SMOOTHING TECHNIQUE AND VARIANCE PROPAGATION FOR ABEL INVERSION OF SCATTERED DATA

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April 1977


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Prepared for
DIRECTORATE OF TECHNOLOGY
ARNOLD ENGINEERING DEVELOPMENT CENTER
ARNOLD AIR FORCE STATION, TENNESSEE 37389
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**SMOOTHING TECHNIQUE AND VARIANCE PROPAGATION FOR ABEL INVERSION OF SCATTERED DATA**

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An analytic method of performing Abel inversions and subsequent analysis of the propagated experimental errors is described. The Abel Inversion is applied to the problem of determining the radial distribution of emission coefficients from measurements of the radiance from a cylindrically symmetric radiating source. The particular scheme investigated is a least-squares polynomial spline fit technique. The spline fit technique involves modeling of the raw data by a series of polynomials, with each polynomial...
applying over a different domain of the data. The polynomials are constrained so that the total data profile is smooth and yet provides the best fit of all the data in the least-squares sense. The resultant polynomial model of the data is inverted analytically. The associated error propagation analysis is developed by casting the numerical equations selected to perform the curve fit and integration into a form whereby the problem can be viewed as a linear transformation from the raw data to the inverted results. In this manner, the variance-covariance matrix of the raw data can be directly transformed to the variance-covariance matrix of the emission coefficients; therefore, the standard deviations of the intensity data points can be related directly to the standard deviations of the resultant emission coefficients. The result is an objective method of determining which series of polynomials yields the best fit to the raw data and the best inversion results.
PREFACE

The work reported herein was conducted by the Arnold Engineering Development Center (AEDC), Air Force Systems Command (AFSC) under Program Element 65807F. The results of the research were obtained by ARO, Inc., AEDC Division (a Sverdrup Corporation Company), operating contractor for the AEDC, AFSC, Arnold Air Force Station, Tennessee, under ARO Project Numbers R33A-00A and R32S-06A. The authors of this report were R. T. Shelby, University of Tennessee Space Institute, and Dr. C. C. Limbaugh, ARO, Inc. The manuscript (ARO Control No. ARO-ETF-TR-76-126) was submitted for publication on October 27, 1976.

The authors wish to express special appreciation to Mr. F. C. Loper, ARO, Inc., whose original work with the least-squares spline fitting technique at AEDC provided the inspiration for the approach used here.
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1.0 INTRODUCTION

1.1 BACKGROUND AND OBJECTIVE

A problem which has long been of importance to the experimentalist is the determination of radially dependent physical properties in axisymmetric sources from measurements of integrated properties along a line of sight. One important example of this type of problem is the calculation of the radial distribution of emission coefficients from observations of the radianc (radiated power per unit area per unit solid angle) from a cylindrically symmetric radiating source. When the source is optically thin, the solution to a problem of this type is usually reached by the use of the Abel transform (Ref. 1). The application of the Abel transform in this instance requires the emission coefficients to have cylindrical symmetry. With the cylindrical symmetry assumption, the measured radianc, I(x), can be written (Fig. 1) as

\[ I(x) = 2 \int_{x}^{R} \frac{c(r) \, r \, dr}{(r - x)^{\frac{3}{2}}} \]  

Here I(x) is the measured radianc (radiated power per unit area per unit solid angle), which is a function of the displacement x, r is the local radius, R is the maximum radius of the source, and c(r) is the emission coefficient (radiated power per unit volume per unit solid angle). Note that I at each x is the result of integrating the emission coefficient c across the extent of the source. The factor 2 arises because of the cylindrical symmetry assumption. Equation (1) is a form of the Abel integral equation (Ref. 2).

The use of Abel's transformation yields

\[ c(r) = -\frac{1}{\pi} \int_{r}^{R} \frac{dI/dx \, dx}{(x^2 - r^2)^{\frac{3}{2}}} \]  

(2)
Equation (2) is the inversion equation which gives the radial emission coefficient in terms of the geometry of the source and the measured radiance distribution. Details of the development of Eqs. (1) and (2) are found, for example, in Ref. 1.

Figure 1. Geometry of the axisymmetric source.

A number of methods have been applied to the solution of Eqs. (1) and (2) to obtain the radial distribution of the unknown emission coefficient $\varepsilon(r)$. An early approach documented by Pearce (Ref. 3) depends upon finding the areas of homogeneous zones in the plasma and replacing the integral by a summation over the zones. Nestor and Olsen (Ref. 4) make a transformation of variables such that $r^2 = v$ and $x^2 = u$ and assume that $I'(u)$ is constant over each small interval. The subsequent series of integrals may then be evaluated, and Eq. (2) is approximated by a sum. Bockasten (Ref. 5) fitted third-degree polynomials exactly to the
data points and approximated Eq. (2) with a sum. Numerical methods such as those described in Refs. 3 through 5, which do not use smoothing of the raw data, have an intrinsic disadvantage in that small errors in the radiance can lead to fairly large errors in the emission coefficient because of inaccuracies in the derivative of \( I(x) \).

Some investigators have used smoothing techniques utilizing curve fitting or other mathematical approximations in an attempt to reduce the effect of experimental error in the data. Freeman and Katz (Ref. 6) least squares fitted a single curve to the raw data. Barr (Ref. 7) used least-squares polynomials to determine the best fit of the data over a number of intervals centered about each data point. Birkebak and Cremers (Ref. 8) used a method similar to that of Barr. The data were least-squares curve fitted to polynomials over a number of data intervals. Dooley and McGregor (Ref. 9) directly applied the integral in Eq. (2) by using the experimental data and a coordinate transformation of the integrand to provide a means of numerically solving the integral.

Comparisons of various techniques (Ref. 8) indicated that, in general, smoothing techniques yield better final results than other methods, particularly when there is appreciable scatter in the data.

A major problem with any method of solving the problem described by Eq. (2) is that of determining the effect of experimental error in the measured radiance values upon the resultant values of the emission coefficient. In an effort to obtain error propagation information, some investigators have applied their methods of solution to simple problems in which analytic solutions could be determined (Refs. 8 and 10). By using scattered and unscattered input data (i.e., radiance values) and by comparing the resultant emission coefficient values with their analytic values, the investigators were able to obtain empirical information on the error propagation characteristics of the techniques as applied to particular test problems. However, at present, no general
analytic method has been developed which, after a prior analysis of the input data, follows the propagation of experimental error throughout the numerical steps of the specific technique. It is the purpose of this investigation to describe a method for smoothing the data for inversion and to develop a concomitant error propagation analysis.

1.2 CRITERIA FOR CURVE FIT CHOICES

Before proceeding to the description of the specific technique, it would be useful to consider some of the more subtle consequences of curve fitting in order to define a criterion by which the curve fit choices used in the work reported herein were made.

Experimental data are subject to random uncertainties and are generated by a physical phenomenon for which one does not necessarily know the functional form. Indeed, specification of the functional form, or an "adequate" approximation thereof, is generally the objective of the analysis. Because of the experimental scatter in the data, one may be uncertain of its correct functional form. The usual approach in such situations is to curve fit the data to some smooth functional form either by analytical or graphical techniques. When the analytic approach is used, the data are often fit to a specific polynomial in the independent variable. However, as is often readily apparent, the first choice of polynomial often cannot do an "adequate" job of fitting the data, and other polynomials are tried. This failure of fit may be caused by the fact that the data are simply not expressible by a polynomial function. Similarly, for the same reason, there are often several choices for polynomials which appear to "adequately" fit the data. Yet, each polynomial is different and will yield somewhat different results. Hence the results of a smoothing process cannot be considered unique insofar as the underlying physical phenomena are concerned, and there can be a large number of possible solutions. (Note that, for a given set of data and a specific polynomial form, the numerical solution is unique; it is the resultant, derived description of the underlying physical phenomena which is not unique).
The observation that it is generally unreasonable to expect a polynomial to provide an adequate representation of some physical phenomenon over a wide range of the independent variable prompted the approach described herein. The data are divided into several intervals, and a different polynomial is assumed valid over each interval. Since physical phenomena are generally smooth, the polynomials are constrained to be smooth and continuous at interval boundaries. Since the data are expected to be randomly scattered, a final least squares constraint is imposed to provide smoothing capabilities.

If the experimental error is propagated through a curve fitting process, one can obtain an estimate for the error, or uncertainty in the results. Even though the correct functional form may not be that chosen for fitting, the errors are propagated as though the chosen function is the correct one. If the chosen function provides an "adequate" representation of the raw data, it is expected that the derived results provide an "adequate" representation of the true physical phenomenon. Thus the propagated errors induced by the data uncertainties are an "adequate" representation of the uncertainties in interpretation of the physical phenomenon. For the application described herein, when several functions appear to give "adequate" curve fits, their results and propagated uncertainties agreed well with each other. This observation makes a strong favorable argument that the results are in fact descriptive of the underlying physical phenomenon. However, it must be remembered that the propagated uncertainties describe the uncertainties in the data fitting the chosen form and not the uncertainties in the chosen function fitting the physical phenomenon.

Since, for a given set of data, there may be a wide choice of possible curve fits, the choice of which curve fit to use is highly subjective and generally must be left to the investigator. The criteria for determining which result to use are as follows:
1. Choose the set of functions which performs an acceptable fit to the data, based on the observed error propagation characteristics.

2. From the set of functions chosen by (1), pick the function which provides the best error propagation characteristics for the problem.

Although the foregoing discussion may appear obvious to the experienced data analyst, the underlying theme has been that there can be no procedure or set of conditions which will unequivocally define the correct curve fits in all cases. The error propagation analysis provides only additional objective data for what is finally a subjective decision.

1.3 TECHNICAL APPROACH

The numerical approach is based on a curve-fitting method which utilizes a least-squares polynomial spline fit technique (Ref. 11) to smooth the data. The Abel inversion is effected by dividing the raw data into several small intervals and obtaining a least-squares polynomial curve fit to the data in each interval. Since the integral is a linear operator, the emission coefficient integral, Eq. (2), is expressed as the sum of several integrals; each integral applies over a different interval of data, and in each case the radiance data are expressed by a different polynomial. The emission coefficient is expressed as

$$
\varepsilon(r) = -\frac{1}{\pi} \int_{r}^{Z_{j-1}} \frac{dI_{j-1}}{(x^2 - r^2)^{1/2}} dx - \frac{1}{\pi} \sum_{i=j}^{n} \int_{Z_{i-1}}^{Z_{j}} \frac{dI_{i}}{(x^2 - r^2)^{1/2}} dx
$$

(3)

where the subscript $i$ identifies the particular interval for evaluation, $Z_{i-1}$ and $Z_{i}$ denote the endpoints of the interval, $n$ is the number of intervals required to span the data, and in all cases $r$ is less than $Z_{j-1}$.
For the values of the emission coefficients to be physically correct, the polynomials must be constrained to be smooth and single valued at interval boundaries and also must satisfy the assumption of cylindrical symmetry. The cylindrical symmetry assumption is easily accounted for by using an even function of the form

\[ I_i(x) = a_{i1} + a_{i2}x^2 + a_{i3}x^4 + a_{i4}x^6 \]  

(4)

where the subscript \( i \) denotes the \( i^{th} \) interval. The functions in adjacent intervals are designed so that their ordinate, slope, and second derivatives have the same value at the endpoints of adjacent intervals. Thus, the coefficients of a sixth-degree, even polynomial over each interval of data are determined such that a best fit in the least-squares sense over the entire set of data is obtained with the condition that the polynomials and their first derivatives are smooth and their second derivatives are continuous at the interval boundaries. With \( I(x) \) represented by a series of polynomials in the form of Eq. (4), the integrals become expressible by direct integration in closed form over each interval of the spline fit.

Another constraint upon the functional representation of the physical data which is imposed by the development of Eq. (2) is that it must have zero slope and ordinate at the boundary of the cylinder. This constraint is imposed artificially by obtaining a new curve over the last interval after the other intervals have been fit with the proper constraints. The constraints of continuous first derivative and smooth ordinate are maintained at the interval boundary for this artificial curve. The construction of the artificial curve reduces the accuracy of the curve fit intensity values in the last interval. However, the curve fit values of all the other intervals are unaffected, and in general the data in the last interval are very near to zero, quite inaccurate, and insignificant with respect to the rest of the data. When the results of the inversion in the last interval are important, the method described herein must be applied cautiously. A computer program to perform the analysis developed herein is described in Appendix A.
2.0 NUMERICAL INVERSION TECHNIQUE AND ERROR ANALYSIS

2.1 NUMERICAL TECHNIQUE

As was indicated in Section 1.0, the problem which prompted the present investigation was the development of a method to analyze the effect of experimental error in observed radiance data upon the emission coefficient values obtained by the solution to the Abel inversion problem. A measure of the experimental error in the radiance data is provided by the standard deviations of the radiance values which are determined from the experimental data.

From statistics theory, it is known that if a matrix vector equation, \( A = MB \), can be written where \( M \) is a transformation matrix relating the vectors \( A \) and \( B \), then a relationship between the standard deviations of the elements of vectors \( A \) and \( B \) can be derived. The first step in the error analysis is the development of the matrix-vector equation, \( E = FY \), where \( Y \) is a vector containing the radiance data values, and \( F \) is a matrix relating the vectors \( Y \) and \( E \).

Since the data are to be divided into intervals, denote the displacement values which are endpoints for the \( k^{th} \) interval by \( Z_{k-1} \) and \( Z_k \) and \( k = 1, \ldots, n \). Note that for \( n \) intervals there are \((n + 1)Z_j\) values, \( j = 0, \ldots, n \). The polynomial in each interval can be written as

\[
P_k(x) = \sum_{i=1}^{4} a_{ki} x^{2i-2} \quad ; \quad k = 1, 2, \ldots, n
\]

\( Z_{k-1} < x < Z_k \)

where \( n \) is the number of intervals into which the data points have been separated. The coefficients \( a_{ki} \) will in general be different for the polynomials representing each interval.
Since the raw data in each interval are to be least-squares curve fit to polynomials of the form of Eq. (5), an expression for the sum of the squares of the deviations of the curve fit values from the actual (input) radiance values is needed for each interval. For this purpose, let

\[ S_k(a_{k1}, a_{k2}, a_{k3}, a_{k4}) = \sum_{i=1}^{m_k} \left[ I(x_j) - P_k(x_j) \right]^2 \quad : k = 1, 2, \ldots, n \]

where \( n \) is the number of intervals, \( m_k \) is the number of data points in the \( k \)th interval, and the displacements \( x_j \) are numbered independently in each interval. The value of \( S_k \) in each interval must now be minimized subject to the constraints:

\[
\begin{align*}
\phi_{k,1}(a_{k1}, a_{k2}, a_{k3}, a_{k4}) &= P_k(Z_k) - P_{k+1}(Z_k) = 0 \\
\phi_{k,2}(a_{k1}, a_{k2}, a_{k3}, a_{k4}) &= P_k'(Z_k) - P_{k+1}'(Z_k) = 0 \\
\phi_{k,3}(a_{k1}, a_{k2}, a_{k3}, a_{k4}) &= P_k''(Z_k) - P_{k+1}''(Z_k) = 0 \\
\end{align*}
\]

where \( P_k'(x) \) is the first derivative of the function \( P_k(x) \), with respect to the coordinate \( x \), and \( P_k''(x) \) is the second derivative. These constraints express the conditions required for the ordinate, slope, and second derivatives of the fitting polynomials for adjacent intervals to be continuous at the interval boundary. This continuity constrains the formulation to provide a more nearly correct representation of the real physical data.

Lagrange's method of undetermined multipliers (Refs. 12 and 13) can be used to minimize Eq. (6) subject to the constraints expressed by Eqs. (7). Let
\[ F_1(a_{11}, a_{12}, a_{13}, a_{14}) = S_1 + \sum_{j=1}^{3} \lambda_{1,j} \phi_{1,j} \]

\[ F_k(a_{k1}, a_{k2}, a_{k3}, a_{k4}) = S_k + \sum_{j=1}^{3} (\lambda_{k-1,j} \phi_{k-1,j} + \lambda_{k,j} \phi_k) \quad (8) \]

\[ F_n(a_{n1}, a_{n2}, a_{n3}, a_{n4}) = S_n + \sum_{i=1}^{3} \lambda_{n-1,j} \phi_{n-1,j} \]

where \( n \) is the number of intervals and the \( \lambda_{mj} \) values are the Lagrange multipliers.

The parameters, \( \lambda_{mj} \), and the coefficients, \( a_{ki} \), which effect the necessary minimization are determined from the \( 4n \) equations

\[ (\frac{1}{2}) \frac{\partial F_k}{\partial a_{ki}} = 0; \quad k = 1, 2, \ldots, n \]

\[ i = 1, 2, 3, 4 \quad (9) \]

and the \( 3(n-1) \) equations

\[ (\frac{1}{2}) \phi_{m,j} = 0; \quad m = 1, 2, \ldots (n-1) \]

\[ j = 1, 2, 3 \quad (10) \]

The constant multiplier, 1/2, is introduced in Eqs. (9) and (10) to put the equations in a form which is convenient for subsequent developments.
By expanding Eq. (9) for \( k = 1 \), one obtains

\[
a_{11} m_1 + a_{12} \sum_{i=1}^{m_1} x_i^2 + a_{13} \sum_{i=1}^{m_1} x_i^4 + a_{14} \sum_{i=1}^{m_1} x_i^6 + \frac{1}{2} \lambda_{11} = \sum_{i=1}^{m_1} I_i
\]

\[
a_{11} \sum_{i=1}^{m_1} x_i^2 + a_{12} \sum_{i=1}^{m_1} x_i^4 + a_{13} \sum_{i=1}^{m_1} x_i^6 - a_{14} \sum_{i=1}^{m_1} x_i^8 + \frac{7}{2} \lambda_{11} + Z_1 \lambda_{12}
\]

\[
+ \lambda_{13} = \sum_{i=1}^{m_1} x_i^2 I_i
\]

\[
a_{11} \sum_{i=1}^{m_1} x_i^4 + a_{12} \sum_{i=1}^{m_1} x_i^6 + a_{13} \sum_{i=1}^{m_1} x_i^8 + a_{14} \sum_{i=1}^{m_1} x_i^{10} + \frac{7}{12} \lambda_{11} + 2Z_1^3 \lambda_{12}
\]

\[
+ 6Z_1^2 \lambda_{13} = \sum_{i=1}^{m_1} x_i^4 I_i
\]

\[
a_{11} \sum_{i=1}^{m_1} x_i^6 - a_{12} \sum_{i=1}^{m_1} x_i^8 - a_{13} \sum_{i=1}^{m_1} x_i^{10} + a_{14} \sum_{i=1}^{m_1} x_i^{12} + \frac{7}{12} \lambda_{11}
\]

\[
+ 3Z_1^5 \lambda_{12} + 15Z_1^4 \lambda_{13} = \sum_{i=1}^{m_1} x_i^6 I_i
\]

where the \( x_i \)'s are the displacement values, \( Z_1 \) is the second endpoint, \( m_1 \) is the number of points in the first interval, and \( I_i = I(x_i) \).

The constant multiplier, \( 1/2 \), in Eq. (9) has the effect of eliminating the factor of 2, which is introduced in Eq. (9) by the differentiation. Expanding Eq. (9) for \( k = 2, 3, \ldots, n \) yields more equations similar in form to Eq. (11).
Expanding Eq. (10) for $m = 1$ yields

$$
\frac{Z_1^2}{2} a_{11} + \frac{Z_1^4}{2} a_{12} + \frac{Z_1^6}{2} a_{14} - \frac{Z_1^2}{2} a_{21} - \frac{Z_1^4}{2} a_{22} - \frac{Z_1^6}{2} a_{23} - \frac{Z_1^6}{2} a_{24} = 0
$$

$$
Z_1 a_{12} + 2Z_1^3 a_{13} + 15Z_1^4 a_{14} - Z_1 a_{22} - 2Z_1^3 a_{23} - 3Z_1^5 a_{24} = 0
$$

Equations of similar form are obtained for $m = 2, 3, \ldots, (n-1)$.

Equations (4) and (10), with Eqs. (11) and (12) describing representative algebraic details, are the least-squares spline fit equations and represent a system of equations which can be written in the matrix-vector notation

$$
BA = C
$$

where $B$ is a $(7n - 3) \times (7n - 3)$ symmetric matrix, $A$ is a $(7n - 3) \times 1$ matrix which has as elements the $4n$ coefficients, $a_{ki}$, and the $3n - 3$ multipliers, $\lambda_{mj}$, and $C$ is a $(7n - 3) \times 1$ matrix, where $n$ is the number of intervals into which the data points have been divided.

The matrix $B$ can be partitioned into

$$
B = \begin{bmatrix}
R & N \\
N^T & 0
\end{bmatrix}
$$

where $R$ is a $4n \times 4n$ block diagonal matrix which is defined as

$$
R = \begin{bmatrix}
P_1 & 0 \\
P_2 & \ddots \\
0 & \ddots & P_n
\end{bmatrix}
$$
The matrices $P_j$ are 4 by 4 symmetric matrices which are defined (letting, for convenience, $\ell_j = m_1 + m_2 + \ldots + m_j$ and $S_j = m_1 + m_2 \ldots + m_{j-1} + 1$)

$$P_j = \begin{bmatrix}
    m_j & \ell_j & \ell_j & \ell_j \\
    \sum_{i=S_j} x_i^2 & \sum_{i=S_j} x_i^4 & \sum_{i=S_j} x_i^6 & \sum_{i=S_j} x_i^8 \\
    \sum_{i=S_j} x_i^4 & \sum_{i=S_j} x_i^6 & \sum_{i=S_j} x_i^{10} & \sum_{i=S_j} x_i^{12} \\
    \sum_{i=S_j} x_i^6 & \sum_{i=S_j} x_i^8 & \sum_{i=S_j} x_i^{10} & \sum_{i=S_j} x_i^{12}
\end{bmatrix}$$

(16)

$N$ is a $4n$ by $(3n - 3)$ block band matrix defined as

$$N = \begin{bmatrix}
    Q_1 \\
    -Q_1 & Q_2 \\
    -Q_2 & \ddots & Q_{n-1} \\
    \vdots & \ddots & \ddots & \ddots \\
    -Q_{n-1} & \cdots & -Q_{n-1}
\end{bmatrix}$$

(17)
The matrix $N$ consists of a block diagonal and one block lower co-diagonal. $N^T$ is the transpose of the matrix $N$. The matrices $Q_j$ are 4 by 3 matrices defined as

$$Q_j = \begin{bmatrix} \frac{Z_j^2}{2} & Z_j & 1 \\ \frac{Z_j^4}{2} & 2Z_j^3 & 6Z_j^2 \\ \frac{Z_j^6}{2} & 3Z_j^5 & 15Z_j^4 \end{bmatrix}$$

(18)

Noting the right side of Eqs. (11) and (12), one can write the matrix $C$ as

$$C = GY$$

(19)

where $G$ is a $(7n - 3)$ by $p$ matrix of certain powers of $x$, the $p$-independent data points, with the elements of the last $3n - 3$ rows all zero, and $y$ is a $p$ by 1 matrix of the $p$-dependent variable data points $I(x)$ (that is, the radiance values). Note that $p = m_1 + m_2 + \ldots + m_n$.

For ease of representation the matrix $G$ may be partitioned into a block diagonal matrix

$$G = \begin{bmatrix} s_1 & \vphantom{0} & \vphantom{0} \\ \vphantom{0} & s_2 & 0 \\ \vphantom{0} & \vphantom{0} & \ddots \\ 0 & \vphantom{0} & \vphantom{0} & s_n \end{bmatrix}$$

(20)
where each $S_j$ is a 4 by $m_j$ matrix:

$$S_j = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
2 & 2 & \cdots & 2 \\
x_{m_j-1} & x_{m_j-1} & \cdots & x_{m_j-1} + m_j \\
x_{m_j-1+1} & x_{m_j-1+2} & \cdots & x_{m_j-1+2} \\
x_{m_j-1+1} & x_{m_j-1} & \cdots & x_{m_j-1} + m_j \\
x_{m_j-1+1} & x_{m_j-1} & \cdots & x_{m_j-1} + m_j
\end{bmatrix} \quad (21)$$

Writing Eq. (13) as

$$A = B^{-1} C \quad (22)$$

and substituting Eq. (19) yields

$$A = B^{-1} G Y \quad (23)$$

where $B^{-1}$ is a $(7n - 3)$ by $(7n - 3)$ matrix, which is the inverse of $B$. Equation (23) can be written as

$$A = W Y \quad (24)$$

where $A$ is the $(7n - 3)$ by 1 matrix, which has as elements the coefficients, $a_{k1}$, and the multipliers, $\lambda_{mj}$. $Y$ is the $p$ by 1 matrix of data intensity values, and $W$ equals $B^{-1} G$, a $(7n - 3)$ by $p$ matrix.

Performing the integration of Eq. (3) with the polynomials of the form of Eq. (4) yields

$$\epsilon(r) = -\frac{(x^2 - r^2)^{\frac{3}{2}}}{\pi} \left[ 2a_{2,j-1} + \frac{4}{3} a_{3,j-1} (x^2 + 2r^2) + \frac{6}{15} a_{4,j-1} \right. \right.$$

$$\left. \left. \left. (3x^4 + 4x^2r^2 + 8r^4) \right] Z_{i-1} - \sum_{i=j}^{n} \frac{(x^2 - r^2)^{\frac{3}{2}}}{\pi} \left[ 2a_{2,i} \right. \right.$$

$$\left. \left. \left. \right. + \frac{4}{3} a_{3,i} (x^2 + 2r^2) + \frac{6}{15} a_{4,i} (3x^4 + 4x^2r^2 + 8r^4) \right] Z_{j} \right] Z_{i-1}$$

\[21\]
where $k$ is chosen so that $r$ is always less than $Z_k$. Expanding Eq. (25) for several values of $r$, one can write the resulting system of equations in the matrix-vector form

$$
E = MA
$$

(26)

where $E$ is an $m$ by 1 matrix of values of the emission coefficient $\epsilon(r)$ evaluated at $m$ different values of $r$, $A$ is a $(7n-3)$ by 1 matrix defined by Eq. (13), and $M$ is an $m$ by $(7n-3)$ matrix with the elements of the last $3n-3$ columns all zero.

Substituting into Eq. (26), the expression for the matrix $A$ from Eq. (24) yields

$$
E = M \Psi Y
$$

(27)

By performing the matrix multiplication

$$
F = MW
$$

(28)

Eq. (22) can be written as

$$
E = FY
$$

(29)

where $E$ is the $m$ by 1 matrix of values of the emission coefficient $\epsilon(r)$ evaluated at $m$ different values of $r$, $Y$ is a $p$ by 1 matrix of the $p$-dependent variable data points $I(x)$, and $F$ is an $m$ by $p$ matrix providing the transformation from the $Y$ space to the $E$ space.

### 2.2 ERROR ANALYSIS TECHNIQUE

To complete the development of the technique to include the error propagation analysis, it is necessary to utilize some definitions and results from statistics theory (Refs. 14 and 15). The expected value of a random variable $x$, denoted by $\mathbb{E}(x)$, is obtained by finding the average value of the function over all possible values of the variable. The expected value of a matrix or vector $M$, denoted by $\mathbb{E}(M)$, is the matrix of the expected values of the elements of $M$. The moments of a distribution are the expected values of the powers of the variable which have the given distribution.
Let $\mu_\varepsilon$ denote the first moment or mean of the emission coefficient function $\varepsilon(r)$ and let $\mu_I$ denote the mean of the radiance function $I(x)$. Then

$$\mu_\varepsilon = \bar{\varepsilon}(F) = \bar{\varepsilon}(F') = \bar{I}(\bar{\varepsilon}(Y)) = \bar{F}\mu_I \quad (30)$$

The covariance matrix of $E$, denoted by $[E]_{cv}$, is the $m$ by $m$ symmetric matrix defined as

$$[E]_{cv} = \delta \left[ (E - \mu_\varepsilon)(E - \mu_\varepsilon)^T \right] \quad (31)$$

where $m$ is the number of different values of $r$ at which the emission coefficient $\varepsilon(r)$ is evaluated. The following steps yield an expression which can be used to evaluate the covariance matrix of $E$:

$$[E]_{cv} = \delta \left[ (E - \mu_\varepsilon)(E - \mu_\varepsilon)^T \right]$$

$$= \delta \left[ (FY - F\mu_I)(FY - F\mu_I)^T \right]$$

$$= \delta \left[ F(Y - \mu_I)(Y - \mu_I)^T \right] \quad (32)$$

$$= F\left( \delta \left[ (Y - \mu_I)(Y - \mu_I)^T \right] \right) F^T$$

The term $[Y]_{cv}$ in Eq. (32) denotes the covariance matrix of $Y$, which is an $m$ by $m$ symmetric matrix.

The diagonal elements of the matrix $[Y]_{cv}$ are identified with the squares of the standard deviations of the radiance measurements. The nondiagonal elements are the covariances between the various elements of the matrix $Y$. In the present problem, since the individual observations are independent, then the elements of $Y$ (that is, the radiance data) are assumed to be uncorrelated. Therefore, all of the nondiagonal elements of the matrix $[Y]_{cv}$ are zero, and thus $[Y]_{cv}$ is simply a diagonal matrix containing the variances of the radiance data.
The matrix \( F \) provides the transformation from the \( Y \) space to the \( E \) space. The form of the matrix is determined by the raw data and the choice of numerical technique. The \( [E]_{CV} \) is the covariance matrix of \( E \), and the diagonal elements are the variances of the emission coefficient values which have been calculated by the least-squares spline fit approximation to the data. The off-diagonal elements of the matrix \( [E]_{CV} \) give the covariance between the various emission coefficient values. The standard deviations of the emission coefficient values are the positive square roots of the diagonal elements of \( [E]_{CV} \). Thus, determination of the matrix \( F \), Eq. (28), and its use in Eqs. (29) and (32) provides the formal solution to the inversion problem and the propagation of the random errors associated with the measurements.

3.0 RESULTS AND DISCUSSION

3.1 INTRODUCTION

In the determination of the solution to a specific inversion problem, described formally by Eqs. (27) and (30), there are generally four parameters which may be varied by the user. These four parameters are (1) the total number of data points, (2) the distribution of the data points chosen for analysis, (3) the number of intervals into which the data are divided, and (4) the number of points in each interval. Variation of each of these parameters affects the elements of the matrix \( F \), Eq. (28), and the subsequent inversion and random error propagation results. Included in this section are results of the Abel inversion of several sets of analytic data chosen to illustrate the effect of variations of the above four parameters on the results of the inversion. Also included, as an illustration of the application of the method to typical experimental data, are the results of the inversion of data taken in a recent research experiment.
3.2 ANALYTIC CONTINUOUS TEST DATA

Thirty-one data points are used to illustrate the technique. These data points were generated from the function

\[ I(x) = e^{-x^2}; \quad 0 \leq x \leq 3.0 \quad (33) \]

which may be inverted by Eq. (2) to yield the analytic function

\[ \epsilon(r) = \frac{1}{\sqrt{\pi}} e^{-r^2} \operatorname{erf}\left(\frac{R^2 - r^2}{\sqrt{6}}\right) \quad (34) \]

The number of data points and the displacement distribution were fixed, leaving only the number of intervals and the number of points per interval to be varied. The inversion was applied to several cases with different values of the free parameters. A standard deviation of 10 percent of each radiance value was assumed in each case. The displacement radiance and standard deviations used are listed in Table 1.

The initial data configuration examined was formed by dividing the data into four intervals, with seven points in the first interval and eight points in the remaining three intervals. The results of inverting the data with this configuration of the test data are presented in Table 2 and Figs. 2 and 3. Figure 2 displays the input radiance data as points and the results of the curve fit as a continuous line. Error intervals equal in magnitude to the radiance standard deviations are shown for each data point. Figure 3 displays the profile of the calculated emission coefficients along with the calculated error interval for each emission coefficient value.

Another configuration of the same set of test data was inverted after distributing the 31 data points into six intervals with five points in the first five intervals and six points in the sixth interval. The results of this case are presented in Table 3 and Figs. 4 and 5. Figure 4 shows the radiance and Fig. 5 the emission coefficient profile for this configuration of the data.
A comparison of Tables 1 and 2 reveals that the percentage errors between the curve fit data and the input data are generally smaller in magnitude for the six-interval configuration than for the four-interval configuration. Of the 31 percentage errors, 24 are smaller for the six-interval configuration. This suggests that the curve fit of the radiance data was better for the case of six intervals. Furthermore, of the 31 standard deviation intervals displayed in each of Figs. 3 and 5, 22 are smaller for the six-interval case. This fact is illustrated more clearly by comparing the calculated standard deviations of the emission coefficient values for the two cases which are listed in the last columns of Tables 2 and 3.

Table 1. Checkout Data with $e^{x^2}$ Values Used as Input Data

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<tr>
<th>Displacement</th>
<th>Radiance (Data)</th>
<th>Standard Deviation</th>
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</thead>
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<td>1.0000 E-00</td>
<td>1.0000 E-01</td>
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<tr>
<td>0.1</td>
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<td>9.8005 E-02</td>
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<tr>
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### Table 2. Inversion Results Using Four Intervals of $e^{x^2}$ Values as Input Data

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<tr>
<th>Displacement</th>
<th>Radiance (Calculated)</th>
<th>Percent Error between Radiance Data and Calculated Radiance</th>
<th>Emission Coefficient</th>
<th>Standard Deviation (Emission Coefficient)</th>
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</table>

Number of points: 31
Number of intervals: 4
Number of points per interval: 7 8 8 8
Figure 2. Intensity for four-interval $e^{-x^2}$ test data.

Figure 3. Emission coefficient profile for four-interval $e^{-x^2}$ test data.
### Table 3. Inversion Result with Six-Interval \( \chi^2 \) Values Used as Input Data

<table>
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<tr>
<th>Displacement (E-00)</th>
<th>Radiance (Calculated)</th>
<th>Percent Error between Radiance Data and Calculated Radiance</th>
<th>Emission Coefficient (Emission Coefficient)</th>
<th>Standard Deviation (Emission Coefficient)</th>
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Number of points: 31
Number of intervals: 6
Number of points per interval: 5 5 5 5 6
Figure 4. Intensity for six-interval $e^{-x^2}$ test data.

Figure 5. Emission coefficient profile for six-interval $e^{-x^2}$ test data.
The results of this comparison point to the conclusion that the emission coefficient values obtained using the six-interval configuration are more accurate for the particular set of data tested than are the values obtained using the four-interval configuration. A comparison of the emission coefficient values obtained for the two cases with the analytically determined values obtained by evaluating Eq. (1) at the proper displacement values shows that this conclusion is indeed correct. Table 4 lists the analytical values of the emission coefficients for this particular problem and the percent error between the analytic emission coefficient values and the calculated values for the two cases. Of the 31 percentage errors listed for each configuration, 26 are smaller in magnitude for the six-interval case.

Table 5 and Figures 6 and 7 show the results of inverting the data with a configuration of six intervals with an uneven dispersion of the data points among the intervals—two points in the first, third, and fourth intervals, eight points in the second and fifth intervals, and nine points in the sixth interval. A comparison of these results with the results of the six-interval configuration with an even dispersion of the data (Table 3, Figs. 4 and 5) suggests that the configuration of data with the data points dispersed uniformly among the intervals yields more accurate emission coefficient values for these analytical data. This conclusion is supported by the data listed in Table 6, which shows the percentage error between the analytic emission coefficient values and the calculated values for the two cases. Examination of the standard deviations of the emission coefficients for the six-interval configuration with an even and uneven distribution of the data points among the intervals reveals that the even distribution provides noticeably smaller standard deviations near the centerline. Standard deviations are also generally smaller for larger displacement values for the even distribution, although the differences are not as great.
Table 4. Percentage Error Between Analytic Emission Coefficient Values and Calculated Values

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<th>Percent Error (Six Intervals)</th>
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Table 5. Inversion Results with Unevenly Dispersed $e^{x^2}$ Values Used as Input Data

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<th>Standard Deviation (Emission Coefficient)</th>
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Number of points: 31
Number of intervals: 6
Number of points per interval: 2 8 2 8 9
Figure 6. Intensity for unevenly dispersed $e^{-x^2}$ test data.

Figure 7. Emission coefficient profile for unevenly dispersed $e^{-x^2}$ test data.
Table 6. Percentage Error Between Analytic Emission Coefficient Values and Calculated Values, Even and Uneven Data

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As a further examination of the technique, the 31 ordinates generated from the function of Eq. (33) were randomly scattered by a fractional amount bounded by ±10 percent of the radiance values. The scattered data, listed in Table 7, were tested for all the same parameter configurations used for the unscattered data. Table 8 lists the results of inverting the scattered data in the six-interval configuration, and Figs. 8 and 9 display the results graphically. A comparison of this data with that shown in

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Table 7. Checkout Data with Scattered $e^{-x^2}$ Values Used as Input Data
Table 8. Inversion Results with Randomly Scattered $\text{e}^{-x^2}$ Values Used as Input

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<tr>
<th>Displacement (Calculated)</th>
<th>Percent Error between Radiance Data and Calculated Radiance</th>
<th>Emission Coefficient</th>
<th>Standard Deviation (Emission Coefficient)</th>
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<td>$2.205460 \times 10^{-1}$</td>
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Number of points: 31
Number of intervals: 4
Number of points per interval: 5 5 5 5 6

Table 3 and in Figs. 4 and 5 for the case of the unscattered data in the same six-interval configuration indicates that, as would be expected, the results of the radian curve fit and the emission coefficient calculations were more accurate for the case of the unscattered data. This conclusion is supported further by the data listed in Table 9, which shows the percentage error between the analytic emission coefficient values and the calculated values for the two cases. Of the 31 values listed for each case, 27 are smaller in magnitude for the case of the unscattered data.
Figure 8. Intensity for scattered $e^{-x^2}$ test data.

Figure 9. Emission coefficient profile for scattered $e^{-x^2}$ test data.
### Table 9. Percentage Error Between Analytic Emission Coefficient Values and Calculated Values, Scattered and Unscattered Data

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<th>Percent Error Unscattered Data</th>
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<td>-1.000000E-02</td>
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3.3 TEST DATA WITH DISCONTINUITY

It is the nature of least-squares curve fitting techniques to reduce fluctuations in the data. Indeed, the techniques are designed to possess this characteristic, making them a highly useful tool for reducing the effects of experimental scatter in the input data. However, when a phenomenon such as a sudden change in slope is a vital aspect of the raw data and therefore an important feature of the emission coefficient values to be calculated, care must be taken in manipulating the parameters to achieve a data set configuration to attain optimum accuracy in the resultant emission coefficient values. Since the basis of the inversion technique is a polynomial curve fit, it is reasonable to expect that a good radiance curve fit will be difficult to achieve when the data possess an abrupt change in slope. To test this hypothesis, a set of data was constructed by using the following form for the emission coefficient function:

$$
\epsilon(r) = \begin{cases} 
0.5, & 0 \leq r \leq 15 \\
\frac{25}{r}, & 15 \leq r \leq 20 
\end{cases}
$$

(35)

By evaluating Eq. (1) for the above functional form of $\epsilon(r)$, one finds the associated analytical radiance function to be

$$
I(x) = \begin{cases} 
(400 - x^2)^{1/4}, & 0 \leq x \leq 15 \\
\frac{50}{x} \sec^{-1} \left| \frac{20}{x} \right|, & 15 < x \leq 20 
\end{cases}
$$

(36)

Twenty-one data points were generated for input data using Eq. (36). These input data are listed in Table 10 along with the associated values of the emission coefficient found by evaluating Eq. (35) at the appropriate displacement values. Note that the sharp drop in magnitude of the radiance between the displacement values of 15 and 16 is reflected in the emission coefficient values at the same points. The initial radiance data are shown graphically in Fig. 10, whereas Fig. 11 displays the emission
coefficient profile. The dotted line in Fig. 11 represents the analytic results, Eq. (35). The smoothing process created a curve which gives results quite different from the analytical results, as would be expected from Fig. 10.

Table 10. Input Data Defining a Curve with a Sudden Change in Slope

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<th>Analytic Emission Coefficient</th>
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<td>19.36492</td>
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<tr>
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</tr>
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<td>7</td>
<td>18.73499</td>
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</tr>
<tr>
<td>8</td>
<td>18.33030</td>
<td>0.50000000</td>
</tr>
<tr>
<td>9</td>
<td>17.86057</td>
<td>0.50000000</td>
</tr>
<tr>
<td>10</td>
<td>17.32051</td>
<td>0.50000000</td>
</tr>
<tr>
<td>11</td>
<td>16.70329</td>
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</tr>
<tr>
<td>12</td>
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<td>0.50000000</td>
</tr>
<tr>
<td>13</td>
<td>15.19868</td>
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<tr>
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<td>14.28286</td>
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<tr>
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</tr>
<tr>
<td>16</td>
<td>2.01094</td>
<td>0.09765626</td>
</tr>
<tr>
<td>17</td>
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<td>0.08650519</td>
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<tr>
<td>18</td>
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</tr>
<tr>
<td>19</td>
<td>0.83569</td>
<td>0.06925208</td>
</tr>
<tr>
<td>20</td>
<td>0.00000</td>
<td>0.06250000</td>
</tr>
</tbody>
</table>
Figure 10. Intensity for twenty-one point sudden change in slope data.

Figure 11. Emission coefficient profile for twenty-six point sudden change in slope test data.
In an attempt to better reproduce the steeply sloping portion of the data between the displacement values of 15 and 20, five more points within the interval \(15 \leq x \leq 20\) were generated from Eq. (36). This set of data is listed in Table 11 along with the associated analytic values of the emission coefficients. The initial radiance data are shown graphically in Fig. 12, and Fig. 13 displays the emission coefficient profile obtained by using the 26 data points listed in Table 11 as well as the analytic results, Eq. (35). A comparison of Figs. 11 and 13 shows that the sudden change in slope in the emission coefficient curve is better reproduced using the second set of data. Nevertheless, the overall results of the inversion are quite unacceptable and seem no better than the results illustrated in Fig. 11.

### Table 11. Twenty-Six Input Data Points Defining a Curve with a Sudden Change in Slope

<table>
<thead>
<tr>
<th>Displacement</th>
<th>Radiance (Data)</th>
<th>Analytic Emission Coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>20.00000</td>
<td>0.50000000</td>
</tr>
<tr>
<td>1.0</td>
<td>19.97498</td>
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</tr>
<tr>
<td>2.0</td>
<td>19.89975</td>
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</tr>
<tr>
<td>3.0</td>
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</tr>
<tr>
<td>4.0</td>
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</tr>
<tr>
<td>5.0</td>
<td>19.36402</td>
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</tr>
<tr>
<td>6.0</td>
<td>19.07878</td>
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</tr>
<tr>
<td>7.0</td>
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<td>0.50000000</td>
</tr>
<tr>
<td>8.0</td>
<td>18.33030</td>
<td>0.50000000</td>
</tr>
<tr>
<td>9.0</td>
<td>17.86057</td>
<td>0.50000000</td>
</tr>
<tr>
<td>10.0</td>
<td>17.32051</td>
<td>0.50000000</td>
</tr>
<tr>
<td>11.0</td>
<td>16.70329</td>
<td>0.50000000</td>
</tr>
<tr>
<td>12.0</td>
<td>16.00000</td>
<td>0.50000000</td>
</tr>
<tr>
<td>13.0</td>
<td>15.19808</td>
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<tr>
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<td>14.25249</td>
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<tr>
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</tr>
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<td>15.5</td>
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<td>0.10405830</td>
</tr>
<tr>
<td>16.0</td>
<td>2.01084</td>
<td>0.06765626</td>
</tr>
<tr>
<td>16.5</td>
<td>1.81998</td>
<td>0.04187376</td>
</tr>
<tr>
<td>17.0</td>
<td>1.63180</td>
<td>0.02856518</td>
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<tr>
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<td>1.46489</td>
<td>0.01613265</td>
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<tr>
<td>18.0</td>
<td>1.25265</td>
<td>0.00761648</td>
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<tr>
<td>18.5</td>
<td>1.05341</td>
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<tr>
<td>19.0</td>
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</tr>
<tr>
<td>19.5</td>
<td>0.57655</td>
<td>0.00574683</td>
</tr>
<tr>
<td>20.0</td>
<td>0.00000</td>
<td>0.00250000</td>
</tr>
</tbody>
</table>
Figure 12. Intensity for twenty-one point sudden change in slope data.

Figure 13. Emission coefficient profile for twenty-six point sudden change in slope test data.
It should be noted from examination of Figs. 10 and 12 that the portion of the radiance curve possessing the steep slope is not well reproduced in either case. In fact, portions of the fitted curve in the steep slope region lie entirely outside the standard deviation error bounds on the radiance data.

It is clear from these examples that it is not always possible to adequately fit the data by the use of the least-squares spline fit technique described herein. It is reasonable to expect that when the data are not well fitted the resultant emission coefficients will not be accurate. This conclusion is supported by examination of the data in Tables 10 and 11 and in Figs. 11 and 13, which deal with the emission coefficients for the radiance data of Figs. 10 and 12. Consequently, in such cases, a different approach to the inversion or a different set of fitting functions, something which has the potential of modeling the discontinuity, should be used.

3.4 APPLICATION TO EXPERIMENTAL DATA

As an illustration of the application of the inversion technique to typical experimental data, inversion results for the radiance profile of a selected spectral line from an argon arcjet are presented. The details of the experiment and implications of the results are included in Ref. 16. The radiance profile is shown in Fig. 14. Data from both sides of the centerline are included to indicate the symmetry of the plume and to provide additional data for the least-squares curve fit. The bars shown in Fig. 14 represent a typical two-standard-deviation bound for the data, and the curve represents the results of the least-squares spline fit. The results of the inversion and error propagation are shown in Fig. 15. The error bars represent the two-standard-deviation uncertainty at each of the radii. The largest standard deviation, that on the centerline, represents approximately a 10-percent uncertainty in the corresponding emission coefficient. The results represent the apparent best fit to the data, as determined by the propagated error, and represent physically acceptable results.
Figure 14. Typical lateral radiance scan from argon arcjet at X/D = 2, 415.8 nm.

Figure 15. Typical emission coefficient profile from argon arcjet at X/D = 2, 415.8 nm.
4.0 SUMMARY

A method of performing Abel inversions and a method of determining the propagation of the associated experimental errors has been presented. The Abel inversion is applied to the problem of determining the radial distribution of emission coefficients from observations of the radiance from a cylindrically symmetric radiating source. The particular scheme for solving the Abel inversion problem is a least-squares polynomial spline fit technique. The spline fit technique involves the division of the raw data into several intervals and the least-squares curve fitting of the data points in each interval to a sixth-degree even polynomial. The ordinates, slopes, and second derivatives of the polynomials are required to be continuous at the interval boundaries. Thus the polynomials are constrained so that the total data profile is smooth and yet provides the best fit of all the data in the least-squares sense. The Abel inversion of the resultant polynomial model of the data is obtained analytically.

The associated error propagation analysis is developed by casting the numerical equations selected to perform the curve fit and integration into a form in which the problem can be viewed as a linear transformation from the raw data to the inverted results. In this manner, the variance-covariance matrix of the raw data can be directly transformed to the variance-covariance matrix of the emission coefficients. The result of the error propagation analysis provides an objective basis for the subjective determination of the series of polynomials providing the most nearly correct fit to the raw data and resultant emission coefficient. A computer program to perform the least-squares spline fit and associated error propagation analysis is described in Appendix A.

To determine an acceptable least-squares polynomial spline fit for a particular set of data, there are generally four parameters to be considered: (1) the total number of data points, (2) the number of
intervals into which the data are divided, (3) the number of points per interval, and (4) the displacement distribution of the data points. It has been shown that the parameter configurations of randomly scattered data which yield accurate emission coefficient values are generally the same parameter configurations which yield accurate emission coefficients for data without scatter. However, when the data curve possesses an abrupt change in slope, it is generally not possible, using a polynomial function, to arrange the data parameters into a configuration that yields an acceptable curve fit.

Although the development of the error propagation method has been applied specifically to a particular least-squares spline fit scheme, the method can be applied, with appropriate modifications, to any least-squares polynomial approximation or polynomial spline fit technique. Therefore, the error analysis technique can serve as a means of comparing the applicability of various schemes to the same problem.

It should be noted that each set of experimental data is unique, and the choice of data parameters must be based upon an analysis of the standard deviations generated for each data set. The error analysis process described provides an objective basis upon which a subjective judgement can be made concerning the acceptability of the results obtained by the application of the least-squares polynomial spline fit technique to a particular problem.

REFERENCES


A.1.0 GENERAL INFORMATION

The analytic and numerical approach described in the text of this report has been coded into a computer program to effect the Abel inversion of data from cylindrically symmetric sources and perform the associated propagation of experimental errors. The purpose of this appendix is to provide the description, documentation, and user manual for the computer program. The program described herein is in an "as developed" state and can be readily modified, as required, for more efficient operation for Abel inversion of data other than emission data.

A.1.1 DESCRIPTION OF PROBLEM

The physical problem is the determination of the radial distribution of the emission coefficients from measurements of the radiance from a cylindrically symmetric, optically thin radiating source. The problem, illustrated in Fig. A-1, is generally expressed mathematically as

\[ y(x) = \frac{2}{\pi} \int_0^R \frac{\epsilon(r) \, r \, dr}{\left( r^2 - x^2 \right)^{\frac{1}{2}}} \]

where \( y(x) \) is the measured radiance as a function of the displacement \( x \), \( R \) is the overall radius of the source, and \( \epsilon(r) \) is the radially dependent emission coefficient, to be determined. The quantity \( y(x) \) is the usual experimental measurement. In the situation described, the emission coefficient can be expressed as

\[ \epsilon(r) = -\frac{1}{\pi} \int_0^R \frac{(dy/dx) \, dx}{\left( \frac{2}{r} \right)^{\frac{1}{2}}} \]
However, because \( y(x) \) is subject to experimental uncertainty, there is uncertainty in the derivative \( \frac{dy}{dx} \). Furthermore, variations in \( \frac{dy}{dx} \) can have pronounced effects upon \( \varepsilon(r) \) if the evaluation proceeds directly using raw data. Consequently, a smoothing process for the data coupled with a means of determining the effects of propagating the experimental uncertainty through the smoothing and subsequent inversion is required.

Figure A-1. Illustration of the physical problem.
The smoothing can be accomplished readily by least-squares techniques, and, with the proper choice of fitting function, the determination of the emission coefficient can become analytic. The details of the mathematical development of the equations for smoothing and subsequent emission coefficient and error propagation are given in the body of the report. The problem may be summarized by noting that a set of data points \( \{(x, y)\} \), where \( x \) is the independent variable and \( y \) is the dependent variable, are to be curve fit to a series of sixth-degree even polynomials. Each polynomial is to be valid over a specific range or interval of the data, and adjacent polynomials are to be smooth in both the function and its first derivative; the second derivative is to be continuous at the interval boundaries. The series of polynomials is to be further constrained so that they provide the best least-squares fit to the data. The mathematical problem is solvable by Lagrange's undetermined multipliers. Subsequent to the least-squares curve fitting, the last interval is fitted with a polynomial to insure zero slope and ordinate at the outer edge of the data.

The dependent variable, \( y \), at any point, \( x \), is thus expressible by the equation

\[
y(x) = a_{i1}x + a_{i2}x^2 + a_{i3}x^4 + a_{i4}x^6
\]

where the subscript \( i \) denotes the \( i^{th} \) interval such that

\[
Z_{i-1} \leq x \leq Z_i
\]

where the \( Z \)'s are the interval boundary points. By writing Eq. (A-1) for each of the dependent variables (radiances) measured, one can evolve a system of equations linear in the unknown coefficients, \( a_{i1}, a_{i2}, a_{i3}, \) and \( a_{i4} \). The introduction of the constraints provides additional equations also linear in the unknown coefficients and Lagrange's multipliers. The mathematical curve-fitting problem is thus expressible in matrix vector notation as

\[
BA = C
\]
where the matrix $A$ is the $(7n - 3)$ by 1 matrix containing the coefficients of the polynomials and the Lagrangean multipliers and $n$ is the number of intervals into which the data have been divided; $B$ is a $(7n - 3)$ by $(7n - 3)$ symmetric matrix of functions of the independent variable, $x$, and the interval division points, $z$; and $C$ is a $(7n - 3)$ by 1 matrix containing functions of the independent and dependent variables. The matrices $B$ and $C$ are described in greater detail below. The objective of the curve fitting technique is to obtain the solution of Eq. (A-3) for the column vector $A$.

The matrix $B$ may be partitioned thus:

$$B = \begin{bmatrix} R & N \\ N^T & 0 \end{bmatrix}$$

(A-4)

where $R$ is a $4n$ by $4n$ block diagonal matrix and $N$ is a $4n$ by $(3n - 3)$ block band matrix, each of which may be further partitioned thus:

$$R = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ \vdots \\ P_n \end{bmatrix}$$

(A-5)

$$N = \begin{bmatrix} Q_1 \\ -Q_1 & Q_2 \\ & -Q_2 & Q_3 \\ & & \ddots & \ddots \\ & & & -Q_{n-1} & Q_{n-1} \\ & & & & -Q_{n-1} \end{bmatrix}$$

(A-6)
The matrices $P_j$ are each 4 by 4 symmetric matrices which are defined as

\[
P_j = 
\begin{bmatrix}
  m_j & \ell_j & \ell_j & \ell_j \\
  \sum_{i=S_j}^2 x_i & \sum_{i=S_j}^4 x_i & \sum_{i=S_j}^6 x_i & \sum_{i=S_j}^8 x_i \\
  \sum_{i=S_j}^4 x_i & \sum_{i=S_j}^6 x_i & \sum_{i=S_j}^8 x_i & \sum_{i=S_j}^{10} x_i \\
  \sum_{i=S_j}^6 x_i & \sum_{i=S_j}^8 x_i & \sum_{i=S_j}^{10} x_i & \sum_{i=S_j}^{12} x_i
\end{bmatrix}
\]

and the matrices $Q_j$ are each 4 by 3 and are defined as

\[
Q_j = 
\begin{bmatrix}
  \frac{1}{4} & 0 & 0 \\
  \frac{Z_j^2}{2} & Z_j & 1 \\
  \frac{Z_j^4}{2} & 2Z_j^3 & 6Z_j^2 \\
  \frac{Z_j^6}{2} & 3Z_j^5 & 15Z_j^4
\end{bmatrix}
\]
The matrix $C$ in Eq. (A-3) is expressed as the product of two matrices,

$$C = GY$$  \hspace{1cm} (A-9)$$

where $G$ is a $(7n - 3) \times p$ matrix of certain powers of $x$ (the $p$-independent data points) and $y$ is a $p \times 1$ matrix of the $p$-dependent data points. The matrix $G$ may be partitioned into a block diagonal matrix

$$G = \begin{bmatrix} S_1 & & & \\ & S_2 & & \\ & & \ddots & \\ & & & S_n \end{bmatrix}$$  \hspace{1cm} (A-10)$$

where each $S_j$ is a $4 \times m_j$ matrix

$$S_j = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_{m_j-1+1}^2 & x_{m_j-1+2}^2 & \cdots & x_{m_j-1+m_j}^2 \\ x_{m_j-1+1}^4 & x_{m_j-1+2}^4 & \cdots & x_{m_j-1+m_j}^4 \\ x_{m_j-1+1}^6 & x_{m_j-1+2}^6 & \cdots & x_{m_j-1+m_j}^6 \end{bmatrix}$$  \hspace{1cm} (A-11)$$

With each of the elements of $B$, $G$, and $Y$ thus defined for a given set of data $(x,y)$, solution to Eq. (A-3) is immediate, yielding a vector containing the coefficients of the polynomials satisfying the least-squares spline fit criteria. The solution can be expressed as

$$A = WY$$  \hspace{1cm} (A-12)$$

where

$$W = B^{-1}G$$  \hspace{1cm} (A-13)$$
With the dependent data thus expressed as the evaluation of a polynomial, the emission coefficient,

$$\epsilon(r) = -\frac{1}{\pi} \int_r^R \frac{dy}{dx} \frac{dx}{(x^2 - r^2)^{1/2}}$$  \hspace{1cm} (A-14)$$
is analytic. Since the range of integration is over $x$, and the dependent variable, $y$, is expressed by different polynomials over different ranges of $x$, the integral, Eq. (A-14), is expressed as the sum of integrals, each valid over a different range; i.e.,

$$\epsilon(r) = -\frac{1}{\pi} \left[ \int_r^{Z_{j-1}} \frac{dy}{dx} \frac{dx}{(x^2 - r^2)^{1/2}} + \sum_{i=j}^{n} \int_{Z_{i-1}}^{Z_i} \frac{dy}{dx} \frac{dx}{(x^2 - r^2)^{1/2}} \right]$$ \hspace{1cm} (A-15)$$

Or, substituting the polynomials for the dependent variables,

$$\epsilon(r) = -\frac{(x^2 - r^2)^{1/4}}{\pi} \left[ 2a_{2,j-1} + \frac{4}{3} a_{3,j-1} (x^2 + 2r^2) + \frac{6}{15} a_{4,j-1} \right]$$

$$+ \frac{4}{3} a_{3i} (x^2 + 2r^2) + \frac{6}{15} a_{4i} (3x^4 + 4x^2r^2 + 8r^4) \bigg|_{Z_{i-1}}^{Z_i}$$ \hspace{1cm} (A-16)$$

which is linear in the polynomial coefficients. Thus, the results of evaluating the emission coefficient at several independent data points can be expressed in matrix vector notation as

$$E = MA$$ \hspace{1cm} (A-17)$$

where $E$ is the column vector of emission coefficients evaluated at $m$ values of $r$, $A$ is the coefficient matrix determined as the solution to the least-squares spline problem, and $M$ is an $m$ by $(7n - 3)$ matrix defining the coefficients of the respective elements of $A$. 
Substituting for A, the emission coefficient may be expressed as

\[ E = MB^{-1}G \]  (A-18)

which expresses the emission coefficient as the result of a linear transformation of the data, y.

With the emission coefficient described as the result of a linear transformation from the data space, it is an easy step to provide the transformation of the uncertainties of the data to uncertainties of the emission coefficient.

Let

\[ F = MB^{-1}G \]  (A-19)

so that

\[ E = FY \]  (A-20)

Then

\[ \left[ E \right]_{cv} = F \left[ Y \right]_{cv} F^T \]  (A-21)

where the \([ ]_{cv}\) symbol is used to describe the variance-covariance matrix of the parameter enclosed.

The objective of the computer code presented herein is to calculate the elements of the respective matrices M, B, G, and Y from input data \{\(x, y\)\} so that the coefficient vector A and the transformation vector F are determined. Further, for input \([Y]_{cv}\), the \([E]_{cv}\) values are determined.

**A.1.2 LIMITATIONS**

As written, the computer program is subject to the following restrictions and limitations:
1. There can be no more than 51 data points.

2. There can be no more than 10 intervals.

3. There must be at least one point per interval.

4. There must be at least two intervals.

5. The variance-covariance matrix of the raw data must be diagonal.

6. The input data must be read in by increasing displacement.

A.2.0 PROGRAM DESCRIPTION

The program in its present form requires approximately 150 K bytes of core on the IBM 370/165 computer, is composed of six Fortran subroutines or functions and a main program, and will perform the computations for 31 data points distributed into four intervals in about 1/2 sec. The time required per case, of course, varies according to the number of points, intervals, and points per interval.

A.2.1 SUBPROGRAM DESCRIPTIONS

A short description of each pertinent routine used in the computer program is listed below.

MAIN PROGRAM

The main program provides the logic for the computer program to execute multiple cases, the proper calling logic to the various routines which effect the data input, inversion, and errors propagation analysis, and the program summary output. The program utilizes two output units:
the line printer and logical unit 8. The logical unit 8 provides an additional temporary storage device so that the results of the analysis may be used in subsequent analysis programs by job stepping. Multiple data cases are accomplished simply by putting the proper logic inside a Do-loop.

**SUBROUTINE INPUT**

As the name implies, this subroutine provides for the input of all data on logical unit 5. The subroutine includes calibration calculations to provide for conversion of the input raw data units to physical units. The input data in raw and calibrated form are output on logical unit 6. All communication with this subroutine is through COMMON.

**SUBROUTINE INVERT**

This subroutine provides the logic to perform the least-squares spline fitting of the data and determination of the coefficients and transformation matrix. The bulk of this work is accomplished in SUBROUTINE COVCAL. However, subsequent to the call to COVCAL, the subroutine recalculates the coefficients for the last interval in order to assure meeting the constraints of zero slope and ordinate at the outer edge of the data. The final set of calculated coefficients are output on logical unit 6, and the percentage difference between the input data and the results of the curve fit are calculated. All communication with this subroutine is through COMMON.

**SUBROUTINE COVCAL**

This subroutine provides for the determination of the elements of the matrix $B$ [Eqs. (A-3), (A-4), and (A-5) through (A-8)], the matrix $G$ [Eqs. (A-9) through (A-11)], the matrix $M$ [Eq. (A-17)] and the matrix
F [Eq. (A-19)]. With these matrices, the polynomial coefficients, the emission coefficients, and the propagated variance-covariance matrix are immediate. The calculations proceed calculating sequentially as follows:

1. The array G [Eq. (A-9) through (A-11)], identified in Fortran as XT.

2. The array B and $B^{-1}$ [Eqs. (A-3) through (A-8)], identified in Fortran as (XTX).

3. The coefficients array A [(Eq. (A-12)], identified in Fortran as AV.

4. The array M [(Eq. (A-17)], identified in Fortran as XTX.

5. The array F [(Eq. (A-19)], identified in Fortran as XT.

6. The array $[E]_{cv}$ [(Eq. (A-21)], identified in Fortran as VC.

All communication with this subroutine is through COMMON.

SUBROUTINE EMSCAL

This subroutine provides the logic for calculation of the emission coefficients at each of the radial positions, Eq. (A-16), numerically equal to the input displacement positions. The calculation proceeds by Do-loop, and tracking of the point with respect to the interval boundary points is maintained so that the correct coefficients are used in the integral evaluation. Communication with this subroutine is through the argument list: No. of points, No. of intervals, interval endpoints, displacement array, curve fit coefficient arrays, and resultant emission coefficient array.
FUNCTION EMFUN

This double precision function performs the numerical evaluation of the Abel integral at either the upper or lower limit when the intensity is described by a four-term, sixth-degree even polynomial, Eq. (A-16). Input arguments include the radius, the upper weighted by the exponent of the corresponding independent variable (that is, $2a_2, 4a_3, 6a_4$). These weighting factors arise from the differentiation of the polynomial.

SUBROUTINE MATINV

This is a standard service routine which obtains the inverse of a matrix by Crout reduction with partial pivoting.

A.2.2 LIST OF FORTRAN VARIABLES

A      Spontaneous transition probability (sec$^{-1}$)

A1(I)  Curve fit coefficient for the $i^{th}$ interval, Eq. (A-3)

A2(I)  Curve fit coefficient for the $i^{th}$ interval, Eq. (A-3)

A3(I)  Curve fit coefficient for the $i^{th}$ interval, Eq. (A-3)

A4(I)  Curve fit coefficient for the $i^{th}$ interval, Eq. (A-3)

C1      Recalculated coefficient for last interval:
         A1(NTVL)

C2      Recalculated coefficient for last interval:
         A2(NTVL)
C3  Recalculated coefficient for last interval:
    \( A3(NTVL) \)

C4  Recalculated coefficient for last interval:
    \( A4(NTVL) \)

CAL2  Calibration constant for intensity according to:
    \[ \text{ENTEN}_{\text{cal}} = (\text{ENTEN}_{\text{input}} - \text{ZERO}) \times \text{CAL2} \]

CAL3  Calibration constant for \( x \) according to:

CAL4  \[ \text{DISP}_{\text{cal}} = (\text{DISP}_{\text{input}} - \text{CAL3}) \times \text{CAL4} \]

CAL5, CAL6  Not used

D  Intermediate computational variable

DEN  Intermediate computational variable

DISP  Array of displacements at which data are taken

EMIS  Vector \( E \) of emission coefficients

EN  Energy level (\( \text{cm}^{-1} \)), not used

ENDPT  Array of interval endpoints

ENTEN  Vector \( Y \) of radiances at the \( x \) locations in \( \text{DISP} \)

F  Intermediate variable

F1  Assignment statement subprograms for the computation of Eq. (A-16), Subroutine COVCAL
Assignment statement subprograms for the computation of Eq. (A-16), Subroutine COVCAL

Fractional value of input radiance data which is to be taken as the standard deviation of the radiance

Statistical weight

Alphanumeric header for identification

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Index
<table>
<thead>
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<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>K1</td>
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</tr>
<tr>
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<td>Index</td>
</tr>
<tr>
<td>K3</td>
<td>Index</td>
</tr>
<tr>
<td>K4</td>
<td>Index</td>
</tr>
<tr>
<td>L</td>
<td>Index</td>
</tr>
<tr>
<td>LIN</td>
<td>Line Counter index</td>
</tr>
<tr>
<td>M2</td>
<td>Index</td>
</tr>
<tr>
<td>M3</td>
<td>Index</td>
</tr>
<tr>
<td>MD</td>
<td>$7 \times \text{NTVL} - 3 = 7n - 3$</td>
</tr>
<tr>
<td>MSV</td>
<td>Intermediate variable</td>
</tr>
<tr>
<td>NPNTVL</td>
<td>Number of points in the $i^{th}$ interval, $m_i$</td>
</tr>
<tr>
<td>NPT</td>
<td>Total number of data points, $p$</td>
</tr>
<tr>
<td>NPT1</td>
<td>Computational variable</td>
</tr>
<tr>
<td>NSETS</td>
<td>Number of sets of data</td>
</tr>
<tr>
<td>NTLO</td>
<td>Computational variable</td>
</tr>
<tr>
<td>NTVL</td>
<td>Number of intervals, $n$</td>
</tr>
<tr>
<td>NTVL4</td>
<td>Computational variable</td>
</tr>
</tbody>
</table>
NTVL41  Computational variable

NTVL42  Computational variable

PERROR  Percentage error between curve fit radiances and input data values

PI  3.14159265

R  A particular x value passed to EMFUN from subroutine EMSCAL

RO  Overall radius of source, \( R_o \)

SD  Array of intensity data standard deviations

SUM  Intermediate computational variable

T1  Computational variable used to compute Eq. (A-16)

T2  Computational variable used to compute Eq. (A-16)

T3  Computational variable used to compute Eq. (A-16)

VC  Emission coefficient covariance matrix \([E]_{cv}\)

WL  Wavelength (angstroms), not used

WORK  Array containing the squares of the interval end points
X  Intermediate computational variable in subroutines INVERT and EMSCAL: array used first to store the elements of the matrix W, and next the elements of the matrix F \([Y]_{cv}\) in subroutine COVCAL

XT  Array containing the elements of the matrix G, and then the elements of the matrix F

XTX  Array containing the elements of the matrix B, next the elements of the matrix \(B^{-1}\), and finally the elements of the matrix M

Y  A particular endpoint passed to EMFUN from EMSCAL

Y1  Intermediate variable

YLP  Array of curve fit intensity values

YSPLN  Array containing data intensity values

Z1  Intermediate variable

Z2  Intermediate variable

Z12  Intermediate variable

Z13  Intermediate variable

Z21  Intermediate variable

Z22  Intermediate variable
ZERO Calibration constant for radiance according to:

\[ \text{ENTEN}_{\text{cal}} = (\text{ENTEN}_{\text{input}} - \text{ZERO}) \times \text{CAL2} \]

ZSPLN Array containing squared x values

### A.3.0 INPUT/OUTPUT

The input consists of certain control parameters defining the logical arrangement of the physical data, the radiance data and the corresponding position (measured from zero on the centerline) and standard deviation estimate, and calibration factors. In addition, there are four input parameters which are not used in the inversion but are passed on to the output data unit for subsequent use. When narrow spectral line data are inverted, the four additional parameters can be used for the atomic constants characterizing the radiation. The output consists of input data, calibrated data, certain intermediate calculation steps, and final results. The output is clearly labeled.

The principal input physical parameter is the radiance (radiated power per unit area per unit solid angle). The principal output physical parameter is the emission coefficient (radiated power per unit volume per unit solid angle). The specific units of the calculated emission coefficient will be consistent with the units chosen for the input radiance and displacement. There is no internal unit conversion provided other than with the calibration factors.

### A.3.1 INPUT DATA CARDS

<table>
<thead>
<tr>
<th>Card No. and Format</th>
<th>Fortran Variable</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. (I3)</td>
<td>NSETS</td>
<td>Number of sets of data to be radially inverted.</td>
</tr>
</tbody>
</table>

68
### Header card to provide means of identifying uniquely each set of input data.

#### Four variables not used directly in the calculation but passed through to the output unit for subsequent use. For narrow spectral line emission data these may be wavelength, transition probability, statistical weight, and energy level respectively.

#### Number of data points.

#### Number of intervals.

- If 0 if only the radiances have changed from the previous data set.
- If 1 each time a new set of data is run.

#### NTVL values: each value is the number of data points in the corresponding interval.

<table>
<thead>
<tr>
<th>2. (20A4)</th>
<th>HEAD</th>
</tr>
</thead>
<tbody>
<tr>
<td>3. (6E12.0)</td>
<td>WL,A,G,EN</td>
</tr>
<tr>
<td>4. (3I3)</td>
<td>NPT, NTVL, ITIME</td>
</tr>
<tr>
<td>5. (26I3)</td>
<td>(NPNTVL(I),I=1,NTVL)</td>
</tr>
</tbody>
</table>
6. (6E12.0) DISP(I) The displacement
n. ENTEN(I) radiance and standard
SD (I) deviation data from
I = 1, NPT the centerline to the
outer edge, two sets
per card.

n+1. (6E12.0) ZERO Radiance data calibra-
CAL2 tion according to
I = (ENTEN - ZERO)*CAL2.
CAL3 Displacement data
calibration according
to X = (DISP - CAL3)
*CAL4.
CAL5 Not used.
CAL6 Not used.

For multiple data sets, repeat cards 2 through n+1.

A.3.2 OUTPUT

The printed output, on logical unit 6, consists of five pages/case, and the identification and logic of the output are self-evident. The printed output consists of 1) input data, 2) calibration data, 3) curve fit coefficients, 4) emission coefficient, and 5) propagated errors.

In addition to the printed output, the various quantities are written (formatted) on logical unit 8 for offline storage of results or use by subsequent job steps. This output is listed as follows:

<table>
<thead>
<tr>
<th>Record No. and Format</th>
<th>Output List</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. (I3)</td>
<td>J</td>
<td>Case No.</td>
</tr>
</tbody>
</table>
2. (20A4) HEAD
Case identification.

3. (4D20.13) WL,A,G,EN
Four variables passed from input to this output record; not used in the radial inversion.

4. (26I3) NPT, NTVL,
(NPNTVL,I = 1, NTVL)
No. of points, No. of intervals and No. of points in each interval.

5. (4D20.13) (A1(I),A2(I),A3(I), A4(I),I = 1, NTVL)
The curve fit coefficients for each of the intervals.

6. (4D20.13) (ENDPT(I),I = 1,NTVL+1)
The endpoints of each interval.

7. (4D20.13) (DISP(I),ENTEN(I),I = 1,NPT)
Ordered pairs of displacement and radiance, two pairs to each record.

8. (4D20.13) (PERRØR(I),I = 1,NPT)
Percentage error between input and fitted radiances.

9. (4D20.13) (EMIS(I),I = 1,NPT)
Calculated emission coefficients.
10. (4D20.13) \((SD(I), I = 1, NPT)\)  
Standard deviation of input radiances.

11. (4D20.13) \((VCLI), I = 1, NPT\)  
Emission coefficient standard deviations.

A.3.3 SAMPLE INPUT SHEET

<table>
<thead>
<tr>
<th>JOB CARD FORMAT</th>
<th>PROJECT NO.</th>
<th>PROGRAMMER</th>
<th>PAGE OF</th>
<th>DATE</th>
</tr>
</thead>
<tbody>
<tr>
<td>CHECK I/O DATA</td>
<td>BKG I/O</td>
<td>USER I/O</td>
<td>INPUT DATA</td>
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</tbody>
</table>

<table>
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<th>0.1</th>
<th>0.99</th>
<th>0.099</th>
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</tr>
</tbody>
</table>

I = 1, NPT
A.4.0 FLOW CHARTS

MAIN PROGRAM
SUBROUTINE INPUT

1. Enter
2. Input Raw Data
3. Print Raw Data
4. Calibrate Raw Data
5. Compute Interval Endpoints
6. Print Calibrated Data and Endpoints
7. Return
SUBROUTINE INVERT

Enter

CALL COVCAL

Calculate New Coefficients for Last Interval

Print Coefficients of All Intervals

Calculate Intensities and Intensity Percentage Errors

Return
AEDC-TR-76-163

SUBROUTINE COVCAL

1

Compute
W = B⁻¹G.
Store in X.

Compute
A = WY.
Store in AV.

II

Store in XT
the elements
of the (7n - 3)
by p matrix G.

Define bᵢⱼ, i = 1, ..., 4n
j = 1, ..., 4n, where
B = [bᵢⱼ], a (7n - 3) by
(7n - 3) matrix. Store
these elements in XTX.

Define the remaining
elements of B. Store
in the corresponding
positions of XTX.

Call subroutine
MATINV to
compute the elements
of B⁻¹ which are then
stored in XTX.

3

Jₖ = 1

I = 1, n

A₁(I) = AV(Jₖ)
A₂(I) = AV(Jₖ + 1)
A₃(I) = AV(Jₖ + 2)
A₄(I) = AV(Jₖ + 3)

1

Start

Store in AV.

AZ(I) = AV(Jₖ)
A₂(I) = AV(Jₖ + 1)
A₃(I) = AV(Jₖ + 2)
A₄(I) = AV(Jₖ + 3)

5

Compute
W = B⁻¹G.
Store in X.
Define matrix \( M \). Store in \( XTX \).

Compute \( F = MW \). Store in \( XT \).

Form \( F^T \). Store in \( XTX \).

Define matrix \( M \). Store in \( XTX \).

Compute \( F[Y]_{cv} \). Store in \( X \).

Compute \( [E]_{cv} \). Store in \( VC \).

Compute emission coefficient standard deviations by taking positive square roots of diagonal elements of \( VC \).

Return

SUBROUTINE COVCAL, CONCLUDED
Note: In this subroutine, the subscript j (denoting interval endpoint) is one digit larger than the subscript i in Eq. (A-2).

SUBROUTINE EMSCAL
Evaluate the term

\[
\rho = \sqrt{\frac{x^2 - r^2}{\pi}} \left[ 2a_2 + \frac{4}{3} a_3 \left( x^2 + 2r^2 \right) + \frac{6}{15} a_4 \left( 3x^4 + 4x^2 r^2 + 8r^4 \right) \right]
\]

used in Eq. (A-16)
### A.5.0 OUTPUT

**CHECKOUT DATA USING DEXP(-X*X) FOR INPUT DATA**

**INPUT DATA**

**NUMBER OF POINTS= 31**

**NUMBER OF INTERVALS= 4**

**NUMBER OF POINTS PER INTERVAL: 7 8 8 8**

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<thead>
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<th>1.00000000-02</th>
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**BEGINNING POINT OF EACH INTERVAL**

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</table>
CHECKOUT DATA USING $\exp(-x^2)$ FOR INPUT DATA
CUBIC COEFFICIENTS

<table>
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<tr>
<th>INTERVAL START</th>
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<th>A3</th>
<th>A4</th>
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<td>PERCENT ERROR</td>
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AEDC-TR-76-163

CHECKOUT DATA USING $U\exp(-XeX)$ FOR INPUT DATA
ERROR ANALYSIS RESULT
STANDARD DEVIATION

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A.6.0 FORTRAN LISTING

LEVEL 21 (JAN 73)  05/356 FORTRAN II

COMPILER OPTIONS - NAME= MAIN,OPT=02,LINECNT=54,SIZE=0000K

SOURCE=ERCOIC,NOLIST,NOEDC,LOAD,NOAP,NOEDIT,IO,REF

C THIS PROGRAM PERFORMS RADIAL INVERSIONS BY LEAST SQUARES SPLINE
C FITTING THE RAW DATA AND THEN INVERTING THE RAW DATA
C

ISN 0002  IMPLICIT REAL*814=I-H=O-2)
ISN 0003  INTEGER HEAD(20)
ISN 0004  DIMENSION NPNVL(10),DISP(S1),ENTEN(S1),ENDPT(12),A1(10),A2(10),
ISN 0005  A3(10),A4(10),EMIS(S1),YCALC(S1),PERERROR(S1),SD(S1),VC(S1),S1)
ISN 0006  COMMON DISP,ENTEN,ENDPT,NO,A1,A2,A3,A4,YCALC,PERERROR,EMIS,HEAD,
ISN 0007  SD,VC

ISN 0008  READ(5,1005)NOETS
ISN 0009  DO 100 J=1,NOETS
C
ISN 0010  CALL INPUT
C
ISN 0011  CALL INVERT
C
ISN 0012  CALL EMSCALL(NHT,NTVL,ENDPT,DISP,A1(1),A2(1),A3(1),A4(1),EMIS)

ISN 0013  CALL EMSCALL(NHT,NTVL,ENDPT,DISP,A1(1),A2(1),A3(1),A4(1),EMIS)

ISN 0014  LINSET
ISN 0015  DO 30 I=1,NPT
ISN 0016  IF(LINL.T.500)GO TO 28
ISN 0017  WRITE(6,1003)
ISN 0018  WRITE(6,1003)
ISN 0019  LIN6
ISN 0020  28 CONTINUE
ISN 0021  WRITE(6,1004),DISP(I),ENTEN(I),YCALC(I),PERERROR(I),EMIS(I)

ISN 0022  L,N,LN1+1
ISN 0023  30 CONTINUE
ISN 0024  WRITE(6,2001)J
ISN 0025  WRITE(6,2001)J
ISN 0026  WRITE(6,2002)MLH,AGEN
ISN 0027  WRITE(6,2002)MLH,AGEN
ISN 0028  WRITE(6,2000)HEADD,NTVL,NPNVL(1),I=1,NTVL
ISN 0029  WRITE(6,2000)HEADD,NTVL,NPNVL(1),I=1,NTVL
ISN 0030  WRITE(6,2002)A1(I),A2(I),A3(I),A4(I),I=1,NTVL
ISN 0031  WRITE(6,2002)A1(I),A2(I),A3(I),A4(I),I=1,NTVL
ISN 0032  WRITE(6,2002)ENTEN(I),I=1,NTVL
ISN 0033  WRITE(6,2002)ENTEN(I),I=1,NTVL
ISN 0034  WRITE(6,2002)ENDPT(I),I=1,ENDPT
ISN 0035  WRITE(6,2002)ENDPT(I),I=1,ENDPT
ISN 0036  WRITE(6,2002)DISP(I),ENTEN(I),I=1,NTVL
ISN 0037  WRITE(6,2002)DISP(I),ENTEN(I),I=1,NTVL
ISN 0038  WRITE(6,3000)
ISN 0039  WRITE(6,3000)
ISN 0040  WRITE(6,3001),DISP(I),ENTEN(I),SD(I),EMIS(I),VC(I),I=1,NPT
ISN 0041  100 CONTINUE
ISN 0042  RETURN
C
ISN 0043  1000 FORMAT (*1*,20A4)

85
ISN 0044

C

1003 FORMAT(1X,'INVERSION RESULTS:',/1X,'5TA',1,5X,'DISPLACEMENT',1,
2 5X,'INTENSITY(DATA) ',5X,'INTENSITY(CALC)',5X,'PERCENT ERROR',1,
3 5X,'EMISSION COEF')

C

ISN 0045

1004 FORMAT(1X,'I3+IPSE19.6)

C

ISN 0046

1095 FORMAT(13)

C

ISN 0047

2000 FORMAT(26I3)

C

ISN 0048

2001 FORMAT(20A4)

C

ISN 0049

2002 FORMAT(4D20.1)

C

ISN 0050

3000 FORMAT(1X,'ERROR ANALYSIS RESULTS: STANDARD DEVIATION:',/1)

ISN 0051

3001 FORMAT(1X,'IPSE19.6+)

ISN 0052

3002 FORMAT(6X,'DISPLACEMENT',6X,'INTENSITY(DATA)',2X,'STD DEVI(ENSTI
T IV)',5X,'EMISSION COEF',6X,'STD DEV(EMISSION COEF)')

ISN 0053

END
PROGRAM NAME: HAI~GOPT GRAPHICS
SOURCES: BCO-NOLIST-NODECK-LOAD+MAP+NOEDIT+ID+XREF

COMPILED INPUT

THIS SUBROUTINE DOES THE INPUT OF DATA FOR ABEL INVERSION

DIMENSION NPT(10), DISP(10), ENENT(10), ENOPT(2), A1(10), A2(10),
A3(10), AA(10), ERR(10), ENTR(10), YCALC(10), ERROR(10), SD(10), VC(10), S1

COMMON DISP, ENENT, ENOPT, RD, A1, A2, A3, AA, YCALC, ERROR, EMIS, HEAD,
WL, a, g, en,
SD, VC

COMMON NPT, NPTVL, NTVL

READ(S, 1000) HEAD

HEAU(S, 1007) WL, a, g, en

HEAU(S, 1001) NPT, NTVL, TIME

READ(S, 1000) ERR(10), ERR(10), (NPTVL(I), NTVL(I)

READ(S, 1000) ERR(10), ERR(10), (DISP(I), ENTR(I), SD(I), I=1, NPT)

READ(S, 1000) A1(I), A2(I), A3(I), AA(I), YCALC(I), ERR(I)

IF (CAL2(I), ENTR(I), SD(I), 1=1, NPT)

WRITE(6, 2000) HEAD

WRITE(6, 2002) NPT, NTVL

WRITE(6, 2003) (NPTVL(I), NTVL(I), I=1, NPT)

WRITE(6, 2005) (NPTVL(I), NTVL(I), I=1, NPT)

CONVERT INPUT SCALE READINGS TO INTENSITIES

DO 1 I=1, NPT

1 DISP(I)=ABS(DISP(I)-CAL3)*CAL4

ENDPT(I)=0.0

J=

JS=1+NPTVL(J)

DO 2 I=1, NPT

IF (J, N=JS) GO TO 5

2 J=J

IF (J, N=NTVL(J), JS=JS+NPTVL(J)

IF (I, N=1, NPT) GO TO 4

ENDPT(J)=DISP(I)

IF (J, N=NTVL(J), 1=1, NPT)

WRITE(6, 3000) HEAD

RETURN

4 CONTINUE

ENDPT(J)=0.0

DO 3 I=1, NPT

3 DISP(I)=DISP(I)-DISP(I-1)

CONTINUE

5 CONTINUE

FAC=SD(I)/ENTEN(I)

ENTEN(I)=ENTEN(I)-ZERO*CAL2

SD(I)=FAC*ENTEN(I)

CONTINUE

87
ISN 0048  10 CONTINUE
ISN 0049  WRITE(*,2001) (ENDPT(I),I=1,NTVL)
ISN 0050  WRITE(*,2001) HEAD
ISN 0051  WRlTE(*,2007) writes
ISN 0052  WRITE(*,2002) (DISP(I),I=1,NTVL)
ISN 0053  ENDPT(NTVL)=DISP(NPT)
ISN 0054  NTVL=DISP(NPT)
ISN 0055  RETURN
ISN 0056  100 WRITE(*,4000)
ISN 0057  4000 FORMAT(*,1X,*ERROR IN NPTNL*)
ISN 0058  101 WRITE(6,4001)
ISN 0059  4001 Format(*,1X,*")
ISN 0060  RETURN
C
ISN 0061  1000 FORMAT(20A4)
C
ISN 0062  1001 FORMAT(2A13)
C
ISN 0063  1002 FORMAT(6E12.0)
C
ISN 0064  2000 FORMAT(*,1X,20A4)
C
ISN 0065  2001 FORMAT(1X,*INPUT DATA*)
C
ISN 0066  2002 Format(*,1X,*NUMBER OF POINTS*,1X,*NUMBER OF INTERVALS*,1X)
C
ISN 0067  2003 Format(*,1X,*NUMBER OF POINTS PER INTERVAL*,2A13)
C
ISN 0068  2004 Format(*,1X,*BEGINNING POINT OF EACH INTERVAL*,/1X,1P1,1P1,1P1)
C
ISN 0069  2005 Format(*,1X,*INPUT DATA ARRAY*)
C
ISN 0070  2006 Format(*,1X,3P1,1P1)
C
ISN 0071  2007 Format(*,1X,*INPUT (*,INTENSITY,STD DEV) ARRAY,///)
C
ISN 0072  3000 Format(*,1X,20A4,1X,J DOES NOT MATCH WITH NPTVL*)
C
ISN 0073  END
LEVEL 21.7 (JAN 73)  

COMPILE OPTIONS = NAME= MAIN,OPT=02,LINCNT=50,SIZE=8000K, 
SOURCE= EBCDIC, WO LIST, NO DEBUG, LOAD= MAP, NO EDIT, ID, KREF 

ISN 0002  
SUBROUTINE INVERT 

ISN 0003  
INPLICIT REAL*8 (A-H,O-Z) 

ISN 0004  
INTEGER HEAD!0) 

ISN 0005  
DIMENSION NNPTNL(V,10),DISP(51),ENTEN(51),ENDOPT(12),AI(10),A2(10), 
A3(10),A4(10),EMIS(51),YCALC(51),PEHROR(51),SD(51),VC(51),S1) 

ISN 0006  
DIMENSION YSPLN(V,10),ZSPLN(V,10),WORK(11) 

ISN 0007  
COMMON DISP,ENTEN,ENDOPT,A1,A2,A3,A4,YCALC,PEHROR,EMIS,HEAD, 
S1,6,A,6,4,6, 
COMMON NPT,NNPTNL,NTVL 

ISN 0008  
CALL COVCAL 

ISN 0009  
CALL COVCAL 

ISN 010  
Z1=ENOPT(NTVL)+**2 

ISN 011  
Z2=ENDOPT(NTVL+1)**2 

ISN 012  
ZI+Z2**2 

ISN 013  
ZI3+Z2**2 

ISN 014  
ZI1+Z2**2 

ISN 015  
ZI2+Z2**2 

ISN 016  
CIA(NTVL-1) 

ISN 017  
CIA2(NTVL-1) 

ISN 018  
CIA3(NTVL-1) 

ISN 019  
CIA4(NTVL-1) 

ISN 020  
YMC1+Z1*Z2+Z1*Z3+C4*Z1 

ISN 021  
YMC2*Z2+Z3*Z1+3*Z3*Z1 

ISN 022  
DEMC1=Z2 

ISN 023  
CAYLIP=Z1*Y/DEN*DEN 

ISN 024  
CAYLIP=Z1*Y/DEN*DEN+Z1*Z2*Z2 

ISN 025  
CAYLIP=Z1*Z2*Z2+Z1*Z3*Z3 

ISN 026  
CAYLIP=Z1*Z2*Z2+Z1*Z3*Z3 

ISN 027  
CAYLIP=Z1*Z2*Z2+Z1*Z3*Z3 

ISN 028  
CAYLIP=Z1*Z2*Z2+Z1*Z3*Z3 

ISN 029  
CAYLIP=Z1*Z2*Z2+Z1*Z3*Z3 

ISN 030  
CAYLIP=Z1*Z2*Z2+Z1*Z3*Z3 

C DISPLAY COEFFICIENTS FOR THE INTENSITY CURVES 

ISN 031  
WRITE(A,1000)HEAD 

ISN 032  
1000 FORMAT (1X,C3,F9.5) 

ISN 033  
WRITE(A,1000) 

ISN 034  
1001 FORMAT (1X,C3,C9.6) 

ISN 035  
WRITE(A,1001) 

ISN 036  
1002 FORMAT (2X,INTERVAL START+2X,INTERVAL END+7X,A11+13X,A21+12X, 
1X,A31+11X,A41) 

ISN 037  
WRITE(A,1005) (ENDPT1,ENDOPT1,A41+13X,A21+12X,A31+11X,A41) 

ISN 038  
1005 FORMAT (1PE15.6) 

C C CALCULATE INTENSITIES AND PERCENTAGE ERROR 

ISN 039  
J+1 

ISN 040  
DO 40 INPT 

ISN 041  
40 FORMAT (1X,C9.6) 

ISN 042  
IF (J+1.NE.ENDOPT(J+1)) GO TO 40 

ISN 043  
C1=AI(J) 

ISN 044  
C2=A2(J) 

ISN 045  
C3=A3(J)
I5N 0046  C3=A3(1)
I5N 0047  C4=A4(1)
I5N 0048  Y=CALC([Y]=X*(C2 + X*(C3+C4)*X))
I5N 0049  P1NOR(1)=0.5
I5N 0050  IF (OBEN(1).EQ.0.0) GO TO 40
I5N 0051  OBENOR(1)*100.0+4*(Y=CALC(1)-OBEN(1))
I5N 0052  40 CONTINUE
I5N 0053  41 CONTINUE
I5N 0054  42 RETURN
I5N 0055  END
LEVEL 21.7 (JAN 73)  

COMPILED OPTIONS = NAME= MAIN, DPT=2, LINEDNT=5, SIZE=0000K,   
SOURCE=CONIC, NOLIST=NODECK=LOAD=MAP=NOEDIT=IO=REF

ISN 0002  SUBROUTINE EMSCAL(NPT,NTVL,ENDPT,DISP,A1,A2,A3,A4,EMIS)
ISN 0003  REAL X(J), Y(J), Z(J)
ISN 0004  DIMENSION ENDPT(I),DISP(I),A(I),A2(I),A3(I),A4(I),EMIS(I)
ISN 0005  J=1
ISN 0006  DO 60 K=1,NPT
ISN 0007  EMIS(K)=0
ISN 0008  J=DISP(I)
ISN 0009  IF(J,EQ,NPT) GO TO 60
ISN 0011  IF(K,L.T.EPDPT(J)) J=J+1
ISN 0013  EMIS(K)=EMFUN(2.0*A2(J), 6.0*A3(J), 6.0*A4(J), X,ENDPT(J))
ISN 0014  IF(J,EQ,NTVL) GO TO 60
ISN 0016  IBOT=J+1
ISN 0017  DO 60 K=IBOT,NTVL
ISN 0018  EMIS(K)=EMIS(J)+EMFUN(2.0*A2(K), 6.0*A3(K), 6.0*A4(K), X,ENDPT(K))
ISN 0019  50 CONTINUE
ISN 0020  60 CONTINUE
ISN 0021  RETURN
ISN 0022  END
LEVEL 21,7 (JAN 73)  OS/360 FORTRAN H

COMPILED OPTIONS - NAME= MAIN, OPT=02, LINCNT=50, SIZE=0000K,
SOURCE=ESDCT, NOLIST, NODECK, LOAD, MAP, NOEDIT, ID, XREF

ISN 0002  DOUUNE PRECISION FUNCTION EMFUN(A2,A3,A4,H)
ISN 0003  IMPLICIT REAL*8(A-H,O-Z)
ISN 0004  PI=3.14159265
ISN 0005  FY=1.00-J1 GO TO 10
ISN 0006  IF(X0.5 D2.0) GO TO 10
ISN 0007  F=SQR(DT/PI)
ISN 0008  T1=AFY
ISN 0009  T2=3.27*(F+F2.0)/15.0
ISN 0110  EMFUN=1/FI
ISN 0111  EMFUN=1/FI
ISN 0112  EMFUN=1/FI
ISN 0113  CONTINUE
ISN 0114  T1=1.7+T2+T3
ISN 0115  EMFUN=T2/FI
ISN 0116  CONTINUE
ISN 0117  EMFUN=0.0
ISN 0118  END

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LEVEL 21.7 (JAN 73)  

COMPILER OPTIONS = NAME= MAIN,OPT=2,LINECNT=50,SIZE=0000K,  
SOURCE,ERGIDC,VLIS=NODECK,LOAD=MAP,NOEDIT=10,NIFF  

SUBROUTINE COVCAL  

**** THIS SUBROUTINE PERFORMS THE ERROR ANALYSIS OPERATIONS... ****  

C C C  

ISN 0002  

C C C  

ISN 0003  

IMPLICIT REAL*8(A-H,O-Z)  

ISN 0004  

INTEGER HEAD(20)  

ISN 0005  

DIMENSION NPTVL(10),DISP(51),ENTEN(119),ENDPT(12),A(110),A2(110)  

1 A(10): A(10),EMIS(51),CALC(51),ERR(51),SO(51),VC(51),K(51)  

2 XA(67,67)*XA(67,67)  

ISN 0006  

COMMON DISP,ENTEN,ENDPT,K(119),A2(110),A(110),EMIS,CALC,ERR,SO,VC  

1 WL=0,EN=0,SO=0  

ISN 0007  

COMMON NPTVL,NPTVL+NTVL  

ISN 0008  

COMMON /REX/ A(67,51)  

ISN 0009  

F1(A,B)=(2./3.)159265)*SORT((A*A)+(B*B))  

ISN 0010  

F2(A,B)=(6./4.4)159265)*SORT((A*A)+(B*B))  

ISN 0011  

F3(A,B)=(2./15.3,159265)*SORT((A*A)+(B*B))  

1 (A*2*A)+(B*2*B)  

ISN 0012  

NTVL+NTVL  

ISN 0013  

NTVL=NTVL+1  

ISN 0014  

NTVL(NTVL+1)  

ISN 0015  

NPTVL=NPTVL+3  

ISN 0016  

NPTVL=NPTVL+3  

ISN 0017  

DO31=1,MD  

ISN 0018  

DO33=1,NPT  

ISN 0019  

3 AT(I,J)=0.0  

ISN 0020  

K=1  

ISN 0021  

K=0  

ISN 0022  

K=4  

ISN 0023  

L=1  

ISN 0024  

DO5=1,NPT  

ISN 0025  

K=0  

ISN 0026  

DO4=K+1,K+K  

ISN 0027  

IF(EM,NE,NPTVL)=GOTO200  

ISN 0028  

IF(DISP(J,J),EQ,0.0)=GOTO210  

ISN 0031  

XT(I,J)=OUTPT(I,J)**K  

ISN 0032  

GOTO200  

ISN 0033  

200 XT(I,J)=1.0  

ISN 0034  

GOTO200  

ISN 0035  

210 XT(I,J)=0.0  

ISN 0036  

251 K=M+2  

ISN 0037  

4 CONTINUE  

ISN 0038  

K=M+2  

ISN 0039  

IF(EM,NE,NPTVL)=GOTO205  

ISN 0041  

K=4  

ISN 0042  

K=3  

ISN 0043  

K=0  

ISN 0044  

L=1  

ISN 0045  

5 CONTINUE  

ISN 0046  

DO61=1,MD  

ISN 0047  

DO62=1,MD  

DATE 260271/08,51,20  

DATE 76,271/08,51,20  

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[Program Listing]

0 XT(1,J)=0.0
   K24
1   K1
2   K30
3   K20
4   I=1
5   JC=1
6   M3=1
7   M5VARPNTVL(K)
8   D0N=1,N=10
9   SUM=0.0
10  D0=SUM+1+MSV
11  IF(K2,E0,0)=GOTO202
12  IF(K1,SPL(E0,E0,0),GOTO7)
13  SUM=SUM+1+ISP(I)**2
14  GOTO7
15  SUM=SUMPNTVL(K)
16  L=MSV
17  7 CONTINUE
18  K3=K3+1
19  XT(1,J)=SUM
20  XT(J,J)=SUM
21  IF(K1,E0,E2)=GOTO203
22  JC=JC+1
23  K2=K2+2
24  GOTO7
25  K2=K2+2
26  H3=H3
27  M5VARVU=VNPNTVL(K)
28  9 CONTINUE
29  K3=K3+1
30  XT(1,J)=XT(1,J)+I
31  XT(J,J)=XT(J,J)+I
32  XT(1,J)=XT(1,J)+I
33  XT(J,J)=XT(J,J)+I
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199  XT(J,J)=XT(J,J)+I
200  XT(1,J)=XT(1,J)+I
201  XT(J,J)=XT(J,J)+I
202  KT=KT+1
203  M5VAR=VNPNTVL(K)
204  K2=K2
205  M2=K2
206  H3=H3
207  M5VARU=VNPNTVL(K)
208  9 CONTINUE

94
ISN 0107  XTX(X,J+6) = XTX(X,J+2)
ISN 0108  XTX(J+6) = XTX(A(I), J+6)
ISN 0109  XTX(J+3), I = XTA(A(I), J+3)
ISN 0110  XTX(A(I), J+6) = XTOPT(K)
ISN 0111  XTX(I,J+7) = XTX(I,J+1)
ISN 0112  XTX(I,J+8) = XTA(A(I), J+8)
ISN 0113  XTX(I,J+1), I = XTOPT(K)
ISN 0114  XTX(I,J+1) = XTA(A(I), J+1)
ISN 0115  XTX(I,J+5) = XTA(I,J+5)
ISN 0116  XTX(I,J+1) = XTA(I,J+1)
ISN 0117  XTX(A(I), J+2) = XTOPT(K)
ISN 0118  XTX(A(I), J+1) = XTA(I,J+1)
ISN 0119  XTX(A(I), J+2) = XTA(I,J+2)
ISN 0120  XTX(A(I), J+1) = XTA(I,J+1)
ISN 0121  XTX(I,J+3) = XTOPT(K)
ISN 0122  XTX(A(I), J+2) = XTA(I,J+2)
ISN 0123  XTX(I,J+7) = XTA(I,J+7)
ISN 0124  XTX(I,J+7) = XTA(I,J+7)
ISN 0125  XTX(I,J+7) = XTA(I,J+7)
ISN 0126  XTX(I,J+7) = XTA(I,J+7)
ISN 0127  XTX(I,J+7) = XTA(I,J+7)
ISN 0128  XTX(I,J+7) = XTA(I,J+7)
ISN 0129  XTX(I,J+7) = XTA(I,J+7)
ISN 0130  XTX(A(I), J+1) = XTA(A(I), J+1)
ISN 0131  XTX(A(I), J+1) = XTA(A(I), J+1)
ISN 0132  XTX(A(I), J+1) = XTA(A(I), J+1)
ISN 0133  XTX(A(I), J+1) = XTA(A(I), J+1)
ISN 0134  XTX(A(I), J+1) = XTA(A(I), J+1)
ISN 0135  XTX(A(I), J+1) = XTA(A(I), J+1)
ISN 0136  XTX(A(I), J+1) = XTA(A(I), J+1)
ISN 0137  XTX(A(I), J+1) = XTA(A(I), J+1)
ISN 0138  XTX(A(I), J+1) = XTA(A(I), J+1)
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ISN 0140  XTX(A(I), J+1) = XTA(A(I), J+1)
ISN 0141  XTX(A(I), J+1) = XTA(A(I), J+1)
ISN 0142  XTX(A(I), J+1) = XTA(A(I), J+1)
ISN 0143  XTX(A(I), J+1) = XTA(A(I), J+1)
ISN 0144  XTX(A(I), J+1) = XTA(A(I), J+1)
ISN 0145  XTX(A(I), J+1) = XTA(A(I), J+1)
ISN 0146  XTX(A(I), J+1) = XTA(A(I), J+1)
ISN 0147  XTX(A(I), J+1) = XTA(A(I), J+1)
ISN 0148  XTX(A(I), J+1) = XTA(A(I), J+1)
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ISN 0150  XTX(A(I), J+1) = XTA(A(I), J+1)
ISN 0151  XTX(A(I), J+1) = XTA(A(I), J+1)
ISN 0152  XTX(A(I), J+1) = XTA(A(I), J+1)
ISN 0153  XTX(A(I), J+1) = XTA(A(I), J+1)
ISN 0154  XTX(A(I), J+1) = XTA(A(I), J+1)
ISN 0155  XTX(A(I), J+1) = XTA(A(I), J+1)
ISN 0156  XTX(A(I), J+1) = XTA(A(I), J+1)
ISN 0157  XTX(A(I), J+1) = XTA(A(I), J+1)
ISN 0158  XTX(A(I), J+1) = XTA(A(I), J+1)
ISN 0159  94  Jx,J+4  C
ISN 0160  D031x1,NPT  C
ISN 0161  D031x1,NPT
ISN 0162  31  TX1(1,J)=0.0  C
ISN 0163  NPT1=NPT+1  C
ISN 0164  J=1  C
ISN 0165  J=2  C
ISN 0166  DTOVx1,NPT1  C
ISN 0167  DMISPT(I)  C
ISN 0168  IF(J,J0,ENPTx1)J+1)GOTO16  C
ISN 0169  GOTO17
ISN 0170  16  Jx,J+1
ISN 0171  J2=J+4  C
ISN 0172  17  TX1(1,J)=F(1(ENPTx1)J+1)+DISP(I)  C
ISN 0173  TX1(1,J)=2(ENPTx1)J+1)+DISP(I)  C
ISN 0174  TX1(1,J)=3(ENPTx1)J+1)+DISP(I)  C
ISN 0175  TX1(1,J)=4(ENPTx1)J+1)+DISP(I)  C
ISN 0176  IF(J,J0,NTVLx2)GOTO19  C
ISN 0177  Kx,J+1
ISN 0178  K2=J+4  C
ISN 0179  18  DTOx2x2,NTVLx4,6  C
ISN 0180  D031x1,NPT  C
ISN 0181  D02x1,NPT
ISN 0182  20  TX1(1,J)=0,0  C
ISN 0183  D02x1,NPT
ISN 0184  D02x1,NPT
ISN 0185  21  TX1(1,J)=X(I,K)*X(K,J)*TX(1,J)  C
ISN 0186  D02x1,NPT
ISN 0187  D02x1,NPT
ISN 0188  D02x1,NPT
ISN 0189  D02x1,NPT
ISN 0190  D02x1,NPT
ISN 0191  D02x1,NPT
ISN 0192  D02x1,NPT
ISN 0193  D02x1,NPT
ISN 0194  D02x1,NPT
ISN 0195  D02x1,NPT
ISN 0196  D02x1,NPT
ISN 0197  D02x1,NPT
ISN 0198  D02x1,NPT
ISN 0199  D02x1,NPT
ISN 0200  D02x1,NPT
ISN 0201  D02x1,NPT
ISN 0202  22  TX1(1,J)=X(I,K)+X(K,J)  C
ISN 0203  D02x1,NPT
ISN 0204  D02x1,NPT
ISN 0205  D02x1,NPT
ISN 0206  D02x1,NPT
ISN 0207  D02x1,NPT
ISN 0208  D02x1,NPT
ISN 0209  D02x1,NPT
ISN 0210  23  TX1(1,J)=X(I,K)*X(K,J)+VC(1,J)  C
ISN 0211  D02x1,NPT
ISN 0212  D02x1,NPT
ISN 0213  D02x1,NPT
ISN 0214  D02x1,NPT
ISN 0215  D02x1,NPT
ISN 0216  D02x1,NPT
ISN 0217  D02x1,NPT
ISN 0218  D02x1,NPT
ISN 0219  D02x1,NPT
ISN 0220  D02x1,NPT
ISN 0221  D02x1,NPT
ISN 0222  D02x1,NPT
ISN 0223  D02x1,NPT
ISN 0224  D02x1,NPT
ISN 0225  D02x1,NPT
ISN 0226  D02x1,NPT
ISN 0227  D02x1,NPT
ISN 0210  29 VC(I+1)=DSURFVC(I+1)
ISN 0211  RETURN
ISN 0212  END
AE DC-TR-76-163
LEVEL 217 (JAN 73) 05/360 FORTRAN H

COMPILER OPTIONS - NAME=MAIN+OPT=O2+LINECNT=5+SIZE=40960K,
SOURCE=ERDCIC+NOLIST+NODECK+LOAD+MAP+NOEDIT+IO+XREF

ISN 0002 SUBROUTINE MAIN(NIN (4+N))
ISN 0003 IMPLICIT REAL*8(!18+4+0=2)
ISN 0004 DIMENSION A(6+67)+LOCATE(67,3)
ISN 0005 00 1 NN,NN
ISN 0006 1 LOCATE(4,3)= 0
ISN 0007 DO 14 NN=1,NN
ISN 0008 MAX= 0.00+0
ISN 0009 DO 14 NN=1,NN
ISN 0010 IF I LOCATE(I+3)=EQ.0 GO TO 2
ISN 0011 GO TO 4
ISN 0012 2 DO 5 NN,NN
ISN 0013 IF I LOCATE(J+3)=EQ.0 GO TO 3
ISN 0014 GO TO 5
ISN 0015 3 IFUARDS(A,J,J),GT,AMAX) GO TO 4
ISN 0016 GO TO 5
ISN 0017 4 AMAXDARS (A(I,J))
ISN 0018 IRMD I
ISN 0019 JCOL= J
ISN 0020 5 CONTINUE
ISN 0021 6 CONTINUE
ISN 0022 IF I AMAX,GT,1.0+0=15 GO TO 7
ISN 0023 GO TO 10
ISN 0024 7 LOCATE(4,1)= IRMD
ISN 0025 LOCATE(J,J)= JCOL
ISN 0026 LOCATE(J,J)= 1
ISN 0027 IF I IRMD,NE,JCOL) GO TO 8
ISN 0028 GO TO 10
ISN 0029 8 DO 9 J=1,NN
ISN 0030 SWAP A(IROW,J)
ISN 0031 A(INOW,J)= A(JCOL,J)
ISN 0032 9 A(JCOL,J)= SWAP
ISN 0033 10 PIVNOTE A(JCOL,J)
ISN 0034 A(JCOL,JCOL)= 1.00+0
ISN 0035 DO 11 J=1,NN
ISN 0036 11 A(JCOL,J)= A(JCOL,J) / PIVOTE
ISN 0037 DO 14 NN=1,NN
ISN 0038 IF I I,NE,JCOL) GO TO 12
ISN 0039 GO TO 14
ISN 0040 12 FA A(I,JCOL)
ISN 0041 A(JCOL,JCOL)= 0.00+0
ISN 0042 DO 13 J=1,NN
ISN 0043 A(I,J)= A(I,J),FA(JCOL,J)
ISN 0044 13 CONTINUE
ISN 0045 14 CONTINUE
ISN 0046 15 IF I LOCATE(L+1)=NE,LOCATE(L+2) GO TO 15
ISN 0047 GO TO 17
ISN 0048 15 (IROW) LOCATE(L+1)
ISN 0049 JCOL= LOCATE(L+2)
ISN 0050 00 1A KE1,NN
ISN 0051 SWAP A(I,JROW)
ISN 0052 A(I,JCOL)= SWAP
ISN 0053 1A CONTINUE
ISH 0063  17 CONTINUE
ISH 0064  RETURN
ISH 0065  18 PRINT 1000
ISH 0066  RETURN
ISH 0067  1000 FORMAT (1H * 10X; 15MSINGULAR MATRIX)
ISH 0068  END
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**FRB-LEVEL LINKAGE EDITOR OPTIONS SPECIFIED LETMAP**

**DEFAULT OPTION(S) USED - SIZE=(100352,12288)**

**MODULE MAP**

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<th>NAME</th>
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***USERPROG DOES NOT EXIST BUT HAS BEEN ADDED TO DATA SET***
NOMENCLATURE

A  Matrix of curve fit coefficients

$a_{ki}$  Coefficient of $x^{2(k-1)}$ in the $k^{th}$ interval

B  Linear transformation matrix, defined in Eqs. (13) and (14)

C  Vector representing right-hand side of least-squares equations, defined by Eqs. (11), (13), and (19)

E  Matrix of emission coefficients

$[E]_{cv}$  Variance-covariance matrix for emission coefficients

F  Represents matrix product, MW

$F_1, F_k, F_n$  Functions in developing least-squares equations, defined by Eq. (8)

G  Matrix of powers of $x$, Eqs. (19), (20), (21)

I  Radiance, generally watts/cm$^2$/sr, but for illustrative purposes, arbitrary units

$i, j, k$  Dummy subscripts

M  Matrix of coefficients of the curve-fit coefficients Eqs. (23) and (24)

m  Number of points at which emission coefficient is evaluated
\( m_k \)  
Number of points in the \( k \)th interval

\( N \)  
Submatrix of \( B \), defined in Eqs. (17) and (18)

\( n \)  
Number of intervals

\( P_j \)  
Submatrix of \( R \), defined in Eq. (16)

\( P_k(x) \)  
Polynomial in \( k \)th interval, Eq. (5)

\( p \)  
Number of points

\( Q_j \)  
Submatrix of \( N \), defined in Eq. (18)

\( R \)  
Submatrix of \( B \), defined in Eqs. (15) and (16), or outer radius of emission source

\( r \)  
Radial position, generally cm, but for illustrative purposes, arbitrary units

\( S_j \)  
Submatrix of \( G \), defined in Eq. (21)

\( S_k \)  
Sum of squares of residuals, defined in Eq. (6)

\( W \)  
Matrix product \( B^{-1}G \)

\( x \)  
Displacement, generally cm, but for illustrative purposes, arbitrary units

\( Y \)  
Column vector of radiance; also used in place of \( I \) in Appendix A

\( [Y]_{cv} \)  
Variance-covariance matrix for radiance
$y$ \quad \text{Scalar member of } Y, \text{ radiance}

$z_k$ \quad \text{Right side abcissa of } k\text{th interval}

$\varepsilon$ \quad \text{Emission coefficients, generally watts/cm}^3\text{/sr, but for illustrative purposes, arbitrary units}

$\lambda$ \quad \text{Lagrange's undermined multiplier}

$\mu_I$ \quad \text{Vector of mean values of radiances}

$\mu_\varepsilon$ \quad \text{Vector of mean values of emission coefficients}

$\mathcal{E}$ \quad \text{Expected value operator}

$\pi$ \quad 3.14159265

$\phi_{k,1}$ \quad \text{Constraint on polynomial at } z_k

$\phi_{k,2}$ \quad \text{Constraint on first derivative at } z_k

$\phi_{k,3}$ \quad \text{Constraint on second derivative at } z_k