ON A CLASS OF POLYNOMIALS CONNECTED WITH THE KORTEWEG-DE VRIES EQUATION

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In [1] special classes of solutions of the Korteweg-de Vries equation

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were studied, in particular, all those \( u = u(x,t) \) which are rational functions of \( x \) for each value of \( t \). It turns out that these solutions are rational functions of \( t \) as well and of very special structure. In this paper we give a new construction of these solutions with emphasis on their algebraic properties.

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were studied, in particular, all those \( u = u(x,t) \) which are rational functions of \( x \) for each value of \( t \). It turns out that these solutions are rational functions of \( t \) as well and of very special structure. In this paper we give a new construction of these solutions with emphasis on their algebraic properties.

To describe the family of rational solutions of (1.1), one does well to introduce the sequence of associated Korteweg-de Vries equation

\( u_t = X_k u \quad k = 1, 2, \ldots \)

which are related by

\[ X_k = \frac{\partial}{\partial x} \frac{\delta H_k}{\delta u} \]

to the sequence of conservation laws

\[ H_k = \int P_k (u, u', \ldots) dx \]

associated with (1.1). These \( X_k \) can be recursively defined by

\[ X_{k+1}(u) = \left( u \frac{\partial}{\partial x} + \frac{3}{2} \frac{\partial^3}{\partial x^3} u - \frac{1}{2} \left( \frac{\partial}{\partial x} \right)^3 \frac{\delta H_k}{\delta u} \right) \]

as was shown originally by Lenard, see [3]. For the notation used here we refer to [6,1].

The above nonlinear differential operators commute, and so do the flows \( e(t, X_k) \) generated by them. We ask for the manifold \( M \) of rational functions invariant under all the flows \( e(t, X_k) \). It is one of the results of [1] that \( M \) decomposes into denumerably many manifolds \( M_d \) of dimensions \( d \) for \( d = 1, 2, \ldots \) and, moreover, each \( M_d \) is generated from the single
function

\[ u = \frac{d(d + 1)}{x^2} \]

by the flows \( e(\sum_{k} \tau_k X_k) \) (see [1], Section 3, Theorem 2).

In this paper we want to give a representation of \( M_d \) in terms of a class of polynomials \( \theta_d = \theta_d(\tau_1, \tau_2, \ldots, \tau_d) \) depending on \( d \) variables. These polynomials will be defined recursively in Section 2 and they allow the representation of all \( u \in M_d \) in the form

\[ u_d(x) = -2\left( \frac{2}{3x} \right)^2 \log \theta_d(\tau_1 + x, \tau_2, \ldots, \tau_d) \]

and this representation is one to one, so that \( \tau_1, \tau_2, \ldots, \tau_d \) can be viewed as global coordinates on \( M_d \). The \( X_k \) give rise to vector fields \( \Gamma_k \) on \( M_d \) which are expressible in terms of the \( \tau_j \) and in Section 5 we will determine these \( \Gamma_k \). It turns out that the \( \tau_2, \ldots, \tau_d \) can be subjected to a group of birational transformations

\[ \tau_j^* = a_j \tau_j + g_j(\tau_1, \ldots, \tau_{j-1}) \quad a_j \neq 0, \]

\( g_j \) being polynomials, without affecting the above representation. Moreover, the parameters \( \tau_j^* \) can be introduced so that

\[ \Gamma_k = \frac{\partial}{\partial \tau_k}, \]

i.e. that the solutions of \( u_t = X_k u \) are given by

\[ u_d(x, \tau_2^*, \ldots, \tau_k^* + t, \tau_{k+1}^*, \ldots, \tau_d^*) \]

In other words the \( \tau_k^* \) can be identified with \( t \)-variable \( t_k \) of \( X_k \). This picture was developed already in [1] but here this representation is made more explicit through (1.3) and the construction of the polynomials \( \theta_k \). The representation (1.3) is analogue to that of Its and Matveev [4] for the case of solutions of (1.1) having a fixed period in \( x \) and for which the corresponding Hill's equation has only finitely many simple eigenvalues.
It is conceivable that (1.3) could be obtained by a limit process from the formula [4], but we did not succeed in this way. Similarly one may expect that (1.3) could be obtained as the limit of the $N$-soliton potentials [8], but neither did we succeed in this way.

We mention that the solutions of the type (1.3) for the case $d = 2$ were considered by H. Moses [9].

The construction of the $\phi_k$ as well as the proof of the above statements are based on a transformation of differential operators $L = -D^2 + u$ into each other which is certainly not new (see [2,11,12]) and the so-called Miura transformation [7] for which we give a natural derivation. This derivation is based on the factorization

$$L = A^* A; \quad A = D - v$$

where $u = v' + v^2$. Similarly as in Lax' work [5], where one considers deformations of operators $L$ in the equivalence class of operators of the form $U^{-1} L U$ for unitary $U$, here we consider deformations of operators $A$ in the equivalence class of operators $U_1 A U_2$, where $U_1, U_2$ are two unitary operators. If we apply these ideas to formal differential operators we are lead to the modified Korteweg-de Vries equation which by $u = v' + v^2$ is transformed into (1.1). This follows from the fact that any deformation of $A$ of this kind gives rise to an iso-spectral deformation of $L$. The ideas are explained in Section 3.

On the other hand if $L = A^* A$ one gets a second differential operator $\tilde{L} = AA^* = -D^2 + \tilde{u}$ by exchanging the role of $A$ and $A^*$. Moreover, $\tilde{L}$ is also iso-spectrally deformed under the above deformation. This gives rise to a Backlund transformation of $u = v' + v^2$ into $\tilde{u} = -v' + v^2$ leaving the $X_j$ invariant. Applying this transformation repeatedly we construct the sequence $u_d$ of (1.3). This transformation has also been employed in the construction of $N$-soliton solutions [11].

In [1] the functions $u_d \in M_d$ were described in terms of their poles $x_1, x_2, \ldots, x_n$, $n = \frac{1}{2} d(d + 1)$, which have to satisfy the conditions

H. McKean, via personal communication, informed us of his success in carrying out this approach.
\[ \sum_{j=1}^{n} (x_k - x_j)^{-3} = 0 \quad \text{for } k = 1, 2, \ldots, n. \]

In contrast, here we give preference to the functions

\[ \theta_d = \prod_{j=1}^{n} (x - x_j) \]

which are constructed in Section 2. As a consequence of their properties their roots satisfy (1.4) and all solutions of (1.4) can be so obtained. Thus the \( \tau_1, \ldots, \tau_d \) can be viewed as uniformizing variables for the algebraic variety (1.4).

The second author wishes to express his thanks for the hospitality of the University of Wisconsin. We are grateful to C. Conley at whose seminar these results were presented and to H. McKean for discussions on this subject. We also are indebted to P. Deift for pointing out the relevance of the factorization of second order operators.
§2. Construction of the polynomials \( \theta_k \).

In this section we construct a sequence of polynomials \( \theta_k(\tau_1, \ldots, \tau_k) \) for \( k = 0, 1, \ldots \) of \( k \) variables, which will be considered as polynomials of one variable \( x = \tau_1 \), the others figuring as parameters. As such they have the degree \( n_k = \frac{1}{2} k(k + 1) \). They are defined recursively by

\[
\theta_0 = 1, \quad \theta_1 = x = \tau_1
\]

and the differential equation

\[
\theta_k^{(1)} \theta_{k-1} - \theta_{k+1} \theta_k = (2k + 1) \theta_k^2
\]

which leaves an integration constant available. We fix this constant by the normalization that the coefficient \( x^{n_k-1} \) in \( \theta_{k+1} \) is equal to \( \tau_{k+1} \). This defines the polynomials uniquely and at each recursive step one picks up a new integration constant \( \tau \).

For the first few polynomials one finds

\[
\begin{align*}
\theta_0 &= 1 \\
\theta_1 &= x \\
\theta_2 &= x^3 + \tau_2 \\
\theta_3 &= x^6 + 5\tau_2 x^3 + \tau_3 x - 5\tau_2^2 \\
\theta_4 &= x^{10} + 15\tau_2 x^7 + 7\tau_3 x^5 - 35\tau_2 \tau_3 x^2 + 175\tau_2^3 x - 7\frac{\tau_2^3}{3} + \tau_4 x^3 + \tau_4 \tau_2
\end{align*}
\]

However, it is by no means obvious that the above differential equations can be solved within the class of polynomials. That this is the case is the contents of the following proposition which is purely algebraic. Its proof is based on division properties of polynomials and properties of the Wronskian

\[
[A, B] = A'B - AB'.
\]

**Proposition:** There exists a unique sequence of polynomials \( \theta_k(x, \tau_2, \ldots, \tau_k) \) in \( k \) variables with rational coefficients satisfying (2.1), the recursion

\*prime or \( D \) stands for \( \frac{3}{\partial x} \).
equation (2.2) and the normalization condition mentioned above.

Moreover, the symmetric expression

\[(\theta_k, \theta_{k-1}) = [\theta_k', \theta_{k-1}'] + [\theta_{k-1}', \theta_k']\]

vanishes identically*); \{ \} is defined by this relation.

The \( \theta_k, \theta_k' \) considered as polynomials in \( x \) over the ring \( \mathbb{Q}[\tau_2, \ldots, \tau_k] \) of polynomials in \( \tau_2, \ldots, \tau_k \) with rational coefficients have no common factors. Finally, \( \theta_k = x^{n_k} + \cdots \) with \( n_k = \frac{1}{2} k(k+1) \).

**Proof:** The above statements will be proved together by induction on \( k \). Assuming they have been proven for \( k \leq d \) we set

\[ X = \theta_{d-1}, \quad Y = \theta_d. \]

We aim to solve the recursion equation

\[(2.4) \quad [Z, X] = (2d + 1)Y^2\]

for \( Z \). Instead we solve the equation

\[(2.5) \quad [Z, X] = (2d + 1)Y^2 + PX, \]

where \( P \) is a polynomial in \( x \) of degree \( \deg P < \deg X \) and where the coefficient of \( x^{n_d-1} \) in \( Z \) vanishes. Equation (2.5) represents a system of \( 2n_k + 1 \) linear equations for the \( n_{k+1} + n_{k-1} \) coefficients of \( Z \) and \( P \). From \( n_k = \frac{1}{2} k(k+1) \) we derive

\[ n_{k+1} + n_{k-1} = 2n_k + 1 \]

and the number of equations and unknowns match. Thus the solvability of (2.5) is assured if we show that the homogeneous system has only the trivial solution. Therefore we consider the equation

\[ Z'X - ZX' = PX. \]

Setting \( Z = cx^a + \cdots, \ c \neq 0, \) the highest term on the left hand side is

\[ c(a - n_{d-1})x^{a+n_{d-1}-1} \]

* A similar identity also holds for \( N \)-soliton potentials [8].
while the right hand side has terms of order \( \leq 2n_{d-1} - 1 \) only. Hence
\[
\deg Z = n < 2n_{d-1} - n_{d-1} = n_{d-1}.
\]
By our normalization the coefficient of \( x^{n_{d-1}} \) in \( Z \) vanishes, hence
\[
\deg Z < \deg X = n_{d-1}.
\]
Moreover,
\[
ZX' = (Z' - P)X
\]
and since \( X, X' \) have no common factor \( X \) must divide \( Z \), which is impossible unless \( Z = 0 \), since \( \deg Z < \deg X \). But \( Z = 0 \) implies \( P = 0 \) and the solvability of (2.5) for polynomials \( Z, P \) with coefficients rational in \( \tau_2, \ldots, \tau_d \) is established.

Next we show that \( P = 0 \) in (2.5) so that \( Z \) in fact is a solution of (2.4). Observe that
\[
[[Z,X],X'] = [Z',X']X
\]
is divisible by \( X \). Thus if we take the Wronskian of both sides of (2.5) with \( X' \) we find
\[
[Z',X']X = (2d + 1)[Y^2,X'] + [PX,X'] =
= (2d + 1)[Y^2,X'] + [P,X']X + PX'X
\]
The crucial observation is that
\[
[Y^2,X'] = 2YY'X' - Y^2X = -(Y,X)Y + YY'X
\]
with the notation of (2.3). Thus, since by induction hypothesis, \( [Y,X] = 0 \), we conclude that \( [Y^2,X'] \) is divisible by \( X \), hence by (2.6) also \( PX'X \) is divisible by \( X \). This implies \( P = 0 \) as \( \deg P < \deg X \), and since \( X, X' \) have no common factor.

Thus we have shown that the polynomial \( Z \) is a solution of (2.4), and thus its coefficient of \( x^{n_{d-1}} \) is zero. \( Z \) is uniquely determined by this requirement. So far the coefficients are known to be rational functions of \( \tau_2, \ldots, \tau_d \) and we verify now that actually they are polynomials in \( \tau_2, \ldots, \tau_d \). For this purpose we compare coefficients in (2.4) for
\[ Z = \sum_{a} Z_a x^a. \]

We find that the terms of degree \( a + n_{d-1} - 1 \) in (2.4) are
\[ (a - n_{d-1})Z_a + \]

plus terms which are polynomials in \( Z_{a+1}, \ldots, Z_{n_{d+1}} \) and the known coefficients of \( Y \) and \( X \). Thus we can determine \( Z_a \) recursively as polynomials in the coefficients of \( X, Y \), with rational coefficients. By induction on \( d \) we conclude that the \( Z_a \) are polynomials with rational coefficients in \( \tau_2, \tau_3, \ldots, \tau_d \). The highest coefficient is found for \( a = n_{d+1} \) from \( (n_{d+1} - n_{d-1})Z_a = (2d + 1) \).

Since \( n_{d+1} - n_{d-1} = 2d + 1 \) we have \( Z_a = 1 \) for \( a = n_{d+1} \). Thus we have
\[ Z = x^{n_{d+1}} + \ldots \]

is a polynomial with coefficients \( Z_a \) in the ring \( \mathbb{Q}[\tau_2, \ldots, \tau_d] \) of polynomials in \( \tau_2, \ldots, \tau_d \).

If we set \( \theta_{d+1} = Z + \tau_{d+1} \theta_{d-1} \)

then \( \theta_{d+1} \) is clearly a solution of (2.2) and satisfies the desired normalization condition.

The proof will be complete if we verify (2.3) for \( k = d + 1 \) and show that \( \theta_{d+1}, \theta_{d+1}^2 \) have no common factors.

To do this we use the identity
\[ ([[\theta_{d+1}, X], Y^2] = XY(\theta_{d+1}, Y) - \theta_{d+1} Y(Y, X) \]

where again \( Y = \theta_d, X = \theta_{d-1} \). By construction
\[ [\theta_{d+1}, X] = (2d + 1)Y^2 \]

and the left side of (2.7) vanishes. By induction hypothesis \( \{Y, X\} = 0 \), hence we conclude from (2.7) that
\[ \{\theta_{d+1}, Y\} = 0. \]
Finally we observe that $\theta_1 = x$, $\theta_2 = x^3 + \tau_2$ satisfy the property that $\theta_k$, $\theta_k'$ have no common factor. Our assertion is proved inductively for

$$\theta_{d+1} = Z + \tau_{d+1}\theta_{d-1} = Z + \tau_{d+1}x$$

from the following remark: If $x$, $x'$ have no common factor over $\mathbb{Q}[\tau_2, \ldots, \tau_d]$ and $Z$ is any other polynomial then $Z + \tau x$ and $Z' + \tau x'$ have no common factor over $\mathbb{Q}[\tau_2, \tau_3, \ldots, \tau_d, \tau]$. Such a common factor would have to be linear in $\tau$, and one readily shows that its degree in $x$ must be zero.

This completes the proof of the proposition.

Remark 1. The above polynomials have the homogeneity property

$$\theta_k(\lambda \tau_1, \lambda^3 \tau_2, \ldots, \lambda^{2k-1} \tau_k) = \lambda^n \theta_k(\tau_1, \ldots, \tau_k).$$

We say the $\theta_k$ are "isobaric" of degree $n_k$ if we assign $\tau_j$ the weight $2j - 1$.

Indeed, if we replace $x$, $\tau_j$, $\theta_k$ by

$$x^* = \lambda x, \quad \tau_j^* = \lambda^{2j-1} \tau_j, \quad \theta_k^* = \lambda^n \theta_k,$$

then one verifies that (2.1), (2.2) and the normalization condition are preserved. Hence by uniqueness

$$\theta_k^* = \theta_k^*(\tau),$$

proving the remark.

Remark 2: The parameters $\tau_2, \tau_3, \ldots$ were introduced by a rather arbitrary normalization condition. One can free oneself from this arbitrariness by replacing the $\tau_j$ by

$$\tau_j^* = a_j \tau_j + g_j(\tau_2, \tau_3, \ldots, \tau_{j-1}),$$

where $g_j$ are polynomials with rational coefficients and $a_j \neq 0$ is a
rational number. Moreover, we require that \( g_j \) be isobaric of degree \( j \) which amounts to requiring that the transformation \( \tau_j \rightarrow \tau_j^* \) commutes with \( \tau_j \rightarrow \tau_j^{2j-1} \). In fact, the above birational transformations form a group and we will reserve the freedom to pick an appropriate such transformation (see Section 5).

In the following we need another property of these polynomials.

Lemma 1: For fixed \( d \geq 1 \) let
\[
\theta_d(x + \tau_1, \tau_2, \ldots, \tau_d) = x^n + a_1 x^{n-1} + \cdots + a_n, \quad n = n_d.
\]
Then the \( a_j \) are isobaric polynomials in \( \tau_1, \ldots, \tau_d \) of degree \( j \). Moreover
\[
a_{2j-1} = a_j \tau_j + q(\tau_1, \ldots, \tau_{j-1}) \quad \text{for} \quad j = 1, 2, \ldots, d
\]
where \( a_j \neq 0 \) and \( q \) is a polynomial, isobaric of degree \( 2j - 1 \).

Corollary: This lemma implies that the above relation can be solved for \( \tau_1, \tau_2, \ldots, \tau_d \) and \( \tau_1, \ldots, \tau_d \) expressed as isobaric polynomials with a non-vanishing linear term, of \( a_1, a_3, a_5, \ldots, a_{2d-1} \). Hence \( \tau_1, \tau_2, \ldots, \tau_d \) are in birational, isobaric equivalence with \( a_1, a_3, \ldots, a_{2d-1} \).

Proof: It is obvious from the above proposition that the \( a_j \) are isobaric polynomials in the \( \tau_2, \ldots, \tau_d \) and we just have to verify that \( a_j \neq 0 \). Of course, \( a_j = a_j(d) \) depends on \( d \), and we simply compute it.

Clearly \( a_j \) is the coefficient of \( \tau_j x^{n-2j+1} \) in \( \theta = \theta_d \) and therefore we have
\[
\frac{\partial}{\partial \tau_j} \theta = a_j x^{n-2j+1} \quad \text{for} \quad \tau_1 = 0, \ldots, \tau_d = 0
\]
while
\[
\theta = x^n \quad \text{for} \quad \tau_1 = 0, \ldots, \tau_d = 0.
\]
Hence if we differentiate (2.2) with respect to \( \tau_j \), \( \frac{\partial}{\partial \tau_j} \) denoted by a dot, \( \frac{\partial}{\partial x} \) by a prime we find
\[
(\dot{a}_{d+1} \delta_{d-1} - \dot{a}_d \delta_{d-1} + \dot{a}_d 6_{d-1} - \dot{a}_{d+1} 6_{d-1}) = 2(2d + 1) \theta \delta \theta.
\]
For $\tau_1 = \tau_2 = \cdots = \tau_d = 0$ this gives

$$(n_{d+1} - 2j + 1) a_j(d + 1) + (n_{d+1} - (n_{d-1} - 2j + 1)) a_j(d - 1) = 2(2d + 1) a_j(d).$$

or since $n_{d+1} - n_{d-1} = 2d + 1$

$$(d - j + 1) a_j(d + 1) + (d + j) a_j(d - 1) = (2d + 1) a_j(d).$$

This recursion formula for $a_j(d)$, together with $a_j(d) = 1$ for $d = j$
(by normalization) and $a_j(d) = 0$ for $d > j$, determines $a_j(d)$ uniquely.

In fact one finds explicitly

$$a_j(d) = \begin{cases} 
[d + j] & \text{for } d = j, \\
[d - j] & \text{for } d > j,
\end{cases}$$

which proves the lemma.

This lemma has the following consequence: The polynomials

$$\theta_d(x + \tau_1, \tau_2, \ldots, \tau_d)$$

are uniquely determined by the choice of the $\tau_1, \ldots, \tau_d$.

Indeed, if

$$\theta_d(x + \tau_1, \tau_2, \ldots, \tau_d) = \theta_d(x + \hat{\tau}_1, \hat{\tau}_2, \ldots, \hat{\tau}_d)$$

for all $x$ then the coefficients $\sigma_j(\tau) = \sigma_j(\hat{\tau})$ agree, which implies by

the above lemma that $\tau_k = \hat{\tau}_k$. This remark implies that the representation

(1.3) of $u \in M_d$ is one to one.

Remark 3: For $\tau_2 = \tau_3 = \cdots = \tau_{d-2} = 0$ one can easily compute $\theta_d$
explicitly and finds for $d \geq 3$

$$\theta_d = x^d (x^{4d-6} + (2d - 1) x^{2d-3} - \frac{2d - 1}{2d - 5} \tau_{d-1} x^{2d-5}).$$

This formula contains as special cases those of [1], proposition 2 and 3, Section 5.
§3. A deformation problem for the modified Korteweg-de Vries equation.

In [5] Lax related the Korteweg-de Vries equation - as well as its higher analogues - to the iso-spectral deformation of the operator

\[ L = -D^2 + u \]

where \( u = u(x) \) is a \( C^\infty \)-function. We will describe an analogue problem for the first order operator

\[ A = D - v \]

with \( v = v(x) \).

For motivation of the following we consider at first a bounded linear operator \( A \) in a Hilbert space and call \( A \) equivalent to \( A_0 \) if there exist two unitary operators \( U_1, U_2 \) such that

\[ U_1^{-1}AU_2 = A_0. \]

Clearly the invariants of this equivalence relation are the spectral invariants of \( A^*A \) or of \( AA^* \).

We ask for deformations \( A(t) \) of \( A_0 = A(0) \) which remain in the same equivalence class. Assuming that \( U_1 = U_1(t), U_2 = U_2(t) \) are defined through differential equations

\[ \dot{U}_j = B_j U_j; U_j(0) = I; j = 1,2 \]

with skew Hermitian \( B_j \), we obtain by differentiation of (3.3)

\[ U_1^{-1}(\dot{A} - B_1 A + AB_2)U_2 = 0 \]

or

\[ (3.4) \]

\[ \dot{A} = B_1 A - AB_2. \]

We apply this consideration to \( A = D - v, v = v(x,t) \) and choose

\[ B_j = D^3 + b_j D + Db_j \]

as skew Hermitian operator. We now consider (3.4) as a formal relation for differential operators. The left hand side of (3.4) is a multiplication operator, namely multiplication by \( -v_t \) and \( b_1, b_2 \) have to be so
determined that in $B_1 A - AB_2$ the coefficients of $D^4, D^3, D^2, D$ vanish. The first two coefficients vanish automatically while the other two are

$$-3v' + 2(b_1 - b_2)$$
$$-3v'' + b_1 - 3b_2 - 2(b_1 - b_2)v.$$  

Setting these expressions equal to zero yields two linear equations for $b_1, b_2$ with the solution

$$b_1 = -\frac{3}{4} (-v' + v^2) + c$$
$$b_2 = -\frac{3}{4} (v' + v^2) + c$$

with an arbitrary integration constant $c$. This constant reflects the trivial solution of (4) with $B_2 = 2cD, v_t = v_x$. Therefore we set $c = 0$ and obtain from (3.4) a partial differential equation for $v$ which is computed to be

$$(3.5) \quad v_t = \frac{1}{4} v''' - \frac{3}{2} v^2 v'.$$

This is the so-called modified Korteweg-de Vries equation which was used by Miura [7] in his derivation of the conservation laws for the KdV equation. For this purpose he showed that the function

$$(3.6) \quad u = v^2 + v'$$

satisfies

$$(3.7) \quad u_t = \frac{1}{4} u''' - \frac{3}{2} uu', $$

if $v$ is a solution of (3.5). This remarkable fact has a natural explanation in the following observation:

If $u = v' + v^2$ then the operator (3.1) can be factored as

$$(3.8) \quad L = A^* A$$

where

$$A^* = -D - v$$

is the formal adjoint of $A$. Moreover, the deformation equation (3.4)
gives rise to a deformation equation for $L$

$$\dot{L} = \dot{A}A + A\dot{A} = -A^*B_1A + B_2A^*A + A^*B_1A - A^*AB_2 = [B_2,L];$$

here we used that $B_j^* = -B_j, B_j = B_j(v)$. This is precisely the deformation problem

$$(3.10) \quad \dot{L} = [B,L]; \quad B = B(u) = D^3 + (b(u)D + Db(u))$$

studied by Lax [5], which leads to (3.7) and $B_2(v) = B(v' + v^2)$, provided the arbitrary constant is normalized appropriately. This shows then that any solutions $v$ of (3.5) gives rise to a solution $u$ of (3.7) via (3.6). This appears as a consequence of the factorization (3.8).

If instead we consider the operator

$$\tilde{L} = AA^* = -D^2 + \tilde{v}$$

with $\tilde{u} = -v' + v^2$, then clearly we find analogously to (3.9)

$$\dot{\tilde{L}} = [B_1,\tilde{L}]$$

and $B_1(v) = B(\tilde{u}) = B(-v' + v^2)$, where $B(\tilde{u})$ again denotes the third order operator obtained by Lax. Moreover, $\tilde{u} = -v' + v^2$ is automatically a solution of (3.7). The duality map $u \rightarrow \tilde{u}$ which takes solutions of (3.7) again into solutions of the same equation, a so-called Backlund transformation, is here related to the deformation of two products $AA^*, A^*A$, i.e. to exchanging $A$ and $A^*$.

These considerations can be generalized to the higher Korteweg-de Vries operators $X_j$ by considering real skew Hermitian operators $B_1, B_2$ of degree $2j - 1$. This leads to operators $B_1(v), B_2(v)$ expressible with polynomial coefficients in $v, v', ...$ (this is a difficult point) which are of the form

$$B_1(v) = B(-v' + v^2)$$
$$B_2(v) = B(v' + v^2)$$

where $B = B(u)$ corresponds to the iso-spectral deformation of $L = A^*A$, ...
i.e. \( \mathcal{L} = [B,L] \) associated with the \( X_j \)-flow. For the explicit representation of \( B(u) \) we refer to [6].

We will not present the proof of this statement since we find it easier to introduce the \( X_j \) via a recursion formula and then verify directly their invariance under the Backlund transformation. This will be done in Section 5.
§4. The construction of the rational potentials.

From the factorization
\[ L = A^*A, \quad \tilde{L} = \tilde{A}^* \tilde{A} \]
one reads off that
\[ (4.1) \quad AL = \tilde{L}A. \]
This identity shows that
\[ (4.2) \quad (L - \lambda)\psi = 0 \text{ implies } (\tilde{L} - \lambda)A\psi = 0 \]
which allows us to construct the solutions of \((\tilde{L} - \lambda)\tilde{\psi} = 0\) from those
of \((L - \lambda)\psi = 0\).

The mapping \(u \mapsto \tilde{u}\) is an involution, i.e. \(\tilde{u}\) is mapped back into \(u\).
The following is closely related to [2]. However, if one observes that
the factorization
\[ L = A^*A; \quad u = v' + v^2 \]
depends on the choice of \(v\) one can reach new potentials by repeated
application of this procedure. The various choices of \(v\) can be easily
described by the solutions \(\psi\) of
\[ L\psi = 0, \quad v = \frac{\psi'}{\psi}. \]
Indeed with this choice of \(v\) one has \(L = A^*A\) for \(A = D - v\); as well
as \(A\psi = 0\). In fact, we may write \(A = \psi D\psi^{-1}\) and hence \(A^* = -\psi^{-1}D\psi\).

From \(A\psi = 0\) one has \(A^*\psi^{-1} = 0\) and \(\tilde{L} = -D^2 + \tilde{u}, \quad \tilde{u} = u - 2v'\) satisfies
\[ \tilde{L}\psi^{-1} = 0. \]
Instead of using \(\psi^{-1}\) and \(\tilde{v} = -\frac{\psi'}{\psi}\) as the basis for the factorization of
\(\tilde{L} = \tilde{A}^*\tilde{A}\) we pick any other solution \(\tilde{\psi}\) of \(\tilde{L}\tilde{\psi} = 0\). Since the Wronskian of
\(\tilde{\psi}\) and \(\psi^{-1}\) is a constant we determine \(\tilde{\psi}\) as a solution of
\[ (4.3) \quad [\tilde{v}, \psi^{-1}] = \text{const} \neq 0 \]
where \([a, b] = a'b - ab'\) denotes the Wronskian determinant. Then
\[ \tilde{v} = \frac{\tilde{\psi}'}{\tilde{\psi}}, \quad \tilde{u} = u - 2v' \]
gives rise to a new potential \( \tilde{u} = \tilde{u} - 2\tilde{v}' \). The solution of (4.3) introduces one integration constant at this step.

This simple procedure, started with the potential \( u = 0 \) yields the rational potentials: We set \( u_0 = 0, \varphi_0 = x \) and construct a sequence of potentials \( u_k \) inductively as follows. We construct \( \varphi_k \) from

\[
[\varphi_k, \varphi_k^{-1}] = \text{const} \neq 0 \quad \text{for} \quad k = 1, 2, ...
\]

and set for \( k \geq 1 \)

\[
(4.4) \quad u_k = \frac{\varphi_k}{\varphi_k'}.
\]

Since this transformation which maps \( u_k \) into \( u_{k+1} \) takes solution of the KdV equations again to solutions of the KdV equations, we will obtain in \( u_k \) a \( k \)-parameter family of solutions of these equations since every step introduces one integration constant. It is surprising, that this procedure gives rise to rational functions \( u_k \), in fact, the most general rational solutions of the KdV equations, aside from the trivial one \( u = -\frac{3x}{t} \) of (1.1). The following lemma expresses the result of this construction (4.4), (4.5) in terms of the \( \theta_k \).

**Lemma 2:** If \( \theta_k \) denotes the polynomials of \( \phi_2 \) then the rational functions

\[
(4.6) \quad \left\{ \begin{array}{l}
u_k = -2 \left( \frac{\theta_k'}{\theta_k} \right)' = -2(\log \theta_k)'' \\
\varphi_k = \frac{\theta_{k+1}}{\theta_k} ; \quad v_k = \frac{\varphi_k'}{\varphi_k}
\end{array} \right\} \quad k = 0, 1, ...
\]

satisfy

\[
(4.7) \quad (-D^2 + u_k)\nu_k = 0
\]

\[
(4.8) \quad \left\{ \begin{array}{l}
u_k = \nu_k' + \nu_k^2 \\
u_{k+1} = -\nu_k' + \nu_k^2
\end{array} \right.
\]
Proof: For $k = 0$ the definition of $\theta_0 = 1$, $\theta_1 = x$ leads to $u_0 = 0$, $\varphi_0 = x$ in agreement with our requirement. The recursion formula (2.2) leads to

$$[\varphi_k, \varphi_{k-1}] = \left[ \frac{\theta_{k+1}}{\theta_k}, \frac{\theta_{k-1}}{\theta_k} \right] = 2k + 1.$$ 

The definition of $u_k$, $\varphi_k$ yield

$$u_k = -2\left( \frac{\theta_k'}{\theta_k} \right)' = -2 \frac{\theta_k'}{\theta_k} + 2 \left( \frac{\theta_k'}{\theta_k} \right)^2$$

$$\frac{\varphi_k}{\varphi_k} = \frac{\theta_{k+1}'}{\theta_{k+1}} - \frac{\theta_k'}{\theta_k} - 2 \frac{\theta_k'}{\theta_{k+1}} \left( \frac{\theta_{k+1}'}{\theta_k} - \frac{\theta_k'}{\theta_k} \right)$$

so that

$$\frac{\varphi_k}{\varphi_k} - u_k = \frac{\theta_{k+1}'}{\theta_{k+1}} - \frac{\theta_k'}{\theta_k} + 2 \frac{\theta_k'}{\theta_{k+1}} \left( \frac{\theta_{k+1}'}{\theta_k} - \frac{\theta_k'}{\theta_k} \right) = \frac{\theta_{k+1}}{\theta_k} \frac{\theta_{k+1}}{\theta_{k+1}}$$

which vanishes by (2.3).

The identity

$$v_k' = \left( \frac{\varphi_k'}{\varphi_k} \right)' = \frac{\varphi_k'}{\varphi_k} - \left( \frac{\varphi_k'}{\varphi_k} \right)^2 = u_k - v_k^2$$

proves one of the last relations. Finally $\theta_{k+1} = \theta_k \varphi_k$ implies

$$u_{k+1} = -2 \log(\theta_k \varphi_k)'' = u_k - 2 \left( \frac{\varphi_k'}{\varphi_k} \right)$$

$$= u_k - 2v_k = -v_k + v_k^2$$

proving the lemma.

We summarize: The relations (4.8) show that $u_{k+1}$ is obtained from $u_k$ by the Backlund transformation of § 3 while (4.6) expresses these potentials in terms of the polynomials $\theta_k$. The equation (4.7) implies that

$$L_k \varphi = (-D^2 + u_k) \varphi = 0$$

has one solution $\varphi = \varphi_k = \frac{\theta_{k+1}}{\theta_k}$. The Wronskian relation shows that
\[ \psi = \frac{\theta_{k-1}}{\theta_k} \] is another solution. Thus the most general solution is given by

\[ \psi = (c_1 \theta_{k-1} + c_2 \theta_{k+1}) \theta_k^{-1}. \]

It is interesting to note that also the solutions of the equation

\[ (L_k - \lambda) \psi = 0 \]

are the product of rational functions in \( x \) and \( e^{\pm \omega x}, \lambda = -\omega^2 \). Indeed by (4.2) we see: If \( \psi \) solves the above equation then

\[ \tilde{\psi} = (D - \nu_k) \psi = (D - \frac{\psi_k}{\phi_k}) \psi = A_k \psi \]

is a solution of \( (L_{k+1} - \lambda) \psi = 0 \). For \( k = 0 \) we have \( \psi = c_1 e^{\omega x} + c_2 e^{-\omega x} \), hence

\[ \psi = A_k \cdots A_2 A_1 (e^{\pm \omega x}) \]

is a basis for \( (L_{k+1} - \lambda) \psi = 0 \). This is essentially contained in [2], in a different guise.

As the special example we mention the case \( \tau_2 = \tau_3 = \cdots = \tau_k = 0 \), i.e. \( \theta_k(x) = x^n_k \) or \( u_k = \frac{2n_k}{x^2} \). The equation

\[ D^2 \psi + \left( \lambda - \frac{k(k+1)}{x^2} \right) \psi = 0 \]

has the solutions \( \psi = \sqrt{x} J_{\frac{1}{2}(k+1)}(i\omega x) \) with the Bessel functions \( J_\nu \), which for integer \( \nu = -\frac{1}{2} \) are indeed elementary functions of the indicated nature.
§5. The KdV-flows.

Following Lenard we define the KdV vector fields $X_k$ and the Hamiltonian $H_k$

\begin{equation}
X_k(u) = D \frac{\delta H_k}{\delta u}
\end{equation}

recursively by $X_1 = D$, or $X_1(u) = u'$, and

\begin{equation}
X_k = (u D + Du - \frac{1}{2} D^3) \frac{\delta H_{k-1}}{\delta u} 
\end{equation}

or formally

\begin{equation}
X_k = (u + Du D^{-1} - \frac{1}{2} D^2) X_{k-1} = RX_{k-1},
\end{equation}

where

\begin{equation}
R = u + Du D^{-1} - \frac{1}{2} D^2.
\end{equation}

This definition requires the verification that at each step $X_{k-1}$ can be written as the derivative of $\frac{\delta H_{k-1}}{\delta u}$, which is expressed as a polynomial in $u$ and finitely many of its derivatives. For the proof of this fact, see e.g. [3], [6]. Also $X_k$ is defined up to an arbitrary constant which is normalized by the requirement that $X_k(u)$ is an "isobaric" polynomial in $u, u', u'', \ldots$ of degree $2k + 1$, which means that every term

\begin{equation*}
\sum_{\alpha_0, \alpha_1, \ldots} u^{\alpha_0} (Du)^{\alpha_1} (D^2 u)^{\alpha_2} \ldots
\end{equation*}

in $X_k(u)$ satisfies $\sum_{\nu \geq 0} (2 + \nu) \alpha_\nu = 2k + 1$; i.e. $u$ is assigned the weight 2 and each derivative the weight 1. (This weighting is consistent with the fact that in $L = -D^2 + u$ multiplication with $u$ and $D^2$ are on the same footing.)

For $k = 2$, for example, we find

\begin{equation*}
X_2(u) = (u D + Du - \frac{1}{2} D^3) u = 3uu' - \frac{1}{2} u''',
\end{equation*}

which agrees with (3.7) up to an unessential factor $-2$. 

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Similarly we introduce a sequence of vector fields \( v_t = Y_k(v) \) with 
\[ -\frac{1}{2} Y_2(v) \] corresponding to (3.5). The requirement is that every solution of 
\[ v_t = Y_k(v) \] gives rise, via \( u = v' + v^2 \), to a solution of \( u_t = X_k(u) \).
This requirement leads to the following recursive definition:

Let \( S \) be the formal operator introduced by Olver in [10], namely

\[
S = 2v^2 + 2v'D^{-1}v - \frac{1}{2} D^2
\]

and set

\[
\begin{cases}
Y_1(v) = v' \\
Y_k(v) = SY_{k-1}(v) & \text{for } k = 1, 2, \ldots
\end{cases}
\]

Again it is important to show that \( Y_k(v) \) can be defined as polynomials in \( v, v', \ldots \), which we will do presently. Secondly, the definition is made unique by requiring that \( Y_k(v) \) are isobaric polynomials of degree \( 2k \) if \( v \) and \( \frac{3}{Dx} \) are assigned the weight 1 each. For example, for \( Y_2(v) \) we find

\[
Y_2(v) = (2v^2 + 2v'D^{-1}v - \frac{1}{2} D^2)v' = 3v^2v' - \frac{1}{2} v'''
\]

which agrees with (3.5) up to the factor -2.

Lemma 3: i) The \( Y_k(v) \) are uniquely defined as isobaric polynomials of degree \( 2k \) by the recursion formula (5.6) and ii) satisfy

\[
(2v + D)Y_k(v) = X_k(v^2 + v')
\]

Moreover

\[ Y_k(-v) = -Y_k(v). \]

Proof: We proceed by induction. For \( k = 1 \) the above statement is evident.
We assume it holds for \( k \) and verify it for \( k + 1 \). First it is to be shown that

\[
SY_k = (2v^2 + 2v'D^{-1}v - \frac{1}{2} D^2)Y_k(v)
\]

can be defined as an isobaric polynomial in \( v \) and its derivatives. For this it is sufficient to verify this for \( D^{-1}(vY_k(v)) \) or for

\[
D^{-1}(2v + D)Y_k(v) = 2D^{-1}(vY_k) + Y_k(v).
\]
But by the induction hypothesis (see (5.7)) we have

\[(2v + D)Y_k = X_k(u) = D \frac{\delta H_k(u)}{\delta u}\]

with \(u = v' + v^2\), and therefore

\[D^{-1}(2v + D)Y_k = \frac{\delta H_k}{\delta u}\]

is indeed a polynomial in \(u = v' + v^2\) and its derivatives. Of course, one could add an arbitrary constant to \(D^{-1}(2v + D)Y_k\) but the choice is unique by requiring that \(D^{-1}(2v + D)Y_k(v)\) be isobaric of degree 2k in \(v, v', \ldots\). Note that \(\frac{\delta H_k}{\delta u}\) is isobaric of degree 2k in \(u, u', \ldots\) when \(u = v' + v^2\), and by induction it follows that

\[Y_{k+1} = SY_k = (2v^2 - v' - \frac{1}{2} D^2)Y_k(v) + v' \frac{\delta H_k}{\delta u}\]

is isobaric of degree 2k + 2 in \(v, v', \ldots\).

To prove (5.7) we observe the intertwining identity

\[R(2v + D) = (2v + D)S\]

for \(S\) which is readily verified. It implies

\[(2v + D)Y_{k+1}(v) = (2v + D)S Y_k(v) = R(2v + D)Y_k(v) = RX_k(v' + v^2) = X_{k+1}(v' + v^2),\]

where the isobaric character of the \(X_k, Y_k\) is used implicitly to fix the integration constants. This completes the induction.

Thus the polynomials \(Y_k(v) (k \geq 1)\) in \(v, v', v'', \ldots\) are uniquely defined. Moreover, they are odd in \(v\), i.e. satisfy

\[Y_k(-v) = -Y_k(v)\]

Indeed this follows immediately from the recursion (5.6) since \(S\) is even in \(v\) and \(Y_1(v) = v'\) is odd, which completes the lemma.

Next we show that \(X_k\) leaves the manifold \(u_d = -2(\log \theta_d)''\) invariant. This is the contents of
Theorem 1: There exists a unique choice of rational functions $\gamma_{kj}(\tau_2, \ldots, \tau_j)$ and differential operators

$$\Gamma_k = \sum_{j=1}^{\infty} \gamma_{kj} \frac{\partial}{\partial \tau_j}$$

such that for $d = 0, 1, 2, \ldots$

(5.8) $X_k(u_d) = \Gamma_k u_d$

and

(5.9) $Y_k(v_d) = \Gamma_k v_d$ where $v_d = \frac{\theta_{d+1}}{\theta_d} - \frac{\theta'_d}{\theta_d}$.

(Since $u_d, v_d$ depend only on finitely many variables the sum breaks off.)

In other words, if the $\tau_j$ satisfy

$$\tau_j = \gamma_{kj}(\tau_2, \ldots, \tau_j) \quad j \leq d$$

then $u = u_d(x + \tau_1, \tau_2, \ldots, \tau_d)$ solves the equation

$$\frac{\partial u}{\partial \tau_k} = X_k(u).$$

Proof: We proceed by induction on $d$. For $d = 0$ we have $u_0 = 0$ and therefore $X_k(u_0) = 0$. Assume next that $\gamma_{kj} = \gamma_{kj}(\tau_2, \ldots, \tau_j)$ for $j = 1, 2, \ldots, d$ have been determined such that (5.8) holds. Then we conclude from (5.7) and

$$u_d = v'_d + v_d^2$$

with $v_d = \frac{\varphi_d}{\varphi'_d}$, $v'_d = \frac{\theta_{d+1}}{\theta_d}$ that

$$(2v_d + D)Y_k(v_d) = X_k(u_d) = \Gamma_k u_d = (2v_d + D)\Gamma_k v_d$$

or

$$(2v_d + D)(Y_k(v_d) - \Gamma_k v_d) = 0.$$ 

Since $\psi = v_d^{-2}$ is a solution of $(2v_d + D)\psi = 0$ we conclude that

(5.10) $Y_k(v_d) - \Gamma_k v_d = c\varphi_d^{-2}$

with $c$ being a rational function of $\tau_2, \ldots, \tau_{d+1}$. On the other hand $v_d$ depends on $\tau_{d+1}$ and in
the coefficient \( \gamma_{k,d+1} \) can be uniquely determined so that \( c = 0 \). Indeed,

\[
\begin{align*}
\varphi_d' &= \frac{\theta_d'}{\theta_{d+1}} - \frac{\theta_d}{\theta_{d+1}}
\end{align*}
\]

where \( \theta_d \) is independent of \( \tau = \tau_{d+1} \) while \( \frac{d\theta_{d+1}}{d\tau} = \theta_{d-1} \).

Hence, by (2.2)

\[
\frac{3v_d}{\theta_{d+1}} = \left( \frac{\theta_{d-1}}{\theta_{d+1}} \right) = -(2d + 1) \frac{\theta_{d}^2}{\theta_{d+1}} = -(2d + 1)v_d^{-2}
\]

and the coefficient of \( \gamma_{k,d+1} \) in (5.11) is \( -(2d + 1)v_d^{-2} \); thus we have from (5.10)

\[
\varphi_d^2(\varphi_k(v_d) - \sum_{j \leq d} \gamma_{kj} \frac{3v_d}{\theta_{j+1}}) = -(2d + 1)\gamma_{k,d+1} + c.
\]

By appropriate choice of \( \gamma_{k,d+1} \) as rational function of \( \tau_1, \ldots, \tau_{d+1} \), we obtain \( c = 0 \), i.e.

\[
\gamma_k(v_d) = \Gamma_k v_d
\]

as claimed (5.9). This determines \( \gamma_{k,d+1} = \gamma_{k,d+1}(\tau_2, \ldots, \tau_{d+1}) \) uniquely.

Using that \( \gamma_k(-v) = -\gamma_k(v) \) we conclude from (5.7)

\[
\begin{align*}
\chi_k(u_{d+1}) &= \chi_k(-v_d' + v_d^2) = (-2v_d + D)\gamma_k(-v_d') \\
&= (2v_d - D)\gamma_k(v_d) \\
&= (2v_d - D)\Gamma_k v_d \\
&= \Gamma_k(v_d^2 - v_d) = \Gamma_k u_{d+1}
\end{align*}
\]

which completes the induction and the proof.

**Lemma 4:** The \( \gamma_{kj} = \gamma_{kj}(\tau) \) are polynomials of \( \tau_2, \tau_3, \ldots \) with rational coefficients. If we assign \( \tau_m \) the weight \( 2m - 1 \) the \( \gamma_{kj} \) are isobaric of weight \( 2(j - k) \). In particular, they depend on \( \tau_k \) with \( k \leq [j - k] \leq j - 1 \) only. Moreover, \( \gamma_{kk} \) is a non-vanishing constant.
Proof: First we express the differential equation \( u_t = X_k(u) \) via \( u = -2(\log \theta) \) in terms of

\[
\theta = \theta_d = x^n + a_1 x^{n-1} + \ldots + a_n
\]

with \( n = n_d \). This expresses the differential equation in terms of the \( a_j \) which finally are transferred to the \( \tau_j \) via the corollary of lemma 1 of Section 2.

From

\[
-2 \frac{d}{dt} \left( \frac{\theta'}{\theta} \right) = u_t = X_k(u) = D \frac{\delta H_k}{\delta u}
\]

we conclude that

\[
\theta^{-2}(\delta \theta' - \delta' \theta) = \frac{1}{2} \frac{\delta H_k}{\delta u}
\]

an integration constant being eliminated by the isobaric property. The right hand side depends on \( u, u', \ldots \), and we observe that

\[
D^m u = \theta^{-m-2} p_m(\theta, \theta', \ldots)
\]

with \( p_m \) being a polynomial in \( \theta, \theta', \ldots \). Since \( \frac{\delta H_k}{\delta u} \) is isobaric of degree 2k we find that

\[
\theta^{2k} \frac{1}{2} \frac{\delta H_k}{\delta u} = Q(\theta, \theta', \ldots)
\]

is a polynomial in \( \theta, \theta', \ldots \). Thus the differential equation takes the form

\[
\theta^{2k-2} (\delta \theta' - \delta' \theta) = Q(\theta, \theta', \ldots)
\]

In this equation we compare coefficients of \( x \). The term of highest power in \( x \) containing \( \sigma_j \) is

\[
x^{(2k-2)n + j - j \cdot n - l} (n - (n - j)) \delta_j = x^{2kn-j-1} j \delta_j
\]

i.e. comparing the coefficients of \( x^{2kn-j-1} \) we find

\[
j \delta_j = \sum_{m < j} \sum_{m' m} a_{m'} m + a_j
\]

where \( a_1, a_2, \ldots, a_j \) are polynomials in the \( \sigma \). Thus we find
\[ \dot{\sigma}_j = b_j(\sigma_1, \ldots, \sigma_d) \]

with polynomials \( b_j \). Finally using the corollary of lemma \( 1 \) we express these differential equations in terms of \( \tau_1, \tau_2, \ldots \) in the form

\[ \dot{\tau}_j = \gamma_{kj}(\tau_1, \tau_2, \ldots) \]

One readily checks the homogeneity of the \( \gamma_{kj} \) to be given by

\[ \gamma_{kj}(\lambda \tau_1, \lambda^3 \tau_2, \ldots) = \lambda^{2j-2k} \gamma_{kj}, \]

by using an argument like in Remark \( 1 \), Section \( 2 \), observing the isobaric property of \( X_k \). In particular, \( \gamma_{kj} = 0 \) for \( j < k \) and \( \gamma_{kk} \) is a constant.

To evaluate this constant we use the fact that

\[ u_d = \frac{2\eta_d}{x^2} \quad \text{for} \quad \tau_2 = \tau_3 = \cdots = \tau_d = 0 \]

and for \( d = k \)

\[ X_k u_k = c_k x^{-(2k+1)} \]

with a rational constant \( c_k \neq 0 \) as one computes from (5.3). Thus we have for \( \tau_2 = \tau_3 = \cdots = \tau_k = 0 \)

\[ 0 \neq X_k u_k = \Gamma_k u_k = \sum_{j=1}^{k} \frac{\partial u_k}{\partial \tau_j} \gamma_{kj} = \frac{\partial u_k}{\partial \tau_k} \gamma_{kk}, \]

as \( \gamma_{kj} = 0 \) for \( j < k \). Hence \( \gamma_{kk} \neq 0 \) as was claimed.

Thus the differential equations induced by \( X_k \) are given by \( \Gamma_k \), or equivalently by

\[ \dot{\tau}_j = \gamma_{kj}(\tau_2, \tau_3, \ldots, \tau_{j-1}) \]

with isobaric polynomials \( \gamma_{kj} \). Obviously the equations can be solved recursively as polynomials in \( t \). In fact, more is true: There exists an isobaric birational transformation \( \tau_j^* \) such that for all \( k \geq 1 \) we have

\[ \tau_k = \frac{\partial}{\partial \tau_k^*} \]

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i.e. the above differential equation reduces to
\[ \Gamma_k \tau_j^* = \delta_{kj} \]
and \( \tau_k^* \) can be interpreted as the time variable for the flow \( X_k \). This is a simple consequence of the fact that the \( X_k \) - and hence the \( \Gamma_k \) - commute.
We recall that the \( \tau_k \) were introduced by an artificial normalization and these parameters are actually defined only up to the group of birational transformations which commute with \( \tau_j + \lambda^{2j-1} \tau_j \). Now we make the unique choice of the parameters so that they are adapted to the KdV flows.

**Theorem 2:** There exists a unique birational transformation
\[ \tau_j = a_j \tau_j + g_j(\tau_2, \tau_3, \ldots, \tau_j-1) \]
with \( g_j \) being isobaric polynomials
\[ g_j(\lambda^3 \tau_2, \lambda^5 \tau_3, \ldots, \lambda^{2j-3} \tau_{j-1}) = \lambda^{2j-1} g_j(\tau_2, \ldots, \tau_{j-1}) \]
with rational coefficients and a rational number \( a_j \neq 0 \) such that
\[ \Gamma_j = \frac{\partial}{\partial \tau_j} \]

**Corollary:** If \( u = u_d(\tau_1, \tau_2, \ldots, \tau_d) \) is expressed in terms of \( \tau^* \) as
\[ u = u_d(\tau_1^*, \ldots, \tau_d^*) \] then the function
\[ u = u_d(\tau_1^* + x, \tau_2^*, \ldots, \tau_d^*) \]
is a solution of \( \frac{\partial u}{\partial \tau_k} = X_k u \).

**Proof:** We proceed by induction as follows: Changing the notation of the \( \tau_j \) also we assume that for some \( k \geq 1 \) we have
\[ \Gamma_j = \partial \tau_j + \gamma_j k \partial \tau_k, \quad 1 \leq j \leq k - 1, \]
\[ \Gamma_k = \gamma_{kk} \partial \tau_k \]
up to terms \( \partial \tau_m \) for \( m > k \), which will be suppressed. For \( k = 1 \) this
\[ \frac{\partial}{\partial \tau} \]

We abbreviate \( \frac{\partial}{\partial \tau} \) by \( \partial \tau \).

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is trivially the case and we will construct a transformation

\[
\begin{cases}
  \tau_j^* = \tau_j & (j \neq k) \\
  \tau_k^* = a_k \tau_k + g_k(\tau_2, \ldots, \tau_{k-1})
\end{cases}
\]  

(5.12)

such that

\[
\Gamma_j = \partial \tau_j^* + 0 \cdot \partial \tau_k^*
\]

\[
\Gamma_k = \partial \tau_k^*
\]

up to terms \( \partial \tau_m^* \) with \( m > k \). This is obviously sufficient for the proof, if we also verify that the \( g_k \) and \( a_k \) have the required properties.

To carry out this induction step it is convenient to break up (5.12) into \( k \) steps and effectively make a second induction. First we achieve by \( \tau_k^* = a_k \tau_k \) that \( \Gamma_k = \gamma_{kk} a_k \partial \tau_k^* = \partial \tau_k^* \) by setting \( a_k = \gamma_{kk}^{-1} \)
a rational number. Assume now that we already achieved

\[
\gamma_{jk} = 0 \quad \text{for} \quad s < j < k.
\]

Then for \( s < j < k \), using the commuting of the \( \chi_k \), hence \( \Gamma_k \), we compute

\[
0 = \Gamma_j \Gamma_s - \Gamma_s \Gamma_j = \frac{\partial \gamma_{sk}}{\partial \tau_j} \partial \tau_k^*
\]

hence \( \gamma_{sk} \) depends on \( \tau_2, \tau_3, \ldots, \tau_s \) only. Therefore we construct a transformation

\[
\tau_k^* = \tau_k + g(\tau_2, \ldots, \tau_s)
\]

so that

\[
\Gamma_j = \partial \tau_j^* \quad \text{for} \quad s < j < k
\]

and

\[
\Gamma_s = \partial \tau_s^* + \left( \frac{\partial g}{\partial \tau_s} + \gamma_{sk} \right) \partial \tau_k^*'
\]

while \( \Gamma_m \), \( 1 < m < s - 1 \), maintains its inductively assumed form.
Thus, if we determine the polynomial $g$ such that

$$\frac{\partial g}{\partial \tau} + \gamma_{sk} = 0$$

then we have effectively replaced $\gamma_{sk}$ by 0. The choice of $g$ is unique if we require it to be isobaric. After finitely many steps we achieve

$$\Gamma_j = \tilde{\gamma}_j^*, \quad 1 \leq j \leq k,$$

up to terms $\tilde{\gamma}_m$ with $m > k$. This completes the induction argument. The isobaric character of the $\gamma_j$ follows readily from that of the $\gamma_{kj}$. This completes the proof of Theorem 2.

In conclusion we remark that the manifold $M_d$ of rational function $u_d = -2(\log \theta_d)^*$ agrees with the manifold considered in [1]. Therefore the roots $x_1, x_2, \ldots, x_n$, $n = n_d$ of $\theta_d$:

$$\theta_d = \prod_{j=1}^{n} (x - x_j)$$

i.e. the poles of

$$u_d = 2 \sum_{j=1}^{n} (x - x_j)^{-2}$$

satisfy, by the derivations in [1] the algebraic equations (1.4). This fact could also be verified directly.
§6. Explicit representations for the $\theta_d$.

If one makes use of Crum's formulae (see [2]) one can represent the polynomials $\theta_d$ as well as the transformation $A_{d-1}A_{d-2} \ldots A_0$ of §4 in terms of Wronskians in explicit form. In order to do this we define the Wronskian of $k$ functions $\psi_1, \psi_2, \ldots, \psi_k$ as

$$W_k = W(\psi_1, \psi_2, \ldots, \psi_k) = \det(D^{i-1}\psi_j) \quad i, j = 1, 2, \ldots, k.$$ 

For abbreviation we also set

$$W_k(x) = W(\psi_1, \psi_2, \ldots, \psi_k, x)$$

with another smooth function $x$. Then one has Jacobi's identity

$$[W_k(x), W_{k+1}] = W_{k+1}(x)W_k \quad \text{for} \quad k = 1, 2, \ldots. \tag{6.1}$$

This is readily verified. The left hand side is a linear differential operator of order $k + 1$ in $x$ which clearly vanishes for $x = \psi_1, \psi_2, \ldots, \psi_k$ as well as for $x = \psi_{k+1}$. Thus, if we assume that the $\psi_1, \psi_2, \ldots, \psi_{k+1}$ are linearly independent, the left hand side must be a multiple of $W_{k+1}(x)$.

Comparing the highest coefficient one obtains (6.1).

We apply the above definitions to a system $\psi_j$ satisfying $\psi_0 = 0$, $\psi_1 = x$ and

$$\psi_j = \psi_{j-1}, \quad j = 1, 2, \ldots, k. \tag{6.2}$$

Then one verifies, for $x = 1$ and setting $W_0 = 1$ that

$$W_k(1) = (-1)^k W_{k-1} \quad \text{for} \quad k = 1, 2, \ldots. \tag{6.3}$$

To prove this we write

$$W_k(1) = W(\psi_1, \psi_2, \ldots, \psi_k, 1) = (-1)^k W(1, \psi_1, \ldots, \psi_k)$$

and since $\psi_1 = x$ the last expression reduces to

$$W_k(1) = (-1)^k W(\psi_2, \psi_3, \ldots, \psi_k)$$

and by (6.2) to (6.3).
Thus setting $x = 1$ in (6.1) and using (6.3) we find

\[(6.4) \quad [W_{k+1}, W_{k-1}] = W_k^2 \quad \text{for} \quad k = 1, 2, \ldots\]

and

\[W_0 = 1, W_1 = \psi_1.\]

Thus if we set $\psi_1 = x$ we see that $\theta_k$ and $W_k$ differ only by a multiplicative factor:

\[(6.5) \quad \theta_k = \mu_k W_k\]

where one determines

\[\mu_k = 1^k \cdot 3^{k-1} \cdot 5^{k-2} \cdots (2k - 1)^1 = \prod_{j=1}^{k} (2k - 2j + 1)^j.\]

But this factor is unessential for the following. The choice $\psi_1 = x$ and (6.2) leads to

\[(6.6) \quad \psi_j = \frac{x^{2j-1}}{(2j-1)!} + \sum_{i=0}^{j-2} \rho_{j-i} x^{2i}\]

Equivalently one can define the $\psi_j$ by the generating function

\[\sum_{j=1}^{\infty} \psi_j s^{2j-1} = \sin h(sx) + \sum_{i=2}^{\infty} \rho_1 s^{2i-1} \cos h(sx)\]

where $\rho_2, \rho_3, \ldots$ are arbitrary parameters.

With this choice of $\psi_1, \psi_2, \ldots$ the formula (6.5) gives the desired explicit representation of $\theta_k$. The $\rho_2, \rho_3, \ldots$ are birationally and isobarically related to the $\tau_2, \tau_3, \ldots$ and one finds for the first few values $\tau_2 = -3 \rho_2$, $\tau_3 = 45 \rho_3$; $\mu_0 = \mu_1 = 1$, $\mu_2 = 3$, $\mu_3 = 45$.

Now we express the mapping $T_d = A_{d-1} A_{d-2} \cdots A_0$ with

\[(6.7) \quad A_j = \varphi_j D \varphi_j^{-1}; \quad \varphi_j = \frac{\theta_j+1}{\theta_j} \frac{W_{j+1}}{W_j}\]

in terms of the Wronskians.
Lemma 5: The map
\[ X \rightarrow T_dX = A_{d-1}A_{d-2} \cdots A_0X \]
has the alternate form
\[ T_d = \frac{W_d(x)}{W_d} \quad \text{for} \quad d = 1, 2, \ldots \]

Proof: This formula occurs in [2]. Clearly for \( d = 1 \) it is easily verified and it suffices to check
\[ A_d \frac{W_d(x)}{W_d} = \frac{W_{d+1}(x)}{W_{d+1}}. \]
Indeed
\[ \frac{W_d(x)}{W_d} = \varphi_d \left( \frac{W_d(x)}{W_{d+1}} \right) = \varphi_d \left( \frac{[W_d(x), W_{d+1}]}{W_{d+1}} \right), \]
and using (6.4), (6.7) the induction step is verified.

In order to interpret the spectral properties of the operator
\[ L = L_d = -D^2 + u_d, \quad u_d = -2(\log \theta_d), \]
we choose the parameters so that the roots of \( \theta_d \), i.e. the poles of \( u_d \) lie off the real axis, which requires complex potentials. The spectral problem, as well as the inverse problem for such complex potentials were studied by Marchenko [13]. We are indebted to Marchenko for the following interpretation.

First it is clear on account of
\[ LT_d = T_d(-D^2) \]
that for \( \lambda = -\omega^2 \) that the solutions of \( (L - \lambda)\psi = 0 \) are linear combinations of
\[ \psi = \frac{W_d(e^{i\omega x})}{W_d} = e^{i\omega x}R_d(x) \]
with rational \( R_d(x) \). Thus \( \lambda > 0 \) gives continuous spectrum with reflection coefficient zero. One finds \( [\psi_+, \psi_-] = 2\omega^{2d+1} \).
Secondly $\lambda = 0$ has an eigenspace of $L^N$ of dimension $\geq \left\lfloor \frac{d+1}{2} \right\rfloor$ in the Hilbert space of the absolutely square integrable functions if $N \geq \left\lfloor \frac{d+1}{2} \right\rfloor$.

In fact, from the intertwining relation

$$T_d(-D^2) = L_d T_d$$

and $D^2 \frac{x^{2v}}{(2v)!} = \frac{x^{2v-2}}{(2v-2)!}$ one finds that the rational functions

$$\phi_v = T_d \left( \frac{x^{2v}}{(2v)!} \right) = \frac{1}{(2v)!} \frac{W_d(x^{2v})}{W_d}$$

satisfy

$$L \phi_v + \phi_{v-1} = 0.$$

We remark that $T_d(x^{2v-1})$ are linearly dependent on $\phi_0, \phi_1, \ldots, \phi_v$ but the $\phi_v$ are linearly independent, as

$$\phi_v \sim c_v x^{2v-d} \text{ for } |x| \to \infty; \text{ with } c_v \neq 0.$$

Thus the $\phi_v \in L^2$ if $0 \leq v < \frac{d}{2}$ and $\lambda = 0$ is an eigenvalue of higher multiplicity. All this requires, of course, that the parameters $\tau_1, \ldots, \tau_d$ so chosen that no root of $\xi_d$ is real.

These considerations allow the full discussion of the spectral properties of these potentials.
References


In [1] special classes of solutions of the Korteweg-de Vries equation

\[ u_t = 3uu_x - \frac{1}{2}u_{xxx} = X^2 u \]

were studied, in particular, all those \( u = u(x,t) \) which are rational functions of \( x \) for each value of \( t \). It turns out that these solutions are rational functions of \( t \) as well and of very special structure. In this paper we give a new construction of these solutions with emphasis on their algebraic properties.