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OPTIMALITY CONDITIONS FOR THE AVERAGE COST PER UNIT TIME PROBLE--ETC(U)
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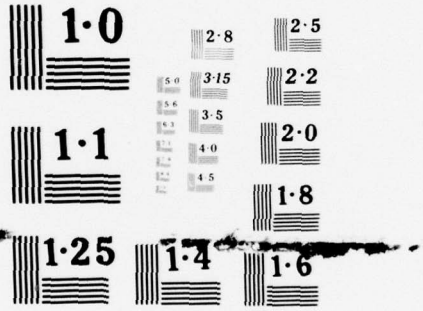

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H. J. KUSHNER

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OPTIMALITY CONDITIONS FOR THE AVERAGE COST PER UNIT TIME PROBLEM WITH A DIFFUSION MODEL

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OPTIMALITY CONDITIONS FOR THE AVERAGE COST PER UNIT TIME PROBLEM WITH A DIFFUSION MODEL[+]

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Abstract

Defining the solution to a stochastic differential equation to be the solution to the Martingale problem of Strook and Varadhan, we obtain results on the existence of an optimal stationary control for the average cost per unit time problem, a necessary and sufficient condition for optimality of a control, and a number of other related results.

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1. Introduction

The purpose of the paper is the development of a necessary and sufficient "Dynamic Programming" like condition, for the average cost per unit time problem. The condition is similar to those developed for other problems by Davis and Varaiya [1] and Bismut [2]. In addition to its intrinsic interest, the criterion appears to be useful for the problem of approximation and computation (see the corollary and remark in Section 6). The main Theorems are 3.1 (characterizing the invariant measure), 4.4 (existence of an optimal stationary control), 5.1 (characterizing the auxiliary $V^u(\cdot)$ function), and 6.1 (necessary and sufficient condition for optimality). Also, a number of auxiliary results are obtained.

We will use conditions (A1) - (A5).

(A1) Let $\sigma(\cdot)$ denote a bounded uniformly continuous and uniformly positive definite $r \times r$ matrix valued function on the Euclidean space R^r.

Let $\mathscr{U}$ denote a compact convex set in some Euclidean space and which contains the origin.

(A2) $f(\cdot), b(\cdot,\cdot), k(\cdot,\cdot)$ are measurable R^r, R^r, and R^1 valued functions on R^r, $R^r \times \mathscr{U}$ and $R^r \times \mathscr{U}$, resp.; b and k are bounded, and are continuous in their second argument for each value of the first argument, and $b(x,0) = 0$. $f(\cdot)$ is bounded on bounded sets.

(A3) The set $\{b(x,\alpha), k(x,\alpha), \alpha \in \mathscr{U}\} \equiv (b(x,\mathscr{U}), k(x,\mathscr{U}))$ is convex and compact for each $x \in R^r$.

Any measurable $\mathscr{U}$ valued function $u(\cdot)$ on R^r is called an admissible control. Functions $b(\cdot,\cdot)$ and $k(\cdot,\cdot)$ are said to be admissible if they satisfy (A2), (A3) and have the form $b(x,u(x)), k(x,u(x))$ for admissible $u(\cdot)$. We will often write $b^u(\cdot) = b(\cdot,u(\cdot))$, $k^u(\cdot) = k(\cdot,u(\cdot))$. Our systems model is the stochastic differential equation

$$(1.1) \qquad dx(t) = [f(x(t)) + b^u(x(t))]dt + \sigma(x(t))dw(t), \; x(0) = x,$$

where $w(\cdot)$ is a standard Wiener process. In particular, the process $x(\cdot)$ will be defined to be the solution to the martingale problem of Strook and Varadhan [3]; hence $w(\cdot)$ may be defined implicitly in terms of $x(\cdot)$. As pointed out by Bismut [2], there are a number of advantages to using the "martingale problem solution" definition of (1.1), particularly when questions of existence are of interest. Here, only feedback (Markov) controls are considered.

The cost functional is

$$(1.2) \qquad \theta(u) = \lim_{T\to\infty} \frac{1}{T} E_x^u \int_0^T k(x(s),u(x(s)))ds = \int k(x,u(x))\mu_u(dx),$$

where $\mu_u(\cdot)$ is the unique invariant measure for (1.1), which will exist under conditions to be imposed. Also, E_x^u denotes expectation under control $u(\cdot)$, and initial condition x.

In order for the problem to be well defined, we need some sort of recurrence for each control. In a sense, we will assume ((A4),(A5)) that the effects of $f(\cdot)$ dominate those of $b(\cdot,\cdot)$ for large $|x|$ and all $u(\cdot)$. Assumption (A4) will be convenient, and (A5), while avoidable, does provide a relatively simple method for obtaining some required estimates. Both are satisfied by a large number of problems.

(A4) <u>There is a non-negative twice continuously differentiable real valued function</u> $W_1(\cdot)$ <u>such that</u> $W_1(x) \to \infty$ <u>as</u> $|x| \to \infty$ <u>and for some</u> $\varepsilon > 0$ <u>and compact</u> K_1,

$$(1.3) \qquad \mathscr{L}^u W_1(x) \le -\varepsilon, \; x \notin K_1, \; \underline{\text{all admissible}} \; u(\cdot),$$

where

$$\mathscr{L}^u = \sum_{i,j} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i [f_i(x) + b_i^u(x)] \frac{\partial}{\partial x_i},$$

the differential generator of (1.1), and $a(\cdot) = \sigma(\cdot)\sigma'(\cdot)/2$.

(A5) Let $W_2(x) = W_1^2(x)$. There are constants $c_2 > 0$, $\alpha > 0$, such that for all admissible $u(\cdot)$,

$$\mathscr{L}^u W_2(x) \leq c_2 - q_2(x), \text{ where } q_2(x) \geq 0, \tag{1.4}$$

and $q_2(x)/W_1(x) \geq \alpha > 0$. Let K_2 denote a compact set such that $q_2(x) \geq c_2$ for $x \notin K_2$.

Remark. Suppose that $f(x) = Ax$ and $\dot{x} = f(x)$ is asymptotically stable. Then we may use $x'Px = W_1(x)$, where P satisfies the Liapunov equation $A'P + PA = -Q < 0$. Also, (A5) holds.

For some additional motivation, let us consider a Dynamic Programming approach. Suppose that there is a smooth function $V(\cdot)$ and a constant γ such that

$$\begin{aligned} &\inf_{u(x)\in\mathscr{U}} [\mathscr{L}^u V(x) + k(x,u(x)) - \gamma] = 0 \\ &\equiv A(x) + \inf_{u(x)\in\mathscr{U}} [V_x'(x) b(x,u(x)) + k(x,u(x)) - \gamma], \text{ each } x. \end{aligned} \tag{1.5}$$

If the solution to (1.1) is well defined for $u(x) = \bar{u}(x)$, the minimizer in (1.5), and if

$$E_x^{\bar{u}} V(x(t))/t \to 0 \quad \text{as} \quad t \to \infty,$$

then

$$\lim_{T\to\infty} \frac{1}{T} E_x^{\bar{u}} \int_0^T k^u(x(s))ds = \gamma.$$

If, in addition, $E_x^u V(x(t))/t \to 0$ as $t \to \infty$, then[+] $\gamma \leq \theta(u)$. If $P^u(x,t,\cdot) \to \mu_u(\cdot)$ $(P^u(x,t,\Gamma) \equiv P_x^u(x(t) \in \Gamma))$ strongly (in variation) as $t \to \infty$, then $\theta(u) = \int k(x,u(x))\mu_u(dx)$.

In general, we do not know whether such a smooth $V(\cdot)$ or an optimal control exists. Part of our aim is to replace (1.5) by a local maximum-principle which does for our problem what the work of Davis and Varaiya [1] or Bismut [2] did for the control problem on a finite time interval or for the discounted control problem.

A one dimensional version - on a finite interval with reflection - was treated by Mandl [5], and an incomplete development of the r-dimensional version of Mandl's result is given in [6].

In Section 2, the solution to (1.1) is defined, and some properties listed. Sections 3, 4, 5, 6 deal with the existence of an invariant measure for each $u(\cdot)$, with certain continuity properties of the measure with respect to $b^u(\cdot)$ and with the

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[+]See Kushner [4] for a formal discussion.

existence of an optimal control, with the existence and properties of an auxiliary $V(\cdot)$ function, and with the maximum principle, resp.

2. The Solution to (1.1)

Let C, $\mathscr{C}_t$ and $\mathscr{C}$ denote the space of R^r valued continuous functions on $[0,\infty)$, and the σ-algebras induced by the coordinate projections $x(s)$, $s \leq t$, and $x(s)$, $s < \infty$, resp., where $x(\cdot)$ is the generic element of C. Define $f^N(\cdot)$: $f^N(x) = f(x)$ if $|x| \leq N$, and is zero otherwise. Then, for each $x \in R^r$, there is a unique measure Q_x^N on $(C, \mathscr{C})$ such that $\{Q_x^N, x \in R^N\}$ solves the martingale problem of Strook and Varadhan [3]. There is also a standard Wiener process $W^{x,N}(\cdot)$ defined on $(C, \mathscr{C}, Q_x^N)$, and adapted to the (completed with respect to Q_x^N) $\{\mathscr{C}_t\}$ and such that

$$dx(t) = f^N(x(t))dt + \sigma(x(t))dW^{x,N}(t), \quad x(0) = x, \text{ w.p.1 } Q_x^N .$$

Define $S_M = \{x: |x| \leq M\}$. We will show that the solution is also well defined for $N = \infty$. The stability condition (A3) yields the following result. In the Lemma, suppose (w.l.o.g) that K_1 is in the interior of S_M.

<u>Lemma 2.1</u>. <u>Assume</u> (A1)-(A2), (A4). <u>Let</u> $x \in S_M$ <u>and</u> $N \geq M$. <u>Define</u> $\sigma_M = \inf\{t: x(t) \notin S_M - K_1\}$, $\sigma(K_1) = \inf\{t: x(t) \in K_1\}$. <u>Then</u> $(P^N, E^N$, correspond to $Q_x^N)$

$$(2.1) \qquad P_x^N\{x(t) \text{ \underline{hits} } K_1 \text{ \underline{before hitting} } \partial S_M\} \geq 1 - \frac{W_1(x)}{k_M},$$

<u>where</u> $k_M = \inf_{|x|=M} W_1(x)$,

$$(2.2) \qquad E_x^N \sigma_M \leq W_1(x)/\varepsilon$$

Proof. By Itô's Lemma, for any $t < \infty$ ($\mathscr{L}^o$ corresponds to $u(\cdot) \equiv 0$),

$$E_x^N W_1(x(t \cap \sigma_M)) = W_1(x) + E_x^N \int_0^{t \cap \sigma_M} \mathscr{L}^o W_1(x(s))ds$$

$$\leq W_1(x) - \varepsilon E_x^N(t \cap \sigma_M).$$

This inequality implies $E_x^N W_1(x(\sigma_M)) \leq W_1(x)$, $W_1(x) \geq \varepsilon E_x^N \sigma_M$, from which both (2.1) and (2.2) follow. *Q.E.D.*

Since $k_M \to \infty$ as $M \to \infty$, it can be shown that (2.1) implies that

(2.3) $\lim_{M\to\infty} \sup_N P_x^N\{\sup_{0\leq t\leq T} |x(t)| \geq M\} = 0$, for each x and T.

By virtue of (2.3), there is a unique solution (P_x^o) to the martingale problem for coefficients (f,σ), and each $x \in R^r$. Similarly, since neither the r.h.s of (1.3) nor K_1 depend on $u(\cdot)$, there is a unique solution (P_x^u) to the martingale problem for all $x \in R^r$ and coefficients $(f+b^u,\sigma)$, where $u(\cdot)$ is admissible. Furthermore,

(2.4a) $E_x^u \sigma(K_1) \leq W_1(x)/\varepsilon$, E_x^u <u>corresponding</u> to P_x^u

(2.4b) $P_x^u\{\sup_{t\leq T} |x(t)| \geq N\} \to 0$ <u>as</u> $N \to \infty$, <u>uniformly in</u> $u(\cdot)$, <u>and in</u> x <u>in bounded sets</u>

(2.4c) $P_x^u\{x(t)$ hits K_1 before hitting $\partial S_N\} \geq 1 - W_1(x)/k_N$, if $S_N \supset K_1$ and $x \in S_N$.

There is a Wiener process $W^{x,u}(\cdot)$, defined on $(C, \mathscr{C}, P_x^u)$ and adapted to $\{\mathscr{C}_t\}$ (completed with respect to P_x^u) and such that

(2.5) $dx(t) = [f(x(t)) + b^u(x(t))]dt + \sigma(x(t))dW^{x,u}(t)$, w.p.1 (P_x^u).

For each real $0 < T < \infty$, define

$$\zeta_0^T(u) = \int_0^T [\sigma^{-1}(x(s))b^u(x(s))]' dW^{x,0}(s) - \frac{1}{2}\int_0^T |\sigma^{-1}(x(s))b^u(x(s))|^2 ds.$$

Then (the proof of (2.6) is the same as that of Theorem 6.2 in [3], where $f(\cdot) \equiv 0$; see also Girsanov [7])

$$dP_x^u = \exp \zeta_0^T(u) \cdot dP_x^o \quad \text{on} \quad (C, \mathscr{C}_T). \tag{2.6}$$

Thus, (at least on each $(C, \mathscr{C}_T)$), for each x all the measures P_x^u are mutually absolutely continuous, so that a.s. statements with respect to one are also a.s. statements with respect to the others on each $(C, \mathscr{C}_T)$. Also[+] (a.s. P_x^o)

$$dW^{x,0}(t) - \sigma^{-1}(x(t))b^u(x(t))dt = dW^{x,u}(t). \tag{2.7}$$

(See Girsanov [7] or Davis and Varaiya [1])

By [3], the solution to the martingale problem with coefficients (f^N, σ) or $(f^N + b^u, \sigma)$, for admissible $b^u(\cdot)$, is a strong Markov and a strong Feller process and in each of these cases the measures of $x(t)$ have densities with respect to Lebesgue measure for all $x = x(0)$ and all $t > 0$. Furthermore, these densities are positive almost everywhere. By the stability condition (A4) and (2.3), (2.4b), these facts are also true for the solution with coefficients $(f+b^u, \sigma)$ for admissible $b^u(\cdot)$. Define $P_x^u\{x(t) \in \Gamma\} \equiv P^u(x,t,\Gamma)$ and denote its density at y by $p^u(x,t,y)$.

[+] Whenever such differentials are equated, we mean to equate the corresponding integrals.

The following Lemma will be useful in the following sections.

Lemma 2.2. Assume (A1) - (A2) and (A4) - (A5). Define

$$\sigma(K_2) = \inf \{t: x(t) \in K_2 \cup K_1\}.$$

Then

$$E_x^u \sigma^2(K_2) \leq \frac{2[W_2(x) + c_2 W_1(x)/\varepsilon]}{\varepsilon\alpha}, \quad x \notin K_1 \cup K_2. \tag{2.8}$$

Proof. By Itô's Lemma and (A5)

$$0 \leq E_x^u W_2(x(t \cap \sigma(K_2))) \leq W_2(x) + E_x^u \int_0^{t \cap \sigma(K_2)} [c_2 - \alpha W_1(x(s))]ds,$$

from which we get (use $E_x^u \sigma(K_2) < \infty$)

$$0 \leq W_2(x) + E_x^u \int_0^{\sigma(K_2)} [c_2 - \alpha W_1(x(s))]ds.$$

The last inequality, (2.4a) and $\sigma(K_2) \leq \sigma(K_1)$ imply that

$$0 \leq W_2(x) + c_2 E_x^u \sigma(K_1) - \alpha\, \varepsilon \int_0^\infty E_x^u I_{\{\sigma(K_2) \geq s\}} E_{x(s)}^u \sigma(K_2) ds.$$

The integrand equals

$$E_x^u I_{\{\sigma(K_2) \geq s\}} (\sigma(K_2) - s).$$

Hence, the integral equals $E_x^u \sigma^2(K_2)/2$, from which (2.8) follows. Q.E.D.

3. The Invariant Measure

Let G and G_1 be spheres in R^r, centered at the origin, with radii γ and γ_1, resp., $\gamma < \gamma_1$, and boundaries Γ and Γ_1, resp., and with $G \supset K_1 \cup K_2$. Define $\tau' = \inf\{t: x(t) \in \Gamma_1\}$, $\tau_1 = \inf\{t: x(t) \in \Gamma\}$, $\tau_1' = \inf\{t: t > \tau_1, x(t) \in \Gamma_1\}$, and define τ_n and τ_n', $n > 1$, recursively by $\tau_n = \inf\{t: t > \tau_{n-1}', x(t) \in \Gamma\}$, $\tau_n' = \inf\{t: t > \tau_n, x(t) \in \Gamma_1\}$. τ will be used for $\tau_2 - \tau_1 = \tau_2$, when $x \in \Gamma$. Define $\tilde{X}_n = x(\tau_n)$. Then, if $x \in \Gamma$, $\{\tilde{X}_n\}$ is a (homogeneous) Markov chain on the state space Γ, and Khasminskii [8] uses it to construct the invariant measure for $\{x(t)\}$. Let $\tau(A)$ denote the amount of time $(\int_0^\tau I_{\{x(t)\in A\}}dt)$ that $x(t)$ spends in a Borel set A during $[0,\tau_2] = [0,\tau]$, when $x(0) = x \in \Gamma$ (if $x(0) \in \Gamma$, then $\tau_1 = 0$).

<u>Theorem 3.1</u>. <u>Assume</u> (A1) - (A2), (A4). <u>Then</u> there is a <u>constant</u> c_3:

$$\sup_{\substack{x\in\Gamma \\ u}} E_x^u \tau \le c_3 < \infty. \tag{3.1}$$

<u>Both</u> $\{\tilde{X}_n\}$ <u>and</u> $x(\cdot)$ <u>have unique finite invariant measures (for each</u> $u(\cdot)$) $\tilde{\mu}_u$ <u>and</u> μ_u, <u>resp., where for each Borel set</u> A <u>(note that</u> $\mu_u(R^r) = 1$)

$$\begin{aligned} \mu_u(A) &= \bar{\mu}_u(A)/\bar{\mu}_u(R^r), \\ \bar{\mu}_u(A) &= \int_\Gamma \tilde{\mu}_u(dx) E_x^u \tau(A). \end{aligned} \tag{3.2}$$

<u>The measure</u> μ_u <u>has a density (with respect to Lebesgue measure) which is positive almost everywhere and the value at the point</u> y <u>is given by</u>

$$\int p^u(x,t,y)\mu_u(dx). \tag{3.3}$$

<u>For any bounded Borel function</u> $F(\cdot)$,

$$\int F(x)\bar{\mu}_u(dx) = \int_\Gamma \tilde{\mu}_u(dx)E_x^u \int_0^\tau F(x(s))ds. \tag{3.4}$$

<u>Also</u>

$$\sup_u E_x^u \tau_1 \leq W_1(x)/\varepsilon, \quad x \notin G. \tag{3.5}$$

<u>For each Borel set</u> A <u>and bounded measurable function</u> $F(\cdot)$,

$$P^u(x,t,A) \to \mu_u(A),\ E_x^u F(x(t)) \to \int F(x)\mu_u(dx),\ \underline{\text{as}}\ \ t \to \infty. \tag{3.6}$$

<u>Proof</u>. Set $\tau_0' = \inf\{t: x(t) \notin G_1\}$. To prove (3.1), we first show that, for fixed $t > 0$ and some real $c < 1$

$$\inf_{x,u} P_x^u\{\tau_0' \leq t\} \geq 1 - c. \tag{3.7}$$

(3.7) follows from the fact that there is a $c < 1$ such that

$$\inf_{x,u} P_x^u\{\sup_{s\leq t}|x(s)| \geq \gamma_1\} = \inf_{x,u} P_x^u\{\sup_{s\leq t}|x + \int_0^s (f(x(s)) + b^u(x(s)))ds$$

$$+ \int_0^s \sigma(x(s))dW^{x,u}(s)| \geq \gamma_1\}$$

$$\geq \inf_{u,x\varepsilon G_1} P_x^u\{\sup_{s\leq t}|\int_0^s \sigma(x(s))dW^{x,u}(s)| \geq \gamma_1 + |x| + Kt\} \geq 1 - c,$$

where K is a bound on $|f + b^u|$ in G_1. Now

$$P_x^u\{\tau_0' > nt\} = E_x^u I_{\{\tau_0' > (n-1)t\}} I_{\{\tau_0' > nt\}}$$

$$= E_x^u I_{\{\tau_0' > (n-1)t\}} E_{x(nt-t)}^u I_{\{\tau_0' > t\}} \leq E_x^u I_{\{\tau_0' > (n-1)t\}} c$$

$$\leq \cdots \leq c^n$$

which implies that $\sum_n nt\, c^{n-1} = c_4$ is an upper bound to $E_x^u \tau_0'$. Hence, $E_x^u \tau' \leq c_4$ for $x \in \Gamma$. Indeed, (to be used later)

$$E_x^u (\tau')^\alpha \leq \sum_n (nt)^\alpha c^{n-1} < \infty,\ x \in G. \tag{3.8}$$

Equation (3.5) follows from (2.4a), since $G \supset K_1 \cup K_2$. Thus, for $x \in \Gamma$,

$$E_x^u \tau = E_x^u \int_0^{\tau'} ds + E_x^u \int_{\tau'}^{\tau} ds$$

$$= E_x^u \tau' + E_x^u E_{x(\tau')}^u \tau_1 \leq c_4 + E_x^u W_1(x(\tau'))/\varepsilon \leq c_3$$

for some real c_3, which gives (3.1).

In [8], Khasminskii proves that there is a unique invariant measure $\tilde{\mu}_u$ under the conditions (i): $P^u(x,t,A) > 0$, all open A, all x and all $t > 0$, and (ii): that $x(\cdot)$ be recurrent (Khasminskii's definition of recurrence is implied by (3.5)) and (iii): $x(\cdot)$ is a strong Feller and a strong Markov process. Under the additional condition (3.1) (for fixed $u(\cdot)$) there is a unique finite invariant

measure μ_u given by (3.2) ([8], Theorems 2.1, 3.2 and 3.3). Equations (3.3) and (3.4) follow by simple calculations.

Since μ_u and $P^u(x,t,\cdot)$ all have densities which are positive almost everywhere (all x, and all $t > 0$), they are mutually absolutely continuous. Equation (3.6) then follows from [8], Theorem 3.4 or Doob [8], Theorem 5 (let his Φ equal our μ_u and his $P_1 = 0$), since μ_u and $P^u(x,t,\cdot)$ are mutually absolutely continuous for $t > 0$. Q.E.D.

4. Existence of an Optimal Control

Theorems 4.1 and 4.2 provide some preliminary results. Theorem 4.3 proves that, in a sense, the invariant measure is continuous in the function $b^u(\cdot)$. This leads directly to the existence Theorem 4.4. Define $g(\cdot,\cdot) = (b(\cdot,\cdot),k(\cdot,\cdot))$.

Theorem 4.1. *Assume* (A1) *to* (A3). *Let* $\{u^n(\cdot)\}$ *denote a sequence of admissible controls, and write* $g^n(\cdot) = (b^{u_n}(\cdot),k^{u_n}(\cdot))$. *If there is a bounded measurable function* $\bar{g}(\cdot)$ *such that*

$$\int_A g^n(x)dx \to \int_A \bar{g}(x)dx, \quad \text{all Borel } A,$$

then $\bar{g}(\cdot)$ *is admissible in the sense that there is an admissible* $u(\cdot)$ *such that* $\bar{g}(\cdot) = (b^u(\cdot),k^u(\cdot))$, *for almost all* x.

Proof. The theorem is a standard existence theorem. See Roxin [10] or McShane and Warfield [11]. By an argument such as that used by Roxin [10], $\bar{g}(x) \in \{\gamma: \gamma = g(x,\alpha), \text{ for some } \alpha \in \mathscr{U}\} \equiv g(x,\mathscr{U})$ for almost all x. We can assume that the conclusion holds for all x. Then the theorem follows by the implicit function theorem in [11]. Q.E.D.

A family $\{\phi_\alpha\}$ of measures on R^r is said to be *tight* if for each $\varepsilon > 0$, there is a compact K_ε such that $\phi_\alpha(R^r-K_\varepsilon) \le \varepsilon$, all α. Let $\ell(\cdot)$ denote *Lebesgue measure*.

Theorem 4.2. Assume (A1) - (A2), (A4) - (A5). Then $\{\mu_u, u(\cdot)$ admissible$\}$ is tight. Also, $\mu_u(A) \to 0$ as $\ell(A) \to 0$, uniformly in $u(\cdot), A$.

Remark. The theorem is true, but harder to prove, without (A5). Since (A5) will be used later anyway, we use it now to simplify the proof.

Proof. Since $\bar{\mu}_u(R^r)$ is bounded uniformly in $u(\cdot)$ (by virtue of (3.1)) to show tightness, we only need to show that (refer to the definition of μ_u in (3.2))

$$\sup_{\substack{x\varepsilon\Gamma \\ u}} E_x^u \tau(S_N') \to 0 \quad \text{as} \quad N \to \infty,$$

where $S_N' = \{y: |y| \geq N\}$. Recall that $G_1 \supset K_1 \cup K_2$, and assume that $\{y: |y| \leq N\} \supset G_1$. (See (A4), (A5) for the definition of K_1, K_2.) Then, for $x \varepsilon \Gamma$,

$$(E_x^u \tau(S_N'))^2 = [E_x^u \tau(S_N') I_{\{\tau(S_N')>0\}}]^2$$

$$\leq \sup_{\substack{x\varepsilon\Gamma_1 \\ u}} E_x^u \tau_1^2 \cdot \sup_{\substack{x\varepsilon\Gamma \\ u}} P_x^u\{\tau(S_N') > 0\} \equiv M_1 M_2^N.$$

By (2.4c), $M_2^N \leq \sup_{x\varepsilon\Gamma_1} W_1(x)/k_N$ and by Lemma 2.2, $M_1 < \infty$. The first assertion of the theorem now follows, since $k_N \to \infty$ as $N \to \infty$.

Fix ε and (by tightness) choose compact K_ε such that $\mu_u(K_\varepsilon) \geq 1 - \varepsilon$, all $u(\cdot)$. Then, for $t > 0$,

$$\mu_u(A) = \int_{R^r - K_\varepsilon} \mu_u(dx) P^u(x,t,A) + \int_{K_\varepsilon} \mu_u(dx) P^u(x,t,A) \leq$$

$$\leq \varepsilon + \sup_{x \varepsilon K_\varepsilon} P^u(x,t,A).$$

Also, by (2.6) (recall that the superscript o corresponds to $u = b = 0$),

$$[P^u(x,t,A)]^2 = [E_x^o I_{\{x(t)\varepsilon A\}} \exp \zeta_0^t(u)]^2$$

$$\leq E_x^o I_{\{x(t)\varepsilon A\}} E_x^o \exp 2\zeta_0^t(u) \leq \text{constant} \cdot P^o(x,t,A).$$

The last assertion of the theorem follows from the last two inequalities since ε is arbitrary and $P^o(x,t,A) \to 0$ uniformly in $x \varepsilon K_\varepsilon$ and in A, as $\ell(A) \to 0$. Q.E.D.

Theorem 4.3. Assume (A1) to (A5). Let $\bar{b}(\cdot)$ be a bounded measurable function, and $\{b^{u_n}(\cdot)\}$ admissible, such that (write sub or superscript n for u_n)

$$\int_A b^n(x)dx \to \int_A \bar{b}(x)dx, \text{ all Borel } A.$$

Then (Theorem 4.1) there is an admissible $u(\cdot)$ such that $\bar{b}(\cdot) = b^u(\cdot)$ a.e. Also,

$$\exp \zeta_0^t(u_n) \to \exp \zeta_0^t(u) \tag{4.2}$$

weakly in L_1 (with respect to P_x^o) as $n \to \infty$, for each x and $t > 0$. In particular,

(4.3) $$P^n(x,t,A) \to P^u(x,t,A)$$

(4.4) $E_x^n F(x(t)) \to E_x^u F(x(t))$, <u>each</u> x, $t > 0$, <u>Borel</u> A, <u>bounded measurable</u> $F(\cdot)$.

(4.5) $$\int \mu_n(dx)F(x) \to \int \mu_u(dx)F(x),\ F(\cdot)\ \underline{\text{bounded and measurable}}.$$

<u>Proof</u>. The convergence (4.2) is proved by[+] Bismut [2], Theorem IV-3, and (4.3), (4.4) follow from that convergence and (2.6).

Since $\{\mu_n\}$ is tight, it is weakly sequentially compact, (Billingsley [12], p. 37). I.e., each subsequence contains a further subsequence $\{\mu_{n_i}\}$ such that, for some probability measure $\hat{\mu}$,

$$\int F(x)\mu_{n_i}(dx) \to \int F(x)\hat{\mu}(dx),\ \text{all bounded continuous } F(\cdot)$$

(Billingsley [12], pp. 35-37). Let n index such a convergent subsequence, with (weak) limit $\hat{\mu}$.

Let $F(\cdot)$ be bounded and continuous. Let $\varepsilon > 0$ and define K_ε as in Theorem 4.2 and write

$$\int \mu_n(dx)F(x) = \int \mu_n(dx)E_x^n F(x(t))$$

$$= \int_{K_\varepsilon} \mu_n(dx)E_x^n F(x(t)) + \int_{R^r - K_\varepsilon} \mu_n(dx)E_x^n F(x(t)).$$

The second term is $\leq \varepsilon \sup_x |F(x)|$. Since $x(\cdot)$ is a Feller process under each control, $E_x^v F(x(t))$ is continuous in x, for each $v(\cdot)$. Then, the function

[+] In [2], $f = 0$, but the proof is exactly the same.

$E_x^n F(x(t))$ is continuous in x and converges to the continuous function $E_x^u F(x(t))$, by (4.4). This convergence implies that, for each $\delta \geq 0$, there is a Borel set $A_\delta \subset K_\varepsilon$, with $\ell(A_\delta) \leq \delta$, such that $|E_x^u F(x(t)) - E_x^n F(x(t))| \leq \varepsilon$ for large n, and $x \notin A_\delta$, $x \in K_\varepsilon$. Those estimates, Theorem 4.2, the arbitrariness of δ and the weak convergence imply that the first term on the right goes to $\int_{K_\varepsilon} \hat{\mu}(dx) E_x^u F(x(t))$, as $n \to \infty$. Since the l.h.s. converges to $\int \hat{\mu}(dx) F(x)$, and $\varepsilon > 0$ is arbitrary, we conclude that

$$\int \hat{\mu}(dx) F(x) = \int \hat{\mu}(dx) E_x^u F(x(t)).$$

This equation together with the arbitrariness of $t > 0$ and $F(\cdot)$, implies that $\hat{\mu}$ is an invariant measure - under control $u(\cdot)$. <u>Thus, the uniqueness Theorem 3.1 implies that</u> $\hat{\mu} = \mu_u$. Since the result does not depend on the selected subsequence, we have that $\mu_n \to \mu_u$ weakly, as $n \to \infty$, and (4.5) holds for bounded and continuous $F(\cdot)$.

Let $F(\cdot)$ be bounded and measurable. Then, for $t > 0$, the invariance of μ_n implies

$$\int \mu_n(dx) F(x) = \int \mu_n(dx) F_t^n(x), \quad F_t^n(x) = E_x^n F(x(t)).$$

By the strong Feller property, $E_x^u F(x(t))$ is continuous in x. Now, as in the proof of (4.5) for continuous $F(\cdot)$, the almost uniform convergence of $F_t^n(x)$ to $E_x^u F(x(t))$, Theorem 4.2 and the tightness and weak convergence of $\{\mu_n\}$ imply

$$\int \mu_n(dx)F_t^n(x) \to \int \mu_u(dx)E_x^u F(x(t)),$$

which must also equal the limit of $\int \mu_n(dx)F(x)$. This implies (4.5), since by the invariance of μ_u, the r.h.s. equals $\int \mu_u(dx)F(x)$. Q.E.D.

Theorem 4.4. Assume (A1) to (A5). Then there is an optimal admissible control.

Proof. Let $\{u_n(\cdot)\}$ denote a minimizing sequence. Then $\underline{\theta} \equiv \lim \theta(u_n) = \inf_{u(\cdot)} \theta(u)$. Let n also index a weak star ($\sigma(L_\infty, L_1)$ topology) convergent subsequence of $\{b^n(\cdot), k^n(\cdot)\}$ with limit $(\bar{b}(\cdot), \bar{k}(\cdot))$, where we let n replace the index u_n. There is an admissible control $u(\cdot)$ such that (Theorem 4.1) $(\bar{b}(\cdot), \bar{k}(\cdot)) = (b^u(\cdot), k^u(\cdot))$. If $k(\cdot)$ does not depend on the control, then $\theta(u_n) \to \theta(u)$ by (4.5). Hence, in this case there is an optimal control.

Now let $k(\cdot)$ depend on the control. Let $F_n(\cdot)$ be a sequence of bounded measurable functions which converges to a function $F(\cdot)$ in the weak star topology. Then[+] (Bismut [2], Proposition IV-4, p. 48)

$$\int_0^t F_n(x(s))ds \to \int_0^t F(x(s))ds$$

in probability (P_x^o, each x), as $n \to \infty$. Note that $E_x^o \exp 2\zeta_0^t(v)$ is bounded uniformly in x and in the control $v(\cdot)$. Let $F_n(\cdot)$

[+]In [2], $f = 0$, but the proof is exactly the same for our case.

be defined by $F_n(x) = k(x,u_n(x))$, and set $F(x) = k(x,u(x))$. By the convergence in probability, the convergence (4.2) and the boundedness of $E_x^o \exp 2\,\zeta_0^t(u_n)$ uniformly in n,x, we have

$$E_x^{u_n} \int_0^t F_n(x(s))ds = E_x^o \exp \zeta_0^t(u_n) \int_0^t F_n(x(s))ds$$

$$\to E_x^o \exp \zeta_0^t(u) \int_0^t F(x(s))ds$$

(4.6)

$$= E_x^u \int_0^t F(x(s))ds \quad \text{(a continuous function of } x\text{)}.$$

Integrating the left and right sides, resp., of (4.6) with respect to μ_n and μ_u, resp., and using the invariance of these measures, yields the two equations

$$t\theta(u_n) = \int \mu_n(dx) \int_0^t E_x^{u_n} k(x(s),u_n(x(s)))ds \tag{4.7a}$$

$$t\theta(u) = \int \mu_u(dx) \int_0^t E_x^u k(x(s),u(x(s)))ds. \tag{4.7b}$$

Now, (4.6) implies that the right hand integral in (4.7a) converges to that in (4.7b) for each x. This, together with the tightness of $\{\mu_n\}$, the last part of Theorem 4.2, and an argument like that used in the proof of Theorem 4.3 to show (4.5) yields that

$$\theta(u_n) \to \theta(u). \qquad \text{Q.E.D.}$$

5. The Auxiliary $V^u(\cdot)$ Function

Our aim is to get a replacement for the $V(\cdot)$ function in (1.5), which will play an important role in the sequel.

In this section the control $u(\cdot)$ is fixed, and we return to the Markov chain $\{\tilde{X}_n\}$ of Section 3. For a measurable set $\gamma \in \Gamma$, let $\pi_u(x,\gamma) = P_x^u\{\tilde{X}_1 \in \gamma\}$, $x \in \Gamma$, and recall that the unique invariant measure for $\{\tilde{X}_n\}$ is denoted by $\tilde{\mu}_u$. Let ϕ be a finite measure on Γ. The chain $\{\tilde{X}_n\}$ is said to be <u>uniformly ϕ recurrent</u> if for each measurable $\gamma \in \Gamma$ such that $\phi(\gamma) > 0$,

$$(5.1) \qquad \sum_{m=1}^{n} P_x^u\{\tilde{X}_m \in \gamma,\ \tilde{X}_i \notin \gamma,\ i < m\} \to 1 \quad \text{uniformly in } x \in \Gamma, \text{ as } n \to \infty$$

(Orey [13], p. 26). A sufficient condition for (5.1) is (Orey [13], p. 29) that if $\phi(\gamma) > 0$ then there is an $n < \infty$ and $\varepsilon > 0$ (perhaps depending on γ) such that

$$(5.2) \qquad \sum_{m=1}^{n} P_x^u\{\tilde{X}_m \in \gamma,\ \tilde{X}_i \notin \gamma,\ i < m\} \geq \varepsilon$$

for all $x \in \Gamma$. If the chain is uniformly ϕ recurrent and a-periodic then there are constants C and $\rho \in (0,1)$ such that

$$(5.3) \qquad |P_x^u\{\tilde{X}_n \in \gamma\} - \tilde{\mu}_u(\gamma)| \leq C\rho^n$$

uniformly in γ and $x \in \Gamma$ (a consequence of equation (6.2) in [13], p. 26, and the invariance of $\tilde{\mu}_u$). Thus, the n-step transition probability $\pi_u^{(n)}(x,\cdot)$ converges to $\tilde{\mu}_u$ in variation, at an exponential rate.

Define, for $x \in R^r$ (see Section 3 for the definition of τ_i, τ and $\bar{\mu}_u$),

$$(5.4)\qquad \tilde{V}^u(x) = E_x^u \int_0^{\tau_1} [k^u(x(s)) - \theta(u)]ds$$

$$+ \lim_n \sum_{m=1}^{n} [E_x^u \int_{\tau_m}^{\tau_{m+1}} k^u(x(s))ds - \int \bar{\mu}_u(dx)k^u(x)]$$

$$(5.5)\qquad V^u(x) = E_x^u \int_0^{\tau_1} \tilde{k}^u(x(s))ds + \lim_n E_x^u \int_{\tau_1}^{\tau_n} \tilde{k}^u(x(s))ds,$$

where $\tilde{k}^u(x) \equiv \tilde{k}(x,u(x)) - \theta(u)$.

<u>Theorem 5.1</u>. <u>Assume</u> (A1) - (A2), (A4) - (A5). <u>Then</u> $\tilde{V}^u(x)$ <u>and</u> $V^u(x)$ <u>are well defined</u>. <u>There are constants</u> C_0, C_1 <u>such that</u>

$$(5.6)\qquad \begin{aligned} |\tilde{V}^u(x)| &\le C_0 + C_1 E_x^u \tau_1 \\ |V^u(x)| &\le C_0 + C_1 E_x^u \tau_1. \end{aligned}$$

<u>The tail of</u> (5.5) $(E_x^u \int_{\tau_n}^{\tau_m} \cdot)$ <u>goes to zero as</u> $n,m \to \infty$, <u>uniformly in</u> x, <u>and</u> $E_x^u \int_0^{\tau_n} \tilde{k}^u(x(s))ds$ <u>is bounded uniformly in</u> n <u>and</u> $x \in \Gamma$.

<u>Proof</u>. Let $\phi = \ell$ = Lebesgue measure on Γ. $\pi_u(x,\gamma) > 0$ if γ is open in Γ. Then $\pi_u(x,\gamma) > 0$ if $\ell(\gamma) > 0$. Since $\pi_u(\cdot,\gamma)$ is continuous (by the strong Feller property - it also follows from the assertion 6° of Khasminski in [8], with a suitable definition of U,Γ there), $\inf_{x\in\Gamma} \pi_u(x,\gamma) > 0$. Thus, by the criterion (5.2), with $n = 1$, $\{\tilde{X}_n\}$ is uniformly ℓ-recurrent.

Let $\hat{F}(\cdot)$ be a bounded measurable function on Γ. By virtue of (5.3),

$$(5.7) \qquad \sum_n |E_x^u \hat{F}(\tilde{X}_n) - \int_\Gamma \tilde{\mu}_u(dx)\hat{F}(x)| \le \text{constant}, \ x \in \Gamma.$$

Let $\hat{F}(x) = E_x^u \int_0^\tau k^u(x(s))ds$, which is bounded on Γ by (3.1). Note that

$$E_x^u \hat{F}(\tilde{X}_n) = E_x^u E_{\tilde{X}_n}^u \int_0^\tau k^u(x(s))ds = E_x^u \int_{\tau_n}^{\tau_{n+1}} k^u(x(s))ds.$$

Then, by using (3.4) and (5.7),

$$(5.8) \qquad \sum_{n=1}^\infty |E_x^u \int_{\tau_n}^{\tau_{n+1}} k^u(x(s))ds - \int_\Gamma \mu_u(dx)k^u(x)\bar{\mu}_u(R^r)| \le \text{constant}, \ x \in \Gamma,$$

This, together with $E_x^u \int_0^{\tau_1} |k^u(x(s))|ds \le \sup_x |k^u(x)| E_x^u \tau_1$, implies both that $\tilde{V}^u(\cdot)$ is well defined and also the first line of (5.6).

Now, redo the above argument with $\hat{F}(x) = E_x^u \tau$, $x \in \Gamma$. Then

$$(5.9) \qquad \sum_{n=0}^\infty |E_x^u E_{\tilde{X}_n}^u \tau - \int_\Gamma \tilde{\mu}_u(dx) E_x^u \tau| \le \text{constant}, \ x \in \Gamma.$$

Noting[+] that $\int_\Gamma \tilde{\mu}_u(dx)E_x^u\tau = \bar{\mu}_u(R^r)$, the convergence in (5.8) and (5.9) allows us to replace $\bar{\mu}_u(R^r)$ in (5.8) by $E_x^u E_{\tilde{X}_n}^u \tau = E_x^u \int_{\tau_n}^{\tau_{n+1}} ds$, and still to get convergence. From this, we get both that $V^u(\cdot)$ is well defined, and the last bound in (5.6). The last two assertions of the theorem follow from (5.8) and (5.9). Q.E.D.

The Lemma gives some useful estimates.

[+] See (3.4), with $F(x) \equiv 1$.

The constant C may have different values in each usage.

Lemma 5.1. Assume (A1) - (A2), (A4) - (A5). Then, for all admissible $u(\cdot), v(\cdot)$ and all x, $s \geq 0$, $t \geq 0$, and Markov times $\alpha \leq t$.

(5.10) $$E_x^v |V^u(x(t))| \leq C[1+W_1(x)+t]$$

(5.11a) $$E_{x(t)}^v |V^u(x(t+s))| \leq C[1+W_1(x(t))+s]$$

(5.11b) $$E_{x(t)}^v |V^u(x((s+t)\cap\rho))| \leq C[1+W_1(x(t))+s], \text{ w.p.1 for any Markov time } \rho \geq t$$

(5.11c) $$E_{x(t)}^v W_i(x(t+s)) \leq C[1+W_i(x(t)) + s] \quad \text{w.p.1.}$$

(5.12) $$E_x^v |V^u(x(\beta))|^2 \leq C[1+t+W_2(x)], \text{ each } u(\cdot), v(\cdot) \quad \beta = \alpha \cap t$$

(5.13) $$\int \mu_v(dx)|V^u(x) < \infty, \int \mu_v(dx)|W_1(x)| < \infty, \text{ each } u,v.$$

Proof. By (5.6),

$$E_x^v |V^u(x(t))| \leq \text{constant}[1+E_x^v E_{x(t)}^u \tau_1].$$

By (A4), there is a constant ε_1 such that $\mathscr{L}^v W_1(x) \leq \varepsilon_1$, all x and $v(\cdot)$. Thus, by an application of Itô's Lemma,

$$E_x^v W_1(x(t)) \leq W_1(x) + \varepsilon_1 t.$$

The last two equations and the bounds (3.5) and (3.8) imply (5.10). Equation (5.11) follows by similar calculations.

We prove (5.12) only for $\alpha = t$. The general proof is similar. Write

$$V^u(x(t)) \le C[1+E^u_{x(t)}\tau_1] \le C[1+W_1(x(t))].$$

Continuing, and using (5.11c) and (A5),

$$E^v_x|V^u(x(t))|^2 \le CE^v_x[1+W_2(x(t))] \le$$

$$C[1+W_2(x)+t].$$

To prove (5.13), define $\sigma_N = \inf\{t: |x(t)| \ge N\}$, and for each integer M define $W_1^M(x) = \min[W_1(x),M]$, and note that (by (A5) and Itô's Lemma)

$$E^v_x W_2(x(t\cap\sigma_N)) \le W_2(x) + E^v_x \int_0^{t\cap\sigma_N} [c_2-q_2(x(s))]ds.$$

Thus, by bounding $q_2(\cdot)$ and letting $N \to \infty$,

$$0 \le W_2(x) + E^v_x \int_0^t [c_2-\alpha W_1^M(x(s))]ds.$$

Divide the last equation by t, let $t \to \infty$, and get (using (3.6), the convergence of $P^v(x,t,\cdot)$ to the invariant measure $\mu_v(\cdot)$)

$$c_2 \ge \lim_{t\to\infty} \frac{1}{t}\int_0^t \alpha E^v_x W_1^M(x(s))ds = \alpha \int \mu_v(dx)W_1^M(x).$$

Hence, $c_2 \geq \alpha \int \mu_v(dx) W_1(x)$. This, together with $V^u(x) \leq C\cdot[1+W_1(x)]$, implies (5.13). Q.E.D.

Theorem 5.2 will be used to obtain the optimality criterion in Theorem 6.1.

Theorem 5.2. Assume (A1) - (A2) and (A4) - (A5). Then $V^u(\cdot)$ is continuous and the process M_t^u given by

$$M_t^u \equiv V^u(x(t)) - V^u(x) + \int_0^t \tilde{k}^u(x(s))\,ds$$

is a continuous square integrable martingale, adapted to $\{\mathcal{E}_t\}$ and under P_x^u, each x.

Proof. First we note several facts. $V^u(\cdot)$ does not depend on G_1. If $\gamma < \gamma' < \gamma_1$, and G_1' is a sphere with radius γ', then several cycles ($\Gamma \to \Gamma'$, etc.) of the process for the G_1' case may be included in one cycle for the G_1 case, but the values of $V^u(\cdot)$ are the same. Also, as $\gamma_1 \downarrow \gamma$, $\sup_{x\in\Gamma} E_x^u\tau \to 0$. Note also that

$$\inf_x P_x^u\{\tau_n \geq T\} \to 1, \text{ as } n \to \infty, \text{ each } T < \infty.$$

Let n_T denote the largest integer i such that $\tau_i \leq T$. Write $\tilde{k}^u(x(s))$ as $\hat{k}(s)$. It will be shown that

$$\lim_{T\to\infty} \lim_{m,n\to\infty} E_x^u \int_{T\cap\tau_n}^{\tau_m} \hat{k}(s)ds = 0, \text{ uniformly in bounded } x \text{ sets.} \tag{5.14}$$

This will imply that

$$V^u(x) = E_x^u \int_0^T \hat{k}(s)ds + \varepsilon(T,x),$$

where $\varepsilon(T,x) \to 0$ as $T \to \infty$, uniformly in bounded x sets. This and the strong Feller property imply that $V^u(\cdot)$ is continuous.

Now, for some constants C_i, and any integer Q,

$$\lim_{m,n} E_x^u \left| \int_{T \cap \tau_n}^{\tau_m} \hat{k}(s)ds \right| \leq E_x^u \left| \int_T^{\tau_{N_T+1}} \hat{k}(s)ds \right|$$

$$+ E_x^u \sum_{i > n_T} \left| \int_{\tau_i}^{\tau_{i+1}} \hat{k}(s)ds \right|$$

$$\leq C_1 E_x^u |\tau_{N_T+1} - \tau_{N_T}| + \sum_{i \leq Q} E_x^u \left| \int_{\tau_i}^{\tau_{i+1}} \hat{k}(s)ds \right| + C_2 E_x^u \tau_Q I_{\{Q \geq N_T\}}.$$

The first and third terms on the right can be made arbitrarily small (uniformly in bounded x sets) by selecting large T and small $(\gamma_1 - \gamma)$, and large T, resp. The central term can be made small, uniformly in x, by choosing Q large. This implies (5.14).

It can be shown that

$$E_{x(t)}^u [V^u(x(t+s)) - V^u(x(t)) + \int_t^{s+t} \tilde{k}^u(x(v))dv] = 0, \text{ all } s \geq 0, t \geq 0,$$

where the conditional expectation above exists by Lemma 5.1. The martingale property follows from this. The continuity and square integrability follow from the continuity of $V^u(\cdot)$ and (5.12), resp. Q.E.D.

6. The Maximum Principle

Let $u(\cdot)$ be admissible. It will be seen in Theorem 6.2 that there is a Borel function (a r-row vector) $\psi^u(\cdot)$ such that for each $x \in R^r$,

$$M_t^u = \int_0^t \psi^u(x(s))\sigma(x(s))dW^{x,u}(s) \quad \text{w.p.1} \quad P_x^u,$$

(6.1)

$$E_x^u \int_0^t |\psi^u(x(s))|^2 ds < \infty.$$

Let $v(\cdot)$ be admissible. Then $x(\cdot)$ satisfies (w.p.1 P_x^v)

$$dx(t) = [f(x(t)) + b^v(x(t))]dt + \sigma(x(t))dW^{x,v}$$

By (2.7), we can suppose that (w.p.1 P_x^v)

$$\begin{aligned} (6.2) \qquad dW^{x,v}(t) &= dW^{x,0}(t) - \sigma^{-1}(x(t))b^v(x(t))dt \\ &= dW^{x,u}(t) + \sigma^{-1}(x(t))(b^u(x(t)) - b^v(x(t)))dt. \end{aligned}$$

Since P_x^o, P_x^u and P_x^v are mutually absolutely continuous, all a.s. statements with respect to one are also a.s. statements with respect to the others.

Theorem 6.1 is the "maximum" or "Hamilton-Jacobi" principle, a natural development for our problem, of some of the ideas in [1] and [2].

Theorem 6.1. Assume (A1) - (A2), (A4) - (A5), and let $u(\cdot)$, $v(\cdot)$ be admissible. If

$$(6.3) \qquad e^{u,v}(x) \equiv (k^u(x) - k^v(x)) + \psi^u(x)(b^u(x)-b^v(x)) > 0$$

<u>on a set</u> A <u>of positive Lebesgue measure, then there is an admissible control</u> $\bar{v}(\cdot)$ <u>such that</u> $\theta(\bar{v}) < \theta(u)$. <u>The condition</u> $e^{u,v}(x) \leq 0$ <u>a.e. for each admissible</u> $v(\cdot)$ <u>is necessary and sufficient for</u> $u(\cdot)$ <u>to be optimal.</u>

<u>Proof</u>. First, we derive the basic formula (6.5). Using (6.1), (6.2) and the definition of M_t^u yields (a.s. P_x^u)

$$0 = V^u(x(t)) - V^u(x) + \int_0^t \tilde{k}^u(x(s))ds$$

$$- \int_0^t \psi^u(x(s))\sigma(x(s))[dW^{x,v}(s) - \sigma^{-1}(x(s))(b^u(x(s)) - b^v(x(s)))ds].$$

Define

$$\sigma_N = \min\{t: \int_0^t |\psi^u(x(s))\sigma(x(s))|^2 ds = N\}.$$

Then

$$(6.4) \qquad 0 = E_x^v V^u(x(t\cap\sigma_N)) - V^u(x) + E_x^v \int_0^{t\cap\sigma_N} \tilde{k}^u(x(s))ds$$

$$+ E_x^v \int_0^{t\cap\sigma_N} \psi^u(x(s))[b^u(x(s)) - b^v(x(s))]ds$$

where the expectations exist by Lemma 5.1.

By the uniform integrability implied by (5.12), the first term on the r.h.s of (6.4) tends to $E_x^v V^u(x(t))$ as $N \to \infty$. Also, by (2.6) (use $W^{x,u}$ in $\zeta_0^t(v-u)$ not $W^{x,0}$),

$$E_x^v \int_{t\cap\sigma_N}^t |\psi^u(x(s))|ds = E_x^u \exp \zeta_0^t(v-u)\cdot \int_{t\cap\sigma_N}^t |\psi^u(x(s))|ds$$

$$\leq [E_x^u \exp 2\zeta_0^t(v-u)]^{1/2}[E_x^u(\int_{t\cap\sigma_N}^t |\psi^u(x(s))|ds)^2]^{1/2} \equiv A_1A_{2N}.$$

$A_1 < \infty$, and $A_{2N} \to 0$ as $N \to \infty$ by (6.1). Thus, we can replace $t \cap \sigma_N$ by t throughout (6.4). Setting $\sigma_N = 0$ in the above equation yields

$$E_x^v \int_0^t |\psi^u(x(s))|ds \leq \text{constant}[E_x^u \int_0^t |\psi^u(x(s))|^2 ds]^{1/2}.$$

But

$$E_x^u \int_0^t |\psi^u(x(s))|^2 \leq \text{constant}\{E_x^u|V^u(x(t)) - V^u(x)|^2 + 1 + t^2\}$$

$$\leq \text{constant}\{1 + t + W_1(x)\}^2.$$

Then by the last inequality, Swartz's inequality, (5.11c), (A5) and (5.13), $E_x^v \int_0^t \psi^u(x(s))[b^u(x(s)) - b^v(x(s))]ds$ is integrable with respect to μ_v.

Furthermore, with $\sigma_N \cap t$ set equal to t, the E_x^v can be put inside all the integral signs in (6.4). Doing this and integrating each term with respect to μ_v, and using the invariance of μ_v (under control $v(\cdot)$), yields $\int \mu_v(dx)[E_x^v V^u(x(t)) - V^u(x)] = 0$ and

$$0 = \int_0^t ds \int \mu_v(dx)E_x^v\{\tilde{k}^u(x(s)) + \psi^u(x(s))[b^u(x(s)) - b^v(x(s))]\}.$$

Now subtract the zero quantity $\int \mu_v(dx)\tilde{k}^v((x))$ from the above equation, and use the invariance of μ_v (under control $v(\cdot)$) to get

$$0 = t\left\{\int \mu_v(dx)[(\tilde{k}^u(x) - \tilde{k}^v(x)) + \psi^u(x)(b^u(x) - b^v(x))]\right\},$$

or, equivalently

$$(6.5) \quad 0 = \int \mu_v(dx)[e^{u,v}(x) + \theta(v) - \theta(u)].$$

Next, let $A = \{x: e^{u,v}(x) > 0\}$ and let $\ell(A) > 0$. Define the admissible control $\bar{v}(\cdot)$ by: $\bar{v}(x) = u(x)$ on $R^r - A$, $\bar{v}(x) = v(x)$ on A. Since (6.5) holds for all $u(\cdot), v(\cdot)$, $e^{u,\bar{v}}(x) \geq 0$ and

$$0 = \int \mu_{\bar{v}}(dx)[e^{u,\bar{v}}(x) + \theta(\bar{v}) - \theta(u)].$$

But $e^{u,\bar{v}}(x) > 0$ on A, and $\mu_{\bar{v}}(A) > 0$ by Theorem 3.1. Thus, $\theta(\bar{v}) < \theta(u)$, proving the first assertion of the theorem. The second assertion follows easily by the same type of argument on (6.5). Q.E.D.

Remark. The reason for inserting the corollary is discussed after the proof.

Corollary. Assume (A1) to (A5). Let $u(\cdot)$ be optimal and $v(\cdot)$ an admissible control. Then

$$(6.6) \qquad \theta(v) = \theta(u) - \int \mu_v(dx)e^{u,v}(x).$$

Let $\psi^u(\cdot)$ be bounded on bounded x-sets. For each $\varepsilon > 0$, let

$\psi_\varepsilon(\cdot)$ <u>denote a (r-row vector) Borel function such that</u>

$$(6.7)\qquad \begin{aligned} &\int_K |\psi_\varepsilon(x) - \psi^u(x)|dx \to 0, \ \underline{\text{each bounded set}}\ K, \ \underline{\text{as}}\ \varepsilon \to 0, \\ &\sup_{\varepsilon, x\varepsilon K} |\psi_\varepsilon(x)| < \infty, \ \underline{\text{each bounded set}}\ K. \end{aligned}$$

<u>Let</u> K <u>denote a fixed compact set in</u> R^r, <u>which is the closure of its interior</u>. <u>Suppose that the function</u> $v_\varepsilon(\cdot)$ <u>is calculated by</u>

$$(6.8)\qquad v_\varepsilon(x) = \arg\inf_{\alpha\varepsilon\mathscr{U}} [k(x,\alpha) + \psi_\varepsilon(x)b(x,\alpha)], \ \underline{\text{for almost all}}\ x \varepsilon K,$$

$$v_\varepsilon(x) = 0, \ x \notin K.$$

<u>Then</u> $v_\varepsilon(\cdot)$ <u>can be assumed to be admissible, and</u>

$$(6.9)\qquad \overline{\lim_{\varepsilon\to 0}}\ \theta(v_\varepsilon) \le \theta(u) - \underline{\lim_{\varepsilon\to 0}} \int_{R^r-K} \mu_{v_\varepsilon}(dx)[(k^u(x) - k^o(x)) + \psi^u(x)b^u(x)].$$

<u>Proof</u>. Equation (6.6) is just (6.5). By the complete lattice property ([14], p. 302) of $L_1(K)$, the inf in (6.8) can be assumed to be in $L_1(K)$. Then, by the properties of $\mathscr{U}, b(\cdot), k(\cdot)$ in (A2) and (A3), we get that the inf (evaluated at x) is in the set $k(x,\mathscr{U}) + \psi_\varepsilon(x)b(x,\mathscr{U})$, for almost all $x \varepsilon K$. Then, the implicit function theorem cited in Theorem 4.1 can be used to show that there is an admissible control which attains the inf almost everywhere. We call this control $v_\varepsilon(\cdot)$.

Note that, if $\psi_\varepsilon(x) \to \psi^u(x)$ for a fixed $x \in K$, then

$$\inf_{\alpha\in\mathscr{U}} [k(x,\alpha) + \psi_\varepsilon(x)b(x,\alpha)] \to \inf_{\alpha\in\mathscr{U}} [k(x,\alpha) + \psi^u(x)b(x,\alpha)], \quad \text{as } \varepsilon \to 0. \tag{6.10}$$

Also, the L_1 convergence (6.7) implies that for each $\delta > 0$, there is an $\varepsilon_\delta > 0$ and a set $A_{\varepsilon\delta} \in K$ with $\ell(A_{\varepsilon\delta}) < \delta$ for $\varepsilon < \varepsilon_\delta$ and such that $|\psi_\varepsilon(x) - \psi^u(x)| \leq \delta$ for $x \notin A_{\varepsilon\delta}$, $\varepsilon < \varepsilon_\delta$. This, together with (6.10), implies that the difference between the sides in (6.10) converges in $L_1(K)$ as $\varepsilon \to 0$. Note that the r.h.s. of (6.10) equals $k^u(x) + \psi^u(x)b^u(x)$ (almost everywhere) by optimality of $u(\cdot)$, and the theorem. Now, by (6.6),

$$\begin{aligned}\theta(v_\varepsilon) = \theta(u) &- \int_K \mu_{v_\varepsilon}(dx)[(k^u(x)-k^{v_\varepsilon}(x)) + \psi^u(x)(b^u(x)-b^{v_\varepsilon}(x))] \\ &- \int_{R^r-K} \mu_{v_\varepsilon}(dx)[(k^u(x)-k^o(x)) + \psi^u(x)b^u(x)].\end{aligned} \tag{6.11}$$

The integrand of the first integral of (6.11) equals

$$[k^u(x) + \psi^u(x)b^u(x)] - [k^{v_\varepsilon}(x) + \psi_\varepsilon(x)b^{v_\varepsilon}(x)] - (\psi^u(x) - \psi_\varepsilon(x))b^{v_\varepsilon}(x).$$

The remarks below (6.10), and the fact that $\mu_v(A) \to 0$ as $\ell(A) \to 0$ uniformly in $v(\cdot)$ and A (Theorem 4.2)) imply both that the first integral on the r.h.s. of (6.11) goes to zero, as $\varepsilon \to 0$, and the theorem. Q.E.D.

Remark on the Corollary. The corollary was given because it will probably be useful when used in conjunction with a procedure for computing or estimating $\psi^u(\cdot)$. Usually, we would not be able to calculate $\psi^u(\cdot)$ exactly, and the corollary asserts that, even if the computation is approximate, its use to get a control may yield good results, since the cost is "continuous in $\psi_\varepsilon(\cdot)$", in a sense, provided that $\int_{R^r - S_N} \mu_v(dx)|\psi^u(x)| \to 0$ as $N \to \infty$, uniformly in $v(\cdot)$. We would expect that this latter condition would hold quite often.

Theorem 6.2. Assume (A1) - (A2), (A4) - (A5). Then M_t^u has the representation (6.1).

Proof. By Theorem 2.3 of Davis and Variaya [1], and the square integrability of M_t^u, there is a process $\xi^{u,x}(\cdot)$ such that $E_x^u \int_0^t |\xi^{u,x}(s)|^2 ds < \infty$ for each t and such that

$$M^u(s) = \int_0^t \xi^{x,u}(s)\sigma(x(s))dW^{x,u}(s), \text{ w.p.1. } P_x^u.$$

Let $\mathscr{A}^u$ denote the class of continuous random functions that are square integrable martingales under P_x^u, each x, and are also homogeneous additive functions of the Markov process $x(\cdot)$, and which are adapted to $\{\mathscr{C}_t\}$. If $N(\cdot) \in \mathscr{A}^u$, then the quadratic variation $\langle N,N\rangle_t$ has a representation which is a homogeneous additive non-decreasing function of $x(\cdot)$. It does not otherwise depend on $x(0) = x$. (See, for example, Mayer [15], Theorem 3, p. 126. The result is also implied

by Kunita and Watanabe [16], Appendix.) The processes $M^u(\cdot)$ and

$$W^{x,u}(t) = \int_0^t \sigma^{-1}(x(s))[dx(s) - (f(x(s)) + b^u(x(s)))]ds]$$

are in $\mathscr{A}^u$. Also $W^{x,u}(\cdot) \pm M^u(\cdot)$ are both in $\mathscr{A}^u$. Then

$$\langle W^{x,u} + M^u, W^{x,u} + M^u\rangle_t - \langle W^{x,u} - M^u, W^{x,u} - M^u\rangle_t = 4\langle W^{x,u}, M^u\rangle_t$$

is a homogeneous additive process. But

$$\langle W^{x,u}, M^u\rangle_t = \int_0^t \xi^{x,u}(s)\sigma(x(s))ds,$$

which must also have a representation as a homogeneous additive function of $x(\cdot)$, and which does not otherwise depend on $x(0) = x$. Thus, there is a Borel function $\psi^u(\cdot)$, not depending on $x = x(0)$, such that $\xi^{u,x}(s)\sigma(x(s)) = \psi^u(x(s))\sigma(x(s))$, for almost all s w.p.1 P_x^u, each x. Q.E.D.

REFERENCES

[1] M.H.A. Davis, P.P. Varaiya, "Dynamic programming conditions for partially observable stochastic systems", SIAM J. on Control, 11, 1973, pp. 226-261.

[2] J.M. Bismut, Théorie probabiliste du contrôle des diffusions, Memoirs Amer. Math. Soc., 4, no. 167, January 1976.

[3] W.W. Strook, S.R.S. Varadhan, "Diffusion processes with continuous coefficients", Comm. Pure and Appl. Math., 22, 1969, pp. 345-400, 479-530.

[4] H.J. Kushner, Introduction to Stochastic Control Theory, Holt, Rinehart and Winston, New York, 1971.

[5] P. Mandl, "On the control of non-terminating diffusions", Theory of Prob. Applic. 9, 1964, pp. 591-603.

[6] Ya. A. Kogan, "On optimal control of a non-terminating diffusion process with reflection", Theory of Prob. Applic., 14, 1969, pp. 496-502.

[7] I.V. Girsanov, "On transforming a certain class of stochastic processes by absolutely continuous transformation of measures", Theory of Prob. Applic. 5, 1960, pp. 285-301.

[8] R.Z. Khazminskii, "Ergodic properties of recurrent diffusion processes, and stabilization of the solution to the Cauchy problem for parabolic systems", Theory of Probability and its Applications, 5, 1960, pp. 179-196.

[9] J.L. Doob, "Asymptotic properties of Markov transition probabilities", Trans. Amer. Math. Soc., 63, 1948, pp. 393-421.

[10] E. Roxin, "The existence of optimal controls", Michigan Math. J., 9, 1962, 109-119.

[11] E.J. McShane, R.B. Warfield, "On Fillipov's implicit function lemma", Proc. Amer. Math. Soc. 18, 1967, pp. 41-47.

[12] P. Billingsley, Convergence of Probability Measures, Wiley, New York, 1968.

[13] S. Orey, Lecture Notes on Limit Theorems for Markov Chain Transition Probabilities, Van Nostrand Reinhold Co., London, 1971.

[14] N. Dunford, J. T. Schwartz, Linear Operators, Wiley, New York, 1958.

[15] P.A. Mayer, "Intégrals stochastiques". Séminaire de Probabilitiés no. 1, pp. 72-141, Lecture Notes in Math 39, Springer-Verlag, Berlin.

[16] H. Kunita, S. Watanabe, "On square integrable martingales", Nagoya Math. J., 30, 1967, pp. 209-245.