An inequality for sums of dyads and tensors.*

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An inequality for sums of dyads and tensors.

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Given finite rank transformation $R$ on Hilbert space where

$$2 \text{rank}(R) \leq r(U) + r(V) \leq \text{rank}(R) + N,$$

where $r(U) = \dim(\text{span}(u_1, u_2, \ldots, u_N))$, $r(V) = \dim(\text{span}(v_1, v_2, \ldots, v_N))$.

Applications to sums of decomposable Kronecker products are given.
An inequality for sums of dyads and tensors*
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ABSTRACT: Given a finite rank transformation \( R \) on Hilbert space with dyadic sum decomposition
\[
\sum_{i} (u_i \times v_i) = R,
\]
then it is shown that
\[
2 \cdot \text{rank}(R) \leq r(U) + r(V) \leq \text{rank}(R) + N,
\]
where \( r(U) = \dim(\text{span}(u_1, u_2, \ldots, u_N)) \) and
\( r(V) = \dim(\text{span}(v_1, v_2, \ldots, v_N)) \).

Applications to sums of decomposable Kronecker products and to sums of dyads are presented.

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Introduction. In previous works, relations between dyadic and Kronecker products of vectors (definitions follow) are explored, cf. [2], [3]. In fact, consider the general situation where finite rank linear transformation $R$ on infinite-dimensional Hilbert space, $H$, is the sum of dyadic products. If the number of terms of this sum is known, then these dyadic terms can be fairly well characterized [3, Thm. 3.2]. In this paper, we consider dyadic sum decompositions for $R$ where $N$, the number of terms, is not known a priori, and present a sharp inequality which ties together

(i) the rank of $R$,

(ii) the ranks (dimension of the spans) of the dyad component vectors, and

(iii) $N$, the number of distinct dyads which sum to $R$.

This inequality proves useful for establishing necessary conditions for certain special questions, e.g., when do $N$ dyads sum to a single Kronecker product, or when do $N$ dyads sum to (another) dyad? These questions, in turn, relate to the complexity question in the computation of matrix products, cf., [4], [1].

2. Definitions and Preliminaries. $L(H,K)$ denotes all bounded linear transformations from Hilbert space $H$ to Hilbert space $K$. Among the elements of $L(H,K)$ are the dyads (rank one transformations) $(x \times y)$ defined for each $y \in H$, $x \in K$ by requiring that for all $z \in H$, $(x \times y)z = \langle z, y \rangle x$, where $\langle , \rangle$ is the inner product on $H$. We proceed to give the Kronecker or tensor product
A \otimes B^t: First, for A \in L(H,K), A^*, the adjoint of A, is that element of L(K,H) given by \langle Ay, x \rangle = \langle y, Ax \rangle for all y \in H, x \in K.

As an example, (x \otimes y)^* = (y \otimes x) for all dyads. H denotes the Hilbert space of linear functionals on H. That is, for x \in H, \overline{x} \in H is defined by \overline{x}: y \to \langle y, x \rangle for all y \in H. This leads to the definition of A^t \in L(K,H) where A \in L(H,K). In fact, for all x \in H, \overline{y} \in K, we define A^t(\overline{y})(x) = \overline{y}(A(x)). Finally, for any A \in L(H_1,K_1), B \in L(H_2,K_2) we define the Kronecker (or tensor) product A \otimes B^t by A \otimes B^t: C \to ACB for all C \in L(K_2,H_1).

We will use \text{rank}(R) to denote the rank of a transformation R, i.e., \text{rank}(R) is the dimension of the range of R. Also, if U = \{x_1, x_2, \ldots, x_N\} \subset H, then we will use \text{rank}(U) to denote the rank of the set U, i.e., \text{rank}(U) is the dimension of \text{span}(U), the linear span of the set U.

Before arriving at our inequality, we will be using the following characterization of dyadic sums:

\textbf{Theorem 2.1} ([3, Th. 3.2]). Given finite-rank linear transformation R \in L(H,K) and the set U = \{u_1, u_2, \ldots, u_n, \ldots, u_N\} \subset K where the range of R is a subspace of span \langle U \rangle. Assume (by re-ordering if necessary) that the first n \leq N elements of U form a basis for span \langle U \rangle (i.e., n = \text{rank}(U), the rank of U). Accordingly the N-n \geq 0 remaining vectors u_{n+1}, u_{n+2}, \ldots, u_N define N-n scalars \{a_i^{(j)}: i = 1, 2, \ldots, n, j = n+1, n+2, \ldots, N\} by the equations

$$u_j = \sum_{i=1}^{n} a_i^{(j)} u_i , \ j = n+1, n+2, \ldots, N.$$
Then for \( N-n \) arbitrary vectors \( \{v_{n+1}, v_{n+2}, \ldots, v_N\} \subset H \) we have the representation

\[
\sum_{i=1}^{N} (u_i \times v_i) = R
\]  

(2.1)

if and only if each "earlier" \( v_1 \) is given by

\[
v_1 = R^*(\alpha_1) - \sum_{j=n+1}^{N} \overline{\alpha}_1(j) v_j, \quad i = 1, 2, \ldots, n=r(U),
\]

(2.2)

where \( \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \in \text{span } \langle U \rangle \) is the unique biorthonormal complement to \( \{u_1, u_2, \ldots, u_n\} \in \text{span } \langle U \rangle \) (i.e., \( \langle \alpha_i, u_j \rangle = \delta_{ij} \), the Kronecker delta). The summation in (2.2) is taken to be zero in case \( n = N \).

3. **The Inequality**

**Theorem 3.1.** Given finite-rank linear transformation \( R \in L(H, K) \) and sets of vectors \( U = \{u_1, u_2, \ldots, u_N\} \subset K, \ V = \{v_1, v_2, \ldots, v_N\} \subset H \) such that

\[
\sum_{i=1}^{N} (u_i \times v_i) = R.
\]

(3.1)

Then

\[
2 \cdot \text{rk}(R) \leq r(U) + r(V) \leq \text{rk}(R) + N,
\]

(3.2)
where \( \text{rk}(R) \) = dimension (range of \( R \)), and

\[
\text{r}(U) = \text{dimension (span } \langle U \rangle) \\
\text{r}(V) = \text{dimension (span } \langle V \rangle).
\]

**Proof:** By re-ordering the terms of sum (3.1) if necessary, we will assume that the first \( n = r(U) \) elements, \( u_1, u_2, \ldots, u_n \) of \( U \), form a basis for \( \text{span} \langle U \rangle \). Thus, the ordered set \( V \) lends itself to characterization (2.2). In fact,

\[
\text{r}(V) = \text{rank}(\text{span}\langle v_1, v_2, \ldots, v_n, v_{n+1}, \ldots, v_N \rangle), \quad (3.3)
\]

where \( v_1 = R^*(a_1) = \sum_{j=n+1}^{N} \alpha_j^{(i)} v_j, \ i = 1, 2, \ldots, n \) (from (2.2)). Equivalently,

\[
\text{r}(V) = \text{rank}(\text{span}\langle R^*(a_1), R^*(a_2), \ldots, R^*(a_n), v_{n+1}, \ldots, v_N \rangle) \quad (3.4)
\]

The equivalence of (3.3) and (3.4) follows by observing that each of the \( N \) vectors in (3.4) belong to the linear span of the \( N \) vectors in (3.3), and vice versa. From (3.4) we now obtain

\[
\text{r}(V) \leq \text{rank}(\text{span}\langle R^*(a_1), \ldots, R^*(a_n) \rangle) + \text{rank}(\text{span}\langle v_{n+1}, \ldots, v_N \rangle) \\
\leq \text{rk}(R^*) + N - n \quad (3.5) \\
= \text{rk}(R) + N - r(U),
\]
which gives us the right-hand side of inequality (3.2). Obtaining the left-hand side of (3.2) is immediate, since from (3.1) we deduce that \( \text{span} \langle U \rangle \supset \text{range } R \), while \( \text{span} \langle V \rangle \supset \text{range } R^* \) (recall \( (u_1 \times v_1)^* = (v_1 \times u_1) \)). Thus, \( r(U) \geq rk(R) \) and \( r(V) \geq rk(R^*) = rk(R) \) implying

\[
2 \cdot rk(R) \leq r(U) + r(V). \tag{3.6}
\]

Finally, (3.5) with (3.6) establishes (3.2) and the proof is done. ■

**Is the inequality sharp?** The left side of (3.2) yields equality whenever the entire \( N \)-element sets \( U \) and \( V \) are linearly independent (i.e., when \( n = N = rk(R) \)). In following the proof of the right-hand inequality for (3.2), we observe the two inequalities in (3.5). The first inequality yields equality if and only if

\[
\text{span}<R^*(\bar{a}_1), R^*(\bar{a}_2), \ldots, R^*(\bar{a}_n)> \cap \text{span}\langle v_{n+1}, v_{n+2}, \ldots, v_N \rangle = \{0\}.
\]

That is, by choosing each of the \( N-n \) arbitrary vectors \( v_{n+1}, \ldots, v_N \) in \( H \) outside the range of \( R^* \). The second inequality of (3.5) becomes equality if and only if the \( N-n \) element set \( \{v_{n+1}, v_{n+2}, \ldots, v_N\} \) is linearly independent.

4. **Final Remarks.** In [3, Th. 4.2, 4.3], it is shown that
\[ \sum (u_1 \times v_1) = R \text{ if and only if } \sum (u_1 \otimes v_1) = R' \] (4.1)

where the passage from \( R \) to \( R' \) is a well-defined linear relationship. This provides a dual form to (3.2) with tensor products replacing the dyads of (3.1) and this \( R' \) replacing \( R \). As an easy special case, let us use (3.2) and dyad-tensor duality to justify the following statements for non-zero \( u_i, v_i, x_i, y_i \in \mathbb{H}, i = 1, 2, 3 \).

**Proposition.** Suppose

\[
(u_1 \times v_1) + (u_2 \times v_2) = (u_3 \times v_3), \quad \text{and} \quad (x_1 \otimes u_1) + (x_2 \otimes y_2) = (x_3 \otimes y_3). \tag{4.2a} \tag{4.2b}
\]

Then all the \( u_i \)'s or else all the \( v_i \)'s are non-zero scalar multiples of each other. Similarly, all the \( x_i \)'s or else all the \( y_i \)'s are scalar multiples of each other.

**Proof.** The proof of this assertion will not appeal to the definitions of the dyad \( (u_1 \times v_1) \) or of the tensor \( (x_1 \otimes y_1) \), since inequality (3.2) applies. In fact, write (4.2a) as

\[
(u_1 \times v_1) + (u_2 \times v_2) - (u_3 \times v_3) = 0 \quad (\text{i.e., } N = 3, R = 0) \tag{4.2a'}
\]

from which we obtain via (3.2) that

\[
2 \cdot 0 \leq r([u_1, u_2, u_3]) + r([v_1, v_2, v_3]) \leq 0 + 3. \tag{4.3}
\]
Since we have assumed no \( u_1 \) or \( v_1 \) is zero, the ranks \( r(u), r(v) \geq 1 \). At the same time, the upper bound of 3 given by (4.3) assures us that both \( r(u) = 2 \) and \( r(v) = 2 \) can not happen, i.e., at least one of the terms \( r(u), r(v) \) in (4.3) equals one, or all the \( u_j \)'s or all the \( v_j \)'s are scalar multiples of each other. By our duality result, (4.1), (4.2a') is equivalent to

\[
(u_1 \otimes v_1) + (u_2 \otimes v_2) - (u_3 \otimes v_3) = 0 ,
\]

and the same conclusion obtains, i.e., in (4.2b), either \( r((x_1, x_2, x_3)) \) or \( r((y_1, y_2, y_3)) \) equals one, or all the \( x_j \)'s or all the \( y_j \)'s are scalar multiples of each other if (4.2b) is given.
References

2. _____, Decomposable tensors as sums of dyads, Linear and Multilinear Algebra 1 (1974), 327-335.
3. _____, Characterizations of sums of dyads and of Kronecker products (submitted).
SUPPLEMENTARY

INFORMATION
Errata

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