TIME INTEGRATION OF UNSTEADY TRANSonic FLOW TO A STEADY STATE SOLUTION BY THE FINITE ELEMENT METHOD,

by

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A secondary investigation applied the steady transonic small disturbance equations to a converging-diverging nozzle.
Time Integration of Unsteady Transonic Flow to a Steady State Solution by the Finite Element Method

by

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ABSTRACT

A finite element method was applied to the unsteady transonic small disturbance equation and integrated until the solution converged to the steady state for a thin non-lifting airfoil. The method of weighted residuals was used to formulate the finite element equations, and Houbolt's method of central differencing in time was used to integrate these equations.

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I. INTRODUCTION

Transonic inviscid flows past a smooth airfoil may be expressed in terms of the velocity potential $\phi$ satisfying the transonic small disturbance equation,

$$
(1 - M_\infty^2 - M_\infty^2(\gamma + 1)\phi_x)\phi_x + \phi_y = 0
$$

This equation presents two major difficulties, 1) it is non-linear and 2) it is of the mixed hyperbolic-elliptic type. Analytical solutions to non-linear equations are difficult to obtain. One must normally resort to numerical methods. When the coefficient $(1 - M_\infty^2 - M_\infty^2(\gamma + 1)\phi_x)$ in equation 1 is negative, the flow is supersonic and the equation is called hyperbolic; otherwise the flow is subsonic and the equation is elliptic. The forms of the two solutions are fundamentally different. Hyperbolic equations admit both discontinuities, which propagate only in characteristic directions, and the presence of shock fronts. Elliptic equations, on the other hand, require that the dependent variables and their derivatives be continuous and that a change in any part of the flow field affects every other part. Many non-linear elliptic equations are solved by appropriate relaxation iteration techniques by casting the equation in Poisson's form with the non-linearity acting as the driving force. Solutions to hyperbolic equations are usually obtained by the method of characteristics or by
finite difference marching techniques which use an artificial viscosity to represent the average jump conditions across the shock wave. In mixed supersonic and subsonic flows, normal numerical procedures break down because the boundary between the two regions is not known a priori.

Finite element numerical techniques have evolved as a powerful tool in obtaining approximate solutions to a wide variety of engineering problems, particularly ones with Neumann-Dirichlet boundary conditions, i.e., elliptical problems. They offer several outstanding advantages. Some of these are:

1) Non-homogeneous problems may be treated with relative simplicity.

2) Complex geometries may be modeled with relative ease since the elements can be graded in size and shape to follow boundaries of arbitrary shape.

3) Once the finite element model has been established, a variety of problems can be solved by supplying the computer with the appropriate data.

Chan and Brashears [Ref. 5] developed a finite element computer program for steady transonic flow over a non-lifting airfoil. This program uses the least squares method of weighted residuals to approximate equation 1 by a system of algebraic equations, and assembles the equations in a special way to account for the hyperbolic region of flow. This technique prevents the influence of downwind nodes from propagating upstream in the supersonic region.

The purpose of this thesis is to investigate the possibility of speeding the convergence of a solution by transforming the steady transonic equation to the unsteady equation
and integrating over time until the time dependent terms vanish and to extend the program of Ref. [5] to the transonic region of a converging-diverging nozzle.
II. DISCUSSION OF THE FINITE ELEMENT APPROACH

In a continuum problem of any dimension, the field variable, whether it is velocity potential, velocity, temperature, displacement or some other quantity, possesses infinitely many values because it is a function of each generic point in the solution region. Consequently, the problem is one of an infinite number of unknowns. The finite element approach subdivides the solution domain into a finite number of subdomains called elements and expresses the unknown field variable in terms of assumed approximating functions within each element. The approximating functions are sometimes called interpolating functions and are defined in terms of the values of the field variables at specified points called nodes or nodal points. Nodes usually lie on the element boundaries where adjacent elements are considered to be connected. In addition to boundary nodes, an element may also contain interior nodes. The nodal values of the field variable and the interpolating function for the elements completely define the behavior of the field variable within the elements. Once these unknowns are found, the interpolating functions define the field variable throughout the assemblage of the element.

Clearly, the nature of the solution and the degree of accuracy of the approximation depends not only on the number and size of the elements used but also on the interpolating
functions selected. Interpolating functions may not be chosen arbitrarily because certain compatibility conditions must be satisfied. Often such functions are selected so that the field variable or its derivatives are continuous across adjoining element boundaries. Once the problem is formulated in terms of individual elements, the contributions of each element may be assembled to define the entire solution domain. This means, for example, that if we are treating a problem in stress analysis we can find the force-displacement or stiffness characteristics of each element and then assemble the elements to determine the stiffness of the whole structure. Finite element solutions are not, of course, restricted to structures problems, but the matrix of equations defined by the interpolating functions and the nodal field variables is still referred to as the stiffness matrix regardless of the field variable in the problem.

Solutions to continuous problems by the finite element approach always follow a systematic step-by-step process. This process is completely general to the finite element method and it is outlined below. [Ref. 4]

1. Discretize the continuum.

The first step is to divide the solution domain into elements. A variety of element shapes may be used and one or more different element shapes may be used in the same region. The type and number of the elements used in a given problem are a matter of engineering judgement.
2. Select the interpolating functions.

The next step is to choose the type of interpolating function to represent the variation of the field variable over the element. The field variable may be a scalar, a vector, or a higher order tensor. Often polynomials are selected as interpolating functions for they are easy to integrate and differentiate. The degree of polynomial chosen depends on the number of nodes and the nature and number of the unknowns and the continuity requirements imposed at the nodes and the element boundaries. The unknown quantities at the nodes may be assigned to the field variable and their derivatives.

3. Find the element properties.

After the elements and their interpolating functions have been selected, the matrix of algebraic equations which express the properties of the individual elements must be determined. Several methods are available for this task. These are: the direct approach, the variational approach, the method of weighted residuals and the energy balance approach. Reference [41] is a good source of information on the various techniques.

4. Assemble the element properties to obtain the system equations.

To find the properties of the over-all system, the matrix equations expressing the behavior of each element must be
added to the matrix equation of all other elements. The basis for this assembly procedure stems from the fact that connecting elements have common nodes and the field variable must be the same for each element sharing that node.

At this point the boundary conditions for the system of equations must be accounted for and the system of equations must be modified before it is ready for solution.

5. Solve the system of equations.

The assembly process of step 4 produces a set of simultaneous equations which can be solved to obtain the unknown field variables. Linear equations have a number of standard solution techniques readily available, but solutions to non-linear equations are more difficult to obtain.

6. Make additional computations if desired.

Important parameters, such as pressure coefficient in aerodynamic problems, may now be calculated from the values of the field variables.
III. THEOREY AND BASIC EQUATIONS

A. STEADY TRANSONIC FLOW

Chan et al. [Ref. 5] developed an algorithm to analyze steady transonic flow over non-lifting thin airfoils. Boundary layer effects were ignored and the imbedded shock wave was assumed to be weak. These assumptions are consistent with transonic small disturbance theory which can be expressed mathematically by the following expressions.

\[ (1 - \frac{M_\infty^2}{\gamma + 1}) \phi_x \phi_{xx} + \phi_{yy} = 0 \]  \hspace{1cm} (1)

Boundary conditions -

\[ \nabla \cdot \phi = 0 \] \hspace{1cm} at infinity  \hspace{1cm} (2)

\[ \nabla = (1 + u) \frac{dg}{dx} \] \hspace{1cm} on the airfoil \hspace{1cm} (3)

\[ u = 0 \] \hspace{1cm} on the line of symmetry \hspace{1cm} (4)

where \( g(x, y) \) defines the airfoil and \( \frac{dg}{dx} \) describes the airfoil slope.

The above expressions appear in their dimensionless form where \( \phi = \) perturbed velocity potential and the perturbed velocity components in the \( x \) and \( y \) directions are respectively defined as

\[ u = \phi_x \]

\[ v = \phi_y \]
\( M_\infty = \text{freestream Mach number} \) and \( \gamma = \text{ratio of specific heats} \), which for air is taken to be 1.4. The physical coordinates \( x' \) and \( y' \) and the velocity potential \( \phi' \) are related to the dimensionless quantities by

\[
x = x'/c, \quad y = y'/c, \quad \phi = \phi'/cU_\infty
\]

where \( c \) is the chordlength of the airfoil and \( U_\infty \) is the freestream velocity.

Once the flowfield solution is determined in terms of the perturbed velocities, the secondary unknowns are subsequently calculated. These include:

\[
a = \left( \frac{\gamma - 1}{2} \right) \left( U_\infty^2 - V^2 \right) + \left( \frac{U_\infty^2}{M_\infty^2} \right)^2 \left( \frac{1}{2} \right) \tag{5}
\]

\[
M = \frac{V}{a} \tag{6}
\]

\[
\frac{p}{P_0} = \frac{1}{\left[ 1 + \frac{\gamma - 1}{2} M_\infty^2 \right] \gamma / (\gamma + 1)} \tag{7}
\]

\[
C_p = -\left[ \frac{2u}{U_\infty} + (1-M_\infty^2) \frac{U_\infty^2}{U_\infty^2} + \frac{V^2}{U_\infty^2} \right] = -\frac{2u}{U_\infty} \tag{8}
\]

In the above, \( U_\infty = 1 \) is the normalized freestream velocity, \( a = \text{local sound speed} \), \( p = \text{local static pressure} \), \( M = \text{local Mach number} \), \( V = \text{total velocity} \), \( P_0 = \text{stagnation pressure} \) and \( C_p = \text{pressure coefficient} \).

B. UNSTEADY TRANSONIC FLOW

Unsteady transonic inviscid flow may be expressed in terms of the velocity potential \( \phi(x,y,t) \) to a first order approximation by
\[
(1 - M_\infty^2 - M_\infty^2(\gamma + 1) \phi_x) \phi_{xx} + \phi_{yy} - 2M^2 \phi_t - M^2 \phi_{tt} = 0 \quad (9)
\]

which has the same non-linear coefficient retained for steady transonic flow in Equation 1.

Boundary conditions require that the disturbances vanish far from the airfoil,

\[
\phi_x = 0 \quad \phi_y = 0 \text{ at infinity} \quad (10)
\]

and that the flow remain attached to the body. Let \( B(x, y, t) = 0 \) define the body at any instant. The surface tangency restraint may now be expressed by the substantial derivative \( DB/DT \) vanishing.

\[
DB/DT = B_t + (1 + \phi_x)B_x + \phi_yB_y \quad (11)
\]

If the body remains stationary \( B_t = 0 \), and the tangency condition becomes the same as in steady flow, namely

\[
v = (1 + u)\frac{dg}{dx} \quad (12)
\]

where \( \frac{dg}{dx} \) represents the airfoil slope.

The pressure coefficient for isentropic unsteady compressible flow is defined by

\[
C_p = \frac{2}{\gamma M_\infty^2} \left\{ [1 - \frac{\gamma - 1}{2} M_\infty^2 (2 \phi_x + 2 \phi_t + \phi_x^2 + \phi_y^2)]^{\gamma/(\gamma - 1)} - 1 \right\} \quad (13)
\]

Expanding by the binomial expansion and retaining only the first order terms gives,

\[
C_p = -2\phi_x - 2\phi_t \quad (14)
\]
IV. FINITE ELEMENT FORMULATION

The method of weighted residuals is a technique for approximating solutions to linear or non-linear partial differential equations and it is the basis for the finite element formulation of the transonic small disturbance equation (Equation 1). This procedure involves assuming the general functional behavior of the field variable which would approximately satisfy the basic equation and boundary conditions. Substituting this approximation into the original differential equation results in some error called a residual. A system of algebraic equations results when a weighted average of the residual is forced to vanish as it is averaged over the entire domain.

A. STEADY FLOW

The approximate solution to equation 1 is assumed to be

$$\hat{\phi} = N_i \phi_i$$  \hspace{1cm} (15)

where $N_i$ are the interpolating functions which exhibits the behavior of equation 1 and $\phi_i$ are the undetermined parameters at the nodal points.

When $\hat{\phi}$ is substituted into equation 1, the resulting residual is

$$R = [1 - M^2 - M^2(\gamma + 1)N_k, x \phi_k]N_j, xx + N_j, yy$$  \hspace{1cm} (16)
The weighted average is determined by multiplying the residual $R$ by $m$ linearly independent weighting functions $W_i$ and integrating over the elemental domain. Forcing this residual to vanish yields,

$$
\int \! W_i R \, dA = 0 \quad i = 1 \text{ to } m
$$

Chan et al. [Ref. 5] found that when the weighting function $W_i$ for equation 1 is chosen to be $\partial R/\partial \phi_i$ the resulting matrix is positive definite and well conditioned. This choice of weighting functions is referred to as the method of least squares because it is equivalent to minimizing the square of the residuals summed over the domain with respect to the undetermined parameters. That is,

$$
\delta x = \int \! R \, dA
$$

$$
\partial \chi / \partial \phi_i = \int \! \partial R / \partial \phi_i R \, dA = 0
$$  \quad (17)

Integrating over the domain produces the system of algebraic equations

$$
S_{ij} \phi_j = 0
$$  \quad (18)

where the elemental matrix $S_{ij}$ is defined as

$$
S_{ij} = \int \! Q_j \, P_i \, dA
$$  \quad (19)

With $Q_j$ and $P_i$ equal to

$$
Q_j = [1 - M^2 - N^2(\gamma + 1)N_k, x \phi_k] N_j, xx + N_j, yy
$$

$$
P_i = Q_i - M^2 (1 + \gamma)N_k, xx \phi_k N_i, x
$$
B. UNSTEADY FLOW

Development of the unsteady flow finite element equations is similar to the procedure used to formulate the finite element equations for steady transonic flow. The least squares method of weighted residuals is again used but \( \phi \) is now a function of time as well as the spatial coordinates \( x \) and \( y \).

The approximate solution has the form,

\[
\hat{\phi} = N_i(x,y)\phi_i(t) \tag{20}
\]

Substituting \( \hat{\phi} \) in the unsteady transonic small disturbance equation, the weighted residual becomes,

\[
\chi = \iint (R_1 + R_2 + R_3)^2 dA \tag{21}
\]

where

\[
R_1 = \left\{ [1 - M_\infty^2 - M_\infty^2(y + 1)\phi_k N_k, x]N_j, xx + N_j, yy \right\} \phi_j
\]

\[
R_2 = -2M_\infty^2 N_j, x \phi_j
\]

\[
R_3 = -M_\infty^2 N_j, \ddot{\phi}_j
\]

Expanding equation 21 yields

\[
\chi = \iint [R_1^2 + R_2^2 + R_3^2 + 2R_1R_2 + 2R_2R_3] dA \tag{22}
\]

and minimizing with respect to the undetermined parameters \( \phi_i \) the following system of algebraic equations result,

\[
\frac{\partial R_i}{\partial \phi_j} = P_i
\]

\[
\phi_j = 0 = 2\iint \frac{\partial R_i}{\partial \phi_j} [R_1 + R_2 + R_3] dA
\]
where $P_i$ has been previously defined in the steady finite element formulation.

The above equation may be rewritten in the form

$$S_{ij} \dot{\phi}_j + S_{Cij} \dot{\phi}_j + S_{Mij} \ddot{\phi}_j = 0 \quad (23)$$

The stiffness matrix $S_{ij}$ is the same as that developed for the steady transonic equation and the damping ($S_{Cij}$) and mass ($S_{Mij}$) matrices are defined below.

$$S_{Cij} = -\int M^2 N_j \phi_i dA \quad (24)$$
$$S_{Mij} = -\int M^2 N_j \phi_i dA \quad (25)$$
V. ELEMENT DESCRIPTION AND ASSEMBLY OF EQUATIONS

A. ELEMENT DESCRIPTION

The basic element used in the finite element program is the non-conforming cubic triangular element developed by Bazeley et al. [Ref. 2]. Also used in the program are the quadrilateral and trapezoidal elements constructed from these triangular elements. These three types of elements can be mixed and used freely in the entire flow region except that only trapezoids should be used to cover the supersonic and mixed region in order to enact the special assembly procedures required by the hyperbolic equation which describes the flow in that region.

The basic triangular element is shown in Fig. 1, which at each vertex has the velocity potential and the velocity components as the undetermined parameters. This type of element was chosen because both Dirichlet and Neumann boundary conditions can be imposed with equal convenience. In addition, because velocity components are used as primary unknowns secondary parameters, such as Mach number and pressure coefficient can be calculated directly without resorting to numerical differentiation, which would produce additional errors.

In the element, the approximate solution is assumed as

\[ \hat{\phi} = N_i \phi_i \quad (i = 1 \text{ to } 9) \]
Figure 1 - Triangular Element
In which $\phi_i$'s are the nine undetermined parameters of $\phi$ and $N_i$ are the interpolation functions which are expressed in terms of the area coordinates.

The shape or interpolating functions and their first and second derivatives are defined below.

Letting,

\begin{align*}
a_i &= x_j y_k - x_k y_j \\
b_i &= y_j - y_k \\
c_i &= x_k - x_j \\
\Delta &= \text{area of triangle 1-2-3} = (b_j c_k - b_k c_j)/2 \\
\alpha &= 0.5 (c_k - c_j) \\
\beta &= 0.5 (b_j - b_k) \\
H &= \zeta_k \zeta_j \zeta_k \\
H_x &= b_i \zeta_j \zeta_k + b_j \zeta_k \zeta_i + b_k \zeta_i \zeta_j \\
H_y &= c_i \zeta_j \zeta_k + c_j \zeta_k \zeta_i + c_k \zeta_i \zeta_j \\
H_{xx} &= 2(\zeta_k b_j b_k + \zeta_j b_k b_k + \zeta_k b_k b_j) \\
H_{yy} &= 2(\zeta_i c_j c_k + \zeta_j c_k c_i + \zeta_k c_i c_j)
\end{align*}

with $i = (1, 2, 3), k = (3, 1, 2)$ then one has for $l = (1, 4, 7), i = (1, 2, 3)$

\begin{align*}
N_1 &= \zeta_i^2(3 - 2\zeta_i) + 2H \\
N_{1,x} &= [6b_i \zeta_i (1 - \zeta_i) + 2H_x]/2\Delta \\
N_{1,y} &= [6c_i \zeta_i (1 - \zeta_i) + 2H_y]/2\Delta
\end{align*}
\[
N_{1,xx} = [6b_i^2 (1 - 2\xi_i) + 2H_{xx}] / (2\Delta)^2 \\
N_{1,yy} = [6c_i^2 (1 - 2\xi_i) + 2H_{yy}] / (2\Delta)^2
\]

for \(1 = (2, 5, 8), \ l = (1, 2, 3)\)

\[
N_{1} = \xi_i^2 (c_k \xi_j - c_j \xi_k) + \alpha H
\]

\[
N_{1,x} = [2b_i \xi_i (c_k \xi_j - c_j \xi_k) + 2\xi_i^2 + \alpha H_x] / 2\Delta
\]

\[
N_{1,y} = [2c_i \xi_i (c_k \xi_j - c_j \xi_k) + \alpha H_y] / 2\Delta
\]

\[
N_{1,xx} = [2b_i^2 (c_k \xi_j - c_j \xi_k) + 4b_i (2\Delta) \xi_i + \alpha H_{xx}] / (2\Delta)^2
\]

\[
N_{1,yy} = [2c_i^2 (c_k \xi_j - c_j \xi_k) + \alpha H_{yy}] / (2\Delta)^2
\]

for \(1 = (3, 6, 9), \ l = (1, 2, 3)\)

\[
N_{1} = \xi_i^2 (b_j \xi_k - b_k \xi_j) + \beta H
\]

\[
N_{1,x} = [2b_i \xi_i (b_j \xi_k - b_k \xi_j) + \beta H_x] / 2\Delta
\]

\[
N_{1,y} = [2c_i \xi_i (b_j \xi_k - b_k \xi_j) + 2\xi_i^2 + \beta H_y] / 2\Delta
\]

\[
N_{1,xx} = [2b_i^2 (b_j \xi_k - b_k \xi_j) + \beta H_{xx}] / (2\Delta)^2
\]

\[
N_{1,yy} = [2c_i^2 (b_j \xi_k - b_k \xi_j) + 4c_i (2\Delta) \xi_i + \beta H_{yy}] / (2\Delta)^2.
\]
Quadrilateral and trapezoidal elements as shown in Fig. 2 are also used in the present program, the former is used in the subsonic region and the latter in the mixed and supersonic region. The element matrix for the quadrilateral element is obtained by combining appropriately the matrices for two triangles, while the matrix for trapezoidal element is obtained by averaging contributions from two left-running and two right-running triangles. The averaging process is designed to remove the bias effects inherent in the quadrilaterals used.

B. ASSEMBLY OF EQUATIONS

Straightforward application of the finite element assembly technique to transonic flows would fail (the solution diverges) because this would allow disturbances to propagate upwind in the supersonic region of flow where the governing equation is hyperbolic. Hyperbolic equations have a time-like dependency in that the solution at the downwind station is affected by the upwind station but not vice-versa. Assembly techniques for a transonic flow finite element program must take into account this time-like dependency. If the x-axis is taken as a time-like direction in the supersonic region, the element matrix may be assembled in a way similar to a backward finite difference operator, which has been successful in solving hyperbolic equations.

Consider the rectangular element sketched below with the upwind station I and the downwind station II, each having two nodal points with the element type chosen. The element matrices can be constructed in the usual manner.
a. Quadrilateral Element


b. Trapezoidal Element

Figure 2 - Quadrilateral and Trapezoidal Elements
However, before assembling the element matrix into the system matrix the non-linear coefficient of equation 1 is evaluated.

\[ C = 1 - M_\infty^2 - M_\infty^2 (\gamma + 1) u \]

The sign of the coefficient being positive, zero, or negative defines the equation as elliptic, parabolic, or hyperbolic. If C is non-positive for all nodes in the element, the rows representing the improper downwind influence on the solution at an upwind station are ignored during assembly. This feature is automatically applied in the program requiring only a little care in arrangement of the nodes of the element. In the anticipated supersonic region, element node points should be arranged in the order as indicated in figure 3, starting with the upper left corner and proceeding in the counter-clockwise direction. In the elliptic region, i.e., where the coefficient is positive, no special assembly technique is invoked.

C. ITERATIVE PROCEDURES

With the equations assembled and the proper boundary conditions imposed, the system of non-linear algebraic equations is solved by iterative procedures in the form

\[ S_{ij}(\tilde{\phi})\phi_j(n) = l_i \quad (23) \]

to solve for the solution in the \( n^{th} \) iteration. The function \( \tilde{\phi} \) is defined as

\[ \tilde{\phi} = \theta \phi(n-1) + (1-\theta)\phi(n-1) \]
Figure 3 - Nodal Arrangement for Supersonic Region
in which the under-relaxation factor $\theta$ is in the range $0 < \theta < 1$. For subsonic flow $\theta = 1$, which is simply a successive approximation, yields good results, but it is necessary to under-relax somewhat with $\theta$ approximately .5 for supercritical flow. Generally, a smaller relaxation factor will make the solution more stable but it will tend to slow down the rate of convergence.

Equation 23 is subject to the convergence criterion that the change in local Mach number between two successive iterations is less than a prescribed value $\epsilon$ at all nodes in the flow field. That is,

$$\left| \frac{M(n) - M(n-1)}{M(n)} \right| \leq \epsilon.$$
VI. INTEGRATION OF UNSTEADY FINITE ELEMENT EQUATIONS

The unsteady transonic small disturbance equation (equation 9) when suitably reduced to a finite element approximation appears in the form,

\[ S_{ij} \dot{\phi}_j + S_{Cij} \dot{\phi}_j + S_{Mij} \ddot{\phi}_j = R_j \]  

This equation is analogous to a damped spring mass system, hence \( S_{ij}, S_{Cij}, \) and \( S_{Mij} \) are respectively referred to as the stiffness, damping, and mass matrices.

Mathematically, equation 26 represents a system of second order differential equations with constant coefficients, which can be solved by standard numerical procedures for differential equations, such as Runge-Kutta or Milne methods. However, this would be a very costly technique if the coefficient matrices are very large. In practical finite element analysis there are a few effective methods which take advantage of the banded matrices usually encountered in finite elements. One such method is the direct numerical integration method.

Direct integration involves a numerical step-by-step procedure aimed at satisfying equation 26 only at discrete time intervals \( \Delta t \) apart and not over all time \( t \). Conceptually, direct integration is a finite element method in space and a finite difference method in time. Examples of direct integration are the central difference method, Houbolt integration and the Wilson method. The first two schemes are finite
difference schemes whereas the latter is a linear acceleration method. Linear acceleration integration assumes a linear variation of acceleration from time $t$ to time $t + \Delta t$.

Central differencing can be very effective in the solution of many dynamic problems especially those that involve a large system of equations. However, this method is unstable for all time steps larger than a critical time step.

Houbolt integration is an implicit finite differencing method related to central differencing, only it has the advantage of being stable for all TIME STEPS.

The Houbolt method was used to integrate the unsteady finite element equations because of this stability.

Houbolt integration uses the following finite difference expansions:

$$\phi_{i,t+\Delta t} = \left[ 2\phi_{i,t+\Delta t} - 5\phi_{i,t} + 4\phi_{i,t-\Delta t} - \phi_{i,t-2\Delta t} \right] / \Delta t^2$$

$$\phi_{i,t+\Delta t} = \left[ 11\phi_{i,t+\Delta t} - 18\phi_{i,t} + 9\phi_{i,t-\Delta t} - 2\phi_{i,t-2\Delta t} \right] / 6\Delta t$$

which are two backward-difference formulas with errors of order $(\Delta t)^2$.

The solution of $\phi_{i,t+\Delta t}$ must satisfy equation 26 and at time $t+\Delta t$ equation 26 becomes

$$S_{ij}\phi_{j,t+\Delta t} + SC_{ij}\dot{\phi}_{j,t+\Delta t} + SM_{ij}\ddot{\phi}_{j,t+\Delta t} = 0$$

Substituting the finite difference formulas for $\phi_{j,t+\Delta t}$ and rearranging all known vectors on the right hand side, the solution for $\phi_{j,t+\Delta t}$ is obtained, namely:
\[(a_0 SM_{ij} + a_1 SC_{ij} + S_{ij})\phi_j, t + \Delta t = R_j, t + \Delta t \]  

\[ + (a_2 SM_{ij} + a_3 SC_{ij})\phi_j, t + (a_4 SM_{ij} + a_5 SC_{ij})\phi_j, t - 2\Delta t \]

Where the constant integration coefficients are:

\[ a_0 = \frac{2}{\Delta t^2} \]
\[ a_1 = \frac{11}{6\Delta t} \]
\[ a_2 = \frac{5}{\Delta t} \]
\[ a_3 = \frac{3}{\Delta t} \]
\[ a_4 = 2a_0 \]
\[ a_5 = -\frac{a_3}{2} \]
\[ a_6 = \frac{a_0}{2} \]
\[ a_7 = \frac{a_3}{9} \]

Equation 27 may be written as

\[ SE_{ij} \phi_j, t + \Delta t = RE_j \]  

where the effective stiffness matrix \( SE_{ij} \) and the effective load vector \( RE_j \) are defined as:

\[ SE_{ij} = S_{ij} + a_0 SM_{ij} + a_1 SC_{ij} \]
\[ RE_j = SM_{ij} (a_2 \phi_j, t + a_4 \phi_j, t - \Delta t + a_6 \phi_j, t - 2\Delta t) \]
\[ + SC_{ij} (a_3 \phi_j, t + a_5 \phi_j, t - \Delta t + a_7 \phi_j, t - 2\Delta t) \]

Accurate knowledge of the vectors \( \phi_j, t - \Delta t \) and \( \phi_j, t - 2\Delta t \) are required to yield an accurate solution for \( \phi_j, t + \Delta t \) and
normally the Houbolt integration scheme requires a special starting procedure to determine the initial two vectors \( \phi_j, \Delta t \) and \( \phi_j, 2\Delta t \). However, since the primary interest of this problem is to integrate the equations until they converge to a steady state solution, it is not necessary to obtain an accurate time history of the flow. Errors induced by the inaccurate starting vectors will vanish as time approaches infinity. Therefore, the starting vectors may be chosen somewhat arbitrarily.
VII. CONVERGING-DIVERGING NOZZLE

S. F. Shen [Ref. 10] demonstrated the feasibility of calculating compressible flows through a converging-diverging (Laval) nozzle by dividing the region of calculations into three patches, a subsonic region, a supersonic region and a transonic one, of course bounded by the other two regions. The locations of the boundaries for each region were chosen arbitrarily provided the sonic line is bracketed by the subsonic and supersonic boundaries.

Two different finite element formulations were used for the subsonic and supersonic regions, but Shen [10] resorted to analytical approximations to cover the transonic patch. This restricted the calculations to nozzles with small throat curvatures because no analytic solutions exist for nozzles with large throat curvatures. It is conceivable that STRANL-II could be adapted to provide a continuous solution throughout all three regions.

Outside the transonic region of flow the governing small disturbance equation is

\[(1 - M^2)\phi_{xx} + \phi_{yy} = 0\]  \hspace{1cm} (29)

This holds for both subsonic and supersonic flow. Comparing equation 29 with equation 1, the transonic small disturbance equation, we notice that only the non-linear coefficient \(M^2\) distinguishes the two equations from each other.
This coefficient becomes negligible when the Mach number becomes less than .8 or greater than 1.2. With this consideration in mind, it was assumed that equation 1 would adequately describe the flow through the nozzle and that the finite element formulation developed for the non-lifting airfoil would apply to the Laval nozzle.

A. BOUNDARY CONDITIONS

Two solutions are possible for a converging-diverging nozzle: 1) Symmetric flow, where the flow is subsonic through the domain, except for a small supersonic region near the wall in the throat, and 2) Asymmetric solution, where the flow accelerates to sonic velocity in the throat and then continues to accelerate to supersonic velocity in the diverging section. Different boundary conditions apply for the two solutions. For the symmetric case, both inlet and exit velocities must be specified. Inlet and exit velocities are equivalent in the subsonic solution. The supersonic solution requires that only the inlet velocities be specified. If the exit velocities are also applied, the problem is overspecified and the solution may not converge.

Velocities at the inlet and exit are not uniform in the y direction, therefore the disturbances cannot be set to zero as in the case of the non-lifting airfoil. Boundary velocities must be calculated by solving equation 29 analytically.

Equation 29 is a linear equation which can be mapped to Laplace’s equation,

\[ \nabla^2 \phi = 0, \]
by letting \( y' = \sqrt{(1 - \frac{v^2}{a^2})} \ y \). Laplace's equation is easily solved for the case of the hyperbolic nozzle by transforming from cartesian coordinates to elliptic coordinates. This transformation simplifies the solution because the stream lines must be hyperbolas to follow the nozzle boundary and therefore follow the hyperbolic coordinate \( \nu = \text{constant} \).

If the elliptic coordinates \( \mu \) and \( \nu \) are chosen such that the curves \( \mu = \text{constant} \) are ellipses and the \( \nu \) curves are hyperbolas, then the velocity potential which satisfies Laplace's equation for a hyperbolic nozzle is simply

\[
\phi = A\mu
\]

where \( A \) is a constant of integration. The stream function is

\[
\psi = A\nu
\]

The transformation \( w = \mu + i\nu = \cosh^{-1}(2z/a) \) gives rise to the elliptic coordinates

\[
y = \frac{1}{2} a \cos \mu \cos \nu, \quad x = \frac{1}{2} a \sinh \mu \sin \nu
\]

\[
r = \frac{1}{2} a \left[ \cosh^2 \mu - \sin^2 \nu \right]
\]

\[
r_1 = \sqrt{(y + a/2)^2 + x^2}
\]

\[
r_2 = \sqrt{(y - a/2)^2 + x^2}
\]

Solving for \( \mu \) and \( \nu \) produces

\[
\mu = \cosh^{-1} \left[ (r_1 + r_2)/a \right]
\]

\[
\nu = \cos^{-1} \left[ (r_1 + r_2)/a \right]
\]

The nozzle boundary is defined by \( \nu_0 = \text{constant} \), which along with the equation for the nozzle wall in cartesian
coordinates, \( y' = f(x) \), implicitly defines the constant \( a \). Substituting for \( \mu \) in the velocity potential produces,

\[
\mu = A \cosh^{-1} \left[ \frac{(r_1 + r_2)}{a} \right]
\]

from which the velocities may be determined.

\[
\begin{align*}
u &= \phi_x = \frac{A}{\sqrt{\left(\frac{r_1 + r_2}{a}\right)^2 - 1}} \left\{ \frac{3r_1}{3x} + \frac{3r_2}{3y} \right\} \\
v &= \phi_y = \frac{A}{\sqrt{\left(\frac{r_1 + r_2}{a}\right)^2 - 1}} \left\{ \frac{3r_1}{3y} + \frac{3r_2}{3y} \right\}
\end{align*}
\]

The constant of integration \( A \) may be determined by specifying the flow rate through the nozzle, but when the inlet velocities are normalized with respect to the freestream velocity \( \left( U_\infty \right) \) \( A \) is factored out of the problem.

Velocities for compressible flow can be solved by mapping back to the physical coordinate system \((x,y)\) plane.

Other boundary conditions are universal to both problems. These are:

\[
\begin{align*}
u &= 0 \quad \text{on the line of symmetry} \\
v &= (1 + u)df/dx \quad \text{at the nozzle wall}
\end{align*}
\]
\( F(x) \) defines the nozzle boundary in terms of a ratio of the throat semi-height as a function of \( x \). The throat semi-height is taken to be 1 for convenience.

Pressure ratio, sound speed, and Mach number are calculated as before by equations 5 through 8.
VIII. DISCUSSION OF RESULTS

A. TIME INTEGRATION TO A STEADY STATE SOLUTION

As stated before, the Houbolt method of integration is stable for all time steps. Results of the test cases bear this out with the larger time steps providing the most rapid convergence to a steady state solution. Time steps were tried from $t = .1$ to $t = 100$. Time steps larger than $t = 100$ were not attempted because as $t$ becomes too large the influence that the damping and mass matrices have on the effective stiffness matrix becomes negligible compared to the stiffness matrix. That is:

$$SE_{ij} = S_{ij} + 2SC_{ij}/\Delta t^2 + 6SM_{ij}/\Delta t$$

as $\Delta t \to \infty$

$$SE_{ij} \approx S_{ij}$$

The starting solutions were chosen somewhat arbitrarily. $\phi_j, 2\Delta t$ was chosen to satisfy the first iteration of the steady solution

$$S_{ij}\phi_j, 2\Delta t = 0$$

when the non-linear term ($u$) in the coefficient

$$1 - M^2 - M^2(\gamma + 1)u$$

was set to zero. $\phi_j, \Delta t$ and $\phi_j, 0$ were chosen as multiples of $\phi_j, 2\Delta t$ and respectively they were
\[ \phi_j, \Delta t = 0.5 \phi_j, 2\Delta t \]
\[ \phi_j, 0 = 0 \]

This starting procedure proved to be superior to choosing the first three vectors closer to the converged solution. If \( \phi_j, 2\Delta t, \phi_j, \Delta t, \phi_j, 0 \) were chosen to be the last three time steps of the previous case, the solution oscillated and converged much slower than with the starting solutions chosen as above.

The stiffness, mass, and damping matrices were recalculated after each time step, using the under-relaxation technique described above. This was necessary to utilize the special assembly procedures invoked by STRANL-II to prevent the inadmissible influence of downwind nodes from propagating upstream in the supersonic region.

For barely critical flow (\( M_a = 0.861 \)) and subsonic flow, an under-relaxation factor \( \theta = 1 \) (successive approximation) resulted in convergence to a steady state solution after only three time steps. Eleven time steps were required for the supercritical solution to converge using the same relaxation factor. Reducing \( \theta \) to 0.5 increased the rate of convergence and the solution achieved steady state after six time steps.

Figure 4 compares the steady state solution for a 6% thick circular arc airfoil at \( M_a = 0.909 \), using the same integration method, with the results obtained in Ref. [5]. Chan's results converged in 10 iterations after using the results from the barely critical flow as an initial guess to the
supercritical solution. Figure 4 is a plot of local Mach numbers at boundary nodes on the airfoil.

B. CONVERGING-DIVERGING NOZZLE

The nozzle chosen for the test cases was the two-dimensional Oswatitsch nozzle with the boundary defined as

\[ y = 1 + \sqrt{\frac{2}{3}}(x - 2.5)^2 \]

where the throat at \( x = 2.5 \) has a semi-height of 1. The inlet was taken to be \( x = 0 \) and the exit was at \( x = 5 \). \( M_\infty = 0.44 \), the inlet Mach number on the nozzle center-line was chosen to yield sonic conditions in the throat.

Two solutions were possible for this inlet condition, the symmetric solution and the asymmetric solution; but neither solution was achieved by the finite element method. Although the solution converged for the subsonic case in three iterations, center-line Mach numbers deviated significantly from both one-dimensional theory and from Oswatitsch's approximation [Fig. 5]. When the local Mach number \( M \) exceeded the inlet Mach number by approximately .2 (\( M > 0.64 \)) the solution was invalid. Differences at the center part of the nozzle are due to an essentially incorrect free stream Mach number. Patching the solution at \( x = 1.5 \) would improve the solution.

A second test case was run for the supersonic section of the nozzle with the inlet boundary on the sonic line. The exit boundary was left free to float. Here the solution was unstable and no meaningful results were obtained.
Figure 4 - Comparison of Time Integration Results with Steady State Results.
Figure 5 - Nozzle Center-Line, Inlet Mach Number 0.4300.
IX. PROGRAM MODIFICATION

The finite element computer program for non-lifting airvoils, as developed in Ref. [5], is separated into two parts. These have been designated STRANL-I and STRANL-II by Lockheed Corporation. STRANL-I generates a finite element mesh to be used as inputs to STRANL-II, which assembles the finite element equations, applies the boundary conditions, and solves the non-linear system of equations. Detailed descriptions and instructions for the use of the two programs can be found in Ref. [5]. Only the modifications to the above programs will be discussed in this section.

A. UNSTEADY EQUATIONS

Modifications to STRANL-II to form and solve the unsteady finite element equations were three-fold:

1) The new elemental matrices $SC_{ij}$ and $SM_{ij}$ were calculated and assembled.

2) All the matrices were stored on an external magnetic disk to be accessed and reassembled later because of the amount of space required to store three large matrices, in core memory.

3) The effective stiffness matrix $SE_{ij}$ and the effective load vector $RE_{ij}$ were assembled, and the system of equations solved.

Several existing subroutines in the original STRANL-II program were modified to assemble the damping and mass matrices. These include subroutines NEWK, EMTC, DERV, and EMQT. Two new subroutines were added to perform the other tasks.
B. MODIFICATIONS TO CALCULATE THE MASS AND DAMPING MATRICES

EMTC in the STRANL-II program calculated the elemental stiffness matrix by numerically integrating the equation,

\[ S_{ij} = \int Q_i P_j dA \]

Equations were added to EMTC to perform the additional numerical integrations for the mass and damping matrices. All three matrices were calculated at the same time. EMQT assembled the elemental stiffness matrices for a quadrilateral and trapezoidal element from the contributions of the triangular elements. Mass and damping matrices were calculated in the same fashion.

Subroutine NEWK, an original subroutine in STRANL-II, which assembled the finite element equations for steady flow, was modified to assemble \( SC_{ij} \) and \( SM_{ij} \). A new calling argument, NMAT was passed to NEWK, which assembled contributions from the triangular, quadrilateral and trapezoidal element matrices into the global matrices \( S_{ij} \), \( SC_{ij} \), and \( SM_{ij} \), depending on NMAT being 1, 2 or 3.

1. Subroutine STORE

Given a non-symmetric matrix stored in a banded node, plus the right hand side vector, subroutine STORE separates this system into two matrices and stores them on a magnetic disk. Figures 6 and 7 show the decomposition of a banded matrix into banded storage, and the further decomposition of this banded stored matrix to two smaller matrices by subroutine STORE. In these figures, D, L, and U represent the
Figure 6 - Decomposition of a Banded Matrix Plus Right Hand Side
Figure 7 - Separation of a Banded Stored Matrix by Store
diagonal matrix, the lower triangular matrix, and the upper triangular matrix respectively. HBW is the half bandwidth and R is the right hand side vector.

STORE requires an additional work area one-half the size of the originally dimensioned matrix which is to be stored.

2. Subroutine TIME

Subroutine TIME integrates the system

\[ S_{ij} \ddot{\phi}_j + SC_{ij} \dot{\phi}_j + SM_{ij} \dot{\phi}_j = R_j \]

by Houbolt integration.

TIME reassembles the three matrices which were stored on the magnetic disk to form the effective stiffness matrix and the effective load vector. Once this system of equations is assembled, a banded equation solver is called to yield the solution for \( \phi_j, t+\Delta t \). Figure 8 is a schematic flow chart of TIME. In Fig. 8 when \( L = 1 \), the lower triangular matrix and the diagonal of the effective stiffness matrix are formed by adding the appropriate contributions from the stiffness, mass and damping matrices. When \( L = 2 \), the upper triangular matrix is formed in like fashion.

C. CONVERGING-DIVERGING NOZZLE

1. Application of the Boundary Conditions

Regardless of the type of problem for which a set of system equations have been assembled, the equations will have the form

\[ K_{ij} x_i = R_i \]
Figure 8 - Flow Chart of Subroutine TIME
Figure 9 - Flow Chart of STANL-II Modification to Integrate Unsteady Equations
in which $K_{ij}$ is an $nxn$ matrix and $x_i$ and $R_i$ are vectors of length $n$. These equations do not take into account the known values of $x_i$ on the boundaries. However, for a unique solution of the above equation, at least one or more nodal variables must be specified and $K_{ij}$ must be modified to render it non-singular. For each equation $i$, either $x_i$ or $R_i$ must be specified but it is physically impossible to specify both $x_i$ and $R_i$. There are a number of ways to apply the boundary conditions to the equations and when they are applied the number of equations is reduced. However, it is convenient to leave the number of equations unchanged to avoid major restructuring of the computer storage. One such method is described below.

If $k$ is the subscript of the prescribed nodal variable, the $k^{th}$ row and the $k^{th}$ column of the original $K_{ij}$ matrix are set to zero, $K_{kk}$ is set to 1 and $R_k$ is replaced by the known value of $x_k$. Each of the $n-1$ remaining terms of $R_i$ is modified by subtracting from it the value $K_{ik}x_k$. This procedure is repeated for all the boundary values. Of course, when the matrix is stored in a banded mode, the algorithm will differ from that for the nxn square matrix, but the procedure is similar.

Subroutine BNDRY applies the boundary conditions for the modified program. Setting the option parameter IOPT(4), in STRANL-II, equal to 1 will call BNDRY which will read the boundary velocities and apply them to a banded stored matrix by the method described above.
When the value of the $x_k$ is zero, as in the non-lifting airfoil problem, the algorithm becomes simpler than the above method because there is no need to either set the $k$th column to zero or subtract the value $K_{ik}x_k$ from the right hand vector.
X. TEST CASES

In computing the flow field for either the non-lifting airfoil or for the Laval nozzle, the following procedures were followed:

1) The desired mesh was sketched with each node assigned a number.

2) Appropriate input cards based on the sketch were prepared and supplied to STRANL-I to generate the data on punched cards for input to STRANL-II.

3) The above punched cards were supplied to STRANL-II with three additional cards as input parameters for each case, plus additional cards for the boundary velocities, if the nozzle solution is desired.

4) Results of the finite element calculations are printed after each iteration, and the converged solution is punched on cards for possible later use.

A. TIME INTEGRATION TO A STEADY STATE SOLUTION

Test cases for the integration of the unsteady transonic finite element equations were conducted to calculate the steady transonic flow over a 6% thick circular arc airfoil. These tests were made using the same airfoil, mesh, and freestream Mach numbers as Chan et al. [Ref. 5] published. These conditions were chosen to provide a source for comparison of the results.

Freestream Mach numbers used in these calculations were:

\[ M_{\infty} = 0.806 \text{ (subcritical)} \]
\[ M_{\infty} = 0.861 \text{ (barely critical)} \]
\[ M_{\infty} = 0.909 \text{ (supercritical)} \].
Each case was treated individually with \( \phi_j = 0 \) used as the initial guess for each case, whereas Chan et al. [Ref. 5] used zero for the initial guess for \( M_\infty = .806 \) and then used the computed results as the initial guess for each subsequent case.

DELT, the value for the time step, is input by a parameter specified in columns 41-45 of the second card following the title card for each case when the unsteady option (IOPT(6) = 1) is selected.

1. **STRANL-I Program**

   a. **Input**

   Input cards used to generate the finite element mesh are listed on the next page. Cards were arranged in accordance with Ref. [5], in the following order:

   - Title card
   - Option card
   - Element cards
   - Card for the total number of nodes
   - Node coordinate cards
   - Card for the number of boundary nodes
   - Card for the boundary nodes at infinity
   - Card for the nodes on the line of symmetry
   - Card for the nodes on the airfoil
   - Cards for the slope of the airfoil.

   Input cards to STRANL-I for these tests are listed on the next three pages.

   b. **Output**

   Output from STRANL-I is in the form of printouts and punched cards. Printouts from STRANL-I are listed on the following eight pages.
## Input to STRANL-I

**Steady Transonic Flow -- Mesh 6 -- 154 Elements, 170 Nodes**

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OUTPUT FROM STRANL-I

TIME INTEGRATION SOLUTION--MESH 6--154 ELEMENTS, 170 NODES

NO. OF ELEMENTS = 154  NO. OF NODES = 170  FULL BAND WIDTH = 84

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2. **STRANL-II Program**

   a. **Input**

      Listed on the next three pages are the input cards to the STANL-II program.

   b. **Output**

      The output from this program is in the form of printouts for each iteration and punched cards for the converged solution. Output from STRANL-II for $M_\infty = .909$ is listed on the following eight pages.
## INPUT TO STRANL-11

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| 1.8COE | 0.0 5.0 3.000E-01 5.184E-01 2.427E-01 9.901E-01 1.906E-01 |</p>
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B. CONVERGING-DIVERGING NOZZLE

Two test cases were run for the converging-diverging nozzle. The first case was for symmetric flow designed to yield sonic conditions in the throat of the nozzle. The second case dealt with supersonic flow in the diverging section by starting with the sonic line as the boundary of the nozzle mesh. Oswatitsch's two-dimensional nozzle [Ref. 9], $y = 1 + \sqrt{2(x - 2.5)^2}$ with a semi-throat height of 1 at $x = 2.5$ was used in both cases. Boundaries for the subsonic nozzle were at $x = 0$ and $x = 5$.

1. Symmetric Solution

This problem was analyzed using the mesh shown in Fig. 10, which consists of 126 elements and 152 nodes. In the second card (the option card) IOPT(4) = 1 indicates that non-zero boundary velocities at the inlet and the exit will be read and applied by subroutine BNDRY. This option requires that the number of inlet and exit boundary nodes be specified in columns 36-40 of the next card. Perturbation velocities at the boundary nodes follow on the subsequent four cards. Subroutine BNDRY reads $u$ and $v$ respectively for the first boundary node and then continues reading $u$ and $v$ for each inlet and then each exit node in the order specified on the appropriate card.

2. Supersonic Case--Diverging Section

The mesh used for this case is sketched in Fig. 11 and input cards to STRANL-II follow on the next page. For this
case the options in effect are IOPT(4) = 1 and IOPT(5) = 1 which cause non-zero boundary velocities to be applied to the sonic line only.
## INPUT TO STRANL-II

### STEADY TRANSONIC FLOW CONVERGING DIVERGING NOZZLE - 3 M = .375

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## INPUT TO STRAN-II

### STEADY SUPERSONIC FLOW, SONIC LINE INPUT, UNIFORM MESH

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Figure 10 - Finite Element Mesh for the Converging-Diverging Nozzle.
STRANL-II

STEADY TRANSSONIC FLOW ANALYSIS BY FINITE ELEMENT METHOD

USING LEAST SQUARES WITH TRIANGULAR AND QUADRILATERAL ELEMENTS

IOPT(1)=1 USE RESULTS OF PREVIOUS CASE AS STARTING SOLUTION

WHILE THE OTHER OPTION IS IGNORED

IOPT(2)=1, READ IN NON-ZERO INITIAL GUESS

IOPT(3)=1, APPLY LINEARIZED BOUNDARY CONDITIONS ON CHORDLINE

IOPT(4)=1, READ IN NON-ZERO BOUNDARY CONDITIONS

IOPT(5)=1, SONIC LINE VELOCITIES AS INPUTS

THE PROGRAM AS PRESENTLY DIMENSIONED ALLOWS THESE MAXIMA

200 ELEMENTS, 180 NODES, 50 NODES FOR EACH TYPE OF BOUNDARY COND

MAX FULL BANDWIDTH = 84

DEVELOPED AND CODED BY STEVENS CHAN OF LOCKHEED-HUNTSVILLE ALA.

DIMENSION TITLE(18), IOPT(20), NGRID(200, 4), S,(540, 84), SLF(540)

DIMENSION WORK(540, 42), UT(540, 3), NDATA(3)

DIMENSION X(180), Y(180), RMLP(180), CCF(180), AJNT(180)

DIMENSION NIDS(3), NID(3, 50), VAF(50), AR(100)

DIMENSION UBS(50), WBS(50)

LOGICAL LR(50)

EQUIVALENCE(NIDS(1), NFARF), (NIDS(2), NWAKE), (NIDS(3), NBODY)

EQUIVALENCE (NDATA(1), NSK), (NDATA(2), NSM), (NDATA(3), NSC)

DATA NDATA/9, 13, 14/

DATA NGRID, NCM, NA/200, 180, 84, 50/

NRM=3*NPM

NFCM=NCM/2

DATA PI/3.1415926/, GAMMA/1.40/

IA=2*NA

CCNST=0.5*(GAMMA-1.0)

EXP=-GAMMA/(GAMMA-1.0)

READ TITLE, CONTROL KEYS AND PROGRAM PARAMETERS

DATA NREAD, NWRITE, NPUNCH/10, 8, 3/

100 READ(NREAD, 805, END=2000) TITLE(I), I=1, 18

101 CONTINUE

READ(NREAD, 820) (IOPT(I), I=1, 20)

READ (NREAD, 830) ITGIV, ZTEST, RMAG, F1, NFARF$

WRITE (NWRITE, 910) (TITLE(I), I=1, 18), ZTEST

WRITE (NWRITE, 820) (IOPT(I), I=1, 20)

WRITE(6, 999) ITGIV, ZTEST, RMAG, F1, NFARF$

995 FORMAT(1HO, I10, I13, 5X, I10, F8.4, 5X, RMAG=*, F8.4, 5X, NFARF=*, 15)

IFES=0

RFT=1.0

SCMAC=RMAG**2

TRA00020

TRA00030

TRA00040

TRA00050

TRA00060

TRA00070

TRA00080

TRA00090

TRA03100

TRA01110

TRA00120

TRA00130

TRA00140

TRA00150

TRA00160

TRA00170

TRA00180

TRA00190

TRA00200

TRA00210

TRA00220

TRA00230

TRA00240

TRA00250

TRA00260

TRA00270

TRA00280

TRA00290

TRA00300

TRA00310

TRA00320

TRA00330

TRA00340

TRA00350

TRA00360

TRA00370

TRA00380

TRA00390

TRA00400

TRA00410

TRA00420

TRA00430

TRA00440

TRA00450

TRA00460

TRA00470
IF (IOPT(4) .EQ. 1) READ(NREAD,840) (UB(I),VB(I),I=1,NFARFS)
IF (IOPT(4) .EQ. 1) CALL BCCNDE(UE,VB,IOPT(5),NFARFS)
IF (IOPT(1) .EQ. 1) GO TO 382
C
READ AND PRINT MESH DATA, BOUNDARY NODES, AND AIRFCIL SLOPE
C
READ (NREAD,825) NELS,NPS,NBW,(NICS(I),I=1,2)
READ (NREAD,825) (NOD(I,J),J=1,NELS)
READ (NREAD,840) (X(I),Y(I),I=1,NP)
CC 110 I=1,3
NS=NIDS(I)
110 READ (NREAD,825) (NOD(I,J),J=1,NS)
READ (NREAD,825) (VAF(I),I=1,NBCCY)
CC 120 READ (NREAD,825) (NJNT(I),I=1,NP)
IF (IOPT(3) .NE. 1) GO TO 116
DO 115 J=1,NBODY
115 Y(I)=0.0
116 CONTINUE
C
WRITE (NWRITE,920) NELS,NPS,NBW
WRITE (NWRITE,930) CC 220 N=1,NELS
WRITE (NWRITE,825) N,(NCD(N,J),J=1,4)
WRITE (NWRITE,935) CC 230 I=1,NP
WRITE (NWRITE,940) I, NJNT(I),X(I),Y(I)
WRITE (NWRITE,951) (NID(I,J),J=1,NFARFS)
WRITE (NWRITE,952) (NID(2,I),I=1,NWAKE)
WRITE (NWRITE,953) (NID(3,I),I=1,NBODY)
WRITE (NWRITE,955) (VAF(I),I=1,NBODY)
C
REDEFINE MESH DATA, ECT. USING NEW NODAL NUMBERING SYSTEM
C
CC 238 N=1,NELS
DO 238 I=1,4
IF (NOD(N,I) .EQ. 0) GO TO 238
KK=NOD(N,I)
NOD(N,I)=NJNT(KK)
238 CONTINUE
DO 239 I=1,3
IS=NIDS(I)
CO 239 J=1,IS
KK=NID(I,J)
239 NJNT(KK)=NID(I,J)
CC 243 I=1,NPS
RMLP(I)=X(I)
243 RML(I)=Y(I)
DC 244 II=1,NPS
I=JNT(II)
X(I)=RMLP(II)
244 Y(I)=RML(II)

C HALF BANDWIDTH AND NUMBER OF EQUATIONS
C
NBW=NBW/2
NEQ=3*NPS
C
C READ NONZERO INITIAL GUESS OR PROCEED WITH ZERO SOLUTION
C
IF (IOPT(2) .NE. 1) GO TO 250
READ (1,840) (S(I,NBW),I=1,NEQ)
CC TO 295
250 DO 260 I=1,NEQ
260 SLP(I)=0.0

C ITERATIONS START HERE AND CHECKED IF ITGIV IS EXCEEDED. IF SO,
C PRINT FAIL TO CONVERGE AND PROCEED TO NEXT CASE. OTHERWISE
C CONTINUE TO ITERATE
C
265 IRES=IRES+1
1 CONTINUE
IF (IRES .GT. ITGIV) GO TO 600
C FORMULATE SYSTEM OF ALGEBRAIC EQUATIONS
DO 266 I=1,NEQ
DC 266 J=1,NEB
266 S(I,J)=0.0
KMAT=1
CALL NEWK(SQM,NRM,NCM,NEQ,NBW,NFM,NELS,NSC,SLP,S,CCF,NPM,
1 X,Y,NMAT)
C IMPOSE P.C. FOR FARFIELD, LINE OF SYMMETRY, AND ON AIRFCIL
IF (IOPT(4) .NE. 0) CALL BNDRY (S,UB,VB,VAF,NID,NARF,NBODY,
1 NWAKE,NHBW,NEQ,IOPT,IRES)
IF(IOPT(4) .NE. 0) GO TO 276
DO 274 I=1,NARF
IE=3*NID(I,1)-3
DC 272 II=1,3
IF(II.EQ.1)
DC 270 K=1,NBW
270 DC 270 K=1,NBW
270 S(IK)=0.0
272 S(IK,NHBW)=1.0
272 CONTINUE
276 CONTINUE
DC 280 I=1,NWAKE
IER=3*NID(2,1)
DC 270 K=1,NBW
278 S(I,E,K)=0.0
280 S(I,E,NHBW)=1.0
DC 285 J=1,NBODY
IER=3*NID(3,1)
DC 282 K=1,NBW
282 S(I,E,K)=0.0
S(I,E,NHBW)=1.0
IF (IDPT(3),NE,1) S(I,E,NHBW-1)=-VAF(J)
C
CALL BANDED SOLVER TO SOLVE THE SYSTEM OF EQUATIONS
C
STORE STIFFNESS MATRIX AND INITIALIZE STARTING TIME DEPENDENT
UNKNOWN
C
CALL STORE(S,WORK,NRM,NCM,NHC,NHS,NBW,NEQ,NSK)
IF (IRES.GT.1.OR.RMAC.GT.91) GO TO 401
CALL BNCEQ(S,NRM,NCM,NEQ,NHBW)
DC 420 I=1,NRM
UT(1,3)=0.0
UT(I,2)=.5*S(I,NBW)
420 UT(I,1)=S(I,NBW)
DC TO 295
C
ASSEMBLE AND STORE THE DAMPING MATRIX
C
ASSEMBLE AND STORE THE DAMP M ATRIX
C
ASSEMBLE AND STORE THE DAMP M ATRIX
C
PRINT COMPUTED RESULTS
C
WRITE (NWRITE,970) RMAC,RFT
WRITE (NWRITE,975) IRES
DC 305 I=1,NP5
TRA01440
TRA01450
TRA01460
TRA01470
TRA01480
TRA01490
TRA01500
TRA01510
TRA01520
TRA01530
TRA01540
TRA01550
TRA01560
TRA01570
TRA01580
TRA01590
TRA01600
TRA01610
TRA01620
TRA01630
TRA01640
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TRA01660
TRA01670
TRA01680
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TRA01820
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TRA01890
TRA01900
TRA01910
TRA01920
J=NJT(I)
I1=3*NJT(I)
PCT=S(I-1,NBW)
UPT=S(I-1,NBW)
V=S(I1,NBW)
U=UPT+1.0
C SQ=U*U+V*V
ASQ=CONST*(1.0-QSQ)+1./SQMAC
RML(J)=SQRT(QSQ/ASQ)
PRATIO=1.0+CONST*RML(J)**2)**EXP
CFE=-2.*UPT
COF(J)=SQMAC*(1.0+2.4*UPT)
DELM=RML(J)-RMLP(J)
305 WRITE (NWRITE,978) I,POT,U,V,COF(J),RML(J),FRATIO,CP,DELM
C
C C
C DISPLAY PRESSURE COEFFICIENT CP
C
C WRITE (NWRITE,985)
ISTOP=0
IFNT=0
DO 320 J=1,NBODY
IF (J .EQ. NBODY) ISTOP = 1
I=NID(3,J)
UPT=S(3*I-1,NBW)
V=S(3*I,NBW)
CFE=-2.*UPT
CC CONTINUE
IF (IRES .LT. 2) GO TO 382
C
C CHECK CONVERGENCE. IF SO, PUNCH CONVERGED SOLUTION AND
C PROCEED TO NEXT CASE. OTHERWISE, UPDATE SOLUTION AND CONTINUE
C TO ITERATE
C
C CO 340 I=1,NPS
PCTE=1.0-RMLP(I)/RML(I)
IF (ABS(PCTE) .LT. ZTEST) GO TO 340
RFT=F1
CC TO 382
C
C CONTINUE
WRITE (NPUNCH,840) (S(I,NBW),I=1,NEQ)
CC TO 100
C
C CO 385 I=1,NPS
RMLP(I)=RML(I)
RFTC=1.0-RFT
CC 390 I=1,NEQ
SLP(I)=RFT*S(I,NBW)+RFTC*SLP(I)
CC TO 265
C
C WRITE (NWRITE,980)
WRITE (1,840) (S(I,NBW),I=1,NEQ)
          GC TO 100
       825 FORMAT (18A4)
       826 FORMAT (40I1)
       827 FORMAT (16I5)
       830 FORMAT(15,F10.0,15)
       840 FORMAT(1P8E10.3)
       910 FORMAT(1H1,2X,18A4//' CONVERGENCE LIMIT =',F6.4//)
       920 FORMAT(1H0,'NO. OF ELEMENTS=',I4,' NO. OF NOTES=',I4,'FULL
BANDWITH =',I4//)
       930 FORMAT('ELE. NO. AND ELEMENT NODES/')
       935 FORMAT('OLD NODE',, 'NEW NODE','X(I)',,'Y(I)')/
       940 FORMAT(16,1I10,2E15.5)
       951 FORMAT('NODES AT FARFIELD'/(2015))
       952 FORMAT('NODES ON THE LINE OF SYMMETRY'/(2015))
       953 FORMAT('NODES ON THE AIRFOIL'/(2015))
       965 FORMAT('SLOPE ALONG NODES ON AIRFOIL'/(8E15.5))
       970 FORMAT(1H1,'MACH NUMBER=',F6.3,'RELAX. FACTOR =',F6.4//)
       975 FORMAT(1H0,4X,'NO. OF ITERATIONS=',I4, /
               1H0,7X,'NODE',8X,'PHIT',11X,'UCOM',11X,'VCOM',11X,'PCOM',11X,'CPOM',11X,'CELM']/
               12X,'COF',11X,'LMAC',11X,'P/F0',13X,'CP',11X,'CELM/')
       980 FORMAT('OFAIL TO CONVERGE IN SPECIFIED NO. OF ITERATIONS')
       985 FORMAT(110,1P8E15.4)
       995 FORMAT (1H1)
      200 STOP
      END
SUBROUTINE BCOND

SLBROUINE BCOND(U, V, IOPT, N)
DIMENSION U(N), V(N), UP(50), VP(50)
DATA NWRITE/6/
C0 10 I=1,N
UP(I)=I+U(I)
VP(I)=V(I)
10 CONTINUE
IF (IOPT .EQ. 1) GO TO 150
WRITE(NWRITE,100)
100 FORMAT ('UCOM AT INLET')
N2=N/2
N3=N2+1
WRITE (NWRITE,110) (UP(I),I=1,N2)
WRITE (NWRITE,120)
120 FORMAT ('VCOM AT INLET')
WRITE (NWRITE,110) (VP(I),I=1,N2)
WRITE (NWRITE,130)
130 FORMAT ('UCOM AT EXIT')
WRITE (NWRITE,110) (UP(I),I=N3,N)
WRITE (NWRITE,140)
140 FORMAT ('VCOM AT EXIT')
110 FORMAT (1H8,(F10.7,5X))
WRITE (NWRITE,110) (VP(I),I=N3,N)
CC TO 160
150 WRITE(NWRITE,121)
WRITE(NWRITE,110)(UP(I),I=1,N)
WRITE(NWRITE,122)
WRITE(NWRITE,110)(VP(I),I=1,N)
121 FORMAT ('UCOM AT SONIC LINE')
122 FORMAT ('VCOM AT SONIC LINE')
160 RETURN
ENC

TRA00010 TRA00020
TRA00030 TRA00040
TRA00050 TRA00060
TRA00070 TRA00080
TRA00090 TRA00100
TRA00110 TRA00120
TRA00130 TRA00140
TRA00150 TRA00160
TRA00170 TRA00180
TRA00190 TRA00200
TRA00210 TRA00220
TRA00230 TRA00240
TRA00250 TRA00260
TRA00270 TRA00280
TRA00290 TRA00300
TRA00310 TRA00320
SUBROUTINE BNDEQ

SLRORUTINE BNDEQ(A,NRMAX,NCMAX,N,ITERM)

CONTINUE

EQUATION SOLVER FOR BANDED NON-SYMMETRIC SYSTEM OF EQUATIONS
SOLUTION STORED IN THE LAST COLUMN AT (1,2*ITERM)

DIMENSION A(NRMAX,NCMAX)
DATA NREAD,NWRITE,NPUNCH/4,8,3/
CERO=1.E-6
PARE=CERO**2
NEND=2*ITERM
NBM=NBM+1

BEGIN ELIMINATION OF THE LOWER LEFT

DC 1000 I=1,N
IF(ABS(A(I,ITERM)) .LT. CERO) GO TO 410
GO TO 420

410 IF(ABS(A(I,ITERM)) .LT. PARE) GO TO 1600
WRITE (6,420) A(I,ITERM)
420 PERMAT (* WARNING ILL-CONDITIONED A-MATRIX. A=',E16.6,' I=1,14)
430 JLAST=MINO(I+ITERM-1,N)
L=ITERM+1

431 CONTINUE
DO 500 J=1,JLAST
L=L-1
IF (ABS(A(J,L)) .LT. PARE) GO TO 500
B=A(J,L)
500 CONTINUE

DC 450 K=L,NBM
AI(J,K)=A(J,K)/B
IF(I .EQ. N) GO TO 1200
500 CONTINUE
L=0
JFIRST=I+1
IF (JLAST .LE. I) GO TO 1000
DC 900 J= JFIRST,JLAST
L=L+1
IF (ABS(A(J,ITERM-L)) .LT. PARE) GO TO 900
DC 600 K=ITERM,NBM
600 AI(J,K-L) = A(J-L,K) -A(J,K-L)
AI(J,NBM) = A(J-L,NBM) - A(J,NEND)
IF (I .GE. N-ITERM+1) GO TO 900
CC 600 K=1,L
800 AI(J,NBM-K) = -A(J,NBM-K)
900 CONTINUE
1000 CONTINUE

TRA00010
TRA00020
TRA00030
TRA00040
TRA00050
TRA00060
TRA00070
TRA00080
TRA00090
TRA01000
TRA01100
TRA01200
TRA01300
TRA01400
TRA01500
TRA01600
TRA01700
TRA01800
TRA01900
TRA02000
TRA02100
TRA02200
TRA02300
TRA02400
TRA02500
TRA02600
TRA02700
TRA02800
TRA02900
TRA03000
TRA03100
TRA03200
TRA03300
TRA03400
TRA03500
TRA03600
TRA03700
TRA03800
TRA03900
TRA04000
TRA04100
TRA04200
TRA04300
TRA04400
TRA04500
TRA04600
C BACK-SUBSTITUTION

120C L=ITERM -1
120D D 1500 I=2,N
120D D 1500 J=1,L
120D D IF (N+1-I+J .GT. N) GO TO 1500
120D D ATEM1=A(N+1-I,NBND)
120D D ATEM2=A(N+1-I+J,NBND)
120D D ATEM3=A(N+1-I,ITERM+J)
120D D A(N+1-I,NBND)=A(N+1-I,NBND)-A(N+1-I,J,NBND)*A(N+1-I,ITERM+J)
1500 CONTINUE
1500 RETURN

C PRINT THE ENTIRE MATRIX IF ZERO CN MAIN DIAGONAL

1600 WRITE (6,1601)
1601 FORMAT ("COMPUTATION STOPPED IN BNEQ BECAUSE ZERC APPEARED ON MAIN DIAGONAL. THE MATRIX FOLLOWS.")
1600 DO 1602 I=1,N
1602 WRITE (NWRITE,1603) (A(I,J), J=1,NBND)
1603 FORMAT (10E12.4)
1603 STOP
1603 ENDE
SUBROUTINE BNDRY

SLBNDRY IN S, UBV, VAF, NID, NFARF, NEDDY, NWAKE, NBW
1, NFW, NEQ, IOPT, IRES

DIMENSION S(450, 60), UBV(50), VAF(50), NID(3, 50), IOPT(40)
NBW=NBW-1

C

IF (IOPT(5) .EQ. 1 .AND. IRES .GT. 1) NFARF=NFARF/2
DC 10 I=1, NFARF
IE=3*NID(I, 1)-2
DC 10 J=1, 2
IE=IE+1
NX=IE-NBW
BC=UB(I)
IF (J .EQ. 2) BC=VB(I)
DO 41 L=1, NBW$
LX=NX+L
IF (LX .LE. 0 .OR. LX .GT. NEQ) GO TO 41
SLX(NBW)=S(LX, NBW)-BC*S(LX, NBW-L)
S(LX, NBW-L)=0.
41 CONTINUE
DC 30 K=1, NBW
31 S(I, J)=0.0
30 CONTINUE
S(I, NBW)=1.0
40 S(I, NBW)=BC
DC 100 I=1, NEDDY
IE=3*NID(3, I)
BC=VAF(I)
NX=IE-NBW
BC=410 L=1, NBW$
LX=NX+L
IF (LX .LE. 0 .OR. LX .GT. NEQ) GO TO 41
SLX(NBW)=S(LX, NBW)-BC*S(LX, NBW-L)
S(LX, NBW-L)=0.0
41 CONTINUE
DC 300 K=1, NBW
31 S(I, J)=0.0
30 CONTINUE
S(I, NBW)=1.0
100 S(I, NBW)=BC
DC 200 I=1, NWAKE
IE=3*NID(2, I)
DO 330 K=1, NBW
330 S(I, K)=0.0
320 S(I, NBW)=1.0
200 IF (IOPT(5) .EQ. 1 .AND. IRES .GT. 1) NFARF=NFARF*2
RETURN
END
SUBROUTINE EMTC

SUBROUTINE EMTC(A, AT, ATT, XL, YL, PEL, SQMAC)       TRA00010
EVALUATE ELEMENT MATRIX FOR A TRIANGLE BY GAUSSIAN QUADRATURE       TRA00020
SUBROUTINE DERIV CALLED TO EVALUATE SHAPE FUNCTION DERIVATIVES       TRA00030
AT THE GAUSSIAN POINTS
DIMENSION A(9,9), P(9), Q(9), NP(5), B(3), C(3), XL(3), YL(3), S(3),    TRA00040
1 DNX(9), DNXX(9), DNYY(9), PEL(9),    TRA00050
2 DN(9), AT(9,9), ATT(9,9)    TRA00060
DIMENSION EINT(3,7,7)       TRA00070
DATA LMAX/7/, WT/0.2255.3*0.13239415,3*0.1253618/       TRA00080
CATA EINT/3*0.33333333,0.05961587,3*0.47014206,0.05961587,    TRA00090
1 3*0.47014206,0.05961587,0.79742699,3*0.1253618/,    TRA01000
2 0.79742699,3*0.10128651,0.79742699/       TRA01010
CATA NP/1,2,3,112/, GAMMA/1,40/    TRA01020
CATA NREAD, NTWRT, NPUNCH/4,8,3/    TRA01030
DC 1 I=1,9       TRA01040
DC 2 J=1,9       TRA01050
ATT(I,J)=0.       TRA01060
ATT(I,J)=0.       TRA01070
ATT(I,J)=0.       TRA01080
ATT(I,J)=0.       TRA01090
2 A(I,J)=0.       TRA01100
CC 4 I=1,3       TRA01110
J=NP(I+1)       TRA01120
K=NP(I+2)       TRA01130
E(I)=YL(J)-YL(K)       TRA01140
C(I)=XL(K)-XL(J)       TRA01150
AREA=0.5*(B(2)*C(3)-B(3)*C(2))       TRA01160
CST1=1.0-SQMAC       TRA01170
CST2=SQMAC*(1.0+GAMMA)       TRA01180
CO 100 L=1,LMAX       TRA01190
DO 10 I=1,3       TRA01200
10 S(I)=EINT(I,L)       TRA01210
CALL DERIV(AREA, B, C, S, DNX, DNXX, DNYY)       TRA01220
U=0.0       TRA00310
UX=0.0       TRA00320
CC 30 I=1,9       TRA00330
U=U+DNX(I)*PEL(I)       TRA00340
3C UX=UX+DNX(I)*PEL(I)       TRA00350
ALPHA=CST1-CST2*U       TRA00360
CO 40 I=1,9       TRA00370
P(I)=ALPHA*DNXX(I)*DNYY(I)       TRA00380
4C Q(I)=P(I)-CST2*UX*DNX(I)       TRA00390
WEIGHT=WT(I)*AREA       TRA00400
CC 60 I=1,9       TRA00410
CST=WEIGHT*Q(I)       TRA00420
TRA00430
TRA00440
TRA00450
SUBROUTINE EMQT

SLBROUTINE EMQT(XQ,YQ,PMQ,SQMAC,EG,ETT,ECTT,NTRS)

GENRATE MATRIX FOR A QUADRILATERAL OR TRIANGLE

SLBROUTINE EMTC CALLED TO GENERATE MTRIX FC A BASIC TRIANGLE

DIMENSION EQ(I2,I2),ET(9,9),XQ(4),YQ(4),XT(3),YT(3),MP(3,4)
DIMENSION PMQ(12),PMT(9),ECTT(I2,12),ETT(12,9);

1 CATA MP/1,2,3,4,1,2,3,4,1,2/ CATA NREAD,NWRITE,NPUNCH/4,8,37
FTCR=1.0 IF (NTRS ,EQ. 4) FTOR=.5
CC 100 I=1,12 CC 100 J=1,12
ECT(I,J)=0.0 ECTT(I,J)=0.0
10C DO 150 II=1,NTRS DO 150 II=1,NTRS
CC 105 I=1,3 CC 105 I=1,3
NI=MP(I,II) NI=MP(I,II)
IT=3*(II-1) IT=3*(II-1)
IG=3*(II-1) IG=3*(II-1)
CC 102 J=1,3 CC 102 J=1,3
IT=IT+1 IT=IT+1
IC=IQ+1 IC=IQ+1
102 PTT(II)=PMQ(IQ) PTT(II)=PMQ(IQ)
XY(I)=XQ(NI) XY(I)=XQ(NI)
1C5 Y(I)=YQ(NI)
CALL EMTC(ET,ETT,ECTT,XT,YT,PMT,SQMAC)
DC 130 KI=1,3 DC 130 KI=1,3
NR=3*(MP(K,II)-1) NR=3*(MP(K,II)-1)
IE=3*(K-1) IE=3*(K-1)
DC 130 KK=1,3 DC 130 KK=1,3
NR=NR+1 NR=NR+1
IE=IE+1 IE=IE+1
DC 130 L=1,3 DC 130 L=1,3
NC=3*(MP(L,II)-1) NC=3*(MP(L,II)-1)
JE=3*(L-1) JE=3*(L-1)
DC 130 LL=1,3 DC 130 LL=1,3
NC=NC+1 NC=NC+1
JE=JE+1 JE=JE+1
ECT(NR,NC)=ECTT(NR,NC)+ETT(IE,JE)*FTOR
ECT(NR,NC)=ECTT(NR,NC)+ETT(IE,JE)*FTOR
130 EC(NR,NC)=EQ(NR,NC)+ET(IE,JE)*FTOR
150 CCNTINUE

TRACTED FORM
SUBROUTINE DERV

SLBRoutines(Area, B, C, S, Dn, Dnx, Dnxx, Dnxy)

EVALUATE SHAPE FUNCTION DERIVATIVES AT GAUSSIAN POINT

DIMENSION B(3), C(3), S(3), Dn(9), Dnx(9), Dnxx(5), Dnxy(9), NP(5)

DATA NP/1, 2, 3, 4/,
DATA NREAD, WRITE, NP, MCH/4, 8, 3/

TCA = 2 * AREA
TCSQ = TWA ** 2
CC = 200 * I = 1, 3
J = NP(I + 1)
K = NP(I + 2)
S = SJ(J)
SK = SK(K)
B1 = B1(I)
B2 = B2(J)
C1 = CJ(I)
C2 = CJ(J)
C3 = C3(K)
SISQ = SI * SI
BISQ = BI * BI
CISQ = CI * CI
ALFA = 0.5 * (CK - CJ)
BETA = 0.5 * (BJ - BK)

S = SI * SJ * SK
+ X = BI * J * SK + BJ * SK * SI + BK * SI * SJ
+ HY = 2 * (S1 * CJ * CK + S2 * CJ * CI + SK + CI * CJ)
+ HYY = 6 * SI * (1.0 - SI)
+ CS = 6 * SI * (1.0 - 2.0 * SI)
+ L3 = 1 - L
+ DL(L) = 2 * ST * (3 - 2 * SI) + 2 * H
+ DNX(L) = BI * CSS + 2 * HX
+ DNX(L) = BISQ * CS + 2 * HXX
+ DNY(L) = CISQ * CS + 2 * HY
+ CS = CK * SJ * CJ * SK

L = L + 1
+ DL(L) = SISQ * CS + ALFA * H
+ DNX(L) = 2 * BI * CS + TWOA * SISQ + ALFA * HX
+ DNX(L) = 2 * BI * CS + 4 * BI * TWOA * ALFA * HXX
+ DNY(L) = 2 * CJ * CS + ALFA * HY
+ ES = BI * SK + BK * SJ
L = L + 1
SUBROUTINE NEWK

SUBROUTINE NEWK(SQMAC,NRM,NCM,NEQ,NBW,NEM,NELS,NOD,SLP,S, TRA00010
           COEF,NPM,X,Y,NMAT)
           TRA00020

C GENERATE SYSTEM MATRIX BY ASSEMBLING CONTRIBUTIONS FROM
C ALL THE ELEMENTS
SUBROUTINE EMQT CALLED TO GENERATE ELEMENT MATRIX
C
DIMENSION COEF(NPM),X(NPM),Y(NPM),XQ(4),YQ(4),PM(12),BB(12,12)
DIMENSION BCC(12,12),BMM(12,12)
DIMENSION NOD(NEM,4),S(NRM,NCM),SLP(NRM)
DATA NREAD,NWRITE,NPUNCH/4,8,3/
NPE=NWB/N2
CC 480 N=1,NELS
II=1
IF (NOD(N,4)) 402,402,404

402 NPEL=3
NTRS=1
GO TO 410

404 NPEL=4
NTRS=4
II=0
CC 408 II=1,NPEL
II=NOD(N,1)

408 IF (COEF(N), GT, 1.00) II=I2+1
IF(I2.EQ.0) NTRS=2
IF(I2.EQ.4) II=3
DO 425 I=1,NPEL
   II=NOD(N,1)
   XQ(I)=X(N)
   YQ(I)=Y(N)
   DO 425 J=1,3
   IS=3*(II-1)+J
   IE=3*(II-1)+J
   CALL EMQT(XQ,YQ,PM,SQMAC,BB,BCC,BEM,NTRS)
DO 450 I=II,NPEL
   II=NOD(N,II-1)
   IE=3*(II-1)
   DO 450 J=1,NPEL
   NR=NR+1
   IE=IE+1
   CC 450 JJ=1,3

104
NC = NC + 1
JE = JE + 1
IF (NMAT - 2) 443, 442, 441
441 S(NR, NC) = S(NR, NC) + BBN(IE, JE)
GO TO 450
442 S(NR, NC) = S(NR, NC) + BBC(IE, JE)
GO TO 450
443 S(NR, NC) = S(NR, NC) + BB(IE, JE)
CONTINUE
450 CONTINUE
480 RETURN
ENC
SUBROUTINE STORE

SUBROUTINE STORE (S, WCRK, NRM, NCM, NHCM, NHBW, NEQ, NDATA)
DIMENSION S(NRM, NCM), WORK(NRM, NHCP)
REIND NDATA
CC 10 I=1, NRM
CC 10 J=1, NHCM
1C WCRK(I,J)=0.0
CC 20 I=1, NEQ
DE 20 J=1, NHBW
20 WCRK(I,J)=S(I,J)
WRITE (NDATA) WORK
CC 30 I=1, NEQ
CC 30 J=1, NHBW
JA=NHBW+J
3C WCRK(I,J)=S(I,JA)
WRITE (NDATA) WORK
REIND NDATA
RETURN
END
SUBROUTINE TIME

SUBROUTINE TIME(S,U,WORK,NRM,NCM,NXCM,NHBW,NEQ) TIM000010
HCULBOLT INTEGRATION FOR UNSTEADY PROBLEMS TIM000020
DIMENSION S(NRM,NCM),WORK(NRM,NXCM),U(NRM,3),NDATA(3) TIM000030
DATA NDATA/9,13,14/ TIM000040
DATA DELT/100./ TIM000050
NRM=2*NHBW TIM000060
NBW=2*NHBW TIM000070
NBW1=NWBW-1 TIM000080
AC=2./DELT**2 TIM000090
A1=1./DELT TIM001000
A2=5./DELT**2 TIM001100
A3=3./DELT TIM001200
A4=-2.*AO TIM001300
A5=-A3/2.* TIM001400
A6=AO/2.* TIM001500
A7=A3/9.* TIM001600
CC 100 I=1,NRM TIM001700
CC 100 J=1,NCM TIM001800
100 S(I,J)=0.0 TIM001900
C FCM LOWER TRIANGULAR PLUS DIAGONAL EFFECTIVE MATRIX TIM002000
DO 25 K=1,3 TIM002100
NCATA=NDATA(K) TIM002200
REAL (NCATA$) WORK TIM002300
IF (K-2) I=1,2,3 TIM002400
1 CCNST=1 TIM002500
CC TO 4 TIM002600
2 CCNST=AC TIM002700
CC TO 4 TIM002800
3 CCNST=A4 TIM002900
4 DC 10 J=1,NHBW TIM003000
CC 10 J=1,NEQ TIM003100
10 S(I,J)=WORK(I,J)*CONST*S(I,J) TIM003200
C MULTIPLY CONTRIBUTING MATRICES BY THE TIME DEPENDENT TIM003300
C VECTORS TO FORM RIGHT HAND SIDE TIM003400
DO 20 J=1,NHBW TIM003500
JW=NHBW+1-J TIM003600
IM=0 TIM003700
CC 20 I=JM,NEQ TIM003800
IM=IM+1 TIM003900
IF (K-2) I=1,12,13 TIM004000
11 UM=0 TIM004100
CC TO 20 TIM004200
12 UM=A2*U(IM,1)+A4*U(IM,2)+A6*U(IM,3) TIM004300
GG TO 20 TIM004400
C
FCRM UPPER TRIANGULAR EFFECTIVE MATRIX
CC 45 K=1,3
NCATA$=NDATA(K)
READ (NCATA$) WORK
IF(K-2) 21,22,23
21 CCNST=1
GO TO 24
22 CCNST=A0
GO TO 24
23 CCNST=A1
24 DC 30 J=1,NHBW
DO 30 I=1,NEQ
JA=J+NHBW
30 S(I,JA)=S(I,JA)+WORK(I,J)*CONST
MAXJ=NHBW-1
CC 40 J=1,MAXJ
JA=NHBW+J
MAXI=NEQ-J
IP=J
CC 40 I=1,MAXI
IP=IM+I
IF (K-2) 31,32,33
31 UM=0
CC TO 40
32 UM=U2*U(IM,1)+A4*U(IM,2)+A6*U(IM,3)
CC TO 40
33 UM=U3*U(IM,1)+A5*U(IM,2)+A7*U(IM,3)
40 S(IM,NBW)=S(IM,NBW)+WORK(I,J)*UM
45 CONTINUE
843 FORMT(1H ,4(F10.5,5X))
CALL BNDSEQ(S,NRM,NCM,NEQ,NHBW)
CC 50 I=1,NRM
L(I,3)=U(I,2)
50 U(I,1)=S(I,NBW)
50 U(I,2)=U(I,1)
888 FORMAT(1H ,6(F10.5,2X))
RETURN
END
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