### ON SCHEFFE'S S-METHOD: A REVIEW

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**Contract or Grant Number:** AF-AFOSR 76-3059-76

**Distribution Statement:** Approved for public release; distribution unlimited.

**Abstract:**
This paper gives in detail the background and derivation of a simultaneous statistical inference in linear models which is commonly known as Scheffe's S-method. It is written for a person with knowledge of linear models and design of experiment. It also gives a list of 46 references related to the subject.

**Keywords:** Scheffe's S-method, simultaneous statistical inference, linear estimation, contrast, connectedness, F test.
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by

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August, 1977

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Research sponsored by the Air Force Office of Scientific Research under Grant AFOSR 76-3050.
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Introduction: We shall assume throughout the following model:

\[ \Omega: Y = X\beta + e \]

where \( Y \) is an \( n \times 1 \) vector of observations, \( X \) is an \( n \times p \) known matrix of rank \( r \), \( \beta \) is a \( p \times 1 \) vector of unknown parameters, and \( e \) is distributed \( N(0, \sigma^2 I) \).

Definition 1. A linear parametric function \( \psi = c'\beta \), where \( c \) is a vector of known constants, is said to be an estimable linear parametric function if there exists an unbiased linear estimator \( a'Y \), i.e., such that \( E(a'Y) = \psi \).

The estimability of \( \psi \) solely depends on the design matrix \( X \) as the following known tests for estimability of \( \psi \) show:

(i) \( \psi \) is estimable if and only if \( c' \) is in the row space of \( X \), i.e., if and only if there exists a vector \( t \) such that \( c' = t'X \).

(ii) \( \psi \) is estimable if and only if there exists a vector \( k \) such that \( c' = k'X'X \).

(iii) \( \psi \) is estimable if and only if \( c' = c'H \), where \( H = (X'X)^{-1}X'X \).

It should be noted that in practice it is not a trivial matter to check for estimability due to complicated nature of the design matrix \( X \). This is why the experimenter is well
advised to specify his set of linear parametric functions of interest and try to collect his data (i.e., chooses his design matrix \( X \)) which guarantees the estimability of his functions of interest.

**Definition 2.** We say the design \( X \) is connected for \( V \) if \( V \) is estimable under \( X \). Otherwise, \( X \) is said to be disconnected for \( V \).

**Estimation, Test and Confidence Interval for \( V \).**

If \( V \) is estimable under \( X \) then it's well known that the best linear unbiased estimator of \( V \) is given by

\[
\hat{V} = c' \hat{\beta} = c' (X'X)^{-1} X' Y,
\]

with

\[
\text{var } \hat{V} = \sigma^2 \phi = c' (X'X)^{-1} c \sigma^2.
\]

An unbiased estimator of \( \sigma^2 \phi \) is \( c' (X'X)^{-1} c \hat{\sigma}^2 \) where

\[
\hat{\sigma}^2 = \frac{1}{n-r} Y'(I-X(X'X)^{-1}X')Y.
\]

Therefore

\[
\hat{\phi} \sim N(\psi, \sigma^2 \phi).
\]

We know that

\[
Q_1 = \frac{1}{\sigma^2} Y'(I-X(X'X)^{-1}X')Y = \frac{n-r}{\sigma^2} \hat{\sigma}^2 \sim \chi^2 (n-r).
\]

\( \hat{V} \) is a linear form in \( Y \) and thus \( \hat{V} \) and \( Q_1 \) are indepen-
dent if

\[ c'(X'X) - X'[I - (X'X)^{-1}X'] = c'(X'X) - c'(X'X)X'X(X'X)^{-1}X' = 0. \]

Now since \( \psi \) is estimable thus \( c' \) can be expressed as \( t'X \). Therefore, by substituting \( t'X \) for \( c' \) in the above expression we obtain:

\[
=t'X(X'X)^{-1}X' - t'X(X'X)^{-1}X'X(X'X)^{-1}X'
= t'X(X'X)^{-1}X' - t'X(X'X)^{-1}X' = 0.
\]

Since \( \hat{\psi} \sim N(\psi, \sigma^2) \), this implies that

\[
\frac{\hat{\psi} - \psi}{\sigma} = \frac{\hat{\psi} - \psi}{\sigma(c'(X'X) - c)^{1/2}} \sim N(0, 1).
\]

Thus

\[
\frac{(\hat{\psi} - \psi)/\sigma(c'(X'X) - c)^{1/2}}{[Q_1/(n-r)]^{1/2}} = \frac{\hat{\psi} - \psi}{\sigma(c'(X'X) - c)^{1/2}} \sim \frac{\hat{\psi} - \psi}{\sigma}
\]

\[
\sim t(n-r).
\]

or equivalently

\[
\left(\frac{\hat{\psi} - \psi}{\sigma}\right)^2 \sim F(1, n-1).
\]

This statistic can be used for testing hypothesis of the form \( H_0: \psi = m \). This statistic can also be used for constructing
confidence intervals for $\Psi$.

$$P \left[ \frac{\hat{V} - \Psi}{\hat{\sigma}^2} \leq F_{a}(1, n-1) \right] = 1 - \alpha$$

or

$$P \left[ \hat{V} - \hat{\sigma} F_{a}(1, n-1)^{1/2} \leq \Psi \leq \hat{V} + \hat{\sigma} F_{a}(1, n-1)^{1/2} \right] = 1 - \alpha.$$

Suppose we have a set of linear parametric functions $\Psi_1, \Psi_2, \ldots, \Psi_t$ and we wish to construct $1 - \alpha$ simultaneous confidence intervals for these $t$ linear parametric functions. The above confidence intervals give $1 - \alpha$ confidence intervals for individual $\Psi$'s. Scheffe's $S$-method answers this problem.

But first we need the following definitions.

**Definition 3.** Two linear parametric functions $\Psi_1$ and $\Psi_2$ are said to be algebraically independent if their corresponding coefficient vectors $c_1$ and $c_2$ are independent.

**Definition 4.** By a $q$-dimensional subspace $L$ of linear estimable functions under the design matrix $X$ we mean the subspace generated by the coefficient vectors of $q$ independent linear estimable functions under $X$. We say $\Psi = c' \beta \in L$ if $c' \in L$. 
The S-method of multiple comparison is based on

**Theorem 1.** Under \( \Omega \) the probability is \( 1 - \alpha \) that all estimable functions \( \Psi \) in a given \( q \)-dimensional space \( L \) simultaneously satisfy

\[
(1) \quad \Psi - S_{\Phi} \leq \Psi \leq \Psi + S_{\Phi},
\]

where \( S = [q F_{\alpha}(q, n-1)]^{1/2} \).

One can rewrite (1) in the following form

\[
P(|\Psi - \Psi| \leq \hat{\sigma}_{\Phi} [q F_{\alpha}(q, n-1)]^{1/2} \text{ for all } \Psi \in L) = 1 - \alpha,
\]

or

\[
P\left(\frac{(\Psi - \Psi)^2}{\sigma^2} \leq \frac{1}{\hat{\sigma}_{\Phi}} (X'X)^{-1} c \right. \text{ for all } \Psi \in L) = 1 - \alpha.
\]

Since \( \Phi = \Psi, \Psi = c' \beta \), therefore it suffices to prove that the maximum value of \( (c' \hat{\beta} - c' \beta)/c'(X'X)^{-1} c \) for all nonzero \( c \in L \) is distributed as \( \sigma^2 \chi^2(q) \); and this maximum is independent of \( (n-1)\hat{\sigma}^2 \) which is distributed as \( \sigma^2 \chi^2(n-r) \).

To prove this we need the following lemmas.

**Lemma 1.** Let \( A \) be a symmetric matrix of order \( n \). The maximum value of \( z'Az/z'z \) over all nonzero \( z \in E_n \) is \( \lambda \), the largest eigenvalue of \( A \), and this maximum is attained when \( z \) is any eigenvector of \( A \) corresponding to the root \( \lambda \).
Proof. First we show that the following two problems are equivalent.

(i) maximize $\frac{z'Az}{z'z}$, (ii) maximize $z'Az$.

Let

$$\max_{z \neq 0} \frac{z'Az}{z'z} = m_1$$

and

$$\max_{z'z = 1} z'Az = m_2$$

and suppose $m_1$ is attained for $z = z_1$ and $m_2$ is attained for $z = z_2$, i.e.,

$$\frac{z_1'Az_1}{z_1'z_1} = m_1$$

and

$$\frac{z_2'Az_2}{z_2'z_2} = m_2.$$

Let

$$\bar{z}_1 = \frac{1}{\sqrt{z_1'z_1}} z_1,$$

then

$$\bar{z}_1'\bar{z}_1 = 1.$$

Thus

$$\frac{\bar{z}_1'\bar{z}_1}{z_1'z_1} = \frac{z_1'Az_1}{z_1'z_1} = m_1.$$

Therefore, $m_1 \leq m_2$. Also since $z_2'z_2 = 1$, $z_2 \neq 0$ and

$$\frac{z_2'Az_2}{z_2'z_2} = \frac{m_2}{1} = m_2$$

thus $m_2 \leq m_1$. Hence $m_1 = m_2$.

Note: Since $\max_{z \neq 0} \frac{z'Az}{z'z}$ is equivalent to $\max_{z'z = 1} z'Az$ and $z'Az$ is a continuous function of $z$ and $\{z: z \in E_n, z'z = 1\}$ is a compact set, it follows that $\max_{z \neq 0} \frac{z'Az}{z'z}$ exists.

We shall now present two methods of finishing the proof of
Lemma 1.

Method 1 (Lagrange multiplier method).

We want to maximize $z'Az$ subject to the constraint $z'z = 1$. Let

$$f(z, \lambda) = z'Az - \lambda(z'z - 1)$$

where $\lambda$ is a Lagrange multiplier. Since $\frac{\partial f}{\partial z} = 2Az - 2\lambda z = 0$, this implies that $Az = \lambda z$ so $\lambda$ must be an eigenvalue of $A$.

On the other hand, if $\lambda$ is an eigenvalue of $A$, then

$$z'Az = z'(Az) = z'((\lambda z) = \lambda z'z,$$

so that

$$\max_{z'z=1} z'Az = \max_{z'z=1} \lambda z'z = \max_{z'z=1} \lambda$$

where $\lambda$ is an eigenvalue of $A$. Therefore

$$\max_{z'z=1} z'Az = \max_{z'z=1} z'Az = \max_{z'z=1} \lambda = \lambda.$$

Now suppose $v$ is any eigenvector corresponding to $\lambda$, then

$$\frac{v'Av}{v'v} = \frac{v'(Av)}{v'v} = \frac{v'\lambda v}{v'v} = \lambda$$

so that the maximum value is attained for any eigenvector associated with $\lambda$.

Method 2. Since $A$ is a real symmetric matrix of order $n$, there exists an orthogonal matrix $P$ such that $P'AP = \Lambda$ where $\Lambda$ is a diagonal matrix with the (real) eigenvalues of $A$ on the diagonal. Let $P_j$ denote the $j$-th column of $P$, i.e., $P = (P_1:P_2:\ldots:P_n)$, then $P_j$ is an eigenvalue
of \( A \) corresponding to \( \lambda_j \) satisfying
\[
P_j'P_k = \begin{cases} 0 & \text{when } j \neq k \\ 1 & \text{when } j = k \end{cases}
\]
It follows that
\[
A = PAP' = \lambda_1P_1P_1' + \lambda_2P_2P_2' + \ldots + \lambda_nP_nP_n'
\]
and
\[
I = PP' = P_1P_1' + P_2P_2' + \ldots + P_nP_n'.
\]
The set \( \{P_1, P_2, \ldots, P_n\} \) is an orthonormal basis for \( E_n \).
Let \( z \in E_n \), then \( z = Pw \), where \( w' = (w_1, w_2, \ldots, w_n) \) and \( w \)'s are the coordinates of the vector \( z \) with respect to the basis \( \{P_1, P_2, \ldots, P_n\} \). Therefore
\[
\frac{z'Az}{z'z} = \frac{(Pw)'PAP'(Pw)}{(Pw)'(Pw)} = \frac{w'PAP'Pw}{w'Pw} \quad \frac{w'PAP'Pw}{w'Pw}
\]
\[
= w'Aw = \frac{\lambda_1w_1^2 + \lambda_2w_2^2 + \ldots + \lambda_nw_n^2}{w_1^2 + w_2^2 + \ldots + w_n^2}.
\]
So maximizing \( z'Az/z'z \) for \( z \neq 0 \), \( z \in E_n \) is equivalent to maximizing
\[
\frac{\lambda_1w_1^2 + \lambda_2w_2^2 + \ldots + \lambda_nw_n^2}{w_1^2 + w_2^2 + \ldots + w_n^2} \quad \text{for } w \in E_n
\]
and \( w \neq 0 \) (since \( z \neq 0 \)).
Suppose \( \max(\lambda_1, \lambda_2, \ldots, \lambda_n) = \lambda \), then
$$\lambda_1 w_1^2 + \lambda_2 w_2^2 + \ldots + \lambda_n w_n^2 \leq \frac{1}{\sum_{i=1}^{n} w_i^2}$$

so

$$\frac{\lambda_1 w_1^2 + \lambda_2 w_2^2 + \ldots + \lambda_n w_n^2}{w_1^2 + w_2^2 + \ldots + w_n^2} \leq \frac{\sum_{i=1}^{n} w_i^2}{\sum_{i=1}^{n} w_i} = 1.$$ 

If \( \lambda = \lambda_j \) then let \( w_i = 0 \) if \( i \neq j \) and \( w_j \neq 0 \). Then

$$\frac{\lambda_1 w_1^2 + \lambda_2 w_2^2 + \ldots + \lambda_n w_n^2}{w_1^2 + w_2^2 + \ldots + w_n^2} = \frac{\lambda_j w_j^2}{w_j^2} = \lambda_j = \lambda.$$

so \( \lambda \) is attainable and the maximum is attained for \( w = \{0, \ldots, 0, w_j, 0, \ldots, 0\} \) or equivalently for \( z = P w, \mathbf{w}_j \neq 0 \), i.e., for any eigenvalue of \( A \) corresponding to \( \lambda \) the largest eigenvalue of \( A \).

**Lemma 2.** Let \( A \) be a symmetric matrix of order \( n \) and let \( B \) be any positive symmetric matrix of order \( n \). The maximum value of \( z' A z / z' B z \) over all nonzero \( z \in \mathbb{E}_n \) is \( \lambda \), the largest eigenvalue of \( B^{-1} A \), and this maximum is attained for any eigenvector of \( B^{-1} A \) corresponding to the root \( \lambda \).

**Proof.** First let us solve the following problem. Maximize \( z' A z \) subject to \( z' B z = 1 \). Using the method of Lagrange's multiplier let

$$f(z, \lambda) = z' A z - \lambda(z' B z - 1)$$

where, \( \lambda \) is a Lagrange multiplier. A necessary condition is

$$\frac{\partial f}{\partial z} = 2Az - 2\lambda Bz = 0$$

so
Az = λBz = Bλz → B^{-1}Az = λz.

So that λ is an eigenvalue of B^{-1} and z is the corresponding eigenvector. Therefore,

\[
\max_{z Bz=1} z'Az = \max_{z Bz=1} z'(Az) = \max_{z Bz=1} z'\lambda Bz
\]

\[
= \max_{z Bz=1} \lambda z' Bz = \max_{z Bz=1} \lambda = 1.
\]

Now we shall show that the following two problems are equivalent:

(i) \(\max_{z Bz=1} z'Az\),  (ii) \(\max_{z Bz=1} z'Az\).

Proof of this is very much like the counterpart proof given in the beginning of Lemma 1.

Let

\[
\max_{z \neq 0} z'Az = m_1' \quad \text{and} \quad \max_{z Bz=1} z'Az = m_2'
\]

and suppose \(m_1'\) is attained for \(z = v_1\) and \(m_2'\) is attained for \(z = v_2\), i.e.,

\[
\frac{v_1'A}{v_1'Bv_1} = m_1' \quad \text{and} \quad \frac{v_2'A}{v_2'Bv_2} = m_2' \quad \text{under} \quad v_2'Bv_2 = 1.
\]

Let

\[
\tilde{v}_1 = \frac{v_1}{(v_1'Bv_1)^{1/2}}
\]
then
\[ \tilde{v}_1' B \tilde{v}_1 = \frac{v_1'Bv_1}{v_1'Bv_1} = 1, \]
also
\[ \frac{v_1'A\tilde{v}_1}{v_1'B\tilde{v}_1} = \frac{v_1'Av_1}{v_1'Bv_1} = \lambda. \]
Therefore, \( m_1 \leq m_2 \). On the other hand, since \( v_2'Bv_2 = 1 \) and since \( B \) is positive definite \( v_2 \neq 0 \),
\[ \frac{v_2'A\tilde{v}_2}{v_2'B\tilde{v}_2} = \frac{v_2'Av_2}{v_2'Bv_2} = m_2, \]
thus \( m_2 \leq m_1 \). Hence \( m_1 = m_2 \).

We shall shortly give a generalization of the preceding results. Let \( L \) be a vector subspace of \( \mathbb{E}^n \) of dimension \( q \). Let the columns of \( C = [c_1, c_2, \ldots, c_q] \) be a basis for \( L \).

**Lemma 3.** The maximum value of \( z'Az/z'z \) over all nonzero \( z \in L \) is \( \lambda \), the largest eigenvalue of \( CC^+A \), and is attained when \( z \) is any eigenvector of \( CC^+A \) corresponding to the root \( \lambda \), where \( C^+ \) is the Moore-Penrose generalized inverse of \( C \).

**Proof.** \( C \) is an \( n \times q \) matrix of rank \( q \) thus \((C'C)^{-1}\) exists and one can check that
\[ C^+ = (CC)^{-1} C', \]
is the Moore-Penrose generalized inverse of \( C \). Also note that matrices \( CD \) and \( DC \) have the same eigenvalues. Now
let \( v \in E_q \), then \( z = Cv \in L \), hence

\[
\max_{z \in L, \ z \geq 0} \frac{z^{'Az}}{z^{'z}} \geq \max_{v \in E_q, \ v \neq 0} \frac{(Cv)^{'}A(Cv)}{(Cv)^{'(Cv)}}
\]

\[
= \max_{v \in E_q, \ v \neq 0} \frac{v^{'}(C^{'AC})v}{v^{'}(C^{'C})v}.
\]

Since \( C^{'C} \) is positive definite we can use Lemma 2 and conclude that

\[
\max_{v \in E_q, \ v \neq 0} \frac{v^{'}(C^{'AC})v}{v^{'}(C^{'C})v}
\]

\[
= \text{largest eigenvalue of } (C^{'C})^{-1} C^{'AC}
\]

\[
= \text{largest eigenvalue of } C^{'AC}
\]

\[
= \text{largest eigenvalue of } C(C^{'A}) \text{ (see the remark about CD and DC)}
\]

\[
= \lambda.
\]

To obtain the inequality in the other direction, let \( z \in L \).
This implies that there exists a \( v \in E_q \) such that \( z = Cv \).
Thus

\[
\max_{z \in L, \ z \geq 0} \frac{z^{'Az}}{z^{'z}} \leq \max_{v \in E_q, \ v \neq 0} \frac{(Cv)^{'}A(Cv)}{(Cv)^{'(Cv)}}
\]

\[
= \text{largest eigenvalue of } CC^{'A} = \lambda.
\]

Now let \( z \) be any eigenvalue of \( CC^{'A} \) corresponding to the root \( \lambda \). Then \( CC^{'A}z = \lambda z \) which implies \( z^{'}CC^{'A}z = \lambda z^{'}z \) which in turn implies \( z^{'}Az/z^{'}z = \lambda \) because \( z \in L \) implies that \( z^{'}CC^{'} = z^{'} \).
Corollary 1. Let $H$ be any $q \times n$ matrix of rank $q$. Then the maximum value of $z'Az/z'z$ over all nonzero $z$ in $\mathbb{E}_n$ satisfying $Hz = 0$ is $\lambda$, the largest eigenvalue of $(I-H'H)A$, and is obtained when $z$ is any eigenvector of $(I-H'H)A$ corresponding to the root $\lambda$. Here $H^+$ denotes the Moore-Penrose generalized inverse of $H$.

Proof. $Hz = 0$ if and only if $z$ belongs to the column space of $I-H'H$. This is seen as follows:

If $Hz = 0$, then $z$ is in the column space of $I-H'H$, i.e., there exists $w$ such that $(I-H'H)w = z$. Set $w = z$ then $(I-H'H)z = z - H'Hz = z - H'(Hz) = z - H'(0) = z$. On the other hand, if $z$ is in the column space of $I-H'H$ then $Hz = H(I-H'H)w = Hw - HH'Hw = Hw - Hw = 0$. Now the proof follows from Lemma 3.

Lemma 4. Let $B$ be a positive definite matrix of order $n$. Then the maximum value of $z'Az/z'Bz$ over all nonzero $z$ in $L$ is $\lambda$, the largest eigenvalue of $C(C'BC)^{-1}C'A$, and is attained when $z$ is any eigenvector of $C(C'BC)^{-1}C'A$ corresponding to the root $\lambda$.

Proof. An argument similar to the one used in the proof of Lemma 3 gives us

$$\max_{z \in L, \, z \neq 0} \frac{z'Az}{z'Bz} = \max_{v \in \mathbb{E}_q', \, v \neq 0} \frac{(Cv)'A(Cv)}{(Cv)'B(Cv)}.$$
By an earlier result one gets
\[
\max_{v \in \mathbb{E}, v \neq 0} \frac{v^\top C' ACv}{v^\top C' BCv} = \text{largest eigenvalue of } [(C' BC)^+ C' AC]
\]
\[
= \text{largest eigenvalue of } [C(C' BC)^+ C' A] = \lambda
\]
(recall the argument about CD and DC).

Now let \( z \) be any eigenvector of \( C(C' BC)^+ C' A \) corresponding to the root \( \lambda \). Then
\[
C(C' BC)^+ C' Az = \lambda z \quad \text{which implies that}
\]
\[
z' BC(C' BC)^+ C' Az = \lambda z = \lambda z' Bz, \quad \text{which implies that}
\]
\[
z' Az/z' Bz = \lambda, \quad \text{since } z \in \mathbb{L} \quad \text{implies that}
\]
\[
z' BC(C' BC)^+ C' = z'.
\]

This latter claim is seen as follows. Since \( z \in \mathbb{L} \) then there exists a \( w \) such that \( Cw = z \), i.e., \( z \) is a linear combination of the columns of \( C \) which generate \( \mathbb{L} \). Then
\[
z' BC(C' BC)^+ C' = w' C' BC(C' BC)^+ C' = w' C' = (Cw)' = z'.
\]

**Lemma 5.** The Moore-Penrose of \( X \) is given by \( (X'X)^+X' \) where \( A^+ \) denotes the Moore-Penrose of the matrix \( A \).

**Proof.** By definition \( K \) is the Moore-Penrose generalized inverse of \( A \) if \( AKA = A \), \( KAK = K \), \( (AK)' = AK \) and \( (KA)' = KA \). Therefore, we shall check these four conditions for \( X^+ \). In what follows we use the following well known facts:
\[ F_1: \quad X(X'X)^{-}X' \text{ is symmetric and } X(X'X)^+X' = X(X'X)^{-}X' \text{ where } (X'X)^{-} \text{ is any generalized inverse of } (X'X). \]

\[ F_2: \quad X(X'X)^{-}X'X = X \quad \text{and} \quad X'X(X'X)^{-}X' = X'. \]

(i) \[ XX'^+X = X(X'X)^+X'X = X(X'X)^{-}X'X = X, \]

(ii) \[ X'^+XX'^+ = (X'X)^+X'X(X'X)^+X' = (X'X)^+X'X(X'X)^{-}X' = (X'X)^+X'. \]

(iii) \[ (XX'^+) = (X(X'X)^+X')' = (X(X'X)^{-}X')' = X(X'X)^{-}X(X'X)^+X' = XX'^+. \]

(iv) \[ (X'^+) = ((X'X)^+X'X)' = (X'X)^+ \text{ since } (X'X)^+ \text{ is the Moore-Penrose inverse of } X'X \text{ and thus } (X'X)^+X'X \]

is symmetric.

**Lemma 6.** The Moore-Penrose generalized inverse of \( X' \) is \( (X'^+)' \).

**Proof.**

(i) \[ X'(X'^+)'X' = [XX'^+] = [X] = X', \]

(ii) \[ (X'^+)'X'(X'^+) = [X'^+X'^+] = [X'^+]', \]

(iii) \[ [X'(X'^+)]' = [(X'^+X')]' = [X'^+X'] = X'(X'^+)', \]

(iv) \[ [(X'^+)'X'] = [(XX'^+)]' = [XX'^+] = (X'^+)'. \]

**Lemma 7.** If \( X'^+ \) is the Moore-Penrose of \( X \), then \( XX'^+(X'^+)' = (X'^+)' \).

**Proof.** From Lemma 6 \( X'^+ = (X'X)^+X' \). Thus

\[
XX'^+(X'^+)' = X(X'X)^+(X'X)^+X' [(X'X)^+X']' \\
= X(X'X)^+ [X'X]^+X'X' \\
= X(X'X)^+ [X'X[(X'X)^+]' ] \\
= X(X'X)^+ [X'X[X'X]' ]^+ \text{ by Lemma 6}
\]
\[ X(X'X)^+ [X'X(X'X)^+] \] by a property of Moore-Penrose

\[ = X(X'X)^+ \] generalized inverse

\[ = [(X'X)^+] X' = [(X'X)'] X' \]

\[ = [(X'X)^+X'] = (X^+)'. \]

Lemma 8. The set of nonzero eigenvalues of \( DC \) coincides with the set of nonzero eigenvalues of \( CD \).

**Proof.** Let \( \Lambda_1 \) and \( \Lambda_2 \) be the set of nonzero eigenvalues of \( DC \) and \( CD \) respectively.

If \( (DC)x = \lambda x \Rightarrow C(DC)x = \lambda Cx \)

\[ \Rightarrow CD(Cx) = \lambda (Cx) = CDy = \lambda y, \]

so if \( \lambda \) is an eigenvalue of \( DC \) it is an eigenvalue of \( CD \),

\[ \Rightarrow \Lambda_1 \subseteq \Lambda_2. \] Similarly, \( \Lambda_2 \subseteq \Lambda_1. \) Thus \( \Lambda_1 = \Lambda_2. \)

**Proof of Theorem 1.**

(1). \[ \max \left( \frac{c' \beta - c' \tilde{\beta}}{c'} \right)^2 \]

\[ = \max_{c \in L} \left( \frac{a'X \tilde{\beta} - a'X \beta}{a'X(X'X)^+Xa} \right)^2 \]

\[ \text{Reason: } c' \tilde{\beta} \text{ is estimable } \Rightarrow \exists \text{ an } a \text{ such that } \]

\[ c' = a'X. \] But \( c \in L \Rightarrow c' = \Sigma t_i c_i, c_i = b_i X, c_i \)'s \[ \text{were chosen } \Rightarrow c' = \Sigma t_i b_i'X = [\Sigma t_i b_i']X = a'X \text{ so } a' \]

is a linear combination of \( b_i \)'s. But from \[ c_i = b_i X = [\Sigma t_i b_i']X = a'X \text{ so } a' \text{ is a combination } \]

of \( b_i \)'s. But from \( c_i = b_i' X = X' b_i = c_i \) or \( b_i = (X')^+ c_i. \)
(2) \[
\left( \alpha' X^\prime - a X^{} \beta \right)^2 = \frac{(a' X' X^' X - a X^{} \beta a X^{} X a)}{a X' X^' a} = \frac{(a' X(X' X)^' X - a X^{} \beta a X^{} X a)}{a X(X' X)^' X a}
\]
\[
= \frac{(a' X X^+ Y' - a X^{} \beta)}{a X X^+ a}.
\]
Reason. See Lemmas 5 and 6.

(3) \[
\left( \alpha' X^+ Y' - a X^{} \beta \right)^2 = \frac{(a' Y' - a X^{} \beta a X^{} a)}{a a} = \frac{a' (Y-X\beta)(Y-X\beta)' a}{a a}
\]
Reason. Since \(a \in \mathbb{F}[X^'+C] = a \in \text{column space of } (X')^+, \) i.e., \(a \) an \(f \) such that \(a = (X')^+ f = (X^')' f = a' = f' X^+ . \) Thus \(a' X X^+ = f' X^+ X X^+ = f' X^+ = a' .\)

(4) From (1) and (3)
\[
\max_{c \in L} (c \alpha - c \beta)^2 = \max_{c \in L} \frac{a' (Y-X\beta)(Y-X\beta)' a}{a a}
\]
\[
= \max_{a \in \mathbb{F}[X^'+C]} \frac{a' A a}{a a}, \quad A = (Y-X\beta)(Y-X\beta)'
\]
\[= \text{largest eigenvalue of } [(X')^+ C] [(X')^+ C]' (Y-X\beta)(Y-X\beta)'
\]
by Lemma 3.
\[= \text{largest eigenvalue of } (Y-X\beta)' [(X')^+ C] [(X')^+ C]' (Y-X\beta)
\]
by Lemma 8, but this is a scalar,
\[= (Y-X\beta)' [(X')^+ C] [(X')^+ C]' (Y-X\beta) = Q_1 \] which is a quadratic in \((Y-X\beta) \sim N(0, \sigma^2 I)\).

The claim is that \(Q \sim \sigma^2 x^2(q) . \) This will be the case if we prove that \([(X')^+ C] [(X')^+ C]' \) is idempotent and its rank is \(q . \) The idempotency is obvious since in
general \((BB^+)(BB^+) = BB^+BB^+ = BB^+\). We shall now show that rank \([((X')^+C][(X')^+C]^+] = q\). This can be seen as follows:

\[
\begin{align*}
\text{r}[(X')^+C][(X')^+C]^+ & \leq \text{r}[(X')^+C] \leq \text{r}[C] = q,
\end{align*}
\]
on the other hand,

\[
\begin{align*}
\text{r}[(X'X)^+C][(X')^+C]^+ & \geq \text{r}[(X')^+C][(X')^+C]^+[X']^+C
\geq \text{r}[X'(X'X)^+X]'C = \text{r}[(X'X)^+X]'C
\geq \text{r}[X'(X'X)^+]'C = \text{r}[X'(X'X)^+]'X'K \text{ since } C = X'K
\geq \text{r}[(X'X)^+]'X'K = \text{r}[(X'X)^+]'X'K = \text{r}[X'(X'X)^+]'X'K
\geq \text{r}[X'K] = \text{r}[C] = q.
\end{align*}
\]

The proof of Theorem 1 will be complete if we show that \(Q_1\) and \(Q_2\) are independent where,

\[
Q_2 = (n-r) \hat{\beta}^2 = Y'(I-X)X'X^{-1}X'X
\]

\[
= (Y-X\hat{\beta})'(I-X)(X')^{-1}(X'-X)(Y-X\hat{\beta}).
\]

It is sufficient to prove that

\[
[I-X(X'X)^-X'][(X')^+C][(X')^+C]^+ = [I-X(X'X)^-X'][(X')^+C][(X')^+C]^+] = 0,
\]

by Lemma 6

\[
\text{LHS} = [(X^+)'C][(X^+)'C]^+ - X(X'X)^-X'(X^+)'C[(X^+)'C]^+
= [(X^+)'C][(X^+)'C]^+ - X(X'X)^+X'(X^+)'C[(X^+)'C]^+
= [(X^+)'C][(X^+)'C]^+ - XX^+(X^+)'C[(X^+)'C]^+
= [(X^+)'C][(X^+)'C]^+ - (X^+)'C[(X^+)'C] = 0.
\]
The relation of the S-method or S-intervals for $\bar{y}$ in $L$ and the standard F-test of the hypothesis

$$H_0: \bar{y}_1 = \bar{y}_2 = \ldots = \bar{y}_q = 0$$

is stated in

Theorem 2. Under $\Omega$ the $\alpha$-level F-test of $H_0$ will accept $H_0$ if and only if for all $\bar{y}$ in $L$ the intervals (1) in Theorem 1 cover zero.
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