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ON PLATOON FORMATION ON TWO-LANE ROADS

by

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1. Introduction and Summary

This communication studies a simplified model for platoon (bunch) formation on two-lane two-way highways.

The behavior of vehicles on two-lane two-way highways is a very complex process and several models have been proposed by different authors under various simplified assumptions. Usually the objectives of such studies are to derive the distribution function of the number of vehicles in a platoon and to find the average speed of a fast test car moving in a stream of slow vehicles.

The models for traffic flow on roads may be divided, according to their method of study, into two groups, microscopic models and macroscopic models. In the micro approach a detailed description of the behavior of individual vehicles is the basis for the construction of the model. The macro approach, on the other hand, studies the behavior of sizable groups of vehicles without specifying the behavior of any single vehicle.

Several models have been proposed for traffic flow on two-lane roads. Tanner [9] assumes that traffic in one direction is moving at a constant speed $v$ while traffic in the opposite direction moves at a
constant speed $V$. Spacings between bunches are assumed to be exponentially distributed. The purpose of this study is to determine the average speed of a single vehicle having a free speed of $u$ ($u > v$) and traveling in the $v$ stream. Miller [7] discussed a macro model for the determination of the average passing rate from bunches. He assumed exponential spacings between platoons and that the platoons are in a state of equilibrium in the sense that the average rate at which vehicles join a bunch equals the rate at which they leave it. Taylor, Miller, and Ogden [10] compared numerical results of several bunching models using both experimental and simulated bunch size data. Galin and Epstein [5] studied the steady-state situation on a road in which passing is possible only in passing points located at equal distances along the road. Models for traffic flow in a no-passing zone are proposed by Cowan [2], Hodgson [6], and Epstein, Galin, and Shlifer [4].

The present study is an extension of the models proposed by Galin and Epstein [5], Cowan [2], Hodgson [6], and Epstein, Galin, and Shlifer [4]. We assume that the road consists of alternating free-passing zones and no-passing zones in each traffic direction. The free-passing zones are named Type I sections and the no-passing zones are named Type II sections. It is also assumed that all road sections of the same type have the same length. What we in fact have is a sequence of no-passing zones of fixed length separated by a sequence of free-passing zones of fixed length.

In reality the lengths of the Type I and Type II sections are random variables dependent on traffic in the opposite direction. However, we assume that the Type I and Type II sections have constant lengths because it enables us to analyze traffic in one direction independently of traffic in the opposite direction. Even under this simplified assumption, the analysis is quite complex. We believe that the present model may give some insight into the mechanism of platoon formation. Moreover, there are situations in which this model may in fact provide an approximation to the behavior of vehicles in a road. This will be the case when traffic intensity is low and passing is frequently prohibited due to sight and road conditions.
A summary of the paper now follows. In Section 2 we give a detailed description of our model and its underlying assumptions. We also derive several properties which will provide the basis for the study of platoon formation in Section 3. Section 4 contains a numerical example and Section 5 is a discussion of the model.

2. The Model and Some Preliminary Results

Consider a two-lane two-way highway and assume that vehicles enter and leave it only at its end points. The highway consists in each traffic direction of alternating Type I and Type II sections. As traffic in one direction is (assumed) independent of traffic in the opposite direction, we will focus our attention on traffic moving in a traffic direction that will be named "our" direction. Let $l_1$ denote the length of a Type I section in our direction, while $l_2$ denotes the length of a Type II section.

Assume that there are two types of zero size vehicles moving in the highway; slow vehicles have a free speed $v_1$ and fast vehicles have a free speed $v_2$ ($v_2 > v_1$). Input processes of slow and fast vehicles are independent Poisson processes with arrival rate $\lambda_1$ for the slow vehicles and $\lambda_2$ for the fast vehicles. Regarding the movement of vehicles, we shall assume the following. A slow vehicle always maintains its free speed $v_1$. A fast vehicle moves at its free speed $v_2$ except when it comes up against a slow vehicle in a Type II section. When this happens the fast vehicle slows down immediately and follows the slow one at a zero distance up to the end of the section. At the end of the Type II section it immediately passes the slow vehicle and resumes its free speed $v_2$.

We assume that fast vehicles do not disturb one another. Hence, the analysis of the movement of fast vehicles along the highway can be carried out through analyzing the movement of a typical fast vehicle — a "test vehicle." Let $t_0$ denote the time at which the test car arrives.
at the highway, and let \( \{ \tau_n : n=1,2,\ldots \} \) be the interarrival times of the slow vehicles which precede it. Thus, the slow vehicle closest to the entrance arrives \( \tau_1 \) time units prior to \( t=0 \), the slow vehicles in front of it arrives \( \tau_1 + \tau_2 \) time units prior to \( t=0 \), etc. By assumption and by the well-known properties of Poisson processes, it is clear that \( \tau_1, \tau_2, \ldots \) are i.i.d. exponential random variables with mean \( \lambda_1^{-1} \).

The fast test vehicle now starts its trip on the highway. The following are some fundamental observations regarding its movement and interactions with other vehicles:

(a) Let \( D_m(u) \), \( (m=1,2,\ldots ; u \leq \ell_1) \) denote the distance between the test car and the closest slow vehicle ahead of it when the test car is at a distance \( u \) from the beginning of the \( m \)th Type I road section. Then \( D_m(u) \) are i.i.d. exponential random variables, all with mean \( \lambda_1^{-1} \).

(b) Consider two fast vehicles, say No. 1 and No. 2, and suppose No. 1 is the one that arrives first at our highway. Suppose also that No. 1 is impeded by a slow vehicle (henceforth Vehicle A) at the \( i \)th Type II section. Then if No. 2 is impeded by A at the \( (i+j) \)th Type II section (\( j \geq 1 \)), then No. 1 and No. 2 will never be in the same platoon.

We first establish (a). Let

\[
D_n = \tau_n v_1, \quad n=1,2,\ldots
\]

and denote by \( f_n(\cdot) \) the probability density function of \( S_n \). By definition, \( D_1(0) = D_1 \), and since \( D_n \) are independent and exponentially distributed (parameter \( \lambda_1/v_1 \)) it follows that \( f_n(\cdot) \) is a gamma density with parameters \( (n, \lambda_1/v_1) \). The distance between the test car and the \( n \)th preceding slow vehicle at \( t=0 \) is \( S_n \), and when the test car is at a distance \( u \) (\( u \leq \ell_1 \)) from the entrance, this distance reduces to
If the test car has already passed the \( n \)th slow vehicle, \( S'_n \) is negative.

Let \( F_{m,u}(\cdot) \) denote the distribution function (d.f.) of \( D_m(u) \); then

\[
F_{1,u}(x) = \Pr \left[ 0 \leq D_1 - u(1 - v_1/v_2) \leq x \right] 
\]

\[
+ \sum_{n=2}^{\infty} \Pr \left[ S_{n-1} - u(1 - v_1/v_2) < 0, 0 \leq S_n - u(1 - v_1/v_2) \leq x \right]
\]

\[
= e^{-\lambda_1/v_1}u(1 - v_1/v_2) - e^{-(\lambda_1/v_1)}(u(1 - v_1/v_2) + x)
\]

\[
+ \sum_{n=1}^{\infty} \int_{y=0}^{\infty} f_n(y) \Pr \left[ u(1 - v_1/v_2) - y < D_n \leq u(1 - v_1/v_2) - y + x \right] dy
\]

\[
= 1 - e^{-\lambda_1/v_1}y.
\]

Now set

\[ k = \min \left\{ n : S_n - \ell_1 (1 - v_1/v_2) \geq 0 \right\}, \]

and obtain

\[ D_2(0) = \begin{cases} 
\ell_{k+1}v_1, & \text{if } D_1(\ell_1) \leq \ell_2(1 - v_1/v_2) \\
D_1(\ell_1) - \ell_2(1 - v_1/v_2), & \text{if } D_1(\ell_1) > \ell_2(1 - v_1/v_2).
\end{cases} \]

From this we conclude that \( D_2(0) \) is an exponential random variable (r.v.) due to the exponentiality of \( \ell_{k+1}v_1 \) and \( D_1(\ell_1) \), and the lack of memory property of this distribution. Thus starting with \( D_1(0) \) as exponential (with parameter \( \lambda_1/v_1 \)) we find that \( D_1(u) \) has the same distribution for \( u \leq \ell_1 \), which in turn leads to \( D_2(0) \) following also the same exponential distribution. By induction, it follows then that \( D_m(u), (0 \leq u \leq \ell_1) \) is exponential with parameter \( \lambda_1/v_1 \) for any \( m = 1,2,3,\ldots \).
Now we establish (b). Let \( x_{mn}^k \) denote the length of time Vehicle No. \( k \), \( k=1,2 \), maintains a speed \( v_n \), \( n=1,2 \), for the \( m \)th time after passing Vehicle A. The values of \( x_{mn}^k \) are determined by \( v_1, v_2, x_1, x_2 \), and the distances between A and the slow vehicles preceding it. None of the values of these parameters changes, hence,

\[
x_{mn}^1 = x_{mn}^2, \quad n=1,2; \quad m=1,2,3,\ldots.
\]

Now we assume that No. 1 passes A at time \( t \), and therefore it passes the \( i \)th impeding vehicle at

\[
t_{i}^{1} = t + \sum_{m=1}^{i} x_{m1}^1 + \sum_{m=1}^{i} x_{m2}^1.
\]

No. 2 passes the \( j \)th impeding vehicle at

\[
t_{j}^{2} = t + \sum_{m=1}^{j} \frac{x_{m1}^2}{v_1} + \sum_{m=1}^{j} x_{m2}^2 = t + \sum_{m=1}^{j} \frac{x_{m1}^1}{v_1} + \sum_{m=1}^{j} x_{m1}^1 + \sum_{m=1}^{j} x_{m1}^1 > t_{i}^{1};
\]

consequently, No. 1 and No. 2 will never simultaneously pass any slow vehicle, which means that they will never move in the same platoon. This proves Statement (b).

From here we can make two further conclusions essential to the analysis of the next section:

(c) Vehicle No. 2 will never move in a platoon with No. 1 if it passes Vehicle A later than the end of the \((i+1)\)st Type II section.

(d) If the distance between No. 1 and No. 2 exceeds \( (l_{1} + l_{2}) \times (1/v_1 - 1/v_2) \), then No. 1 and No. 2 will never move in the same platoon.

To see why (c) is true, imagine that there is a (fictitious) vehicle in a platoon which is moving after A in the \((i+1)\)st Type II section. Then by (a) the fictitious vehicle will never come up against No. 1. Hence, since fast vehicles never pass one another, then fast vehicles moving behind the fictitious one will never come up against No. 1. Assertion (d) is now
obvious since No. 2 will be unable to pass a slow vehicle, which impedes No. 1 at a Type II section before the beginning of the next consecutive Type II section.

3. The Platoon Formation

We will now determine the distribution function of the number of fast vehicles in platoons which arrive at the ends of Type II sections. A platoon is formed when several fast vehicles together are impeded by a slow vehicle. When such a platoon arrives at the end of the Type II section in which it is formed, the fast vehicles pass the slow leader, continue moving as a fast platoon, and the zero relative distances among the constituent fast vehicles never change. The platoon may join (or be joined by) other platoons later. In the discussion here we do not differentiate between platoons moving at their free speed \( v_2 \) and platoons moving at a speed \( v_1 \) (although this separation may be added).

We begin with the determination of the distribution of the time spent in a Type II section. From (a) of Section 2 we deduce that the times spent by fast vehicles in Type II sections are i.i.d. random variables. Let \( T \) denote the length of a time period spent by a fast vehicle in the \( m \)th Type II section, and let \( F_T(*) \) denote the d.f. of \( T \). Clearly,

\[
T = \max \left\{ \frac{\ell_2}{v_2} ; \left( \frac{\ell_2 - \frac{N}{m}(\ell_2)}{v_1} \right) \right\},
\]

hence

\[
F_T(t) = \begin{cases} 
0 & , t < \frac{\ell_2}{v_2} \\
-\lambda_1 \left( \frac{\ell_2}{v_1} - t \right) & , \frac{\ell_2}{v_2} \leq t \leq \frac{\ell_2}{v_1} \\
1 & , \frac{\ell_2}{v_1} < t 
\end{cases}
\]

and

\[
F_T(dt) = \begin{cases} 
-\lambda_1 \left( \frac{\ell_2}{v_1} - 1/v_2 \right) e & , t = \frac{\ell_2}{v_2} \\
-\lambda_1 \left( \frac{\ell_2}{v_1} - t \right) e^{\lambda_1 dt} & , \frac{\ell_2}{v_2} \leq t \leq \frac{\ell_2}{v_1} \\
0 & , \text{ otherwise.}
\end{cases}
\]

(3.1)
Using a different approach, $F_T(t)$ was calculated previously in [4].

Now we define:

A Type A interval of order $m$ as a time interval $(t, t+x]$ satisfying (i) all fast vehicles which arrive at the highway in this interval are unimpeded in the first $m$ Type II sections, and (ii) the fast vehicles which arrive at the highway at $t-$ (an instant before $t$) or at $(t+x)+$ (an instant after $t$) are impeded in at least one of the first $m$ Type II sections; and

A Type B interval of order $m$ as a time interval $(t, t+x]$ satisfying (i) the fast vehicles which arrive at the highway in $(t, t+x]$ form a platoon while arriving at the end of the $m$th Type II section, and (ii) vehicles that do not arrive at the highway in this interval do not move in this platoon at that point.

Since the arrivals of fast vehicles at the highway are assumed independent and Poisson, one can calculate the distribution function of the number of fast vehicles in a platoon if the distribution function of the Type B intervals is known. We therefore begin with the determination of this distribution.

We notice that the Type A and Type B intervals of order $m$ (formed) in any time interval $(t_1, t_2]$ constitute a partition of this interval.

Furthermore, this partition is a fixed function of $\ell_1, \ell_2, v_1, v_2$ and of the arrivals at the highway of slow vehicles in $(t_1 - m(\ell_1 + \ell_2)(1/v_1 - 1/v_2), t_2]$. (The subtraction of $m(\ell_1 + \ell_2)(1/v_1 - 1/v_2)$ results from the fact that while moving in a stretch of road of length $m(\ell_1 + \ell_2)$, a fast vehicle may pass slow vehicles which arrived at the beginning of this section no earlier than $m(\ell_1 + \ell_2)(1/v_1 - 1/v_2)$ ahead of it.) Since the slow vehicles arrive according to a Poisson process and since $\ell_1, \ell_2, v_1,$ and $v_2$ are constants, then the partition is a stationary process. Moreover, the Type A and Type B intervals in $(t_1, t_2]$ are not independent due to the cyclical effect caused...
by the entrance of any given slow vehicle to consecutive Type II sections every fixed time period -- \((\ell_1 + \theta_2) / v_1\). We will determine therefore the marginal density of the Type B intervals of order \(m\).

To determine the marginal density of a Type B interval, we calculate \(S(u,m)\), the probability that two fast vehicles, arriving at the highway \(u\) unit of time apart, move in one platoon at the end of the \(m\)th Type II section. Let \(T_j\) denote the time spent by the leading fast vehicle in the \(j\)th Type II section. Denote by \(U_j\) the interarrival time of these fast vehicles at the end of the \(j\)th Type II section, and define \(U_0 = u\). Let \(F_{U_1,T_1}\) and \(G_{U_j,T_j}\) be the joint distribution functions of \((U_1, T_1)\) and \((U_j, T_j)\), \(j > 1\), respectively. While calculating \(S(u,m)\) the following \([(3.2) - (3.5)]\) must be taken into account:

\[
S(u,m) = P[U_m = 0 \mid U_0 = u] \tag{3.2}
\]

\(U_k = 0\) implies that \(U_i = 0\) for all \(i \geq k\). \(\tag{3.3}\)

Let \(D_{S,j}\) denote the distance between the second fast vehicle and the first slow vehicle preceding it when the fast vehicle is at the entrance of the \(j\)th Type II section, and let

\[
H = 1/v_1 - 1/v_2 .
\]

Then

\[
(U_m = 0 \text{ and } T_j = \ell_2 / v_2 \text{ for } j \leq m) \text{ implies that } \tag{3.4} \]

\[
(U_j < (\ell_1 + \ell_2)H \text{ and } D_{S,j+1} \geq v_1[U_j - \ell_1 H]^+),
\]

where \([a]^+\) is the positive part of \(a\). The condition on \(U_j\) results from (d) of Section 2 and the information on \(D_{S,j+1}\) is due to the leading fast vehicle having been unimpeded in the \(j\)th Type II section. Finally,

\[
(U_m = 0 \text{ and } T_j > \ell_2 / v_2 \text{ for } j \leq m) \text{ implies that } U_j < \ell_1 H . \tag{3.5}
\]
Unless (3.5) holds, No. 2 is not able to pass the slow vehicle that impeded No. 1 at the jth Type II section before the beginning of the (j+1)st Type II section. From (a) of Section 2 and Equations (3.4) and (3.5), we deduce that 

\[ G_{\text{II}}^{T}(\cdot, \cdot), \quad j=2,3,\ldots \]

are identical. Furthermore, we realize that the reason for the difference between \( F_{\text{II}}^{T}(\cdot, \cdot) \) and 

\[ G_{\text{II}}^{T}(\cdot, \cdot) \]

is that when \( T_{j} = \ell_{2}/v_{2}, \quad j \geq 1 \), we have prior information on \( DS_{j+1} \).

The probability \( S(u,1) \) has a simple expression (see Equation (A.1) in the appendix). The determination of \( S(u,m), \quad m \geq 2 \), is carried out as follows. Define

\[
Q_{m-1}(u_{m-2}, t_{m-2}) = \int_{t_{1}}^{t_{2}} \int_{u_{0}}^{u_{1}} dG_{m-1}^{T}(u, t \mid U_{m-2} = u_{m-2}, T_{m-2} = t_{m-2})
\]

\[
\cdot P[U_{m}=0 \mid U_{m-1}=u, T_{m-1}=t], \quad m=2,3,\ldots
\]

(3.6)

and

\[
Q_{j}(u_{j-1}, t_{j-1}) = \int_{t_{1}}^{t_{2}} \int_{u_{0}}^{u_{1}} dG_{j}^{T}(u_{j}, t \mid U_{j-1}=u_{j-1}, T_{j-1}=t_{j-1})Q_{j+1}(u_{j}, t),
\]

\[ j=2,3,\ldots,m-2 . \]  

(3.7)

Hence

\[
Q_{1}(u,m) = S(u,m) = \int_{t_{1}}^{t_{2}} \int_{u_{0}}^{u_{1}} dF_{\text{II}}^{T}(u_{1}, t \mid U_{0}=u)Q_{2}(u_{1}, t), \quad (3.8)
\]

where

\[
t_{0} = (\ell_{1}+\ell_{2})(1/v_{1} - 1/v_{2}), \quad t_{1} = \ell_{2}/v_{2}, \quad \text{and} \quad t_{2} = \ell_{2}/v_{1}. \quad (3.9)
\]

(3.9)

The conditional distributions of \( G \) and \( F \) and the probability

\[ P[U_{m}=0 \mid U_{m-1}=u, T_{m-1}=t] \]

are derived in the appendix.
Now we determine the distribution function of a Type B interval. Let \( A(m) \) denote the event that the first of the two fast vehicles is impeded in at least one of the first \( m \) Type II sections, and let \( B(u,m) \) denote the event that the time from a random arrival to the end of the first Type B interval of order \( m \) is longer than \( u \). Clearly,

\[
S(u,m) = P[A(m) \cap B(u,m)] = P[A(m)]P[B(u,m) | A(m)] , \tag{3.10}
\]

\[
1 - P[A(m)] = P[T_1 = \ell_2/v_2, T_2 = \ell_1/v_2, ..., T_m = \ell_2/v_2] ,
\]

and due to the independence of \( T_i \), \( i=1,...,m \), one obtains

\[
1 - P[A(m)] = \left( P[T = \ell_2/v_2] \right)^m = e^{-m \lambda \ell_2 H} . \tag{3.11}
\]

Now let \( X(m) \) denote a Type B interval of order \( m \), let \( X'(m) \) denote a Type B interval of order \( m \) containing a random arrival, and let \( F_{X(m)}(\cdot) \) and \( F_{X'(m)}(\cdot) \) denote the respective distributions. It is known that

\[
dF_{X'(m)}(x) = \begin{cases} \frac{x dF_{X(m)}(x)}{E[X(m)]} , & 0 \leq x \leq (\ell_1 + \ell_2)H \\ 0 , & \text{otherwise} \end{cases} \tag{3.12}
\]

hence, because the allocation of a random point is uniformly distributed in \( X'(m) \), we obtain

\[
P[B(u,m) | A(m)] = \int_{x=u}^{x_0} \frac{x dF_{X(m)}(x)}{x E[X(m)]} . \tag{3.13}
\]

We insert (3.13) and (3.11) into (3.10) and get

\[
S(u,m) = 1 - e^{-m \lambda \ell_2 H} \left[ \int_{x=u}^{x_0} x dF_{X(m)}(x) - u \left( 1 - F_{X(m)}(u) \right) \right] . \tag{3.14}
\]

Differentiating (3.14) with respect to \( u \) and denoting

\[
S'(u,m) = \frac{d}{du} S(u,m)
\]

yields
Using the fact that $F_{X(m)}(0) = 0$, we get from (3.15) that

$$E[X(m)] = \frac{1 - e^{-m\lambda u}}{S'(0,m)} \tag{3.16}$$

and

$$F_{X(m)}(u) = \frac{S'(u,m)}{S'(0,m)} \tag{3.17}.$$ 

The expression $S'(u,m)$ is simple for $m=1$ and becomes more messy as $m$ increases.

We are now able to calculate the probability function of the number of fast vehicles in a platoon. Let $N(m)$ denote the number of fast vehicles in a platoon arriving at the end of the $m$th Type II section, let $P_{N(m)}(n)$ denote its probability function, and let $C(m)$ denote the event that the platoon has been impeded in at least one of the first $m$ Type II sections. Clearly,

$$P_{N(m)}(n) = P[C(m)]P[N(m)=n | C(m)] + P[\bar{C}(m)]P[N(m)=n | \bar{C}(m)],$$

where $\bar{C}(m)$ is the complement of $C(m)$. The arrival of fast vehicles at the highway is Poisson, hence

$$P[N(m)=n | \bar{C}(m)] = \begin{cases} 1, & n = 1 \\ 0, & \text{otherwise.} \end{cases} \tag{3.18}$$

Denote by $q_{N(m)}(n)$ the probability that $n$ fast vehicles arrive at a Type B interval.

$$q_{N(m)}(n) = \int_{0}^{\frac{n}{\lambda}} \frac{(\lambda x)^n}{n!} e^{-\lambda x} dF_{X(m)}(x), \tag{3.19}$$

and therefore

$$P[N(m)=n | C(m)] = \frac{q_{N(m)}(n)}{1 - q_{N(m)}(0)}, \quad n=1,2,3,\ldots. \tag{3.20}$$
The platoons that are impeded in at least one of the \( m \) first Type II sections arrive at the end of the \( m \)th Type II section at a rate of 
\[
\lambda^2 \left( 1 - e^{-\lambda_1 t_2^2 H} \right) / E[N(m) | C(m)]
\]
the platoons that are not impeded arrive at the same point with rate \( \lambda_2 e^{-\lambda_2 t_2^2 H} \); hence

\[
P[C(m)] = \frac{\left( 1 - e^{-\lambda_1 t_2^2 H} \right)}{E[N(m) | C(m)]} 
\]

We calculate the conditional expectation of \( N(m) \) using (3.19) and (3.16), and the result we insert into (3.21) to obtain

\[
P[C(m)] = \frac{-\left( 1 - q_{N(m)}(0) \right) S'(0,m) + \lambda_2 e^{-\lambda_2 t_2^2 H} \cdot}{-\left( 1 - q_{N(m)}(0) \right) S'(0,m) + \lambda_2 e^{-\lambda_2 t_2^2 H}}
\]

Combining (3.22), (3.20), (3.19), (3.18), (3.17), (3.6), (3.7), and (3.8) yields the desired probability function, from which we obtain

\[
E[N(m)] = \frac{\lambda_2 E[X(m)]}{\left( 1 - e^{-\lambda_1 t_2^2 H} \right) - \lambda_2 \int_0^{t_0} e^{-\lambda_2 x} dF_{X(m)}(x) + \lambda_2 E[X(m)] e^{-\lambda_1 t_2^2 H}}
\]

The expectation of \( N(m) \) can be bounded without calculating \( dF_{X(m)}(x) \). For this we use Jensen's inequality to get

\[
\lambda_2 \int_0^{t_0} e^{-\lambda_2 x} dF_{X(m)}(x) > e^{-\lambda_2 t_0} E[X(m)]
\]

and as

\[
e^{-\lambda_2 x} \leq 1 - \frac{-\lambda_2 t_0}{t_0} x, \text{ for any } 0 \leq x \leq t_0,
\]

we obtain

\[
\lambda_2 \int_0^{t_0} e^{-\lambda_2 x} dF_{X(m)}(x) \leq 1 - \frac{-\lambda_2 t_0}{t_0} E[X(m)]
\]

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Using (3.24) and (3.25) we get
\[
E_1[N(m)] = \frac{\lambda_2 E[X(m)]}{\left(1 - e^{-m\lambda_1 t_0} \right) \left(1 - e^{-m\lambda_2 t_0} \right) + \lambda_2 E[X(m)] e^{-m\lambda_1 t_0}} 
\]

It is obvious that \( E[N(m)] \) is nondecreasing, hence its upper and lower bounds, \( E_1[N(m)] \) and \( E_2[N(m)] \), respectively, can be established from \( E_1[N(m)] \) and \( E_2[N(m)] \) as follows:
\[
E_1[N(m)] = \max\{E_2[N(m)], E_1[N(m-1)]\},
\]
and
\[
E_2[N(m)] = \max\{E_1[N(m)], E_2[N(m-1)]\}.
\]

The expected number of fast vehicles in a platoon behind a slow vehicle can be calculated without applying the procedure outlined above. Let \( J(l) \) denote the number of fast vehicles in a platoon moving behind a slow vehicle at a distance \( l \) from the entrance, and let \( r_1(l) \) be the probability that a fast vehicle is traveling at a speed \( v_1 \) at that point. We now show that
\[
E[J(l)] = \frac{\lambda_2}{\lambda_1} r_1(l).
\] (3.26)

To prove (3.26), suppose we observe the arrival process of vehicles at the point located at a distance \( l \) from the entrance. Let \( t \) denote the length of the observation period and let \( J_1(t,l) \) and \( J_2(t,l) \) denote the number of slow and fast vehicles, respectively, which arrive at that point in the time interval considered. Denote by \( y_i \), \( i=1,2,...,J_1(t,l) \), the number of fast vehicles in the platoon behind the \( i \)th slow vehicle. Using the strong law of large numbers, one may obtain
The left-hand side of (3.27) can be rewritten as

\[
\lim_{t \to \infty} \frac{\sum_{i=1}^{\infty} Y_i}{J_1(t, \xi)} = \lim_{t \to \infty} \frac{\sum_{i=1}^{\infty} Y_i}{J_2(t, \xi)} \frac{J_2(t, \xi)}{t} \frac{t}{J_1(t, \xi)} = \lim_{t \to \infty} \frac{J_1(t, \xi)}{J_2(t, \xi)} \lim_{t \to \infty} \frac{\sum_{i=1}^{\infty} Y_i}{t} \lim_{t \to \infty} \frac{t}{J_1(t, \xi)}.
\]

(3.28)

Using the strong law of large numbers for the three series in (3.28) yields the desired result. The probability \( r_1(\xi) \) is obtained from (3.1) and is given by

\[
r_1(\xi) = \begin{cases} 
0 & \text{if } \xi - \left[ \frac{\xi}{\ell_1 + \ell_2} \right] (\ell_1 + \ell_2) \leq \ell_1 \\
- \lambda_1 x (1/\nu_1 - 1/\nu_2) & \text{if } 0 < x = \xi - \left[ \frac{\xi}{\ell_1 + \ell_2} \right] (\ell_1 + \ell_2) - \ell_1 < \ell_2, \\
1 - e^{-\lambda_1 x (1/\nu_1 - 1/\nu_2)} & \text{if } \ell_1 < x < \ell_2 
\end{cases}
\]

(3.29)

where \([a]\) is the biggest integer which is equal to or smaller than \(a\).

4. Numerical Examples

We calculate here \( E[X(1)] \), \( E[X(2)] \), \( E[N(1)] \), and the upper and lower bounds of \( E[N(1)] \) and \( E[N(2)] \). Using the procedure outlined in the previous section, we obtain

\[
E[X(1)] = \frac{1 - e^{-\frac{\lambda_1 \ell_2 H}{\lambda_1}}}{\lambda_1},
\]

\[
E[X(2)] = \frac{1 - e^{-\frac{2\lambda_1 \ell_2 H}{\lambda_1}}}{\lambda_1 + 2\lambda_1 e^{-\frac{\lambda_1 \ell_2 H}{\lambda_1}} - \frac{1}{2} \lambda_1 e^{-\frac{2\lambda_1 \ell_2 H}{\lambda_1}}} - 15 - 2\lambda_1 \ell_2 H.
\]
and
\[
E[N(1)] = \frac{1}{\lambda_1 + \lambda_2} \left( \frac{1}{1 - e^{-\left(\lambda_1 + \lambda_2\right)\bar{v}_2 h}} + e^{-\lambda_1 \bar{v}_2 h} \right).
\]

(note, \(E[X(1)] < E[X(2)]\)). The following are the numerical values assumed: \(v_1 = 60\text{ km/h}\), \(v_2 = 80\text{ km/h}\), \(\lambda_2 = 225\text{ vehicles per hour}\), and \(\ell_1 = \ell_2 = 1.0\text{ Km}\) (see graphs).
5. Discussion

This study discusses the platoon formation in a two-lane two-way highway under low to moderate traffic intensity, i.e., up to 4000 vehicles per day in each traffic direction. Under this traffic load, it is reasonable to assume Poisson arrivals of vehicles at the highway (see Taylor et al. [10] and Breiman [1]). As for the assumptions on the sizes of the vehicles, distances between vehicles in platoons, and the passing mechanism, none of them seems to be too restrictive under this traffic intensity. We believe that the most restrictive assumption here is that the highway is divided into Type I and Type II sections having constant lengths. Actually, it would have been more realistic to assume that the lengths of these sections are random variables, since they depend on sight and road conditions and on traffic in the opposing direction. Nevertheless, we preferred this assumption because it enabled us to analyze the movements of vehicles in one lane independently of the traffic in the other lane, and consequently to study the process of platoon formation, a difficult and complex process under any set of reasonable assumptions.

We note that the lengths of Type I and Type II road sections may be determined as functions of the traffic in the opposite lane according to the following procedure. Let $Y$ denote the interarrival times of consecutive vehicles at the entrance of the opposite lane, let $F_Y(\cdot)$ be the distribution of $Y$, and let $d$ denote the minimal interarrival time which enables safe passing. Hence,

$$
\ell_1 = (E[Y>d] - d)v_2,
$$

and $\ell_2$ satisfies

$$
\frac{\ell_2}{v_2} = \int_{x=0}^{d} (y + \frac{\ell_2}{v_2})dF_Y(y),
$$

which yields

$$
\ell_2 = \frac{\int_{0}^{d} ydF_Y(y)}{1 - F_Y(d)} v_2.
$$
Since platoon formation increases the gaps between consecutive platoons, we may actually assign to $\ell_1$ a larger value and to $\ell_2$ a smaller value than the ones calculated from the expression above.

We would also like to point out that even though the result that the length of a Type B section is bounded from above by $(\ell_1 + \ell_2)H$, and consequently the expected length of the platoon is bounded by $\lambda_2(\ell_1 + \ell_2)H$ is derived under the assumption that $\ell_1$ and $\ell_2$ are constants, we expect that under more general assumptions it can be shown that the expected length of a platoon is bounded by an equivalent expression.

Our final remark is in regard to three-lane highways, which are not too common. For such highways the present model may provide a very good fit if the center lane is assigned alternately to one of the traffic directions as a passing lane.
REFERENCES


We calculate here some distributions needed for the evaluation of $S(u,n)$. Let $R_j$ denote the time spent by No. 2 at the $j$th Type II section, and let $M_j(t)$ be the number of slow vehicles which arrive at the $j$th Type II section in the time interval which begins when No. 1 enters the Type II section and ends $t$ units of time later. To simplify the notations define $T_0 = 0$. Clearly,

$$P[U_0 = 0 | U_0 = u, T_0 = t] = P[M_0(t) = 0, T_1 = \frac{t_2}{v_2} + u] = P[M_0(t) = 0] P[T_1 = \frac{t_2}{v_2} + u].$$

$$= \begin{cases} 
1, & \text{if } u = 0 \\
-\lambda_1(1 - e^{-\lambda_1(t_2-u)}) / \lambda_1, & \text{if } u > 0 \\
0, & \text{otherwise.}
\end{cases}$$

For the evaluation of $F$ we use the relation

$$U_j = U_{j-1} + (R_j - T_j).$$

Hence,

$$F_{u_0}(u,t | U_0 = 0) = P[U_1 \leq u_1, T_1 \leq t | U_0 = u] = P[R_1 = u_1 - u + T_1, T_1 \leq t | U_0 = u].$$

$$= \frac{e^{-u_1 - ut} \cdot \int_{u_1}^{t} e^{-u_1} dR, 0 
\text{ otherwise.} \quad (A.1)$$

For the evaluation of $F$ one can derive $dF$, which yields, in the case $u \leq t_2 H$:

$$dF_{u_0}(u_1,t | U_0 = u) = \begin{cases} 
-\lambda_1 u - \lambda_1 t, & u_1 = 0, t \leq \frac{t_2}{v_2} \\
-\lambda_1 t, & u_1 = 0, t > \frac{t_2}{v_2} \\
-2\lambda_1 \frac{t_2}{v_2} \cdot \lambda_1 (u_1 - u) \cdot du_1 dt, & u_1 > 0, t < \frac{t_2}{v_2} \\
-2\lambda_1 \frac{t_2}{v_2} \cdot \lambda_1 (2t_2 - u_1 - u) \cdot du_1 dt, & u_1 > 0, t > \frac{t_2}{v_2} \\
-2\lambda_1 \frac{t_2}{v_2} \cdot \lambda_1 (2t_2 - u_1 - u) \cdot du_1 dt + \lambda_1 \frac{t_2}{v_2} + u, & u_1 = 0, \frac{t_2}{v_2} + u \leq t \leq \frac{t_2}{v_2} \\
-\lambda_1 u - \lambda_1 (t_2 - t) \cdot dt, & u_1 = 0, t \leq \frac{t_2}{v_2} \\
0, & \text{otherwise.} \quad (A.2)$$

When $u > t_2 H$, then $T_1$ and $R_1$ are i.i.d. distributed according to (3.1). If $u \leq t_2 H$, then the distribution of $T_1$ is as before, but here $T_1$ and $R_1$ are not independent and the calculation of $F$ is based on the fact that $M_0(t)$ is a Poisson random variable with parameter $\lambda_1 u$. After evaluation of $F$ one can derive $dF$, which yields, in the case $u \leq t_2 H$:

$$dF_{u_0}(u_1,t | U_0 = u) = \begin{cases} 
-\lambda_1 u - \lambda_1 t, & u_1 = 0, t \leq \frac{t_2}{v_2} \\
-\lambda_1 t, & u_1 = 0, t > \frac{t_2}{v_2} \\
-2\lambda_1 \frac{t_2}{v_2} \cdot \lambda_1 (u_1 - u) \cdot du_1 dt, & u_1 > 0, t < \frac{t_2}{v_2} \\
-2\lambda_1 \frac{t_2}{v_2} \cdot \lambda_1 (2t_2 - u_1 - u) \cdot du_1 dt, & u_1 > 0, t > \frac{t_2}{v_2} \\
-2\lambda_1 \frac{t_2}{v_2} \cdot \lambda_1 (2t_2 - u_1 - u) \cdot du_1 dt + \lambda_1 \frac{t_2}{v_2} + u, & u_1 = 0, \frac{t_2}{v_2} + u \leq t \leq \frac{t_2}{v_2} \\
-\lambda_1 u - \lambda_1 (t_2 - t) \cdot dt, & u_1 = 0, t \leq \frac{t_2}{v_2} \\
0, & \text{otherwise.} \quad (A.3)$$
In the case \( u > E_L \), then
\[
dF_{U, t_1}(u_1, t \mid U_0 = u) =
\begin{cases}
\int_{-21_2} e^{-1_2(1_2/v_2)} & u_1 = u, t = E_L/v_2 \\
\int_{-21_2} e^{-1_2(1_2/v_2)} \lambda_1(u_1 - u) du_1 & u < u_1 \leq u + E_L, t = E_L/v_2, \\
\int_{-21_2} e^{-1_2(1_2/v_2)} \lambda_1(2u - u_1 - u) du_1 dt + \int_{-21_2} e^{-1_2(1_2/v_2)} \lambda_1(u_1 - u) dt & u_1 < u + E_L/v_2 - t, \\
0 & \text{otherwise}.
\end{cases}
\]
\[\text{(A.4)}\]

As for the evaluation of \( G \), from (3.5) and (3.6) we deduce that for any \( j \geq 2 \):

(i) When \( T_{j-1} > E_L/v_2 \) and \( U_{j-1} < E_L \), or when \( T_j = E_L/v_2 \) and \( U_{j-1} < E_L \), then \( G \neq 0 \).

(ii) When \( T_{j-1} > E_L/v_2 \) and \( U_{j-1} \geq E_L \), then No. 2 will never move with No. 1 in the same platoon. Here we define \( dG = 0 \).

(iii) When \( T_{j-1} = E_L/v_2 \) and \( U_{j-1} \geq E_L \), then \( H_j(\xi H) \) is a Poisson random variable with parameter \( \lambda_1(1_2/v_2) \) and
\[
P\{H_j(\xi H) - H_j(\xi H) = 0\} = 1.
\]

Here we have
\[
G_{U, t_1}(u_1, t \mid U_{j-1} = u_{j-1}, T_{j-1} = E_L/v_2) = \int_{-21_2} e^{-1_2(1_2/v_2)} \lambda_1(u_1 - u) du_1.
\]
\[\text{(A.5)}\]

If \( E_L < U_{j-1} < E_L \), then \( R_j \) and \( T_j \) are not independent and (A.5) yields
\[
\begin{align*}
\text{If } \max(\ell_2, \ell_1) < u_j - (\ell_1 + \ell_2)H, \text{ then } R_j \text{ and } T_j \text{ are independent, and (A.5) yields}\n\end{align*}
\[
\begin{align*}
dG_{j, k}(u_j, t_j \mid u_{j-1}, T_{j-1}, \ell_2/v) = & \\
& \begin{cases} \\
-\lambda_1(\ell_1 + \ell_2)H e^{-(u_j - (\ell_1 + \ell_2)H)u_j} \\
-\lambda_1(\ell_2/v) - \lambda_2(\ell_2/v) - \lambda_3(\ell_2/v - u_j - t_j + H) \\
\lambda_1 e^{-\lambda_1(\ell_2/v - u_j - t_j)} e^{-\lambda_1(\ell_2/v - u_j - t_j + H)u_j} \\
\lambda_1 e^{-\lambda_1(\ell_2/v - u_j - t_j)} e^{-\lambda_1(\ell_2/v - u_j - t_j + H)u_j} \\
0
\end{cases}
\end{align*}
\[
\begin{cases} \\
\ell_2 < u_j < (\ell_1 + \ell_2)H, \quad t_j = \ell_2/v \\
u_j < u_{j-1} + \ell_2/v - t \\
\ell_2/v - t < u_j < \ell_1 H + \ell_2/v - t_j, \\
u_j \neq u_{j-1} + \ell_2/v - t \\
\ell_2/v - t < u_j < \ell_2/v + u_{j-1}, \\
u_j \neq u_{j-1} + \ell_2/v - t \\
u_j - (\ell_2/v_2) < t_j < \ell_2/v_2, \quad u_j = 0 \\
otherwise. (A.6)
\end{cases}
\]
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