The m-chromatic number $X_m(G)$ of a graph $G = (V,E)$ is the least integer $k$ such that there exists a mapping $f: V \rightarrow \{S \subseteq \{1,2,\ldots,k\} : |S| = m\}$ having the property that $f(u) \cap f(v) = \emptyset$ whenever $\{u,v\} \in E$. This is a generalization of the standard notion of chromatic number and arises in connection with mobile telephone frequency assignments. Answering a question
of Lovász, our first result shows that for any \( m \geq 1 \) and any \( \varepsilon > 0 \), there exists a graph \( G \) for which \( \frac{X_{m+1}(G)}{X_m(G)} > 2^{-\varepsilon} \). This shows that the known bound of 2 for all \( m \) and \( G \) is essentially best possible.

Our second result shows that the least integer \( m_0 \) for which \( \frac{X_{m_0}(G)}{m_0} = \lim_{m \to \infty} \frac{X_m(G)}{m} \) can be asymptotically as large as \( e^{(n \log n)/2} \) for some \( n \) vertex graphs, though it can never exceed \( e^{(n \log n)/2} \).
TWO RESULTS CONCERNING MULTICOLORING

by

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ABSTRACT

The $m$-chromatic number $\chi_m(G)$ of a graph $G = (V,E)$ is the least integer $k$ such that there exists a mapping $f: V \rightarrow \{S \subseteq \{1,2,\ldots,k\}: |S| = m\}$ having the property that $f(u) \cap f(v) = \emptyset$ whenever $\{u,v\} \in E$. This is a generalization of the standard notion of chromatic number and arises in connection with mobile telephone frequency assignments.

Answering a question of Lovász, our first result shows that for any $m \geq 1$ and any $\epsilon > 0$, there exists a graph $G$ for which $\chi_{m+1}(G)/\chi_m(G) > 2-\epsilon$. This shows that the known bound of 2 for all $m$ and $G$ is essentially best possible. Our second result shows that the least integer $m_0$ for which $\chi_{m_0}(G)/m_0 = \lim_{m \to \infty} \chi_m(G)/m$ can be asymptotically as large as $e^{\sqrt{(\ln n)\ln n}}/2$ for some $n$ vertex graphs, though it can never exceed $e^{(\ln n)\ln n}/2$.

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I. INTRODUCTION

The following generalization of the standard notion of graph coloring has been of recent interest [1,3,4,6,7,8]. A multicoloring of a graph \(G = (V,E)\) is a function \(f\) defined on \(V\) whose values are sets (of "colors") satisfying \(f(u) \cap f(v) = \emptyset\) whenever \(\{u,v\} \in E\). For positive integers \(k,m\), a \((k,m)\)-coloring of \(G = (V,E)\) is a multicoloring \(f\) of \(G\) such that \(|f(v)| = m\) for each \(v \in V\) and \(|\bigcup_{v \in V} f(v)| = k\). The \(m\)-chromatic number \(\chi_m(G)\) is the least integer \(k\) such that there exists a \((k,m)\)-coloring of \(G\). (This last definition differs from that of [6,7] by a factor of \(m\).) Notice that for \(m = 1\) these definitions correspond to the usual graph coloring notions. The purpose of this note is to resolve two questions about multicoloring conveyed to us by P. Erdös [2].

The first question deals with the relationship between \(\chi_m(G)\) and \(\chi_{m+1}(G)\). It is not difficult to see that

\[\chi_{m+1}(G) \leq \chi_m(G) + \chi_1(G) \leq 2 \cdot \chi_m(G)\]
with equality possible in the right-hand inequality only
for $m = 1$. Lovász asked [2] whether, for each value of $m$,
there exist graphs $G$ such that $\chi_{m+1}(G) > (2-\varepsilon)\chi_m(G)$. We
shall answer this question in the affirmative.

The graphs we shall use are defined as follows: for positive integers $n \geq 2m$, the graph $G_m^n$ has vertex set consisting
of all $m$-element subsets of $\{1,2,...,n\}$ and has an edge between
two such vertices exactly when their intersection is empty.
It is easy to see that $\chi_m(G_m^n) \leq n$ merely by considering the
multicoloring provided by the definition of $G_m^n$ and, in fact, it is proved in [7,8] that $\chi_m(G_m^n) = n$. Thus, to answer the
question of Lovász, it suffices to prove the following theorem:

**Theorem 1.** For each $m \geq 2$, there exists a constant $c$ such
that for all sufficiently large $n$

$$\chi_{m+1}(G_m^n) \geq 2n - c.$$ 

In order to prove Theorem 1, we require the following lemma, which is an immediate consequence of a special case of
Theorem 3 in [5].

**Lemma 1.** For fixed $m \geq 2$ and $n$ sufficiently large, there
exists a constant $a_0$ such that the number of $m$-element subsets
of $\{1,2,...,n\}$ which can be chosen so that no two are disjoint
but there is no element common to all is at most $a_0 n^{m-2}$. 
Proof of Theorem 1. Fix $m$. We merely need to show that, for all sufficiently large $n$,

$$x_{m+1}(G_m^n) - x_{m+1}(G_m^{n-1}) \geq 2$$

and the result will follow by induction. So suppose we have a $(k,m+1)$-coloring of $G_m^n$ such that $k = x_{m+1}(G_m^n)$, where $n$ is any integer sufficiently large that the conclusion of Lemma 1 holds and such that $\binom{n-1}{m-1} > ma_0n^{m-2}$, where $a_0$ is the constant of Lemma 1. We first claim that there must be at least $n+1$ colors which each appear on more than $a_0n^{m-2}$ vertices.

Suppose there are $n$ or fewer colors which each appear on more than $a_0n^{m-2}$ vertices. By Lemma 1, each such color can appear on at most $\binom{n-1}{m-1} > a_0n^{m-2}$ vertices since they must all share a common element. Thus, since each of the $\binom{n}{m}$ vertices receives exactly $m+1$ colors, we must have

$$(m+1) \binom{n}{m} \leq n \binom{n-1}{m-1} + (k-n)a_0n^{m-2}$$

$$\leq m \binom{n}{m} + (k-n)a_0n^{m-2}$$

or, rewriting,

$$\binom{n}{m} \leq (k-n)a_0n^{m-2}$$

Since

$$k = x_{m+1}(G_m^n) \leq x_m(G_m^n) + x(G_m^n)$$

$$\leq 2x_m(G_m^n) = 2n$$
it follows that we must have

\[ \binom{n}{m} \leq a_0 n^{m-1}. \]

However this is a contradiction, since \( n \) was chosen sufficiently
large that \( \binom{n}{m} = \frac{n}{m} \binom{n-1}{m-1} > \frac{n}{m} a_0 n^{m-2} = a_0 n^{m-1} \), and the claim
follows.

Thus there are at least \( n+1 \) colors which each appear
on more than \( a_0 n^{m-2} \) vertices. The set of vertices on which
any color \( i \) appears must form a collection of pairwise-inter-
secting \( m \)-element subsets of \( \{1, 2, \ldots, n\} \), by definition of \( G^m_n \).

Thus, by Lemma 1, whenever color \( i \) appears on more than \( a_0 n^{m-2} \)
vertices, all those vertices must contain some common element
\( e_i \). Since there are more than \( n \) such colors, we must have
\( e_i = e_j \) for some \( i \) and \( j \). If we delete from \( G^m_n \) all the vertices
containing \( e_i = e_j \), we obtain a copy of \( G^{n-1}_m \) and a \((k-2, m+1)\)
-coloring of it, since colors \( i \) and \( j \) have disappeared. Therefore

\[ x_{m+1} (G^{n-1}_m) \leq k-2 = x_{m+1} (G^n_m) - 2 \]

and the theorem is proved. \( \square \)

The second question involves what we call the
\textbf{ultimate multichromatic number} \( \chi^*(G) \) defined by

\[ \chi^*(G) = \inf_{m} \chi_m(G)/m. \]
It is proved in [1,7] that the value of $\chi^*(G)$ is always achieved for some finite $m$. One easy way to see this is to formulate the problem of determining $\chi^*(G)$ as a linear programming problem (as done in [4]): Let $v_1, v_2, \ldots, v_n$ be an ordering of the vertices of $G$ and let $S_1, S_2, \ldots, S_k$ be an ordering of the independent sets of $G$. Define $x_{ij}$ to be 1 whenever $v_i \in S_j$ and 0 otherwise. Then the value of $\chi^*(G)$ is given by

$$\chi^*(G) = \min \sum_{j=1}^{k} r_j$$

subject to: $r_j > 0, 1 \leq j \leq k$;

$$\sum_{j=1}^{k} x_{ij} r_j = 1, 1 \leq i \leq n.$$ 

One can show easily, using Hadamard's Theorem, that no basis matrix for this problem can have determinant exceeding $n^{n/2}$ and this is an upper bound on the value of $m$ required.

This upper bound however seems ridiculously large. Erdös asked [2] (as did the authors, independently) whether $\chi^*(G)$ could always be achieved for an $m$ not exceeding the number of vertices of $G$. We answer this in the negative, constructing graphs for which extremely large values of $m$ are necessary to achieve $\chi^*(G)$. 
Let $C_p$ denote the graph which is a cycle on $p$ vertices. The join $G_1 + G_2$ of two graphs $G_1$ and $G_2$, having disjoint vertex sets, consists of all edges and vertices in the two given graphs along with all edges joining a vertex from $G_1$ to a vertex from $G_2$. We use the following two lemmas in our construction:

**Lemma 2.** \([4,7]\) For all integers $p > 1$,

$$\chi^*(C_{2p+1}) = 2 + \frac{1}{p}.$$ 

**Lemma 3.** \([7]\) For all graphs $G_1$ and $G_2$,

$$\chi^*(G_1 + G_2) = \chi^*(G_1) + \chi^*(G_2).$$

Let $p_i$ denote the $i$th prime and define the graph $G(i)$ to be $C_{2p_1+1} + C_{2p_2+1} + \cdots + C_{2p_i+1}$. The number of vertices $n$ of $G(i)$ is given by

$$n = i + 2 \sum_{j=1}^{i} p_j.$$ 

Applying Lemmas 2 and 3, we obtain

$$\chi^*(G(i)) = 21 + \sum_{j=1}^{i} \left( \frac{1}{p_j} \right).$$

Since $\chi_m(G(i))$ must always be an integer, it follows that the least value of $m$ for which $\chi^*(G(i)) = \chi_m(G(i))/m$ can be no less than $\prod_{j=1}^{i} p_j$ (and in fact that value of $m$ will work). Using the Prime Number Theorem and expressing this lower bound in
terms of \( n \), we obtain the asymptotic lower bound of

\[ e^{\sqrt{(n \log n) / 2}}. \]

Thus, though this is still quite far from the upper bound of

\[ n^{n/2} = e^{(n \log n)/2}, \]

we see that extremely large values of \( m \) can be required in order to achieve \( \chi^*(G) \).
REFERENCES


