MULTI-VALUED STATE COMPONENT
RELIABILITY SYSTEMS

by
SHELDON M. ROSS

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(SEE ABSTRACT)
MULTI-VALUED STATE COMPONENT RELIABILITY SYSTEMS

by

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JUNE 1977

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ABSTRACT

Consider a reliability system that is composed of $n$ components each of which is operating at some performance level. We suppose that there exists a nondecreasing function $\phi$, called the structure function, such that $\phi(x_1, \ldots, x_n)$ denotes the performance level of the system when the $i$th component's performance level is $x_i$, $i = 1, \ldots, n$.

Whereas almost all previous work assumed that both $x_i$ and $\phi(x_1, \ldots, x_n)$ were binary variables we shall allow both to be arbitrary nonnegative numbers and we extend many of the important results of the usual binary model to this more general framework. In particular we obtain a fundamental inequality for $E[\phi(X_1, \ldots, X_n)]$ when $\phi$ is binary which can, among other things, be used to generate a host of inequalities concerning IFRA distributions including, as a special case, the IFRA convolution theorem.

In Section 2 we define the concept of an IFRA stochastic process and prove the analog of the IFRA closure theorem; and in Section 3 we do the same for NBU stochastic processes. In the final section we present some applications to stochastic networks.
MULTI-VALUED STATE COMPONENT RELIABILITY SYSTEMS

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0. INTRODUCTION AND SUMMARY

Consider a reliability system that is composed of \( n \) components each of which is operating at some performance level. We suppose that there exists a nondecreasing function \( \phi \), called the structure function, such that \( \phi(x_1, \ldots, x_n) \) denotes the performance level of the system when the \( i^{th} \) component's performance level is \( x_i \), \( i = 1, \ldots, n \).

Whereas almost all previous work has assumed that both \( x_1 \) and \( \ldots, x_n \) were binary variables we shall allow both to be arbitrary nonnegative numbers. In the next few sections we extend many of the important results of the usual binary model to this more general framework. In particular we obtain, in Section 1, a fundamental inequality for \( E[\phi(X_1, \ldots, X_n)] \) when \( \phi \) is binary which can, among other things, be used to generate a host of inequalities concerning IFRA distributions including, as a special case, the IFRA convolution theorem. In Section 2 we define the concept of an IFRA stochastic process and prove the analog of the IFRA closure theorem; and in Section 3 we do the same for NBU stochastic processes. In the final Section we present some applications to stochastic networks.
1. THE STRUCTURE FUNCTION

Suppose now the performance level of component \( i \) is a random variable \( X_i \) having distribution \( F_i \) where \( F_i(x) = P(X_i > x) \), and suppose that the \( X_i \) are independent. We define the function \( r(F_1, \ldots, F_n) \) by

\[
r(F_1, \ldots, F_n) = E[\phi(X_1, \ldots, X_n)]
\]

and call \( r \) the reliability function of the system. It immediately follows from the monotonicity of \( \phi \) that

**Proposition 1:**

If \( F_i \) and \( G_i \) are distributions such that \( F_i(x) \leq G_i(x) \) for all \( x \) then

\[
r(F_1, \ldots, F_n) \geq r(G_1, \ldots, G_n).
\]

We shall need the following lemma which is a slight variation of a lemma used by Block and Savits [3] to prove the IFRA convolution theorem.

**Lemma 1:**

Let \( r(s) \) be a nonnegative nondecreasing function of \( s \), \( s \geq 0 \) and let \( G \) be a distribution function with \( G(0) = 1 \). Then for \( 0 < a \leq 1 \),

\[
\int_0^\infty (r(s))^a d(1 - G^a(s)) \geq \left[ \int_0^\infty r(s) d(1 - G(s)) \right]^a.
\]

**Proof:**

Lemma 4.1 (p. 217) of Ross [5] (also given as Lemma 2.3 (p. 84) of...
Barlow and Proschan [1] generalizes to give that for

\[ 0 \leq x_1 \leq x_2 \leq \ldots \leq x_n, \quad y_i \geq 0, \quad \sum_{i=1}^{n} y_i > 0 \]

\[ \left( \sum_{i=1}^{n} r(x_i) y_i \right)^{\alpha} \leq \sum_{i=1}^{n} (r(x_i))^{\alpha} \left[ \left( \sum_{k=i}^{n} y_k \right)^{\alpha} - \left( \sum_{k=i+1}^{n} y_k \right)^{\alpha} \right]. \]

From this the conclusion follows from a standard limiting argument (as in [3]).

The following theorem is of fundamental importance.

Theorem 1:

If \( \phi \) is a binary function then

\[ (*) \quad r(\bar{F}_1, \ldots, \bar{F}_n) \geq [r(\bar{F}_1, \ldots, \bar{F}_n)]^{\alpha} \]

for all \( 0 \leq \alpha \leq 1 \).

Proof:

The proof is by induction of \( n \). When \( n = 1 \) it follows from the monotonicity of \( \phi \) that it must be of the form \( \phi(x) = \begin{cases} 1 & x > c \\ 0 & \end{cases} \) for some \( c \). Hence \( \mathbb{E}[\phi(X_1)] = \bar{F}_1(c) \) and so both sides of the inequality (*) are equal. So assume (*) for all binary structures of \( n-1 \) components, and consider the \( n \) component case. Conditioning on \( X_n \) yields

\[ (1) \quad r(\bar{F}_1, \ldots, \bar{F}_n) = \int r_s(\bar{F}_1, \ldots, \bar{F}_{n-1}) \, d(1 - \bar{F}_n(s)) \]

where

\[ r_s(\bar{F}_1, \ldots, \bar{F}_{n-1}) = \mathbb{E}[\phi(X_1, \ldots, X_{n-1}, s)] \]

with \( X_i \) having distribution \( \bar{F}_i \). By the induction hypothesis we see that
\[ r_s(\bar{F}_1^a, \ldots, \bar{F}_{n-1}^a) \geq [r_s(\bar{F}_1, \ldots, \bar{F}_{n-1})]^a \]

and so from (1),

\[ r(\bar{F}_1^a, \ldots, \bar{F}_n^a) \geq \int (r_s(\bar{F}_1, \ldots, \bar{F}_{n-1})^a d(1 - \bar{F}_n^a(s)). \]

As it follows from the monotonicity of \( \phi \) that \( r_s \) is nondecreasing in \( s \) we can apply Lemma 1 to the above to obtain that

\[ r(\bar{F}_1^a, \ldots, \bar{F}_n^a) \geq \left[ \int_0^1 r_s(\bar{F}_1^a, \ldots, \bar{F}_{n-1}^a) d(1 - \bar{F}_n^a(s)) \right]^a \]

\[ = (r(\bar{F}_1^a, \ldots, \bar{F}_n^a))^a. \]

**Definition:**

The distribution function \( \bar{F} \), with \( \bar{F}(0) = 1 \), is said to be an IFRA distribution if \( \bar{F}(ax) \geq \bar{F}^a(x) \) for all \( 0 \leq a < 1, x \geq 0 \).

**Corollary 1:**

If \( X_1, \ldots, X_n \) are independent random variables, each having an IFRA distribution, then for all nondecreasing binary functions \( \phi \),

\[ E\left[ \frac{\phi(X_1, X_2, \ldots, X_n)}{a} \right] \]

\[ \geq (E[\phi(X_1, \ldots, X_n)])^a \] for \( 0 \leq a < 1 \).

**Proof:**

If \( X_1 \) is IFRA with distribution \( F_1 \), then \( \frac{X_1}{a} > x \)

\[ \geq \bar{F}_1^a(x). \]
and so from Proposition 1

\[ E \left[ \frac{\phi(X_1, \ldots, X_n)}{\alpha} \right] \geq r \left( \tilde{F}_1, \ldots, \tilde{F}_n \right) \]

and the result follows from Theorem 1.\( \blacksquare \)

The above Corollary provides a host of inequalities concerning IFRA random variables. For instance we have

Corollary 2:

If \( X_1, \ldots, X_n \) are independent IFRA random variables then

(a) \( \sum_{i=1}^{n} X_i \) is IFRA.

(b) \( P \left\{ \prod_{i=1}^{n} X_i > aa^n \right\} \geq \left( P \left\{ \prod_{i=1}^{n} X_i > a \right\} \right)^{\alpha} \quad 0 \leq \alpha \leq 1. \)

Proof:

Part 1 follows from Corollary 1 by using the function

\[ \phi(X_1, \ldots, X_n) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} X_i > a \\ 0 & \text{otherwise} \end{cases} \]

to obtain

\[ P \left\{ \prod_{i=1}^{n} X_i > aa^n \right\} \geq \left( P \left\{ \prod_{i=1}^{n} X_i > a \right\} \right)^{\alpha} \]

similarly (b) follows by using

\[ \phi(X_1, \ldots, X_n) = \begin{cases} 1 & \text{if } \prod_{i=1}^{n} X_i > a \\ 0 & \text{otherwise}. \end{cases} \]
2. THE GENERALIZED IFRA CLOSURE THEOREM

In this section we suppose that the component performance levels vary with time and we let \(X_i(t)\) denote the level of component \(i\) at time \(t\). Thus, for instance, \(\varphi(X(t)) = \varphi(X_1(t), \ldots, X_n(t))\) denotes the system's performance level at time \(t\).

**Definition:**

The real-valued stochastic process \(\{X(t), t > 0\}\) is said to be an IFRA process if \(T_a\) is an IFRA random variable for every \(a\), where

\[T_a = \inf \{t : X(t) \leq a\}\]

is the first time the process reaches or goes below \(a\).

**Theorem 2:** The Generalized IFRA Closure Theorem

If \(\{X_i(t)\}, i = 1, \ldots, n\), are nonincreasing independent IFRA processes then \(\{\varphi(X(t))\}\) is also IFRA whenever \(\varphi\) is nondecreasing.

**Proof:**

Let \(\bar{F}_{i,s}(x) = P(X_i(s) > x)\), and suppose first that \(\varphi\) is a binary function. Let \(T\) denote the first time \(t\) that \(\varphi(X(t)) = 0\). Now

\[P(T > at) = P(\varphi(X(at)) = 1) \text{ by monotonicity}\]

(2)

\[= E[\varphi(X(at))] = r(\bar{F}_{1,at}, \ldots, \bar{F}_{n,at})\]

Now
where \( T_{1,b} \) denotes the first time that \( X_1(t) \) hits or goes below \( b \). Hence from the IFRA hypothesis on \( \{X_1(t)\} \) we see that

\[
P(T_{1,b} > at) \geq (P(T_{1,b} > t))^\alpha
\]

\[
= \bar{F}_{i,t}^\alpha(b).
\]

Thus

\[
\bar{F}_{i,at}(b) \geq \bar{F}_{i,t}^\alpha(b)
\]

and so from (2) and Proposition 1

\[
P(T > at) \geq \tau(\bar{F}_{i,t}, \ldots, \bar{F}_{n,t})
\]

\[
\geq (\tau(\bar{F}_{i,t}, \ldots, \bar{F}_{n,t}))^\alpha \quad \text{by Theorem 1}
\]

\[
= (P(T > t))^\alpha
\]

which proves the result when \( \phi \) is binary. For an arbitrary nondecreasing \( \phi \) we can show that the time to go below \( b \) is IFRA by using the result in the binary case on the binary function defined by

\[
\phi_b(x) = \begin{cases} 
1 & \text{if } \phi(x) > b \\
0 & \text{if } \phi(x) \leq b.
\end{cases}
\]

When \( \phi \) is a binary function we usually say that the system fails at the first \( t \) such that \( \phi(X(t)) = 0 \). Thus if each component process has the property that the time it takes to reach or go below any given
level is an IFRA random variable then so is the time to system failure. The following are two examples for which the component processes are IFRA.

Example 1: A Semi-Markov IFRA Process

If \( \{X(t), t \geq 0\} \) is a semi-Markov process such that \( X(0) = m \), \( P_{i,i-1} = 1, i > 1 \), \( P_{00} = 1 \), and the time in state \( i \) is an IFRA random variable, then it follows from the IFRA convolution theorem (Corollary 2) that \( \{X(t)\} \) is an IFRA process. This would be a model for a component that gradually went to lower states until it died (reached state 0), spending an IFRA amount of time in each state.

Example 2: A Poisson Shock Model

Suppose the component's level remained constant between times of extreme stress which occurred in accordance with a Poisson process. If at these moments its level decreased according to a given distribution then the component process would be IFRA. That is, if

\[
X(t) = \max \left\{ C - \sum_{i=1}^{N(t)} X_i, 0 \right\}
\]

where \( X_i, i \geq 1 \) are independent and identically distributed (i.i.d.) nonnegative random variables that are also independent of the Poisson process \( N(t) \), then \( \{X(t)\} \) is IFRA. This result was first proven by Esary, Marshall and Proschan [4] who also showed that the same result could be obtained under weaker conditions than the i.i.d. assumption on the \( X_i \).
3. AN NBU CLOSURE THEOREM

We start with some definitions.

**Definition:**

The distribution \( F \) with \( F(0) = 1 \) is said to be NBU if

\[
\frac{F(s + t)}{F(s)} \leq F(t) \quad \text{for all } s, t \geq 0.
\]

**Definition:**

The nonincreasing stochastic process \( \{X(t), t \geq 0\} \) is said to be NBU if, with probability 1,

\[
P(T_a > s + t \mid X(u), 0 \leq u < s) \leq P(T_a > t)
\]

for all \( s, t, a \geq 0 \), where \( T_a \) denotes the first time the process hits or goes below \( a \).

**Theorem 3:**

If the component processes are independent NBU processes then

\( \{\phi(X(t))\} \) is also NBU.

**Proof:**

Suppose first that \( \phi \) is binary and let \( T \) denote the first time the process \( \phi(X(t)) \) hits 0. Now consider

\[
P(T > s + t \mid X_i(u), 0 \leq u \leq s, i = 1, \ldots, n) =
\]

\[
E[\phi(X(s + t)) \mid X_i(u), 0 \leq u \leq s, i = 1, \ldots, n].
\]

Now it follows from the definition of an NBU process that the conditional distribution of \( X_i(s + t) \), given \( X_i(u), 0 \leq u \leq s \), is stochastically
smaller than the distribution of $X_i(t)$. Hence, from Proposition 1 we see that

$$E[\phi(X(u + t)) | X_i(u), 0 \leq u \leq s, t = 1, \ldots, n] \leq E[\phi(X(t))] = P(T > t)$$

which proves the result when $\phi$ is binary. As before we can reduce the nonbinary case to the above by defining $\phi_a(x) = \begin{cases} 1 & \text{if } \phi(x) > a \\ 0 & \text{if } \phi(x) \leq a \end{cases}$.

**Example 3: A Semi-Markov NBU Process**

If $\{X(t), t \geq 0\}$ is a semi-Markov process such that $X(0) = m$, $P_{i,i-1} = 1$, $i > 0$, $P_{00} = 0$ and the time in state $i$ is NBU then $\{X(t)\}$ is NBU. This follows from the fact that NBU is preserved under convolution.

**Example 4: An NBU Renewal Shock Model**

Suppose that $X(0)$ is fixed and that at random time points $X(t)$ is decreased by an i.i.d. nonnegative amount. If the random time points occur in accordance with a renewal process whose interarrival distribution is NBU then it is easy to show that $\{X(t)\}$ is NBU. Formally

$$X(t) = \max \left\{ \sum_{i=1}^{N(t)} X_i, 0 \right\}$$

where $X_i$ are i.i.d., independent of $\{N(t)\}$ which is a renewal process with an NBU interarrival distribution.

**Example 5: An NBU Poisson Shock Model**

See Barlow-Proshan [1], p. 160.
4. APPLICATIONS TO STOCHASTIC NETWORKS

The preceding theory can be used in the study of stochastic networks. For instance consider a network and number the arcs. If we let $x_i$ denote the capacity of the $i^{th}$ arc then $\phi(x)$ could be set equal to such quantities of interest as

(i) $\phi_1(x) = \max \text{ flow from } s \text{ to } t$

(ii) $\phi_2(x) = \max \min_{P \in P} x_i$

where $s$ and $t$ are two given nodes of the network and $P$ denotes the set of all paths from $s$ to $t$. Thus $\phi_1(x)$ equals the maximal flow that can be sent from $s$ to $t$ subject to the arc capacities, and $\phi_2(x)$, the maximal flow along a single path (previously studied by Barlow [2]). If we interpret $x_i$ as a cost of sending a unit flow along the $i^{th}$ arc we could be interested in

$$\phi_3(x) = \min_{P \in P} \sum_{i \in P} x_i$$

the minimal cost path from $s$ to $t$. Clearly there are many possible functions $\phi$ of potential interest, and as it is reasonable to suppose that arc capacities (or costs) could change stochastically with time the results of this paper should be applicable. [For $\phi_3$ it may be reasonable to suppose that the $x_i$ increase over time].
REFERENCES


