ON AN OPERATOR THEORY OF LINEAR SYSTEMS WITH PURE AND DISTRIBUTED DELAYS

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Convolution
Linear systems
Control theory

A representation theory based on convolution operations is developed for a large class of linear systems containing pure and distributed delays in state and control. In terms of this framework a necessary and sufficient condition and a sufficient condition are given for functional (null) controllability. The conditions involve the generation of modules defined over a convolution ring of functions.
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Abstract

A representation theory based on convolution operations is developed for a large class of linear systems containing pure and distributed delays in state and control. In terms of this framework a necessary and sufficient condition and a sufficient condition are given for functional (null) controllability. The conditions involve the generation of modules defined over a convolution ring of functions.

1. Introduction

In many control problems the systems under consideration contain pure and distributed time delays in state and control (examples are given by MANITIUS [7]). Such systems are usually referred to as hereditary systems since the rate of change of the present state depends on past values of the state and control or input.

In this paper we consider the class of linear systems given by a first-order functional differential equation of the form

\[ \dot{x}(t) = \int_{-c}^{0} A(\theta)x(t+\theta)d\theta + F_0x(t) + \sum_{i=1}^{r} F_i x(t-a_i) + \int_{-d}^{0} B(\theta)u(t+\theta)d\theta + G_0u(t) + \sum_{i=1}^{s} G_i u(t-b_i) \]

where \( c, d \) and the \( a_i, b_i \) are positive real numbers, the \( F_i, G_i \) are \( n \times n \) (\( n \times m \)) matrices over the reals \( \mathbb{R} \), \( A(\theta) \) (resp. \( B(\theta) \)) is a \( n \times n \) (\( n \times m \)) matrix of (Lebesgue) measurable and integrable functions on \([0,0)\) \([-d,0)\), \( x(t) \in \mathbb{R}^n \) is the "instantaneous state," and \( u(t) \in \mathbb{R}^m \) is the input.

Systems of the form (1) have been studied using mainly functional-analytical methods applied to a state space setting defined in terms of the product space \( \mathbb{R}^n \times L^1(-h,0;\mathbb{R}^m) \) where \( h = \max\{c,d\} \).

In particular, numerous results on controllability and optimal feedback control can be found in the work of DELFOUR-MITTER [5], and DELFOUR-McCALLA-MITTER [6] (see also the survey by MANITIUS [7]).

In contrast to existing methods, our approach to the study of (1) is based on an algebraic setting defined in terms of convolution operators. More precisely, in the next section it is shown that (1) can be written in the form

\[ \dot{x}(t) = (F \ast x)(t) + (G \ast u)(t) \]

where \( \ast \) denotes convolution and \( F \) and \( G \) are matrices whose elements belong to a convolution ring of functions and impulses (Dirac distributions). The convolution representation (2) is a special case of the time-domain operator framework developed by KAMEN [7].

In the latter part of the paper the representation (2) is applied to the problem of driving initial functions to the zero function in finite time (functional null controllability).

New algebraic criteria for controllability are given in terms of modules defined over the convolution rings.

2. Representation by Convolution Operators

Let \( L^1_{\text{loc}} \) denote the space of all real-valued Lebesgue measurable functions \( f(t) \) that are locally integrable, i.e.

\[ \int_{K} |f(t)| dt < \infty \]

for any compact subset \( K \) of \( \mathbb{R} \). Let \( L^1_{\text{loc}}^+ \) denote the subspace of \( L^1_{\text{loc}} \) consisting of all functions with support bounded on the left. It is easily verified that \( L^1_{\text{loc}}^+ \) is a ring with pointwise addition and with convolution defined by

\[ (g \ast f)(t) = \int_{\mathbb{R}} g(\theta)f(t-\theta)d\theta \]

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Note that if the support of \( g \) is contained in \([0,a]\), \( a > 0 \), then

\[
(g * f)(t) = \int_{0}^{a} g(\theta)f(t - \theta)d\theta
\]

and by a change of variables, we have

\[
(g * f)(t) = \int_{-a}^{0} g(\theta)f(t + \theta)d\theta
\]

where \( \hat{g}(\theta) = g(-\theta) \).

Letting \( \delta_{a} \) denote the unit impulse (Dirac distribution) concentrated at the point \( [a] \), define

\[
J = \left\{ \sum_{i=1}^{q} b_{i} \delta_{a_{i}} : \sum_{i} b_{i} \leq 1, a_{i}, b_{i} \in \mathbb{R}, \quad q = \text{positive integer} \right\}.
\]

The addition in the definition of \( J \) can be taken in a formal sense or it can be viewed as addition in some space of distributions. The set \( J \) is an overring of \( L_{\text{loc}}^{+} \) with the convolution operation given by

\[
(f * g + \sum_{j} c_{j} f(t - d_{j}) + b_{j} g(t - a_{j})) = f * g + \sum_{j} c_{j} f(t - d_{j}) + b_{j} g(t - a_{j})
\]

(5)

Note that \( \delta_{0} \) is the identity element of the ring \( J \).

Finally, let \( L_{+}^{1} \) denote the space of all real-valued measurable functions \( f(t) \) defined on \( \mathbb{R} \) with support bounded on the left, such that \( \int_{-\infty}^{t} |f(t)|dt < \infty \). The space \( L_{+}^{1} \) is a subring of \( L_{\text{loc}}^{1} \).

Via the above constructions we can now characterize (1) in terms of convolution operations.

Theorem 1: With \( x \in (L_{+}^{1})^m \) and \( x \in J_{c}^{+} \), the system equation (1) can be written in the form

\[
x'(t) = (A * x)(t) + (B * u)(t)
\]

where \( A(t) = A(-t) \) and \( B(t) = B(-t) \).

By definition of \( A, B \) and the \( F_{i,j}, G_{i,j} \), the coefficient matrices of \( x \) and \( u \) are over \( J_{c} \).

J. Initial Conditions

In this section we show how to "handle" nonzero initial conditions in solving the operator equation \( \dot{x} = F*x + G*u \). This result will be utilized in the next section to study functional controllability.

Suppose that we are given

\[
x(0) = x_{0} \in \mathbb{R}^{n}
\]

(7)

\[
x(t) = \phi(t), \quad t \in [-a,0), \quad \mathbf{g} \in \left\{ \frac{1}{t-a} \right\}_{n}^{\infty}
\]

(8)

\[
u(t) = u(t), \quad t \in [-b,0), \quad u_{0} \in \left\{ \frac{1}{t-b} \right\}_{m}^{\infty}
\]

where \( \mathcal{L}_{[c,d]}^{\mathbb{R}} = \{f \in L_{\text{loc}}^{1} : \text{supp} f \subset \{c,d] \} \} \).

Given \( f \in \mathcal{L}_{[c,d]}^{\mathbb{R}} \), define

\[
x|_{[c,d]}(t) = \begin{cases}
(f(t), t \in [c,d] \\
0, \text{ otherwise}
\end{cases}
\]

(6)

By the results of DELFOUR [4], with initial data (7) and \( u_{0} \in \left\{ \frac{1}{t-a} \right\}_{n}^{\infty} \), the convolution equation (6) has a unique solution \( x(t) \) with \( x|_{(0,m)} \in \left\{ \frac{1}{t-a} \right\} \).

Theorem 2: Given \( u_{0} \in \left\{ \frac{1}{t-a} \right\}_{n}^{\infty} \) with initial data (7), the solution of (6) for \( t > 0 \) is equal to the solution for \( t > 0 \) of

\[
x(t) = (F*x)(t) + (G*u)(t) + v(t)
\]

with initial data equal to zero, where

\[
v = x_{0} + (F* \phi + G* u_{0})|_{(0,m)} \in \mathbb{R}^{n}
\]

(9)

Proof: Clearly \( x_{0} \) sets up the initial value \( x_{0} \in \mathbb{R}^{n} \). Since \( x = \phi + x_{0} \in \left\{ \frac{1}{t-a} \right\} \) and \( u = u_{0} \in \left\{ \frac{1}{t-a} \right\} \), with \( x_{0} = \phi \), we have that

\[
F*x + G*u = F*x|_{(0,m)} + G*u|_{(0,m)} + F* \phi + G* u_{0}
\]

Hence

\[
(F*x + G*u)|_{(0,m)} = F*x|_{(0,m)} + G*u|_{(0,m)}
\]

and

\[
(F* \phi + G* u_{0})|_{(0,m)}.
\]

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As a consequence of Theorem 2, operational methods, such as those developed in [7], can be used to solve operational differential equations of the form (6) with nonzero initial conditions. We shall now apply this result to the study of controllability.

4. Controllability

Let \( L_1 \) denote the set of all \( f \in \mathbb{L}_1 \) such that supp \( f \) is compact and contained in \([0, \infty)\). With the induced operations \( L_1 \) is a convolution ring contained in the ring \( J_1 \).

**Definition:** The system given by (6) is said to be (null) controllable if for any initial condition (7), there exists a control \( u \in L_1^m \) such that the solution of (6) is zero for all \( t > h \), some \( h > 0 \).

In terms of the following constructions we derive a necessary and sufficient condition for controllability.

Let \( p^n \) denote the \( n^{th} \) derivative of \( \delta \) in the sense of distributions. The element \( p^n \) belongs to \( \mathcal{L}_1 \), the convolution ring of Schwartz distributions on \( \mathbb{R} \) with support bounded on the left (see [7]). Given \( f \in \mathbb{L}_1 \), the derivative of \( f \) in the sense of distributions is equal to \( p^n f \) (with the convolution \( p^n f \) carried out in \( \mathcal{L}_1 \)). Therefore (6) can be expressed completely in terms of convolution operations:

\[
(pI - F) * x = G * u
\]

where \( I \) is the \( n \times n \) identity matrix.

Now let \( J_1[p] \) denote the set of all finite sums \( \sum a_i * p^i \) where \( a_i \in J_1 \). With the standard operations, \( J_1[p] \) is a convolution ring of distributions (contained in \( \mathcal{L}_1 \)). The rings \( J_1 \) and \( L_1 \) are subrings of \( J_1[p] \).

Letting \( J_1[p]^n \) denote the space of \( n \)-element column vectors over \( J_1[p] \), we have that \( J_1[p]^n \) is a free finite module over \( J_1[p] \) with componentwise addition and with multiplication defined by

\[
\pi(\beta_1, \beta_2, \ldots, \beta_n)^{TR} = (\pi * \beta_1, \pi * \beta_2, \ldots, \pi * \beta_n)^{TR}
\]

where \( \pi, \beta \in J_1[p] \), and \( TR \) denotes the transpose.

Since \( L_1 \) is a subring of \( J_1[p] \), by restricting the multiplication operation to \( L_1 \) we have that \( J_1[p]^n \) is a (nonfinite) \( L_1 \)-module. Given \( \gamma_1, \gamma_2, \ldots, \gamma_n \in J_1[p]^n \), let \( (\gamma_1, \gamma_2, \ldots, \gamma_n) \)

\( \gamma_1, \gamma_2, \ldots, \gamma_n \in J_1[p]^n \), let \( (\gamma_1, \gamma_2, \ldots, \gamma_n) \)

denote the \( L_1 \)-submodule of \( J_1[p]^n \) generated over \( L_1^1 \) by \( \gamma_1, \gamma_2, \ldots, \gamma_n \). That is, \( (\gamma_1, \gamma_2, \ldots, \gamma_n) \)

is the set of all sums \( \sum \pi_i \gamma_i, \pi_i \in L_1 \).

**Theorem 3:** The system given by (6) is controllable if and only if \( J_1^n \) is contained in the \( L_1 \)-module generated over \( L_1^1 \) by the columns of the \( n \times n \) matrix \((pI - F)\) and the \( n \times m \) matrix \( G \); that is,

\[
J_1^n \subseteq ((pI - F), G)_1 \tag{9}
\]

**Proof:** Suppose that (9) holds and the initial condition (7) is given. Then there exist \( \sigma \in (L_1)^m \) and \( \beta \in (L_1)^n \) such that

\[
(pI - F) * \beta - G * \sigma = x_0^\delta + (F * \sigma + G * u) |_{(0, \infty)} \in J_1^n.
\]

Therefore by (8) and Theorem 2, \( \beta \) is the solution of (6) with the given initial data and with input \( u \left|_{(0, \infty)} \right. = \sigma \). Hence the system is controllable.

Conversely, suppose that the system is controllable.

Define \( e_i = (0 0 \ldots 1 0 \ldots 0)^{TR} \in \mathbb{R}^n \) for \( i = 1, 2, \ldots, n \) in the \( i^{th} \) place.

By Theorem 2, \( e_i^\delta \) sets up the initial value \( x_0 = e_i \). Then since the system is controllable, by (8) and Theorem 2 there exist \( u_1 \in (L_1)^m \) and \( x_1 \in (L_1)^n \) such that

\[
(pI - F) * x_1 - G * u_1 = e_i^\delta, \quad i = 1, 2, \ldots, n.
\]

But \( e_0^\delta \), \( e_2^\delta \), \ldots, \( e_n^\delta \) is a basis of \( J_1^n \) as a \( J_1 \)-module, so that given \( \gamma \in J_1^n \), there exist \( \gamma_1, \gamma_2, \ldots, \gamma_n \in J_1 \) such that \( \gamma = \sum \gamma_i e_i^\delta \). Hence

\[
\gamma = (pI - F) * (\sum \gamma_i x_i^\delta) - G * (\sum \gamma_i u_i). \tag{10}
\]

Then since \( \sigma \in \mathbb{L}_1 \) for any \( \sigma \in J_1 \), \( f \in \mathbb{L}_1 \), it follows from (10) that the columns of \((pI - F)\) and \( G \) generate \( J_1^n \) over \( L_1^n \).

**Corollary:** The system (6) is controllable if and only if

\[
e_i^\delta \in M ((pI - F), G)_1, \quad i = 1, 2, \ldots, n
\]

where \( \epsilon_i = (0 0 \ldots 1 0 \ldots 0) \in \mathbb{R}^n \) in the \( i^{th} \) place.

In the finite-dimensional case where \( F \) and \( G \) are over \( \mathbb{R}^n \), it is well known (KALMAN [8]) that the system given by (6) is controllable if and only if the rank of the \( n \times n \) matrix \((G, FG, \ldots, F^{n-1}G)\) is equal to \( n \). If \( F \) and \( G \) are
viewed as matrices over $\mathbb{R}$ (rather than $\mathbb{R}_0$), this is equivalent to requiring that

$$\begin{align*}
(G, F, \ldots, F^{n-1} G) & = \mathbb{R}^n .
\end{align*}$$

(11)

Let $N$ be a fixed subring of $\mathbb{J}$ with $\xi_0 \in N$. In view of (11) it is reasonable to ask if the condition

$$\begin{align*}
(G, F, \ldots, F^{n-1} G) & = N^n
\end{align*}$$

(12)

is necessary and sufficient for controllability when $F$ and $G$ are over $N$. The answer is that (12) is sufficient but not necessary. The proof of sufficiency will be given in an expanded version of this paper. The simple example below shows that (12) is not necessary.

Example: Suppose that $\dot{x}(t) = x(t) + u(t - a), a > 0$. In this case $F = \frac{\partial}{\partial t}$ and $G = \frac{\partial}{\partial x}$ are over the subring $a$

$$\begin{align*}
R[\frac{\partial}{\partial t}] & = \left\{ \sum b_i \frac{\partial}{\partial t} \bigg| b_i \in \mathbb{R} \right\} \subset \mathbb{J}.
\end{align*}$$

Since the inverse of $\frac{\partial}{\partial t}$ is $\int_a^t \frac{\partial^2}{\partial s^2} \bigg|_{s=b} \bigg|_{s=t} \bigg|_{s=a} \bigg|_{s=t}$ for $N = R[\frac{\partial}{\partial t}]$, or $N = \mathbb{J}$.

However the system is controllable since there exist $\sigma, \theta \in \mathbb{J}$ such that $(p - \frac{\partial}{\partial t})^{\sigma} \circ \frac{\partial}{\partial x} \circ \frac{\partial}{\partial t} = \delta_0$.

For instance we could take

$$\begin{align*}
\sigma(t) = \begin{cases}
1, & 0 \leq t < a \\
\frac{a}{b-a}(t-a), & a \leq t < b \\
0, & \text{otherwise}
\end{cases}
\end{align*}$$

$$\begin{align*}
\theta(t) = \begin{cases}
\frac{a}{b-a}(t+a-b-1), & 0 \leq t < b-a \\
0, & \text{otherwise}
\end{cases}
\end{align*}$$

where $b$ is any fixed real number greater than $a$. The function $\sigma$ given above is a control for the initial condition $x(0) = 1$.

5. Further Applications

In addition to function controllability, there exist many problem areas that can be investigated using the convolution representation (6).

For example, the results of SONTAG [9] can be carried over to this framework giving a necessary and sufficient algebraic criterion for Euclidean reachability.

A particularly interesting topic is the study of feedback control systems with $u = K^* x$, $K = m \times n$ matrix over a subring $N$ of $\mathbb{J}$. Practical problems include the development of algebraic procedures for the design of $K^*$ to achieve stabilization or pole allocation. Results on this are already available in the case that $F$ and $G$ are over a subring $N$ that is a principal ideal domain (see MORSE [10] and SONTAG [9]).

References


