HIGH GAIN FEEDBACK SYSTEMS AS SINGULAR SINGULAR-PERTURBATION PROBLEMS

R. E. O'MALLEY, JR.

Department of Mathematics
University of Arizona
Tucson, Arizona 85721

We shall consider the linear high gain system

\[
\begin{align*}
\dot{x} &= A(t)x + B(t)u, \quad t \geq 0 \\
u &= gC(t)x
\end{align*}
\]

where the state vector \( x \) is \( n \)-dimensional, the control vector \( u \) is \( m \)-dimensional, and the scalar gain factor \( g \) is large. Introducing the small parameter

\[
u = 1/g,
\]

then, the feedback system (1) takes the form

\[
\dot{x} = (B(t)C(t) + A(t))x, \quad t \geq 0.
\]

If \( BC \) remains a stable matrix, classical singular perturbations theory implies the existence of a unique asymptotic solution to the initial value problem for (3) of the form

\[
x(t, \mu) = x(t, \mu) + \sum_{j=0}^{\infty} \mu_j x_j(t),
\]

Here, the terms \( \mu_j \) each tend to zero as the stretched variable \( t = t/\mu \).

This work was supported in part by the Office of Naval Research, Contract Number N00014-74-C-0126.
Then, we shall obtain a unique asymptotic solution to the initial value problem for (3) of the form
\[ x(t,\mu) = X(t,\mu) + \eta(t,\mu) \]  
where the outer solution \( X(t,\mu) \) and the boundary layer correction \( \eta(t,\mu) \) both have asymptotic series expansions in \( \mu \), with the boundary layer terms decaying to zero as the stretched variable \( \tau = t/\mu \) tends to infinity.

**2. Preliminary Linear Algebra**

Under hypothesis (H), the matrix \( BC \) can be put into its row-reduced echelon form by use of an orthogonal matrix \( E \), i.e. we'll obtain
\[ E(t)B(t)C(t) = \begin{pmatrix} U(t) \\ 0 \end{pmatrix} \]

where \( U(t) \) is a \( k \times n \) matrix of rank \( k \). Indeed, \( E \) can be readily obtained via Householder transformations (cf. Golub (1965)). Writing
\[ E = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} \]
where \( E_1 \) is \( k \times n \), we have
\[ E_2BC = 0 \]  
while the orthogonality of \( E \) implies that
\[ E_1E_1^T = E_2E_2^T = I_n \]

Further,
\[ P = E_1^T E_2 \]  
\[ Q = E_2^T E_2 \]  
are complementary projections with \( P + Q = I_n \), and \( Q \) projects into the null space of \( (BC)^T \) since
\[ (BC)^TQ = (BC)^T E_2 = (E_2BC)^T E_2 = 0 \]  
by (10). Since the dimensions of \( R(Q) \), the range of \( Q \), and \( N((BC)^T) \), the null space of \( (BC)^T \), are both \( n-k \), we have
\[ R(Q) = N((BC)^T) \]

and
\[ R(P) = R(BC). \]  
Finally, (9) and (10) imply that
\[ EBCE^T = \begin{pmatrix} E_1^T & E_2^T \\ 0 & 0 \end{pmatrix} \]  
so the \( k \times k \) matrix
\[ S = E_1^TBCE_1 \]  

is stable
\[ \text{since it has the } k \text{ stable eigenvalues of } BC \]  
guaranteed by hypothesis (H).

**3. A Solution via a Transformed Problem**

If we make the \( 1-1 \) transformation
\[ \nu = Ex, \]
the variable \( \nu \) satisfies the singularly perturbed system
\[ \mu \dot{\nu} = (EBC^T + \mu(\hat{E} + EA)E^T)\nu. \]

This, however, is also a singular singular-perturbation problem since the matrix \( EBC^T \) has rank \( k < n \).

The structure of the solution becomes obvious upon splitting \( \nu \) into vector components
\[ \nu = \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \begin{pmatrix} E_1x \\ E_2x \end{pmatrix}. \]

They satisfy the "nonsingular" singularly perturbed system
\[ \dot{\nu}_1 = S \nu_1 + E_1BC^T E_2 \nu_2 + \mu E_1 \nu_1 + \mu E_2 \nu_2 \]
\[ \dot{\nu}_2 = E_1 \nu_2 + E_2 \nu_2 \]

where we've used the decomposition (14) for \( EBC^T \) and set
\[ (\hat{E} + EA)E^T = \begin{pmatrix} E_1 & E_2 \\ E_3 & E_4 \end{pmatrix}. \]

Standard singular perturbation theory shows that the initial value problem for (19) will have a unique solution of the form
\[ \begin{cases} \nu_1(t,\mu) = \nu_{10}(t,\mu) + \nu_{11}(t,\mu) \\ \nu_2(t,\mu) = \nu_{20}(t,\mu) + \nu_{21}(t,\mu) \end{cases} \]

where all terms have asymptotic expansions in \( \mu \) and \( P_1 \) and \( P_2 \) tend to zero as \( \tau = \tau. \) Further, the leading terms \( \nu_{10} \) and \( \nu_{20} \) of the outer expansion satisfy the reduced problem.

\[ 0 = SW + E_1BC^TW \]

where \( W_{10} \) uniquely satisfies
\[ \dot{W}_{10} = -S^{-1}E_1BC^TW_{20} \]

while \( W_{20} \) uniquely satisfies
\[ \dot{W}_{20} = (E_4 - E_3S^{-1}BC^TW_{20})W_{20}(0) = E_2(0)x(0). \]
The leading terms of the boundary layer correction satisfy
\[
\begin{align*}
\frac{dP_{10}}{dt} &= S(0)P_{10}, \quad P_{10}(0) = E_1(0)x(0) - \nu_{10}(0) \\
\frac{dP_{20}}{dt} &= E_3(0)P_{10}.
\end{align*}
\]
Thus, the decaying solutions are given by
\[
\begin{align*}
P_{10}(t) &= e^{S(0)t}(E_1(0)x(0) - \nu_{10}(0)) \\
P_{20}(t) &= -\int \frac{E_3(0)}{\nu_1}(s)\,ds + E_3(0)s^{-1}(0)P_{10}(t).
\end{align*}
\]
Higher order terms follow analogously. Finally, \(x = E\) implies that the unique solution of the original initial value problem is represented in the additive form (7) with the outer solution
\[
X(t,\nu) = E_1^*(t,\nu)X_1(t,\nu) + E_2^*(t,\nu)X_2(t,\nu)
\]
and the boundary layer correction
\[
\Pi(t,\nu) = E_1^{**}(t,\nu)\Pi_1(t,\nu) + E_2^{**}(t,\nu)\Pi_2(t,\nu)
\]
which decays to zero as \(t = \infty\).

4. A Direct Solution

Let us now try to directly obtain a solution of the form (7). Since the boundary layer correction \(X_1\) decays to zero as \(t = \infty\), it's necessary that the outer solution \(X_2\) satisfy the system (3) for all \(t > 0\). Thus the outer expansion
\[
X(t,\nu) = \sum_{j=0}^{\infty} X_j(t)\nu^j
\]
must satisfy
\[
\dot{X}_j = \nu \dot{X}_j - \nu_2 X_j - A\nu \nu_2^j X_j
\]
as a power series in \(\nu\). This requires that
\[
BCX_j = \dot{X}_{j-1} - A\nu \nu_2^j X_j
\]
for each \(j \geq 1\) with \(\nu_2 \equiv 0\). Multiplying this equation by \(E_1\) and manipulating, we obtain
\[
\Pi_j = ABC(X_j) + A\nu_1
\]
for \(A = E_1^* s^{-1}(0)\). But \(P + Q = I\) then implies that
\[
x_j = BQX_j + A\nu_1
\]
where \(B\) is the projection \(B = I - ABC\). Thus each \(X_j\) has been successively determined up to its projection \(QX_j\) onto \(N((BC)^*)\).

Since \(QBC = 0\), consistency of (30) requires that \(QX_j = 0\). Since \(C = 0\), this Fredholm alternative argument requires that \((QX_j)^* = (Q + QA)X_j\) for each \(j \geq 0\). Using (32), then, we finally obtain the linear differential equation
\[
(QX_j)^* = (Q + QA)B(QX_j) + (Q + QA)X_{j-1}\tag{33}
\]
for \(QX_j\). We therefore find the outer expansion by solving algebraic equations (31) for \(P\), and the differential equation (33) for its complement \(QX_j\). Thus, we're able to formally obtain \(X(t,\nu)\) termwise up to specifying the initial values \(Q(0)X(0)\). Since the initial value \(x(0)\) won't generally satisfy (31) at \(t = 0\), we need the boundary layer correction to account for the resulting initial jump (i.e. a nonuniform convergence at \(t = 0\) occurring in \(X(0)\)).

By linearity, \(\Pi(t,\nu)\) must satisfy
\[
\frac{d\Pi}{dt} = [B(0)C(0) + \mu A(0)]\Pi
\]
as a power series
\[
\Pi(t,\nu) = \sum_{j=0}^{\infty} \Pi_j(t)\nu^j
\]
in \(\nu\). Thus, we must successively obtain decaying solutions of
\[
\frac{d\Pi_j}{dt} = B(0)C(0)\Pi_j + \nu_1\Pi_{j-1}, \quad j \geq 0,
\]
where \(\nu_1 \equiv 0\) and \(\nu_1 \equiv 0\) is generally a linear combination of preceding terms \(\Pi_j\) with polynomial coefficients in \(t\). Since \(QBC = 0\), it follows that \(\frac{d}{dt}(Q(0)\Pi_j(t)) = Q(0)\nu_1\Pi_{j-1}(t)\), so
\[
Q(0)\Pi_j(t) = -Q(0)\int_0^t \nu_1\Pi_{j-1}(s)\,ds, \quad j \geq 0
\]
It remains to find the complementary \(P(0)\Pi_j(t)\)’s. Multiplying (36) by \(E_1(0)\), we have
\[
\frac{d}{dt}(E_1(0)\Pi_j) = S(0)(E_1(0)\Pi_j) + E_1(0)(B(0)C(0)\Pi_j(t) + \nu_1\Pi_{j-1})
\]
Integrating, then, and using (37), we obtain
\[
\Pi_j(t) = E_1(0)e^{S(0)t}E_1(0)\Pi_j(0) - Q(0)\int_0^t \nu_1\Pi_{j-1}(s)\,ds + E_1(0)e^{S(0)(t-s)}E_1(0)\int_0^s \nu_1\Pi_{j-1}(r)\,dr
\]
mic system (39) in an \( n - k \) dimensional space where \( k = \text{rank (BC)} \). The results continue to hold for all \( t > 0 \) provided (H) remains valid and \( X(t) \) decays exponentially to zero as \( t \to \infty \). Finally, observe that (44) displays the two time scale nature of the limiting solution on \( 0 \leq t \leq T \).

Our results should be further related to earlier work in the literature. We note, in particular, Young et al.'s result that \( CX = 0 \) when \( k = m - \text{rank (CB)} \). It implies that the control would stay bounded away from \( t = 0 \).

Acknowledgment: Thanks should be given to Petar Kokotovic and his co-authors for emphasizing the singular perturbation nature of large gain systems. It fit well with the theory of singular perturbation problems, currently under study by this author.

6. References


## REPORT DOCUMENTATION PAGE

<table>
<thead>
<tr>
<th>Field</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>REPORT NUMBER</td>
<td>15</td>
</tr>
<tr>
<td>TITLE</td>
<td>High Gain Feedback Systems as Singular Singular-Perturbation Problems</td>
</tr>
<tr>
<td>AUTHOR(s)</td>
<td>R. E. O'Malley, Jr.</td>
</tr>
<tr>
<td>PERFORMING ORGANIZATION NAME AND ADDRESS</td>
<td>University of Arizona, Department of Mathematics, Tucson, Arizona 85712</td>
</tr>
<tr>
<td>CONTROLLING OFFICE NAME AND ADDRESS</td>
<td>Mathematics Branch, Office of Naval Research, Arlington, Virginia</td>
</tr>
<tr>
<td>REPORT DATE</td>
<td>Apr 1977</td>
</tr>
<tr>
<td>NUMBER OF PAGES</td>
<td>4</td>
</tr>
<tr>
<td>SECURITY CLASS. (of this report)</td>
<td>Unclassified</td>
</tr>
<tr>
<td>DISTRIBUTION STATEMENT (of this Report)</td>
<td>Approved for public release, distribution unlimited</td>
</tr>
<tr>
<td>KEY WORDS</td>
<td>singular arcs, singular control, singular perturbations, optimal control, high gain, feedback control</td>
</tr>
<tr>
<td>ABSTRACT</td>
<td>As an example of a singular perturbation problem, we solve the high gain system ( \dot{x} = A(t)x + B(t)u ), ( u = gC(t)x ) as ( g \to 0 ). Our primary assumption requires ( BC ) to have fixed rank ( k &lt; n ) and no unstable eigenvalues. We find an asymptotic solution of the form ( x(t,\tau) = X(t,\tau) + U(t,\tau) ) where the fast transient ( \tau \to 0 ) goes to infinity as ( \tau = t/\tau ).</td>
</tr>
</tbody>
</table>