^{-6 \int_{\beta}^{\beta^*} d\beta e(\beta)} \right] 
\]

(4.10)
\[ \int I \, d\omega \, \sin \theta \cos \varphi = e^{-2 \int_{\beta_1}^{\beta_2} dB_1 a(\beta_1)} \left[ \delta(r_0 - r) \sin \theta \cos \varphi_o \right. \\
+ 2c \delta(r_0 - r) \int_{\beta_1}^{\beta_2} dB_1 a(\beta_1) \left\{ S_1 \left( \frac{2}{3} e \int_{\beta_1}^{\beta_2} dB_2 a(\beta_2) + \frac{1}{3} e \int_{\beta_1}^{\beta_2} dB_3 a(\beta_3) \right) \\
- \sin \theta \cos \varphi_o \left( C_1 \cos \theta_o + S_1 \sin \theta \cos \varphi_o \right) e^{-4 \int_{\beta_1}^{\beta_2} dB_2 a(\beta_2)} \right\} \\
+ \frac{1}{5} c^2 \delta(r_0 - r) \int_{\beta_1}^{\beta_2} dB_1 a(\beta_1) \left( \left( C_1^2 + S_1^2 \right) \sin \theta_o \cos \varphi_o \left( 1 + 4c \int_{\beta_1}^{\beta_2} dB_2 a(\beta_2) \right) \\
- 2S_1 \left( C_1 \cos \theta_o + S_1 \sin \theta \cos \varphi_o \right) \left( 1 - e^{-10 \int_{\beta_1}^{\beta_2} dB_2 a(\beta_2)} \right) \right\} \\
- 5 \sin \theta \cos \varphi_o \left( C_1 \cos \theta_o + S_1 \sin \theta \cos \varphi_o \right)^2 e^{-10 \int_{\beta_1}^{\beta_2} dB_2 a(\beta_2)} \right\} \\
+ \frac{4}{5} c^2 \delta(r_0 - r) \int_{\beta_1}^{\beta_2} dB_1 a(\beta_1) \int_{\beta_1}^{\beta_2} dB_2 a(\beta_2) \left\{ 2S_2 \left( C_1 \cos \theta_o + S_1 \sin \theta \cos \varphi_o \right) \left( 1 - e^{-10 \int_{\beta_1}^{\beta_2} dB_2 a(\beta_2)} \right) \\
- \left( C_1 S_2 + S_1 \right) \sin \theta_o \cos \varphi_o \left( C_2 \cos \theta_o + S_2 \sin \theta \cos \varphi_o \right) \left( 1 + 4c \int_{\beta_1}^{\beta_2} dB_2 a(\beta_2) \right) \right\} \\
\left. + 10 \sin \theta_o \cos \varphi_o \left( C_1 \cos \theta_o + S_1 \sin \theta \cos \varphi_o \right) \left( C_2 \cos \theta_o + S_2 \sin \theta \cos \varphi_o \right) e^{-10 \int_{\beta_1}^{\beta_2} dB_2 a(\beta_2)} \right\} \right] \\
(4.11) \]
The next step is to replace the variables \( r_0, \theta_0, \varphi_0 \) by \( r, \theta, \varphi \) using Eq. (3.13). The integrations which remain to evaluate the current and the ionization density are over the variables \( r, \beta, \theta, \varphi, \beta_- \). The variable \( r \) appears in the argument of a delta function, \( r-r^-+c \) \((c \cos \theta - S \sin \theta \cos \varphi)\), its derivatives, and \( f \). The derivative with respect to \( r \) of the delta function goes over into \( \partial/\partial (r^-) f \) after the \( r \) integration, where the argument of \( f \) is \( t^-r^/-c - (\gamma-\gamma^-)/1.16 \rho + c \cos \theta - S \sin \theta \cos \varphi \). Since \( t^-r^/-c \) is the retarded time at the electron location, we have now shown that, to the approximation of neglecting the variation of \( \rho \) and \( g \) over the electron motion, the current and ionization are functions of retarded time only, as is of course to be expected.

The variable \( \beta \) appears as the argument of a delta function \( \beta - h(\cos \theta) \) and in integration limits. Since \( \beta \) must exceed \( \beta^- \), the \( \beta^- \) integration must be brought inside the \( \theta \) integration, and then the limits on \( \beta^- \) become \( 0, h \). Referring back to Eqs. (3.15b,c), when \( \beta \) is replaced by \( h \), we shall regard \( C, S \) as functions of the lower limit \( \beta^- \). We then define:

\[
C_1 = \tilde{C}(\beta^-) - \tilde{C}(\beta^-_1) = \int_{\beta^-}^{\beta^-_1} d\beta_2 q(\beta_2) e^{-2 \int_{\beta_2}^{h} d\beta_3 a(\beta_3)} \cos \int_{\beta_2}^{h} d\beta_4 e(\beta_4)
\]  

(4.12)

and a corresponding equation for \( S_1 \). The function \( \tilde{C}(\beta^-) \) differs from Eq. (4.12) only in that the outermost upper limit is \( h \). We have the relations:

\[
C_1 = e^{2 \int_{\beta^-}^{\beta^-_1} d\beta_2 a(\beta_2)} (C_1 \cos \zeta + S_1 \sin \zeta)
\]  

(4.13a)

\[
S_1 = e^{2 \int_{\beta^-}^{\beta^-_1} d\beta_2 a(\beta_2)} (C_1 \sin \zeta - S_1 \cos \zeta)
\]  

(4.13b)
\[ \mathcal{C}_1 \cos \theta + S_1 \sin \theta \cos \varphi = e^{\frac{2}{\beta_1}} \int_{\beta_1}^{\beta} d\beta_2 a(\beta_2) \]

\[ (\mathcal{C}_1 \cos \theta - S_1 \sin \theta \cos \varphi) \quad (4.13c) \]

\[ C_1^2 + S_1^2 = e^{\frac{4}{\beta_1}} \int_{\beta_1}^{\beta} d\beta_2 a(\beta_2) \]

\[ (C_1^2 + S_1^2) \quad (4.13d) \]

\[ \mathcal{C}_1 C_2 + S_1 S_2 = e^{\frac{2}{\beta_1}} \int_{\beta_1}^{\beta} d\beta_3 a(\beta_3) + 2 \int_{\beta_2}^{\beta} d\beta_3 a(\beta_3) \]

\[ (C_1 C_2 + S_1 S_2) \quad (4.13e) \]

The point of introducing these functions is that \( \mathcal{C}(\beta') \) and \( S(\beta') \), functions of only one variable, are easy to compute. The underscored functions are then obtained by subtraction, or the polynomials in \( \mathcal{C}(\beta') - \mathcal{C}(\beta_1) \) may be expanded and the integrations thereby separated into products of integrals. This procedure greatly shortens the computing time.

These discussions show how to integrate over the variables \( r, \beta \). We shall now assume that \( f(t - r/c) \), the gamma-ray time history, is a delta function whose argument is:

\[ t' - r'/c - \int_{\beta'}^{h} \frac{\beta d\beta}{\beta' 1.16 \rho (1-\beta^2)^{3/2}} \left\{ -2 \int_{\beta}^{h} a(\beta_1) d\beta_1 \right\} \]

\[ \left( \cos \theta \cos \int_{\beta}^{h} d\beta_2 \varepsilon(\beta_2) - \sin \theta \cos \varphi \sin \int_{\beta}^{h} d\beta_2 \varepsilon(\beta_2) \right) \quad (4.14) \]

The derivatives of the delta function with respect to \( t-t' \) which arise from the \( r_0 \) derivatives in \( I \) may be taken outside the integrations.
Thus, Eq. (4.14) indicates how we may integrate over \( \beta' \). Choose values of \( t' - r' /c \), \( \theta \), and \( \varphi \). Find the value of \( \beta' \) such that Eq. (4.14) is zero. Set \( \beta' \) equal to that value in Eqs. (4.10) and (4.11). Divide by the derivative of Eq. (4.14) with respect to \( \beta' \), which is of course the integrand of Eq. (4.14). The result will be the integral over \( \beta' \).

The structure of Eq. (4.14) is such that the integral increases monotonically as \( \beta' \) decreases from \( h \). At \( \beta' = h \), expression (4.14) is \( t' - r' /c \) and is positive. When \( \beta' = 0 \), the integral has a maximum value which is a function of \( \theta \) and \( \varphi \). For sufficiently small values of \( t' - r' /c \), there will be a zero for Eq. (4.14) as a function of \( \beta' \) for all values of \( \theta \) and \( \varphi \). As \( t' - r' /c \) increases, it reaches a value which exceeds the maximum value of the integral for some values of \( \theta, \varphi \), which then do not contribute to the integration over \( \beta' \). Those values of \( \theta, \varphi \) correspond to electrons which have slowed to zero velocity, so of course they should not contribute. Eventually, \( t' - r' /c \) becomes so large that it exceeds the integral for all \( \theta, \varphi \) and there is no further contribution to the current or ionization.

We have reached this point in the solution of the transport equation and the calculation of the current and ionization without utilizing the specific forms of the functions \( q, e, \) and \( a \). We shall now employ Eq. (3.11) to simplify the expressions. We first evaluate \( \zeta \) and the integral of \( a \). With Eqs. (3.11b) and (3.15a), and \( \beta \) set equal to \( h \), we have:

\[
\zeta' = \int_{\beta'}^{h} \frac{eB}{m 1.16 \rho} \int_{\beta'}^{h} \frac{\beta_1 \beta_2}{1 - \beta_2^2} d\beta_1 d\beta_2 = \frac{eB}{m 1.16 \rho} \log \frac{1 - h^2}{1 - h^2} \tag{4.15b}
\]

\[
= \frac{1}{K} \log \frac{\gamma_0}{\gamma} \tag{4.15c}
\]
where the constant $K$ is equal to:

$$K = 6.595 \frac{\rho}{B} \quad (4.16)$$

and $\gamma_0$ is the initial electron total energy, including the rest energy in units of $mc^2$.

We can invert Eq. (4.15c) to express $\beta'$ and $\gamma'$ in terms of $\zeta'$, yielding:

$$\gamma' = \gamma_0 e^{-K\zeta'} \quad (4.17a)$$

$$\beta' = \left(1 - \frac{e^{2K\zeta'}}{\gamma_0^2}\right)^{\frac{1}{2}} \quad (4.17b)$$

Thus, as the electron turns in the magnetic field through an angle $\zeta$ in the absence of scattering, the energy decreases exponentially with $\zeta$. Since the smallest possible value of $\beta$ is zero, we see from Eq. (4.15c) that the maximum angle of turn of the electron is directly proportional to the magnetic field, inversely proportional to the density, and proportional to the logarithm of the initial energy. For electrons with the energy distribution of Fig. 7, the logarithm has a maximum value 1.306 at $\theta = 0$ and is down to 0.7 at 40 deg. We shall consider $\log \gamma_0 = 1$ as representative, corresponding to electrons slightly below the mean energy, 0.8 MeV. With a magnetic field of 0.6 gauss, the maximum angle of turn will be 72 deg at an altitude of 20 km, 347 deg at 30 km, and 1568 deg (4.36 revolutions) at 40 km. This cutoff of the angle of turn is caused entirely by the energy loss process.

The integral of $a$ can be evaluated using Eq. (3.11c). We shall arrange the limits in the exponents of Eqs. (4.10) and (4.11) so that the upper limit is always $h$. After simplification, we obtain:

$$\int_{\beta}^{h} d\beta a(\beta) = 1.082 \log \left[ \frac{\gamma_0 - 1}{\gamma_0 + 1} \frac{\gamma + 1}{\gamma - 1} \right] \quad (4.18a)$$
We shall introduce $\zeta$ as a new independent variable. We have, for the function $\bar{C}$ (see Eq. (4.12)), the expressions:

$$e^{-2 \int_0^h d\beta_1 a(\beta_1)} = \left[ \frac{(\gamma - 1)}{\gamma + 1} \right] \left[ \frac{\gamma_o + 1}{\gamma_o - 1} \right]^{2.164}$$

(4.18b)

We shall introduce $\zeta$ as a new independent variable. We have, for the function $\bar{C}$ (see Eq. (4.12)), the expressions:

$$\bar{C}(\zeta') = \frac{m}{e\bar{B}} \int_0^\infty d\zeta \left[ \gamma_o e^{-2K\zeta} - 1 \right]^{1/2} \left[ \begin{array}{c} \gamma_o e^{-K\zeta} - 1 - 1 \\ \gamma_o e^{-K\zeta} + 1 - \gamma_o \end{array} \right]^{2.164} \cos \zeta$$

(4.19)

For $\bar{S}$, $\cos \zeta$ is replaced by $\sin \zeta$. This representation depends directly on $\zeta'$, related to $\beta'$ through Eq. (4.17b), and indirectly on $\theta$, which appears in $\gamma_o$. Equation (4.19) is in precise form for step-by-step numerical integration.

If there were no scattering, we would set $\alpha = 0$. The functions $\bar{C}, \bar{S}$ then represent the forward and sideward components of position of an electron with initial energy $\gamma_o$ which has turned through an angle $\zeta'$ from its initial direction. The first effect of scattering is to insert the factor (4.18b) into $\bar{C}$ and $\bar{S}$. This corresponds to a change in the velocity of a mean position. If we were to keep only the first terms in the expansions (4.10), the motions of the aggregate of electrons would be fully described by a moving mean position.

The scattering expression (4.18a) corresponds closely to Longmire's obliquity theory, which sets the right side of (4.18a) equal to $(\eta - 1)/2$ (see Eq. (22) of Ref. 8, with a slight change in constants). Thus, in some sense, the obliquity theory is a zeroth approximation to the theory we have developed in this report, with the electron velocity distribution Green's function reduced to the first term of $I$. We have not attempted to work out the full details of the comparison.

We shall now write out the full expressions for current and ionization density as integrals over $\theta, \phi, \zeta_1, \zeta_2$. Let $\tau = (\gamma' - r)/c$, the retarded time. We define the integral of Eq. (4.14), the equivalent of
the arrival time of the electron, as \( \zeta(\tau', \theta, \varphi) \). Then the condition that Eq. (4.14) vanishes determines a value \( \zeta'(\tau, \theta, \varphi) \) such that:

\[
\tau = T(\zeta'(\tau, \theta, \varphi), \theta, \varphi) = \frac{m}{eB} \int_0^{\zeta'} \gamma d\zeta \left[ 1 - \beta \left( \frac{\gamma - 1}{\gamma + 1} \frac{\gamma_0 + 1}{\gamma_0 - 1} \right)^{2.164} \left( \cos \theta \cos \zeta - \sin \theta \cos \varphi \sin \zeta \right) \right] \tag{4.20a}
\]

\[
= T_1(\zeta', \theta) + T_2(\zeta', \theta) \cos \varphi \tag{4.20b}
\]

The value of \( \zeta' \) to be used in the current and ionization is to be found by inverting Eq. (4.20). In practice, this must be done by interpolation. The unspecified limits on \( \theta \) and \( \varphi \) are for all values such that Eq. (4.20) has a solution. We define \( U(\zeta) \) as the function of Eq. (4.18b), which we can call the scattering reduction factor, and will write \( U' \) for \( U(\zeta') \), \( U_1 \) for \( U(\zeta_1) \), and \( U_2 \) for \( U(\zeta_2) \). A function \( D \) is defined by:

\[
D(\zeta', \theta, \varphi) = 1 - \beta' U'(\cos \theta \cos \zeta' - \sin \theta \cos \varphi \sin \zeta') \tag{4.21}
\]

\( D \) is the ratio between retarded time and local time of the aggregate of electrons. We also define a function \( A(\zeta) = a(\beta) \frac{d\beta}{d\zeta} \) as:

\[
A(\zeta) = \frac{2.164 \ KY}{\gamma^2 - 1} \tag{4.22}
\]

We introduce the shorthand:

\[
w = \cos \theta \sin \zeta' + \sin \theta \cos \varphi \cos \zeta' \tag{4.23a}
\]

\[
Z_1 = \zeta_1 \cos \theta - \frac{S_1}{\gamma_1} \sin \theta \cos \varphi \tag{4.23b}
\]

We now have the final result for the current and ionization rate:
\[
\frac{u_c}{2} J_c(\tau) = 0.90467 \times 10^{-8} \int \int \frac{\sin \theta \, d\theta d\phi \, K(\cos \theta) \beta \, U^{-1}}{2\pi D(\tau, \theta, \phi)}
\]
\[
\begin{align*}
\left. \begin{cases}
0
\end{cases} \right\}
\end{align*}
\]
\[
\begin{align*}
&+ \frac{1}{3} \frac{\partial}{\partial \tau} \int_0^\xi d\xi_1 A(\xi_1) \left\{ S_1 \left( U_1 + \frac{2}{U_1^2} \right) - 3z_1 U_1 \right\} \\
&+ \frac{4}{3} \frac{\partial}{\partial \tau} \int_0^\xi d\xi_1 A(\xi_1) \int_0^{\xi_1} d\xi_2 A(\xi_2) U_2 \left\{ 2U_1^2 + \frac{3}{U_1^3} \right\} (2u^2 + 3U_1^3)
\end{align*}
\]
\[
(4.24)
\]
\[
\dot{N}_s(\tau) = 17435 \, g \int \int \frac{\sin \theta \, d\theta d\phi \, K(\cos \theta)}{2\pi D(\tau, \theta, \phi)}
\]
\[
\begin{align*}
1 + \frac{1}{3} \frac{\partial}{\partial \tau} \int_0^\xi d\xi_1 A(\xi_1) \left\{ \left( C_1 + S_1 \right) \left( U_1 + \frac{2}{U_1^2} \right) - 3z_1 U_1 \right\} \\
+ \frac{4}{3} \frac{\partial}{\partial \tau} \int_0^\xi d\xi_1 A(\xi_1) \int_0^{\xi_1} d\xi_2 A(\xi_2) U_2 \left\{ \left( C_2 + S_2 \right) z_1 \left( U_1^2 - \frac{1}{U_1^3} \right) - 5z_1 z_2 U_1^2 \right\}
\end{align*}
\]
\[
(4.25)
\]
The time derivatives in these equations should be placed outside the integral sign, since the limits are functions of \( \tau \). We have written them inside to avoid having to repeat the first line of Eqs. (4.24) and (4.25) for each term, but in calculation the time derivatives must be treated last.

This is as far as we have been able to proceed analytically. We have written a computer program to calculate the current and ionization from Eqs. (4.24) and (4.25). This program is purely an exercise in numerical integration, since all the differential equations have been solved. We next discuss the construction of the program.
V. DEVELOPMENT OF THE COMPUTER PROGRAM

Although the programming of Eqs. (4.24) and (4.25) is fairly straightforward, there are some rather tricky numerical points, mostly associated with the zero-velocity cutoff. The program takes as inputs the magnetic field, gamma-ray energy, time-step, and angle-step. Most calculations have been done with a time-step of 0.1 shake and an angle-step of 1 deg in the turn-angle $\zeta$. Double precision arithmetic has been employed throughout.

The program uses different numerical integration routines for the several variables. The integration over $\theta$ is a simple summation, since the end-point terms of the trapezoidal rule vanish because of the factor $\sin \theta$. We have used an integration step in $\theta$ of 3 deg for $\theta$ below 30 deg and 6 deg above. This is a good match to the rate of variation of $K(\cos \theta) \sin \theta$ (see Fig. 6). The innermost integrals over $\zeta_1$ and $\zeta_2$ use the modified trapezoidal rule. This is a four-point rule, which must be specially modified at the end intervals of the integration range or if there are fewer than four points available (as can occur for low-energy electrons). The rule reads

\begin{align}
\int_0^{\Delta x} f(x)dx &= \frac{\Delta x[9f(0) + 19f(1) - 5f(2) + f(3)]}{24} \quad (5.1a) \\
\int_{i\Delta x}^{(i+1)\Delta x} f(x)dx &= \frac{\Delta x[- f(i-1) + 13f(i) + 13f(i+1) - f(i+2)]}{24} \quad (5.1b) \\
\int_{(N-1)\Delta x}^{N\Delta x} f(x)dx &= \frac{\Delta x[f(N-3) - 5f(N-2) + 19f(N-1) + 9f(N)]}{24} \quad (5.1c)
\end{align}

where the argument $i\Delta x$ of $f(i\Delta x)$ has been abbreviated $i$ and similarly for all other terms. Equation (5.1a) represents the first interval of the integration range, (5.1c) represents the last interval, and it is
assumed that \( N \) is at least 3. The error term in Eq. (5.1b) is 11(\( \Delta x \))\(^5\)f\(^{(4)}\)(\( \xi \))/720, where f\(^{(4)}\)(\( \xi \)) denotes the fourth derivative of \( f \) evaluated at some point in the interval. The functions we have to deal with are sufficiently smooth that the error term can be neglected throughout.

The integration over \( \varphi \) is performed by gaussian integration procedures. First, we reconsider the limits of integration. Referring to Eq. (4.20), we define a function \( T_m(\theta, \varphi) \) by:

\[
T_m(\theta, \varphi) = \frac{m}{eB} \int_0^{\cos \theta \cos \zeta - \sin \theta \cos \varphi \sin \zeta} Y_0/K e^{-K\zeta} \left[ 1 - \left( 1 - \frac{2K\zeta}{Y_0} \right)^2 \right]^{2.164} \left( \frac{Y_0 e^{-K\zeta} - 1}{Y_0 e^{-K\zeta} + 1} \right) \]

\[
(\cos \theta \cos \zeta - \sin \theta \cos \varphi \sin \zeta) \quad (5.2a)
\]

\[
Y_0 = \frac{(1+k)^2 + k^2 \cos^2 \theta}{(1+k)^2 - k^2 \cos^2 \theta} \quad (5.2b)
\]

This is the time for an electron to slow to zero velocity from the initial energy \( Y_0 \) appropriate to the direction \( \theta, \varphi \). It is clear that \( T_m \) can be split into the form:

\[
T_m(\theta, \varphi) = T_{m1}(\theta) + T_{m2}(\theta) \cos \varphi \quad (5.3)
\]

This lies between the limits \( T_{m1} \pm T_{m2} \). Suppose now we choose a value of \( \theta \), and let \( \tau \) increase from zero. As long as \( \tau \) is less than \( T_{m1} - T_{m2} \), there will be a solution of Eq. (4.20), \( \tau = T(\zeta', \theta, \varphi) \) for all values of \( \varphi \). If \( \tau \) is between \( T_{m1} - T_{m2} \) and \( T_{m1} + T_{m2} \), there will be a solution for \( \cos \varphi = (\tau - T_{m1})/T_{m2} \), and not otherwise. If \( \tau \) exceeds \( T_{m1} + T_{m2} \), the integral is zero. Since the integrand depends only on \( \cos \varphi \), the integral from 0 to \( 2\pi \) is twice the integral from 0 to \( \pi \). We thus have the integration forms:

\[
\int_0^{\pi} f(\cos \varphi) d\varphi, \quad \int_0^{\varphi_m} f(\cos \varphi) d\varphi \quad (5.4a)
\]
\[ \cos \varphi_m = \frac{(\tau - \tau_{m1})}{\tau_{m2}} = X_m \]  
\hspace{1cm} (5.4b)

For the first integral, we have the gaussian approximation,

\[ \frac{1}{n} \int_0^{\pi} f(\cos \varphi) d\varphi \sim \frac{1}{n} \sum_{n=1}^{n} f \left( \cos \frac{(2i-1)\pi}{n} \right) \]  
\hspace{1cm} (5.5)

whereas the second integral leads to a modified version of Eq. (5.5):

\[ \frac{1}{n} \int_0^{\varphi_m} f(\cos \varphi) d\varphi \sim \frac{1}{n} \sum_{n=1}^{n} \left[ 1 + \frac{2(1 + X_m)}{(1 - X_m)} \left( 1 + \cos \frac{(2i-1)\pi}{n} \right) \right] \]  
\hspace{1cm} (5.6)

Here \( n \) is the order of the gaussian approximation. We have tested the program with \( n = 5 \) and \( n = 9 \). The results were indistinguishable to three significant figures, so we used \( n = 5 \) to reduce the running time.

Other difficulties at the low-speed range are caused by the function \( U \). As can be seen from its definition (Eq. (4.18b)), near the cut-off turn-angle \( \zeta_m = \log \gamma_0/K \), \( U \) vanishes like \( (\zeta_m - \zeta)^{\frac{1}{2}} \). Inverse cubes of \( U \) appear in Eqs. (4.24) and (4.25) multiplied by combinations of \( C \)'s and \( S \)'s which make the integrals convergent. However, the calculation requires division of small quantities by small quantities, with attendant loss of accuracy. To avoid this problem, we cut off the \( \zeta \) integrations one step before the last interval, and then extrapolate to obtain the last value. Testing has shown this procedure to be sufficiently accurate.

With these mathematical problems clarified, we describe the general nature of the program. The integration rule (Eq. (5.1)) has been

*Reference 17, formula 25.4.38, p. 889.
packaged as a subroutine (Part 7) which integrates many functions simultaneously. The atmospheric density $\bar{\rho}$ is stored for altitudes between 18 and 60 km in 2 km steps. Above 60 km, the current per primary gamma is essentially independent of altitude, and the ionization is negligible. The program can take any altitude for which the density has been stored as an input initial altitude.

The program begins at the initial altitude (18 km if not otherwise specified) and the first value of $\theta$ (3 deg). It calculates $K(\cos \theta)$ and $h(\cos \theta) = \gamma_0$. Using Part 7, the functions $\bar{C}, \bar{S}, T_1,$ and $T_2$ (see Eqs. (4.19) and (4.20)) are found and stored in Part 2 for all $\zeta$, in 1 deg steps, up to either $\zeta = \log \gamma_0/K$ or $\zeta = 120$ deg, whichever occurs first. The value 120 deg was found to cover the full time range of 0 to 10 shakes for all electrons. Next, the functions which appear as single integrals over $\zeta_1$ in Eqs. (4.24) and (4.25) are calculated in Part 3. The expressions $C_1, S_1$, and their powers which involve values of $\bar{C}, \bar{S}$ at two values of $\zeta$, are expanded using Eq. (4.12), the separate single-value-of-$\zeta$ functions integrated via Part 7, and then the various functions recombined to form the indicated integrals. This procedure, which permits direct step-by-step integration, is much faster than forming the two-values-of-$\zeta$ functions, which requires multiple cycling. (If there are $N$ values of $\zeta$, and the relative number of functions after expansion is $m$, the computing time is reduced by a factor $N/2m$. Since $m$ is about 2, for $N = 120$, the usual value, this part of the computation is speeded up by a factor of 30.) In Part 4, the double integrals of Eqs. (4.24) and (4.25) are calculated by the same procedure, which this time uses the already calculated single integrals, some of which appear as the inner integrands in the double integrals. At this point, the factors in brackets in Eqs. (4.24) and (4.25), exclusive of the time differentiations, have been calculated and stored for all $\zeta$.

In Part 5, the time $\tau$ is set equal to the first time step, and $i$ (see Eqs. (5.5) and (5.6)) set equal to 1. Equation (4.20) is solved by inverse interpolation to find the value of $\zeta'$ to be inserted into Eqs. (4.24) and (4.25). The value is used for direct interpolation to find the values of the terms in the bracketed expressions of Eqs. (4.24) and (4.25), which are then multiplied by the exterior factors $\beta U' D$ or
The factor $2\pi$ disappears—the $2$ from halving the $\varphi$ integration range and $\pi$ from its use in Eq. (5.5) or (5.6). The value of $i$ is iterated from 1 to 5 and the expression summed using Eq. (5.5) or (5.6), whichever is appropriate to the selected values of $\theta$ and $\tau$. The time is stepped up to either 11 shakes (the extra shake is provided for leeway in later differentiations), or to the last value such that there is contribution to the integral $(T_{m_1}(\theta) + T_{m_2}(\theta))$, whichever occurs first. The time derivatives are then calculated using five-point Lagrange differentiation.* The results are printed and stored. There are many delicate refinements in these procedures, mostly associated with the transition from Eq. (5.5) to Eq. (5.6) and the angle and time cutoffs.

The angle $\theta$ is then stepped to its next value and the calculation returned to the beginning of Part 2. After calculation through Part 5, the results are summed with the previous expressions to form the integration over $\theta$. The successive values are weighted by $\sin \theta \cos(\cos \theta)$ and the variable value of the $\theta$ increment. This process is continued to the last value of $\theta$ (84 deg), at which point we find the current and ionization rate as functions of time at the indicated altitude. The ionization rate is then inserted into Eq. (2.44) and integrated in Part 6 to obtain the total ionization. A special integration routine is required here because Eq. (2.44) cannot be integrated stepwise but must be cycled in $t$ to take account of the explicit appearance of $t$ in the integrand. Finally, the altitude $h$ is incremented and the entire procedure repeated until the highest altitude is reached. The mean running time is about 70 sec per altitude step on the IBM 370/158.

This completes the description of the computer program, which is included as an appendix to this report. We have executed the program for the representative values $B = 0.6$ gauss, $E_{\gamma} = 1.6$ MeV, and shall present and discuss the results.

*Reference 17, formula 25.3.6, p. 883.
VI. RESULTS AND CONCLUSIONS

We shall first discuss the convergence of the series of integrals in Eqs. (4.24) and (4.25). In Figs. 8a, 8b, and 8c, we have plotted the ionization per primary gamma ray versus time at altitudes of 20, 30, and 40 km. The quantity actually plotted is $10^{-3} N_s(\tau)/g(r)$, and the three figures have different vertical scales. The terms corresponding to time derivatives of various order in Eq. (4.25) are denoted by subscripts, thus $N_2$ is the contribution to the total ionization from the second derivative in Eq. (4.25). The curve $N$ is the sum of the three contributions. In Figs. 9a, 9b, and 9c, we have corresponding plots for $10^8 u c J_c(\tau)/2$, designated as $J$ on the figures.

First consider Fig. 8, the ionization. It is apparent that the term $N_0$ is by far the major contributor. Numerically, at 20 km, $N_0$ is at least 84.5 percent of $N$, the minimum occurring at $\tau = 1$. The second derivative term, $N_2$, makes a maximum contribution of 9 percent at $\tau = 1$, and $N_3$ provides a maximum of 7 percent at $\tau = 0.5$. At 30 km, $N_0$ is at least 95.9 percent, $N_2$ at most 3 percent, and $N_3$ at most 1.2 percent. The time integration of Eq. (2.44a), combined with the time differentiations of Eq. (4.25), makes the contributions of $N_2$ and $N_3$ approach zero at large $\tau$. Figure 8 displays very clearly the rapid convergence of the expansion used in Eqs. (3.19a–d) as far as ionization is concerned.

The convergence of the expansion for the current, presented in Fig. 9, is less good, but still satisfactory. At the lowest altitude, 20 km, the early peak of the current is raised and sharpened by the higher order terms, and the curve decreases more rapidly thereafter. The same effect, only less pronounced, occurs at the higher altitudes. At 20 km, the higher order terms can be as much as 40 percent of the total near the peak, but the qualitative behavior is not changed significantly from $J_0$ and there is no physical reason to suspect major contributions from still higher order terms. In any case, the convergence of the series is somewhat obscured by the coupling between the time derivatives and the delta function gamma source. If a smooth source were used, the series would converge much better.
Fig. 8b—Contributions to the ionization; \( \beta = 0.6 \), \( E_y = 1.6 \text{ MeV} \), \( H = 30 \text{ km} \).
Fig. 8c—Contributions to the ionization: $B = 0.6, E_H = 1.6$ MeV, $H = 40$ km.
Fig. 9b—Contributions to the current, $B = 0.6, E_x = 1.6$ MeV, $H = 30$ km
It is of interest to see what effect the Klein-Nishina distribution of initial velocities has had on the current. In Figs. 10a, 10b, and 10c, we have plotted the current versus initial angle at \( \tau = 1, 2, 5, \) and 10 shakes for altitudes of 20, 30, and 40 km, with corresponding curves in Figs. 11a, 11b, and 11c for ionization rate (not ionization). The vertical scales are arbitrary but compatible. These curves represent the contribution to current and ionization rate produced by the number of Compton electrons per unit angle at the indicated initial angle. All curves peak near 10 deg, corresponding to the peak of Fig. 6, but the varying behavior at larger angles is interesting. At the highest altitude, the current is most strongly peaked at 10 deg at early times, whereas the peak is considerably broader as time increases and some of the electrons which started at large angles are deflected forward. At lower altitudes, scattering produces a greater broadening of the distribution. Energy loss effects at the lowest altitudes cause the widest angle (lowest energy) electrons to disappear, as can be seen most clearly in Fig. 11a. The changing shape of the angular dependence as time increases makes it unlikely that it would be a reasonable approximation to replace the Klein-Nishina distribution with purely forward-directed electrons, as has been done by other codes.

The principal results of the computer program are presented in Figs. 12 and 13. Figure 12 depicts the current per incident gamma ray \( (10^8 \mu \text{cJ/2g is plotted}); \) Fig. 13 shows the ionization in thousands of secondary electrons per incident gamma \( (10^{-3} \text{N}_s/\text{g is plotted}). \) These are shown versus time at altitudes between 18 and 60 km. Below 18 km, the number of gammas is essentially negligible (see Fig. 4). Above 60 km, the ionization is negligible (Fig. 13), and the current is virtually independent of altitude (Fig. 12).

At high altitudes, the effects of scattering and energy loss on the current are slight. Accordingly, the current reduces to that produced by magnetic deflection acting on the Klein-Nishina initial distribution. From Fig. 12, this condition essentially holds at altitudes above 45 km for times up to 10 shakes. As the altitude decreases, the scattering begins to cause the electron distribution to decohere. The peak of the current does not rise as high, and it occurs somewhat earlier
Fig. 10a — Current versus initial angle;

$B = 0.6 \quad E_y = 1.6 \text{ MeV} \quad H = 20 \text{ km}$
Fig. 10b—Current versus initial angle,
B = 0.6  E_y = 1.6 MeV  H = 30 km
Fig. 10c — Current versus initial angle;
8 = 0.6  Eγ = 1.6 MeV  H = 40 km
Fig. 11a—Ionization rate versus initial angle;

\( B = 0.6 \quad E_\gamma = 1.6 \text{ MeV} \quad H = 20 \text{ km} \)
Fig. 11b—Ionization rate vs initial angle;
B = 0.6 Eγ = 1.6 MeV H = 30 km
Fig. 1c—Ionization rate vs initial angle;
B = 0.6  $E_y = 1.6$ MeV  $H = 40$ km
Fig. 12 — Current per primary gamma ray; $B = 0.6 \ E_\gamma = 1.6 \ MeV$
Fig. 13—Ionization per primary gamma ray; $B = 0.6 \ E_{\gamma} = 1.6 \text{ MeV}$
in time. From a peak of 1.267 at 2.2 shakes at 60 km, it drops to 1.255 at 2.0 shakes at 40 km, 1.06 at 1.6 shakes at 30 km, and 0.42 at 0.8 shake at 20 km. The peak not only becomes lower and earlier but narrower, with the decay at later times being faster. These effects are exactly what we would qualitatively expect.

A measure of the effectiveness of the scattering in reducing the current is shown in Fig. 14. Here, we have plotted the ratio of the current at a given altitude to the current at high altitude (60 km). The rapid and steady decrease of the relative current with time is evident. Thus, at 20 km, the current is down by more than a factor of four for times later than 2 shakes. This reduction of current at low altitudes is especially important because the fields radiated by these currents are least affected by attenuation. The initial rise of the relative current at high altitudes is real. It corresponds to the fact that those electrons which are scattered forward spend a greater retarded time radiating, with corresponding field enhancement, than those which are scattered away and have their effective radiation reduced. The effect is soon washed out by the general decoherence.

The ionization versus time at various altitudes is displayed in Fig. 13. The horizontal line at 23,600 represents the asymptotic limit of all the curves, corresponding to complete conversion of all the energy of the Compton electrons to secondary ionization. (The mean energy of these electrons produced by the first scattering of the 1.6 MeV gamma rays from the bomb is 800 KeV.) Although it is obvious that the electron energy is lost most rapidly at low altitudes, because the energy loss is proportional to density, the curves do not display any striking behavior, except to note the large range over which the ionization density varies.

If we refer to Eq. (2.38), we see that the ionization density affects the fields by its appearance in conductivity, where it is divided by the density. Therefore, we have plotted $10^{-4} N_s/\rho g$ in Fig. 15. We call this quantity the equivalent conductivity, since there is no convenient way to characterize the dependence of the collision frequency on field strength, which is required to completely specify the conductivity. Figure 15 is quite remarkable. At low altitudes, the change in
Fig. 14—Relative current; $B = 0.6$, $E_\gamma = 1.6$ MeV
Fig. 15—Equivalent conductivity;
B = 0.6   $E_\gamma = 1.6 \text{ MeV}$
the Compton electron paths due to scattering and energy loss causes
the secondary electrons to be produced less rapidly in retarded time
at higher densities. Thus, the equivalent conductivity increases with
altitude from 20 km up to 35 km. Above 35 km, the ionization lag
(Eq. (2.44)) becomes the dominant effect, and the equivalent conduc-
tivity decreases as the altitude increases further. The combination
of these effects collapses the curves of Fig. 13 into a very limited
region. Thus, if the curves of Fig. 15 were replaced by a mean curve,
which is essentially the 25 km curve, the departure from this mean of
all the curves of Fig. 15 for all altitudes and times is less than 20
percent. It accordingly would be reasonable to replace the equivalent
capacity per electron, regarded as a function of altitude, by an
average value which is, of course, a function of time. This approxi-
mation clearly is not permissible for the current.

Figures 12 and 13 represent the current and ionization per gamma
ray. The total current or ionization at a point is found by multiplying
these curves by the function g(r) (Eq. (2.32) in general, Fig. 4 for
these values of B and E). We shall not plot these, since the resultant
curves display multiple crossings and are very difficult to follow. We
note that the curves for total current and total conductivity, regarded
as a function of altitude, peak at about 32 to 34 km, corresponding to
the peak in g(r).

From the extensive analysis of this report, we draw the following
conclusions:

1. The development of the electron velocity distribution with time
has been determined in a consistent and satisfactory manner
that includes the phenomena of magnetic deflection, energy
loss, atmospheric scattering, and the initial Klein-Nishina
velocity distribution.

2. The current and ionization have been calculated and both dis-
play the proper qualitative and quantitative physical behavior.
For a delta function gamma-ray source, the peak of the current
is reduced and made both earlier and narrower by the effects
of scattering, whereas the ionization is determined primarily
by the proportionality of energy loss to density. The conductivity is insensitive to altitude.

3. A computer program prepared to provide numerical values for the current and ionization is presented in the appendix to this report. The program operates rapidly compared with Monte Carlo simulations, thus demonstrating the virtue of the analytic approach. If the program is augmented by an integration, arbitrary gamma-source time dependence can be included. The current and ionization as calculated here can be used as inputs for calculating fields.
APPENDIX: THE COMPUTER PROGRAM

IMPLICIT REAL*8 (A-H,K,O-Z)
DIMENSION E(24), SEC(111), PP(60)
COMMON F(50, 121), ENN(111), AJJ(111), PP(5, 121)
       F(500), CC(121), SS(121), DT(5, 121), Q(50)
       ENM(3, 111), AJMK(3, 111), CYN(5, 121), EXK(5, 121)
       ENM(3), AJMK(3), AJJ(5, 121), AJD(5, 121), AJL(5, 121)

6,  E(3), EK(3), AJK(3), EMA(3, 9), AJA(3, 9)
       DCA(9), QI(9), JC(9), JS(9), OS(9), BU(121), DEO(5), DEE(5)
6,  CK(121), FP(14, 121), SP(5), DE(5), IN(5), DE(5, 4), AK(5, 4)
6,  AK(5, 4), AP(5, 4), F(5, 4), S(5, 4), SM(4), SX(4), EX(4)
6,  ENG(111), AFP(111)
COMMON H(9), DELH, YY, YDEL, YV, H, 4, JJ, KK, BP, EMV, X, XSN, RHH, B, C, YO
6,  TTP, TTO, TTT, TVV, TTV

6,  MLIN, MLIN, ILI, LV, YML, IT, IY, L5, IY, MVX
G(ZDM) = GG * JEXP(-FK * ZDM)
U(ZDM) = (j(ZDM) - 100) * (j+100) / (2 * ZDM + 100)

A(ZDM) = 2.164D0 * K * (ZDM) / (12 * ZDM) ** 2 - 100
O(ZDM) = 5.6569D0 * DE/G0 * ZDM / (12 * ZDM) ** 2 - 100
U(ZDM) = (j(ZDM) + 100) / BB

ELL(MDM) = 3.15494 * 370209.3 ** (1D0 + 600 ** 1003 * 3.15494 * RHH ** 100)
DATA P(3), 105010, 725040, 5, 560570, 0.54931E0, 0.74971D0, 0.02480
6,  0.0159D3, 0.100D0, 0.0512D0 - 2, 0.024D2 - 2, 0.046H2 - 2, 0.323D2 - 2, 0.7496D - 2
6,  0.187D0 - 2, 0.149D0 - 2, 0.189D0 - 2, 0.614D3 - 3, 0.648D3 - 3, 0.566D3 - 3, 0.193D3 - 3
6,  0.312D3 - 3, 0.240D3 - 3, 0.139D3 - 3, 0.354D3 - 3
DATA E(0), 25, 9510506516295150, 0.567752552292470

DATA P(3), 40D0, 10DC, 1.74354504D0
DCON = 1.74354504D0

C **** B = MAGNETIC FIELD, EE = GAMMA ENERGY,
C **** B = TIME-STEP, C = ANGLE-STEP
READ 1, BB, EE, B, C, YM1

YML = YML
1 FORMAT(6D12.4)
MLIN=11D0/B+1.01D0
*MLIN=MLIN**2
K=EE/0.511D0
K=K+100
K=K+200
K=K+300
K=K+1
K=K+2
K=K+3
S=(2D0*K*(K0+K*(9D0+K)))/(K*(10D0+2D0*K))**2
1 -(1D0*K*(1D0+5D0*K)) *(6D0+2D0*K)/K**3

DELH=2DC
READ 1, HHG, YY, HHH, YY
HH=HH
HH=HH
IH=IH
IP(INWH,WE.0) IH=IH-1
10 IH=IH+1
T=IH+1
K=6.959D0 * HH/EE
PRINT 11, HH, BB, EE, B, C
11 FORMAT(1H1,' ALTIMET = ',5.4,1H,,' KM,, MAGNETIC FIELD = ',F6.2,1H,
& GAMMA ENERGY = ',F6.2,' TIME-STEP = ',F6.2,1H,
& SEC., ANGLE-STEP = ',F6.2,1H)
10 512 = 1,MLIN
AJJ(1)=ZERO
ENM(1)=ZERO
ENP(1)=ZERO
AJM(1)=ZERO

PRECEDING PAGE BLANK—NOT FILMED
CONTINUE

\[
T_TV = 15D0
\]
\[
M_VX = 151
\]
\[
Y_YDEL = 3D0
\]
\[
Y_Y = Y_YDEL
\]
\[
20 \text{ CONTINUE}
\]

\textbf{C****** PART 2}

\textbf{EMO=ZERO}

\[
Z_Z = Y_Y \times Q
\]
\[
X_SN = D_SIN (Z_Z)
\]
\[
X = D_COS (Z_Z)
\]
\[
X_S = X \times X
\]
\[
X_SM = 1D0 - X_S
\]
\[
X_SM S = 1D0 - 3D0 \times X_S
\]
\[
X_SM S S = 1D0 - 5D0 \times X_S
\]
\[
Q I (1) = X_S \times Q\]
\[
Q I (2) = X_SN \times Q\]
\[
Q I (3) = Z_EPO
\]
\[
Q I (4) = -Q I (2)
\]
\[
Q I (5) = -Q I (1)
\]
\[
X_K X = (K_1 X / K_1) \times 2
\]
\[
X_S K 2 = K_1 X \times K_2
\]
\[
G_S = (1D0 + K_1 X) / (1D0 - K_1 X)
\]
\[
S_S (1) = Z_EPO
\]
\[
C_C (1) = Z_EPO
\]
\[
B_U (1) = D_SQRT (1D0 - 1D0 / (G_S \times G_S))
\]
\[
Q K (1) = Z_EPO
\]
\[
D_0 = 21 I = 1,5
\]
\[
T_T (I, 1) = Z_EPO
\]
\[
D_D (I, 1) = 1D0 - X \times B_U (1)
\]
\[
A_J 1 (I, 1) = Z_EPO
\]
\[
A_J 2 (I, 1) = Z_EPO
\]
\[
A_J 3 (I, 1) = Z_EPO
\]
\[
E_N 2 (I, 1) = Z_EPO
\]
\[
E_N 3 (I, 1) = Z_EPO
\]
\[
Q I I = Q I (1)
\]
\[
Q I (1) = Q I I / X_SN
\]
\[
W_I (1) = 2D0 / (1D0 + Q I 1)
\]
\[
Q I 2 = Q I I \times Q I 1
\]
\[
Q S (1) = 1D0 - 5D0 \times Q S 1
\]
\[
Q S (1) = 1D0 - 3D0 \times Q S 1
\]
\[
Q C (1) = Q I I \times (2D0 + Q R 1)
\]
\[
E M M 2 = (K_1 S - X_S K 2) / (K_1 S - X_S K 2)
\]
\[
E M M 1 = E M M 2 \times 1C0 / E M M 2 - (1D0 - X_S) \times (2D0 \times K_1 X / (K_1 S - X_S K 2)) \times 2
\]
\[
E M Y = 2D0 \times K_1 S \times X_S \times D_S QRT (1D0 - X_S) \times E M M 1 / S / (K_1 S - X_S K 2) \times 2
\]
\[
F_N 1 T 12, I_Y, E_M Y
\]
\[
12 \text{ FORMAT ('/ ' INITIAL ANGLE = ', F4.0, ' DEG., REL INITIAL NUMBER ='}
\]
\[
1, F9.5)
\]
\[
V_V = D_SQRT (G_G) / K_K / 0
\]
\[
V_V = V_V - C
\]
\[
I_F (V_V, GT, 119D0) V_V = 119D0
\]
\[
I_Y M_X = V_V / C + 1D0
\]
\[
I_Y M_X = I_Y M_X - 1
\]
\[
I_F (I_Y M_X, LE, 5) G_O T O 96
\]
\[
Y_O = Z_EPO
\]
\[
H = C
\]
\[
W = V_V
\]
\[
I_Y M_L = I_Y M_X
\]
\[
L_V = 2
\]
C
ASSIGN 602 TO LGO
CALL PAPT7
502 CONTINUE
Y=H
IY=2
25 CONTINUE
CC(IY)=EI(I,IY)
SS(IY)=EI(2,IY)
Z=Y*Q
GO=6(Z)
OK(IY)=.96207D0*(53-30Z)/FHI+X*CC(IY)
BU(IY)=SQRT(1.0D0-GOZ**2)*UU(Z)
Y=Y+H
IY=IY+1
IF(IY.LE.YV) GO TO 25
IYV=IY
IYV=IYV-1
IY2=IY-2
IY3=IY-3
IY4=IY-4
IY5=IY-5
OK(IYV)=OK(IY5)-500*(OK(IY4)-OK(IYV))
BU(IYV)=BU(IY5)-500*(BU(IY4)-BU(IYV))
CC(IYV)=CC(IY5)-500*(CC(IY4)-CC(IYV))
CC(IYV)=CC(IY5)-500*(CC(IY4)-CC(IYV))
TTP=OK(IYV)
TTO=OK*SS(IYV)
TTL=TTP-170
TTU=TTP+TTO

C
ASSIGN 640 TO LGO
CALL PAPT7
540 CONTINUE
C********* PAPT 3
DO 301 J=1,121
DO 301 J=1,5
301 PP(I,J)=Z*FO
DO 302 I=1,14
302 PF(I,1)=Z*FO
Y=H
IY=2
IY1=120
TH=OK(IY)+X*SM*SS(IY)
34 CONTINUE
TJ=TH
Z=Y*Q
SINZ=DSIN(Z)
COSZ=DCOS(Z)
I=1
36 CONTINUE
CC(I)=CC(I)
SS(I)=SS(I)
IIY=EI(I,IY)
EI1IY=EI(I*1,IY)
EI2IY=EI(I*2,IY)
EI3IY=EI(I*3,IY)
EI4IY=EI(I*4,IY)
F(I) = EIIY * CCI - EIIY
F(I+1) = EIIY * SSI - EIIY
F(I+2) = SSI * FI - CCI * EIIY + EIIY
F(I+3) = CCI * (FI - EIIY) + EIIY
F(I+4) = SSI * (FI - EIIY) + EIIY
F(I+5) = SSI * (FI - EIIY) + EIIY
F(I+6) = CCI * (FI - EIIY) + EIIY
F(I+7) = SSI * (FI - EIIY) + EIIY
F(I+8) = SSI * (FI - EIIY) + EIIY
F(I+9) = ZERO
I = I+10
IF (I.LT.42) GO TO 36
PP 1 = F(I)
PP(1,1) = PP1
PP2 = F(2)
PP(2,1) = PP2
PP3 = F(3)
PP4 = F(4)
PP(4,1) = PP4
PP5 = F(5)
PP(5,1) = PP5
DO 31 I = 33, 49
31 IF (I+20) = F(I) - F(I-20)
F*(1,1) = (2DO * (F(14) + F(15)) + XSN*PP4) / 3DO
F*(2,1) = 2DO * PP3
PP(3,1) = PP3
PP4 = F(4)
PP(4,1) = PP4
PP5 = F(5)
PP(5,1) = PP5
DO 31 I = 31, 49
31 print 853, I, IT, U, TTL, TTU, TW, TQ
853 FORMAT(1X, IYL = 'IY', IYU = 'IY', TTL = 'TTL', TTU = 'TTU')
&F9.4,
TTP =',F9.4,', TTQ =',F9.4)
LV=42
LGO=2
C
ASSIGN 632 TO LGO
CALL PART7
532 CONTINUE
C******** PART 4
DO 401 I=1,200
401 P(I)=ZF*0
Y=H
IY=2
70 CONTINUE
Z=Y*0
SIMZ=DSIN(Z)
COSZ=DCOS(Z)
I=1
43 CONTINUE
SSI=SS(IY)
CCI=CC(IY)
E11Y=E1(I,1Y)
E11Y=E1(I+1,1Y)
E12Y=E1(I+2,1Y)
E13Y=C1*E11Y
E14Y=SSI*E11Y
E15Y=E1(I+6,1Y)
E16Y=CCI*E11Y
E17Y=E1(I+7,1Y)
E18Y=E1(I+8,1Y)
F(I)=E13Y-E11Y
F(I+1)=E14Y-E12Y
F(I+2)=CCI*F(I+1)-SSI*F11Y+F1(I+3,1Y)
F(I+3)=CCI*F(I+1)-E11Y+EI(I+4,1Y)
F(I+4)=SSI*F(I+1)-E12Y+EI(I+5,1Y)
F(I+5)=E15Y-E13Y
F(I+6)=E16Y-E14Y
F(I+7)=CCI*F(I+6)-SSI*E11Y+EI(I+8,1Y)
F(I+8)=CCI*F(I+5)-E11Y+EI(I+8,1Y)
F(I+9)=SSI*F(I+4)-E12Y+EI(I+9,1Y)
FIC1Y=EI(I+11,1Y)
F(I+10)=CCI*FIC1Y-EL11Y+EI(I+12,1Y)
F(I+11)=SSI*FIC1Y+EI(I+13,1Y)
E12Y=EI(I+14,1Y)
F(I+12)=CCI*E12Y-EL11Y+EI(I+15,1Y)
F(I+13)=SSI*E12Y+EI(I+16,1Y)
E13Y=EI(I+17,1Y)
F(I+14)=CCI*E13Y-EL12Y+EI(I+18,1Y)
F(I+15)=SSI*F11Y+EI(I+19,1Y)
I=1+21
IF(I,LT,23) GO TO 43
FF(4,IY)=(-X*(410*F(4)+200*F(13)+300*(F(5)+F(12)))*F(8)-F(15)
6
+X55*(260*F(25)+F(34))*FF(4,IY))*.8D0
F(F,5,IY)=(300*F(9)+F(11))*F(I+3)+400*F(I+10)-F(14)+200*F(16)
6
+F54*400*F(24)+200*F(35))*FF(5,IY))*.8D0
F(F,6,IY)=(-X*(260*F(26)+F(33)+400*F(29))*FF(6,IY))*.8D0
F(F,7,IY)=(100*F(31)+F(37)+FF(7,IY))*.8D0
F(F,11,IY)=(-X*(400*F(1)+300*F(7)+200*F55*F(22))
6
+COS*300*F(2)-200*F(6)))*.8D0+FF(11,IY)
F(F,12,IY)=(SINZ*300*F(6)-200*F(2)+F(3)+F54*F(23)+F(27))
-86-

6  \(-\cos Z \times (4D0 \times P(7) + 3D0 \times F(1) + 2D0 \times X)\) \(\times 0.8D0 \times FF(12, IY)

FF(13, IY) = (2D0 \times (-\sin Z \times F(28) + \cos Z \times (F(23) + F(27))) \times 0.8D0 \times FF(13, IY)

FF(14, IY) = 2D0 \times \cos Z \times F(28) \times 0.8D0 \times FF(14, IY)

Y = Y + 1

IY = IY + 1

IF (Y .LE. YY) GO TO 70

DO 411 I = 1, 14

411 FF(I, IY) = FF(I, IY5) - 5D0 \times (FF(I, IY4) - FF(I, IYV))

& + 1DD \times (FF(I, IY3) - FF(I, IY2))

C******* PART 4A

IY = 2

Y = H

Z = Y * Q

SI_NZ = DSIN(Z)

COS Z = DCOS(Z)

QK = QK(IY)

QL = SS(IY)

DO 49 I = 1, 5

QII = OI(I)

QPI = QR(I)

QSI = QS(I)

QC1 = QC(I)

TT(I, IY) = QKIV + OL * QII

AJ1(I, IY) = (X * SI_NZ + QII * COS Z) \times BUIY

DD(I, IY) = T0 - RUIY \times (X \times COS Z - QII * SI_NZ)

SR2(I, IY) = FF(I, IY) + QII \times FF(2, IY) + QSI \times FF(3, IY)

EN3(I, IY) = FF(4, IY) + QII \times FF(5, IY) + QRI \times FF(6, IY) + QC1 \times FF(7, IY)

AJ2(I, IY) = FF(8, IY) \times QII \times FF(9, IY) + QSI \times FF(10, IY) \times BUIY

49 AJ3(I, IY) = (FF(11, IY) + QII \times FF(12, IY) + QRI \times FF(13, IY) + QC1 \times FF(14, IY))

& + BUIY

Y = Y + H

IY = IY + 1

IF (Y .LE. YY) GO TO 26

CALL PART5

IF (YY .GE. 29, 9D0) YYDEL = 6D0

YY = YY + YYDEL

IF (YY .LT. 89, 9D0) GO TO 2C

PRINT 912, YY

912 FORMAT(/' INITIAL ANGLE = ', F4.1, ' DEG.')

96 CONTINUE

M = 1

T = ZERO

906 CONTINUE

TVI = .5D0 \times (TTV-T) / TTV

ENN(M) = ENN(M) + TVU \times ENP(M) \times FR3

AJJ(M) = AJJ(M) + TVU \times AJP(M)

IF (2 \times ((M-1)/2) \times EQ. M = 1) .AND. T .LE. 2.01D0 .OR.

& 6 \times ((M-1)/5) \times EQ. M = 7) .AND. T .GT. 2.01D0)

EPRINT 855, T, AJJ(M), ENN(M)

855 FORMAT(F8.2, 2(37X, F10.5))

T = MB

M = M + 1

IF (M .LE. MX) GO TO 906

574 FORMAT(1H )

C******* PART 6
D=DCON* rl~H*E
PRINT 661
661 FORMAT(/'TIME',8X,'PREDN',8X,'SECDEL',8X,'JPHIDEL')
M=1
93 CONTINUE
M=M+1
MAX=(M-1)/2+1
DO 95 I=1,MAX
ELL=ELL(I-1)
IF(ELL.GT.100) ELL=100
ELL=ELL(M-I)
IF(ELL.GT.100) ELL=100
IF(I).EQ.1) ELL*FNN(I)+ELL*FNN(M-I+1)
95 CONTINUE
IF(M.GT.7) GO TO 106
IF(M.GT.6) GO TO 105
IF(M.GT.5) GO TO 104
IF(M.GT.4) GO TO 103
IF(M.GT.3) GO TO 102
IF(M.GT.2) GO TO 101
FNNJM=1200*EP(1)
GO TO 120
101 FNNJM=800*EP(1)+1600*EP(2)
GO TO 120
102 FNNJM=900*EP(1)+2700*EP(2)
GO TO 120
103 FNNJM=800*EP(1)+4000*EP(2)+EP(3)
GO TO 120
104 FNNJM=900*EP(1)+3100*EP(2)+2100*EP(3)
GO TO 120
GO TO 120
IF(M.GT.9) GO TO 120
IF(110 I=M,MAX)
FNNJM=FNNJM+2400*EP(I)
110 CONTINUE
IF(2*(M/2).NE.M) FNNJM=KNNJM-1200*EP(MAX)
120 CONTINUE
SPC(M)=D*ENNMJ
PRINT 664,T,EN(N),SEC(M),AJJ(M)
664 Format(F12.2,F15.5,F15.2,F15.5)
IF(M.LT.MIN(M)) GO TO 93
IF(HH.LE.6000) GO TO 10
CALL EXIT
END

SUBROUTINE FACTS
C*       PART 5 REVISED
C*       IMPLICIT FZAL* (A-H,K,O-Z)
C*
C* COMMON EL(50,121),EN(M),AJJ(M),PP(5,121)
C*
C* IF(200),CC(121),SE(121),IT(5,121),PD(5,121),QQ(50)
C* IF(5,ENNMJ,AJJ(M),AJJ(3),AJJ(3),AJJ(3),AJJ(3),AJJ(3),AJJ(3)
C* IF(5,ENNMJ,PP(5,121),PP(5,121),PP(5,121),PP(5,121),PP(5,121))
TTF = (TTA - 2700 * (TIB - TIC) - TTD) / 2400
TTG = (TTA - TIE - TIC + TTD) / 400
TTH = (TIE - TTE) / TTF
TTI = (TTA + TDD * (TIR - TIC) + TTD) / 600
TTJ = TTT / TTF
TTK = 2000 * TTT * TJI / TTF
AP = .500 + TTH * (1000 - TTT + (TTJ - TTP + TTH))
APM2 = AP - 200
APM = AP * AP - 100
AL1 = AP * (AP - 100) * APM2 / 600
AL2 = 0.500 * APM * APM2
AL3 = 0.500 + AP * (AP + 100) * APM2
AL4 = AP * APM * 500
DDA(I) = AL1 * D(I, IY - 1) + AL2 * D(I, IY) + AL3 * D(I, IY + 1) + AL4 * D(I, IY + 2)
ENK(I) = ENK(I) + SIU * ENA(J, I) / DDA1
AJK(J) = AJK(J) + SIU + AJA(J, I) / DDA1
J = J + 1
IF (J .LE. 1) GO TO 1
I = I + 1
IF (T .LE. 5) GO TO 900
DO 97 J = 1, 3
AJJMK(K, M) = ENM(1) + ENK(I) - ENM(1)
ENNM(1) = FACOM / (1D0 * X * BD(1))
GO TO 85
90 CONTINUE
T = TFO
M = 1
ENNM(1) = ENNM(1) / (1D0 * X * BD(1))
GO TO 85
91 M = 2
T = TFO
ENMM(1) = ENNM(1)
AJJMK(K, 1) = AJJMK(K, 1)
ENMM(2) = (1D0 * ENNM(2, 1) - 2D00 * ENNM(2, 2) + 0D00 * ENNM(2, 3)
6 * D00 * ENNM(2, 4) - ENNM(2, 5)) / BDN3
ENMM(3) = (3D00 * ENNM(3, 1) - 16D0 * ENMM(3, 2) + 12D00 * ENNM(3, 3)
6 * D00 * ENNM(3, 4) - ENNM(3, 5)) / BDN3
AJJMK(2) = (3D00 * AJJMK(2, 1) + 10D00 * AJJMK(2, 2) - 18D00 * AJJMK(2, 3)
6 + 6D00 * AJJMK(2, 4) - AJJMK(2, 5)) / BDN3
AJJMK(3) = (11D00 * AJJMK(3, 1) - 20D00 * AJJMK(3, 1) - 20D00 * AJJMK(3, 1) - 20D00 * AJJMK(3, 1) / BDN3
6 + 4D00 * AJJMK(3, 4) - AJJMK(3, 5)) / BDN3
GO TO 85
92 T = TFO
M = 1
ENMM(1) = ENNM(1)
AJJMK(K, 1) = AJJMK(K, 1)
IF (T .GT. TLE) GO TO 901
ENNM(2) = (-ENNMK(2, M-2) + 16DO* (ENNMK(2, M-1) + ENNMK(2, M+1))
6 -30DO*ENNM(2, M) - ENNMK(2, M+2))/BDN2
ENNM(3) = (ENNMK(3, M-2) - 2DO* (ENNMK(3, M-1) - ENNMK(3, M+1))
6 -ENNMK(3, M+2))/BDN3
AJJM(2) = (-AJJM(K(2, M-2) + RD0* (AJJM(2, M-1) - AJJM(2, M+1))
6 +AJJM(2, M+2))/BDN1
AJJM(3) = (-AJJM(3, M-2) + 16DO* (AJJM(3, M-1) + AJJM(3, M+1))
6 -30DO*AJJM(3, M) - AJJM(3, M+2))/BDN2
GO TO 85
901 IF(T.GT.TTLD ) GO TO 902
ENNM(2) = (-ENNMK(2, M-3) + 4DO*ENNMK(2, M-2) + 6DO*ENNMK(2, M-1)
6 -20DO*ENNMK(2, M) + 11DO*ENNMK(2, M+1))/BDN2
ENNM(3) = (ENNMK(3, M-3) - 6DO*ENNMK(3, M-2) + 12DO*ENNMK(3, M-1)
6 -10DO*ENNMK(3, M) + 3DO*ENNMK(3, M+1))/BDN3
AJJM(2) = (-AJJM(2, M-3) + 6DO*AJJM(2, M-2) - 14DO*AJJM(2, M-1)
6 +10DO*AJJM(2, M) + 3DO*AJJM(2, M+1))/BDN1
AJJM(3) = (-AJJM(3, M-3) + 4DO*AJJM(3, M-2) + 6DO*AJJM(3, M-1)
6 -20DO*AJJM(3, M) + 11DO*AJJM(3, M+1))/BDN2
GO TO 85
902 CONTINUE
ENNM(2) = (11DO*ENNM(2, M-4) - 56DO*ENNMK(2, M-3) + 114DO*ENNMK(2, M-2)
6 -104DO*ENNMK(2, M-1) + 35DO*ENNMK(2, M))/BDN2
ENNM(3) = (3DO*ENNMK(3, M-4) - 14DO*ENNMK(3, M-3) + 24DO*ENNMK(3, M-2)
6 -19DO*ENNMK(3, M-1) + 5DO*ENNMK(3, M))/BDN3
AJJM(2) = (3DO*AJJM(2, M-4) - 16DO*AJJM(2, M-3) + 36DO*AJJM(2, M-2)
6 -48DO*AJJM(2, M) + 15DO*AJJM(2, M+1))/BDN1
AJJM(3) = (11DO*AJJM(3, M-4) - 56DO*AJJM(3, M-3) + 114DO*AJJM(3, M-2)
6 -104DO*AJJM(3, M) + 35DO*AJJM(3, M))/BDN2
GO TO 85
801 CONTINUE
T = T + 1
IF(T.GT.TTUB ) GO TO 903
DO 888 J = 1, 5
888 CONTINUE
DO 889 J = 1, 3
889 CONTINUE
X = (T-TTP)/TTQ
TLU = (T-TTL)/(TTU-7)
QII = (1+1)*J
SLU = 2DO/DSQRT(1DO*WI(J)*TLU)
QII = XSN*(X*(1DO-XM)*GII+1DO)/2DO
QII = QII*GII
QII = 1DO-5DO*QII
QII = 1DO-3DO*QII
QII = QII*(2DO*ORI)
IM = 3
IF(T.GT.TTUB2 ) IM = 4
IF(T.GT.TTUE ) IM = 5
IL = 1
802 IY = IY+1
TL = T+(IL-IM)*B
803 IY = IY+1
T = QK(IY)+SS(IY)*QII
IF(TL.GE.IYU ) IY = IY+1
IY = IY+1
T = QK(IY)+SS(IY)*QII
IWA = IY+2
IWB = IY - 1
IWC = IY * 1
TWA = QK (IWA) + SS (IWA) * U II
TWH = QK (IWB) + SS (IWB) * 0 I I
TWC = QK (IWC) + SS (IWC) * U II
TWW = (TWA * 90) * (TWH * TWI) - TWC / 1600
TWF = (TWA + 2700) * (TWH * TWI) - TWC / 2400
TWI = (TWA + TWH + TWC) / 400
TWW = (TWA + TWH + TWC) / 1600

TWA = -AW * (IWA) + SS (IWA) * U II
TWH = -AW * (IWB) + SS (IWB) * 0 I I
TWC = -AW * (IWC) + SS (IWC) * U II
TWW = (TWA * 90) * (TWH * TWI) - TWC / 1600
TWF = (TWA + 2700) * (TWH * TWI) - TWC / 2400
TWI = (TWA + TWH + TWC) / 400
TWW = (TWA + TWH + TWC) / 1600

AL1 = -AP * (AP - 100) * AP * 2 / 600
AL2 = 0.5D0 * AL * AM2
AL3 = -0.5D0 * AP * (AP + 100) * AP * 2
AL4 = AP * AM / 100

DO 804 J = 1, 4

JY = IY + J
Z = (JY - 1) * 110
SINZ = SIN (Z)
COSZ = COS (Z)

BUJY = BU (JY)

DE (IL, J) = DC - BUJY * (X * COSZ + QII * SINZ)
AK1 (IL, J) = X * SINZ + QII * COSZ + BUJY
AK2 (IL, J) = EE (J, JY) + QII * FF (10, JY) + QII * FF (10, JY) + QII * FF (10, JY)
AK3 (IL, J) = EE (J, JY) + QII * FF (10, JY) + QII * FF (10, JY) + QII * FF (10, JY)

E = BUJY

EM2 (IL, J) = EE (1, JY) + QII * FF (2, JY) + QII * FF (3, JY)

DDM (JL) = AL1 * DL (IL, 1) + AL2 * DE (IL, 2) + AL3 * DE (IL, 3) + AL4 * DE (IL, 4)

DDA1 = DDA (IL)

ENA (1, I1) = ID2 / DDA1
ENM (2, I2) = (AL1 * EN (IL, 1) + AL2 * EN (IL, 2) + AL3 * EN (IL, 3) + AL4 * EN (IL, 4)

E = NDDA1

ENA (1, I1) = (IL1 * EN (IL, 1) + AL2 * EN (IL, 2) + AL3 * EN (IL, 3) + AL4 * EN (IL, 4)

E = NDDA1

AJA (1, I1) = (IL1 * AK1 (IL, 1) + AL2 * AK1 (IL, 2) + AL3 * AK1 (IL, 3) + AL4 * AK1 (IL, 4)

E = NDDA1

AJA (2, IL) = (IL1 * AK1 (IL, 1) + AL2 * AK1 (IL, 2) + AL3 * AK1 (IL, 3) + AL4 * AK1 (IL, 4)

E = NDDA1

AJA (3, IL) = (IL1 * AK1 (IL, 1) + AL2 * AK1 (IL, 2) + AL3 * AK1 (IL, 3) + AL4 * AK1 (IL, 4)

E = NDDA1

E = NDDA1

IL = I1 + 1
IF (IL, IL, 5) GO TO 802

GO TO 805

ENK (J) = ESK (J) + SLU * NAA (J, J)

IF (IN, IF, 4) GO TO 906

DEO (1) = -ELA (2, 1) + 15D0 * EN (2, 2) + EN (2, 4) - 10D0 * EN (2, 3)
6 - ENA (2, 5) / DDM2

DEO (2) = -ELA (3, 1) + 2D0 * EN (3, 2) + EN (3, 4) + EN (3, 5) / DDM3
DEO (3) = -ELA (3, 1) + 10D0 * EN (3, 2) - ENA (3, 3) / DDM1

DEO (4) = -ELA (3, 1) + 10D0 * EN (3, 2) - ENA (3, 3) - 30D0 * ENA (3, 3)
E = -ELA (3, 5) / DDM2

GO TO 808

IF (IN, IF, 5) GO TO 907

DEO (1) = -ELA (2, 1) + 4D0 * ENA (2, 2) + 6D0 * ENA (2, 1) - 20D0 * ENA (2, 4)
6 + 11D0 * ENA (2, 5) / DDM2
DEQ(2) = (ENA(3, 1) - 6 DO*ENA(3, 2) + 12 DO*ENA(3, 3) - 10 DO*ENA(3, 4)
         + 3 DO*ENA(3, 5))/BDN3
DEQ(3) = (-AJA(2, 1) + 6 DO*AJA(2, 2) - 18 DO*AJA(2, 3) + 10 DO*AJA(2, 4)
         + 6 DO*AJA(2, 5))/BDN1
DEQ(4) = (-AJA(3, 1) + 6 DO*AJA(3, 2) + 6 DO*AJA(3, 3) - 2 DO*AJA(3, 4)
         + 11 DO*AJA(3, 5))/BDN2
GO TO 808
807 CONTINUE
DEQ(1) = (11 DO*ENA(2, 1) - 5 DO*ENA(2, 2) + 114 DO*ENA(2, 3) - 104 DO*ENA(2, 4)
         + 35 DO*ENA(2, 5))/BDN2
DEQ(2) = (-3 DO*ENA(3, 1) - 14 DO*ENA(3, 2) + 24 DO*ENA(3, 3) - 18 DO*ENA(3, 4)
         + 5 DO*ENA(3, 5))/BDN1
DEQ(3) = (-3 DO*AJA(2, 1) - 16 DO*AJA(2, 2) + 36 DO*AJA(2, 3) - 48 DO*AJA(2, 4)
         + 25 DO*AJA(2, 5))/BDN1
DEQ(4) = (11 DO*AJA(3, 1) - 5 DO*AJA(3, 2) + 114 DO*AJA(3, 3) - 104 DO*AJA(3, 4)
         + 35 DO*AJA(3, 5))/BDN2
808 CONTINUE
DO 809 J = 1, 4
809 DEF(J) = DEF(J) + SLU*DEQ(J)
I = I + 1
IF(I.LE.5) GO TO 812
DO 810 J = 1, 3
ENNMK(J, M) = FACO*ENNE(J)
810 AJMK(J, M) = DEF(J) * FACQ
ENNM(1) = ENNM(1, M)
ENNM(2) = DEF(1) * FACQ
Ennm(3) = DEF(2) * FACQ
AJMK(1) = AJMK(1, M)
AJMK(2) = -0.90467 DO*DEF(3) * FACQ
AJMK(3) = -0.90467 DO*DEF(4) * FACQ
85 CONTINUE
SUM = ENNM(1) + ENNM(2) + ENNM(3)
ENN(M) = ENNM(M) + 5 DO* (SUM + ENF(M))
ENF(M) = SUM
IF(Y.EQ.3010) ENFM(M) = SUM + SUMN
SUM = AJMK(1) + AJMK(2) + AJMK(3)
AJM(M) = AJM(M) + 5 DO* (SUM + ENF(M))
AJP(M) = SUM
IF(Y.EQ.3010) AJP(M) = SUM + SUMN
IF((2*((M-1)/2).EQ.1.AND.T.LE.2,0100)) OR.
6 (5*((M-1)/5).EQ.1.AND.T.GT.2,0100))
5PRINT 552, T, (AJMK(1), I = 1, 3), SUMJ, AJM(M)
6 , (ENNM(I), I = 1, 3), SUMN, ENN(M)
552 FFORMAT(FA.2,2(1X,4F9.5, F1.5))
IF(M.EQ.1) GO TO 91
IF(M.EQ.MLIMK) GO TO 801
IF(M.LT.MLIMK) GO TO 92
903 CONTINUE
IF(MUX.EQ.MLIMK) GO TO 905
904 CONTINUE
TV = 5 DO* (TIV - T)/(TIV - TV)
ENN(M) = ENNM(M) + TV*ENF(M) * FRC
AJM(M) = AJMK(M) + TV*AJP(M)
IF((2*((M-1)/2).EQ.1.AND.T.LE.2,0100)) OR.
6 (5*((M-1)/5).EQ.1.AND.T.GT.2,0100))
5PRINT 855, T, AJM(M), ENN(M)
855 FFORMAT(FA.2,2(37X, F10.5))
T = M + 1
M = M + 1
IF(M.LE.MUX) GO TO 904
SUBROUTINE PART7
C******** PART 7 REvised
C******** 4-POINT VERSION

IMPLICIT REAL*8 (A-H, K, O-Z)
COMMON CT (50, 121), SXN (111), AJ1 (111), FX (5, 121)
& E, F (203), CC (121), CS (121), TI (5, 121), DD (5, 121), CO (5)
& ENM (1), AJ1 (5, 121), AJ2 (5, 121), EN3 (5, 121)
& E, ENM (1), AJ1 (3), AJ1 (5, 121), AJ2 (5, 121), AJ1 (5, 121)
& E, E, E, E, E, E, E, E, E, E, E, E, E, E, E, E, E, E, E, E,
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IF (ILN.LT.2) GO TO 703
I = 1
41 XI (I, 2) = H * Q * (F(I) + F(I + LV)) * 5 * D0
I = I + 1
IF (I .LE. LV) GO TO 41
RETURN
703 Y = Y + H
IGO = 3
IY = IY + 1
GO TO (602, 632, 640), LGC
40 CONTINUE
I = 1
42 F(I + LV2) = F(I + LV3)
I = I + 1
IF (I .LE. LV) GO TO 42
IF (ILN.LT.3) GO TO 704
I = 1
44 XI (I, 2) = H * Q * (5 * D0 * F(I) * (I + LV) - F(I + LV2)) / 12 * D0
XI (I, 3) = H * Q * (F(I) * 4 * D0 * F(I + LV) * F(I + LV2)) / 3 * D0
I = I + 1
IF (I .LE. LV) GO TO 44
RETURN
704 Y = Y + H
IGO = 3
IY = IY + 1
GO TO (602, 632, 640), LGC
45 CONTINUE
I = 1
47 XI (I, 2 ) = H * Q * (9 * D0 * F(I) * (I + LV) - 5 * D0 * F(I + LV) + F(I + LV3)) / 24 * D0
I = I + 1
IF (I .LE. LV) GO TO 47
50 CONTINUE
I = 1
48 XI (I, IY -1) = XI (I, IY - 2) + H * Q * (- F(I) + 13 * D0 * (F(I + LV) + F(I + LV2))
   + F(I + LV3)) / 24 * D0
I = I + 1
IF (I .LE. LV) GO TO 48
I = 1
IF (Y + H .GT. W) GO TO 60
52 F(I) = F(I + LV)
F (I + LV) = F (I + LV2)
F (I + LV2) = F (I + LV3)
I = I + 1
IF (I .LE. LV) GO TO 52
Y = Y + H
IGO = 5
IY = IY + 1
GO TO (602, 632, 640), LGO
55 GO TO 50
60 CONTINUE
62 XI (I, 1) = XI (I, IY - 1) + H * Q * (F(I) - 5 * D0 * F(I + LV) + 19 * D0 * F(I + LV2)
   + 9 * D0 * F(I + LV3)) / 24 * D0
I = I + 1
IF (I .LE. LV) GO TO 62
RETURN

C*********

PART 8

602 CONTINUE
I = 1
Z = Y * Q
002 = 0(Z)
F(I+6) = 00Z*DCOS(Z)
F(I+7) = 00Z*DSIN(Z)
GO TO (37,37,40,45,45),140

CONTINUE
Z=Y*0
AOZ= A(Z)
UUO= UU(Z)
UUOZ= UUOZ
W0 = AOZ/(UUOZ+UU0)
WQ = AOZ*UUOZ
I=1

CONTINUE
WQ= WQ(I)
CC1= CC(IY)
SSI= SS(IY)
F(I+126) = WQ*PP(1,1Y)
F(I+127) = F(I+126)*CCI
F(I+128) = F(I+126)*SSI
F(I+129) = F(I+127)*CCI
F(I+130) = F(I+127)*SSI
F(I+131) = F(I+128)*SSI
F(I+132) = CC1*PP(2,1Y)
F(I+133) = F(I+132)*CCI
F(I+134) = F(I+132)*SSI
F(I+135) = F(I+133)*CCI
F(I+136) = F(I+133)*SSI
F(I+137) = F(I+134)*CCI
F(I+138) = WQ*PP(3,1Y)
F(I+139) = F(I+138)*CCI
F(I+140) = F(I+138)*SSI
F(I+141) = WQ*PP(4,1Y)
F(I+142) = F(I+141)*CCI
F(I+143) = F(I+141)*SSI
F(I+144) = WQ*PP(5,1Y)
F(I+145) = F(I+144)*CCI
F(I+146) = F(I+144)*SSI
I=I+1
IF(I.LT.23) GO TO 138
GO TO (37,37,40,45,45),130

CONTINUE
Z=Y*0
AOZ= A(Z)
UUO= UU(Z)
UUOZ= UUOZ
W0 = AOZ/(UUOZ+UU0)
WQ = AOZ*UUOZ
I=1

CONTINUE
WQ= WQ(I)
CC1= CC(IY)
SSI= SS(IY)
CC1= CC(IY)
F(I+150) = WQ
F(I+151) = WQ*CCI
F(I+152) =QQ1 *SSI
F(I+153) =F(I+151)*SSI
F(I+154) =F(I+151)*CCI
F(I+155) =F(I+152)*SSI
F(I+156) =F(I+154)*CCI
F(I+157) =F(I+155)*CCI
F(I+158) =F(I+154)*SSI
F(I+159) =F(I+155)*SSI
I=I+10
IF(I.LT.42) GO TO 46
GO TO (30,35,40,45,55),IGC
END
REFERENCES


