SHEAR STABILIZATION OF DRIFT WAVES IN NONCIRCULAR CROSS SECTION

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# Shear Stabilization of Drift Waves in Noncircular Cross Section Axisymmetric Configurations

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**Abstract**
A calculation is given of the shear induced damping of drift waves for noncircular cross section axisymmetric devices in the electrostatic approximation. Comparison of damping rates for circular and elliptical cross section tokamaks shows only a weak dependence on the ellipticity.

**Key Words**
Tokamak
Noncircular tokamak
Drift waves
Shear stabilization
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SHEAR STABILIZATION OF DRIFT WAVES IN NONCIRCULAR CROSS SECTION AXISYMMETRIC CONFIGURATIONS

I. INTRODUCTION

The role of shear in the stabilization of drift waves has received a great deal of attention (1-11). (For example, the treatment of the dissipative trapped electron mode instability has recently been studied because of its interest for tokamak transport.) In the most simplified approaches the growth rate may be expressed as

\[ \gamma = \gamma_y - \gamma_s \]

where \( \gamma_y \) is a mode growth term usually due to an electron dissipation mechanism (e.g., scattering of trapped electrons for the dissipative trapped electron mode, electron Landau resonance for the universal instability, finite electron thermal conductivity for the drift dissipative mode), and \( \gamma_s \) is a shear induced damping rate. Shear stabilization results when \( \gamma_s > \gamma_y \). The calculation of \( \gamma_s \) was given by Pearlstein and Berk for slab geometry \( B = B_0 (z_0 + y_0 x/L_s) \) where \( L_s \) is the "shear length" in their study of the universal instability. They took the electrostatic potential in the form \( f(x) \exp(ik_y y) \) and found a mode localized about \( x=0 \), which propagates energy outward in \( x \) until, as a result of the shear, \( \omega/v_i \approx |k_y| = k_y |x|/L_s \), where the wave energy is damped via ion Landau resonance (\( v_i \) is the ion thermal velocity). This calculation is trivially extended to a circular plasma column equilibrium which models a large aspect ratio tokamak of major radius \( R \) if we impose mode periodicity; along the column length, \( z=0 \) to \( z = a \). The reason for the essential similarity between the slab and circular column geometries is that they both have two directions of uniformity, \( y \) and \( z \).

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in the slab model, and \( \Theta \) and \( z \) in the circular cross section case. On the other hand, there is much interest in tokamaks with vertically elongated cross sections due to their potential advantage with respect to achieving higher \( \beta \) and higher current densities. In addition, multipole devices also exhibit magnetic surfaces with noncircular cross section. In these devices the geometrical \( \Theta \) symmetry of circular cross section is lost.

It is the purpose of this paper to obtain the shear induced damping rate of drift waves in such noncircular cross section devices. We make the following simplifying assumptions: (i) large aspect ratio, (ii) electrostatic perturbation, (iii) modal wavelengths larger than \( \rho_i / \rho_i \) (\( \rho_i \) is the ion Larmor radius), and (iv) a longitudinal magnetic field much larger than the poloidal field.

Section II describes the geometry and obtains the fundamental two dimensional partial differential equation which governs this problem. Section III presents the solution of the fundamental equation for the shear induced damping under the assumption that mode rational surfaces are decoupled and discusses this assumption. As an example, these results are applied to a particular tokamak equilibrium in Section IV, and it is shown that the shear induced damping rate is very insensitive to the amount of vertical elongation (at least for the example given).
Related work has been reported by Connor and Hastie\textsuperscript{7} who use a different method and do not include plasma current (applicable to levitrons and multipoles but not tokamaks). The main difference with their result is that the $d\omega/d\psi$ terms in the frequency shift $d$, given in our Eq. (14), would be absent since without plasma current $\alpha$ is constant. Talor\textsuperscript{8} has treated a model equation to determine the effect of strong coupling between mode rational surfaces. Here we only treat situations where mode rational surfaces are decoupled by ion Landau damping (cf. discussion at the end of Section III).
II. GEOMETRY AND FUNDAMENTAL EQUATION

We assume a noncircular cylinder of length $2\pi R$ with periodic boundary conditions applied at the column ends (cf. Fig. 1(a)) to simulate a large aspect ratio axisymmetric device. The coordinates are $(\psi,x,z)$ and are illustrated in Fig. 1, where constant $\psi$ corresponds to a magnetic surface $d\psi = B_\perp d\psi$, $x$ specifies the crosssectional position of a point on a flux surface $dx = a B_\perp d\chi$, $z$ is the length along the cylinder, $B_\perp$ is the poloidal component of $B$ (component $\perp$ to $z$), and $dl_\psi$ and $dl_\chi$ are differential lengths in the $\psi$ and $\chi$ directions. The function $a$ is chosen to make the $(\psi,\chi)$ coordinates orthogonal,

$$\nabla \times a = 0.$$  \hspace{1cm} (1)

In order to see that this is so assume $\psi$ and $\chi$ are orthogonal as in Figure 1 (b) and take the line integral $\int a B_\perp dl = 0$ around the closed path (a,b,c,d) indicated in Figure 1(b). We see that for an orthogonal coordinate system $\int a B_\perp dl = 0$ and so (1) follows.

We now set down the basic equations describing the system. In doing this we make the following assumptions to simplify the analysis:

(i) $\omega >> k_n v_i$, (ii) $T_e >> T_i$, (iii) $\omega << k_n v_e$, (iv) $k_\perp^2 \nu_i ^2 << 1$.

$$\nu_i = \nu_e = \tilde{\nu},$$ \hspace{1cm} (2)

$$\tilde{\nu}_e = N(\psi) e^\psi / T_e.$$ \hspace{1cm} (3)

$$-i \omega \tilde{\nu}_i + \nabla \cdot (\tilde{\nu}_i \nabla) = 0,$$ \hspace{1cm} (4)

$$\nabla = \nabla_\perp + \nabla_\parallel$$ \hspace{1cm} (5)

$$\nabla_\perp = -B^{-2} \varphi \nabla_b + i \omega |B_{\varphi_c}|^{-1} \nabla_\perp \phi$$ \hspace{1cm} (6)
\[ -i\omega \psi_n = -\omega_{ci}^{-1} B \cdot \nabla \psi \]  

(7)

where \( \omega_{ci} = eB/M_i \) and \( E = -\nabla \psi \).

Eliminating \( \bar{n} \) and \( v \) and using tokamak ordering \( (B_\perp/B) \ll 1 \) and taking \( \phi \propto \exp(ikz) \), \( k=n/R \), we obtain a single second order partial differential equation for \( \phi \) in terms of the coordinates \( \psi \) and \( x \). The details leading to this equation appear in Appendix A. The result is

\[
\begin{align*}
\left[ \omega &- \frac{icT_e}{eB} \frac{\partial n N}{\partial \psi} \frac{\partial}{\partial \psi} + \frac{c_s^2}{\omega B^2} \left( \frac{\partial}{\partial \psi} \frac{1}{\alpha} \frac{\partial}{\partial \psi} + \frac{\partial}{\partial x} \alpha \frac{\partial}{\partial x} \right) \right] \varphi = 0, \\
- \frac{cT_e}{eB} &\omega_{ci} \frac{\partial}{\partial \psi} \alpha B_\perp + \frac{\partial}{\partial x} \frac{\partial}{\partial \psi} \left( \frac{\partial}{\partial \psi} \frac{1}{\alpha} \frac{\partial}{\partial \psi} + \frac{\partial}{\partial x} \alpha \frac{\partial}{\partial x} \right) \right] \varphi = 0,
\end{align*}
\]

(8)

where \( \varphi = e\psi/T_e \) and \( c_s = \left( T_e/M_i \right)^{1/2} \). Equation (8) is our basic equation to be studied in detail in the next section. The first term in (8) is due to \( \bar{n}_e \); the second term is due to the ion density perturbation produced by the \( E \times B \) drift and background density gradient; the third term is a result of finite ion inertia along field lines; and the last two terms in (8) represent finite ion inertia across \( B \) (the polarization drift).

If the destabilizing sources of electron dissipation were included Eq. (3) would take on the general form \( \hat{n}_e = N(\psi) \left[ 1 + i\hat{M} \right] \left( e\psi/T_e \right) \) where \( \hat{M} \) can be an operator. Different forms of \( \hat{M} \) result from different electron dissipation mechanisms (as discussed in Section I). Here we set \( \hat{M} = 0 \) (so that no instability mechanism is present) and solve for the mode structure and shear induced damping rate. The effect of \( \hat{M} \) for specific instabilities can then be included as a perturbation on this basic solution as demonstrated in References 4 and 5.
III. ANALYSIS OF BASIC EQUATION

To analyze Eq. (8) we expand \( \varphi \) in a set of orthogonal basis functions \( u_m(x,\psi) \)

\[
\varphi(x,\psi) = \sum_m u_m(x,\psi) \varphi_m(\psi) .
\]

(9)

It is possible to choose the \( u_m \) so that the first three terms in (8), which are larger than the last two, are diagonal for all values of \( \psi \). This will be the most convenient choice for the \( u_m \). We choose \( (n=\alpha B_\perp) \)

\[
u_m = \exp \left[ -i \frac{m}{nq(\psi)} k_B \int x \, n^{-1} \, dx \right] ,
\]

(10)

so that

\[
\delta_{pm} u_m^* \frac{dx}{\eta} = \delta_{pm} \delta \frac{dx}{\eta},
\]

(11)

where \( \delta_{pm} \) is the Kronecker delta, \( * \) denotes the complex conjugate, and the safety factor is \( q(\psi) = (2\pi n)^{-1} k_B n^{-1} \, dx \). Note that the \( u_m(x,\psi) \) also satisfy the periodicity condition

\[
u_m(x,\psi) = u_m(x+\Phi,\psi)
\]
on all magnetic flux surfaces. If we represent Eq. (8) symbolically as \( L \varphi = 0 \) then use of (9) converts (8) to a matrix operator equation

\[
\sum_p L_{pm} \varphi_m(\psi) = 0 ,
\]

(10)

where each of the \( L_{pm} \) is a second order ordinary differential operators in \( \psi \) gotten from the relation
\[ L_{pm} \varphi_m = \left[ \int dx \left( \alpha B_{\perp}^2 \right)^{-1} u_p^* L(u_m \varphi_m) \right] \left[ \int dx \left( \alpha B_{\perp}^2 \right)^{-1} \right]^{-1}. \] \quad (11)

From (11) and (8) we obtain

\[ L_{pm} = D_{pm} + N_{pm} + P_{pm} \] \quad (12)

where

\[ D_{pm} = \delta_{pm} \left\{ \omega - \frac{m/n}{q(\psi)} \omega^* e^{-\frac{k^2 c_s^2}{\omega q^2(\psi)}} \left[ q(\psi) - m/n \right]^2 \right\}, \]

\[ N_{pm} = \frac{cT_e}{eB} \frac{\omega}{\omega_c} \left( \frac{kB}{q(\psi)} \right)^2 \left( \frac{m}{n} \right)^2 \left[ \int dx \frac{e^{i(F_p - F_m)}}{\left| \int dx \right|} \right]^{-1}, \]

\[ P_{pm} = -\frac{cT_e}{eB} \frac{\omega}{\omega_c} \left[ \int dx \right]^{-1} \omega^{-1} \frac{\partial^2}{\partial \psi^2} \frac{1}{\left[ \left( \frac{\partial F_m}{\partial \psi} \right)^2 + \left[ \frac{\partial F_m}{\partial \psi} \right]^2 \right]} \frac{d^2}{d\psi^2} \left\{ \frac{d}{d\psi} \right\}^2 \left[ \left[ \frac{\partial F_m}{\partial \psi} \right] - i \frac{\partial \alpha}{\partial \psi} \left( \frac{-1}{\partial \psi} \right) \right] \right\} dx, \]

where \( F_m = \frac{kBm}{(nq(\psi))} \int dx \left[ \frac{d}{d\psi} \right] \right\} dx, \]

\( \omega^* e^{-\frac{k^2 c_s^2}{\omega q^2(\psi)}} \), and \( D_{pm}, N_{pm} \) and \( P_{pm} \) represent respectively the first three terms of (8), the fourth term in (8), and the last term in (8). Note that \( D_{pm} \) is diagonal while \( N_{pm} \) and \( P_{pm} \) are non-diagonal, and \( P_{pm} \) is a second order differential operator in \( \psi \). Here we only consider the case where it is valid to consider the matrix operator as diagonal. We discuss the conditions under which this is valid later. Ignoring the off diagonal components of \( L_{pm} \) we have \( L_{mm} \varphi_m = 0 \) which using the approximation \( q(\psi) \approx m/n + q'(\psi_0) (\psi - \psi_0) \)

\( (q' \equiv dq/d\psi \) and \( q(\psi_0) \equiv m/n) \) may be written as
\[
\left\{ \frac{d^2}{d\psi^2} - \frac{\omega_c^2}{c_s^2} <B_\perp^2> \left[ 1 - \frac{\omega}{\omega_c} + d \left( \frac{k c_s}{\omega} \right)^2 \left( \frac{q_0}{q_{\psi=\psi_0}} \right)^2 (\psi - \psi_0)^2 \right] \right\}_{\psi = 0} = 0 \tag{13}
\]

where \(d\) is a frequency shift term due to the finite cross-field ion inertia terms

\[
d = \frac{c_s^2}{\omega_c^2} \left\{ k^2 <B_\perp^2> + <B_\perp^2>^{-1} \left[ <B_\perp^2 \left( \frac{\alpha m}{\partial L} \right)^2 > <B_\perp^2 > \right. - <B_\perp^2 \left( \frac{\alpha m}{\partial L} \right) >^2 - \frac{1}{4} \left( <B_\perp^2 \frac{d n_a}{d\psi} > \right)^2 + \frac{1}{2} <B_\perp^2 > <\alpha B_\perp \frac{d^2 \alpha^{-1}}{d\psi^2} > \right\}, \tag{14}
\]

the flux surface average of a function \(G\) is

\[
\langle G \rangle = \left[ \frac{\mathcal{F}_G \ B_\perp^{-1} d\lambda \mathcal{F}_B^{-1}}{d\lambda} \right]_{\psi = \psi_0}.
\]

and the new dependent variable \(\bar{\varphi}_m\) has been introduced to eliminate the first derivative term in \(P_{nm}\).

\[
\bar{\varphi}_m = \exp \left( -j \int g d\psi \right) \varphi_m(\psi).
\]

\[
g(\psi) = \left( \mathcal{F}_\varphi^{-1} d\lambda \right)^{-1} \left( -\frac{1}{2} \frac{d}{d\psi} \mathcal{F}_\varphi^{-1} d\lambda + j \mathcal{F}_\varphi^{-1} \frac{\alpha m}{\partial \psi} d\lambda \right).
\]

Equation (13) is analogous to the equation of Pearlstein and Berk.  

Looking for solutions satisfying an outgoing wave boundary conditions we obtain the eigenfunction and wave eigenfrequency,
\[ \bar{w}_m(\psi) = \exp\left[ -\sigma(\psi - \psi_0)^2/2 \right] \cdot H_n(\sigma^{1/2}(\psi - \psi_0)) \]

where \( H_n \) are Hermite polynomials.

For \( d < 1 \) we may approximate the shear induced damping rate for the least damped mode (\( n=0 \)) as

\[ \gamma_{sh} = -k^2 c_s^2 \omega_c^{-1} \left\langle B_\perp^2 \right\rangle^{1/2} \frac{q'(\psi)}{q(\psi)} \quad (16) \]

which will be evaluated in the next section for a specific equilibrium configuration.

We now discuss the conditions under which the diagonal approximation to (10) is valid.

First of all, if the plasma has a circular cross section, \( \alpha, \eta \) and \( F_m \) are all independent of \( \lambda \), so that the matrix is diagonal because of the orthogonality relation. For noncircular cross section, the matrix elements clearly approach zero as \( m-p \) approaches infinity. The reason is that the matrix element is the integral of a smoothly varying function (the poloidal field) times a very rapidly oscillating function.

In order to truncate the matrix at a one by one, we must invoke ion Landau damping, which is not explicitly included in Eq. (8). As the wave propagates away from the mode rational surface, the value of \( k_n \) increases. When the distance from the rational surface is large enough that \( \omega/k_n v_i \)
approaches about 2 or 3, the ion Landau damping gets very strong. The energy propagating away from the mode rational surface is absorbed in this region and the eigenfunction decays sharply in space. If the spatial point at which $\omega/k_\nu v_i \sim 3$ is less than halfway between the neighboring rational surfaces, solutions to the diagonal matrix (including ion Landau damping) are spatially isolated from one another. Thus the eigenfunctions of the diagonalized matrix are a good approximation to the solution of Eq. (10).

The distance between rational surfaces, $\delta \psi$, is

$$\delta \psi = (ndq/d\psi)^{-1}.$$  

The distance at which $\omega/k_\nu v_i \sim 3, \Delta \psi$, is given by

$$\Delta \psi = \frac{\omega q R}{3nv_i dq/d\psi}.$$  

where we have used $k=n/R$. Thus the condition that the matrix can be diagonal reduces to $\delta \psi > 2\Delta \psi$ or

$$\frac{2\omega q R}{3v_i} < 1.$$  \hspace{1cm} (17)

The work of Taylor\textsuperscript{8} applies to the limit opposite to (17).
IV. EVALUATION OF SHEAR STABILIZATION FOR A SPECIFIC EQUILIBRIUM

We consider an equilibrium which we partially specify by choosing the axial current density as a function of $\psi$,

\[ J_z(\psi) = J_0 \quad \text{for} \quad 0 < \psi < \psi_0, \]  
\[ J_z(\psi) = 0 \quad \text{for} \quad \psi > \psi_0. \]  

(18a)

(18b)

Thus $J_z$ is a constant inside the flux surface $\psi = \psi_0$ and is zero outside this flux surface. Note that the plasma will be assumed to extend into the current free region, $\psi > \psi_0$. We take the surface $\psi = \psi_0$ to be an ellipse of ellipticity $\kappa$ so that the following equilibrium solution applies in $\psi < \psi_0$ (cf. Ref. 12),

\[ \psi = \frac{KB}{2qR} \rho^2, \quad \rho^2 = x^2 + \frac{y^2}{\kappa^2}, \]  
\[ B_\perp = |\nabla \psi| = \frac{KB}{qR} \left( x^2 + \frac{y^2}{\kappa^2} - 4 \right)^{1/2} \]  

(19a)

(19b)

This equilibrium has all magnetic surfaces elliptical with the same ellipticity. Also, since $J_z$ is constant in $\psi < \psi_0$, $q(\psi)$ is also constant in $\psi < \psi_0$, and there is no shear. Outside the $\psi = \psi_0$ surface $q(\psi)$ is not constant, and there is shear. We evaluate the shear induced damping in $\psi > \psi_0$ for a point close to the $\psi = \psi_0$ surface. The geometry dependent factor in (16) is

\[ \left< \frac{B_\perp}{q'/(q)} \right>^1_{1/2} = \left( \frac{\int B_\perp d\chi}{\int B_{\perp}^{-1} d\chi} \right)^{1/2} \frac{d}{d\psi} \frac{\int B_\perp^{-1} d\chi}{\int (aB_\perp)^{-1} d\chi}. \]  

(20)
The evaluation is performed in Appendix B for the particular equilibrium under consideration and we obtain from (16)

\[ \gamma_{sh} = -2 \frac{k c_s^2}{\omega_{ci} r} f(\kappa), \quad f(\kappa) = \kappa^{-2} \left[ \frac{1}{2} (\kappa^2 + 1) \right]^{3/2}. \] (21)

The variation of \( \gamma_{sh} \) with \( \kappa \) [as given by the function \( f(\kappa) \)] is very weak, as can be seen by the following numbers: \( f(1) = 1 \) (circular), \( f(1.5) = 0.988, \) \( f(2.5) = 1.104, \) \( f(3) = 1.242, \) \( f(3.5) = 1.392. \) Thus for \( \kappa \) between 1 and 2.5 the value of \( \gamma_{sh} \) is within 10% of its value for the circular case.

In particular, from a local theory calculation of the dissipative trapped electron mode stability in vertically elongated tokamaks it was found that the growth rate was reduced by elongation. Since the shear damping appears to be approximately independent of elongation (for this example), this indicates the possibility of complete stabilization by vertical elongation.

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REFERENCES

APPENDIX A

DERIVATION OF EQ. (8)

Equation (8) follows simply from Eqs. (2) - (7). Equations (2) and (3) express \( n_i \) in terms of \( \phi \), while Eqs. (6) and (7) express \( v_i \) in terms of \( \phi \). Inserting these expressions in the ion continuity equation then gives a single equation for \( \phi \). The only problem is to express gradients, divergences, and unit vectors in flux co-ordinates.

The length element is

\[
ds = \frac{1}{B_\perp} \, d\psi + \frac{1}{\alpha B_\perp} \, dx + dz. \tag{A1}
\]

The unit vectors perpendicular and parallel to the magnetic field are then

\[
\begin{align*}
\mathbf{i}_1 &= \mathbf{i}_\psi \\
\mathbf{i}_2 &= \frac{B_\perp i_z - B_z i_\psi}{(B_\perp^2 + B_z^2)^{1/2}} \\
\mathbf{i}_3 &= \mathbf{i}_n = \frac{B_\perp i_\psi + B_z i_z}{(B_\perp^2 + B_z^2)^{1/2}}
\end{align*}
\]

The components of the gradient perpendicular and parallel to the magnetic field are

\[
\nabla_\perp = i_\psi B_\perp \frac{\partial}{\partial \psi} + i_2 (i_2 \cdot \nabla)
\]

\[
= i_\psi B_\perp \frac{\partial}{\partial \psi} + i_2 \left( \frac{B_\perp}{(B_\perp^2 + B_z^2)^{1/2}} \right) \left( \frac{\partial}{\partial x} - \alpha B_z \frac{\partial}{\partial x} \right) \tag{A2}
\]
Finally, the divergence of a vector $A$ is

$$\nabla \cdot \mathbf{A} = \alpha B_\perp \left\{ \frac{\partial}{\partial \psi} \frac{\partial A_\perp}{\partial \psi} + \frac{\partial}{\partial x} \frac{\partial A_\perp}{\partial x} + \frac{1}{\alpha B_\perp} \frac{\partial}{\partial z} A_z \right\}.$$  \hspace{1cm} (A4)

By using Eqs. (A2) - (A4) and also assuming $B_\perp \gg B_\parallel$ (consistent with tokamak ordering), and also $\alpha B_\perp \frac{\partial}{\partial x} \gg k_z$ (consistent with propagation nearly perpendicular to $B$), Eqs. (2) through (7) can be manipulated into Eq. (8).
APPENDIX B

EVALUATION OF $\delta_{sh}$ FOR ELLIPTICAL MODEL

Since $J_z = 0$ in $\psi > \psi_0$, $\nabla \times \mathbf{B}_\perp = 0$, there and consequently Eq. (1) implies that $\alpha$ is constant in $\psi > \psi_0$. Thus

$$d/d\psi \mathbf{B}_\perp (\mathbf{d}A / d\psi) d\psi$$

Also from $\nabla \times \mathbf{B}_\perp = 0$, it follows that $\mathbf{B}_\perp \mathbf{d} \mathbf{A}_\perp / d\psi = -R_c^{-1}$.

where $R_c$ is the radius of curvature of the flux surface in the constant $z$ plane. Thus

$$<B^2_\perp>^{1/2} (q'/q) = 2 \left( \int_0^\rho \frac{d1}{d\psi} \frac{dx}{dx} \right)^{1/2} \int_0^\rho \frac{d1}{d\psi} \frac{dx}{dx} \frac{d1}{d\psi} \frac{dx}{dx} \frac{d1}{d\psi} \frac{dx}{dx} \frac{d1}{d\psi} \frac{dx}{dx}$$

To evaluate this quantity on a surface of constant $\psi$ we note that near $\psi = \psi_0$ in $\psi > \psi_0$, the flux surface is approximately elliptical and $\mathbf{B}_\perp$ is approximately $\mathbf{B}_\perp$ on the $\psi = \psi_0$ surface. To evaluate the integrals we need $\mathbf{B}_\perp$, $R_c$, and $d1/\psi$ as functions of $x$ on the magnetic surface ($\rho = $ constant). From Eqs. (19) we obtain

$$B_\perp = \frac{B_0}{q_0 R} \left[ 1 + (\kappa^2 - 1) (x/\rho)^2 \right]^{1/2} \quad (B2)$$

$$\frac{d1}{dx} = \left[ 1 + (dy/dx)^2 \right]^{1/2} = \left[ 1 + (\kappa^2 - 1) (x/\rho)^2 \right]^{1/2} \left[ 1 - (x/\rho)^2 \right]^{-1/2} \quad (B3)$$

To find $R_c$, we express $\mathbf{i}_\psi$, the unit normal to the magnetic surface,

$$i_\psi = \frac{\mathbf{v}_\rho^2}{\mathbf{v}_\psi^2} \quad (R_c$$ is then given by
\[ R_c^{-1} = \left( \frac{d \psi}{d \chi} \right)^{-1} = \left( \frac{\partial \psi}{\partial x} \right)^{-1} \frac{1}{\left( \frac{d \chi}{d x} \right)^{-1}}, \]

and thus

\[ R_c^{-1} = \kappa \rho^{-1} \left[ 1 + (\kappa^2 - 1) \left( \frac{x}{\rho} \right)^2 \right]^{-3/2}. \]  

(B4)

Putting (B-2) to (B-4) in (B-1) we obtain

\[ \langle B^2 \rangle^{1/2} \left( \frac{q'}{q} \right) = 2 \left( \kappa \rho^2 \right)^{-1} \left[ \frac{1}{2} (\kappa^2 + 1) \right]^{3/2}, \]

from which Eq. (21) follows.
Fig. 1 — (a) Noncircular cylinder, and (b) $\psi$, $\chi$ coordinates in a constant $z$ plane