DISCONNECTED SOLUTIONS

by

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1. Introduction. In the book, Theory of Games and Economic Behavior (1944), J. von Neumann and O. Morgenstern introduced a theory of solutions (or stable sets) for multi-person cooperative games in characteristic function form. A longstanding conjecture has been that the union of all solutions of any particular game is a connected set. (E.g., see [3].) This announcement describes a twelve-person game for which this conjecture fails. The essential definitions for an n-person game will be reviewed briefly before the counterexample is presented. A sketch of the proof is presented here, and the details will appear elsewhere.

2. The Model. An n-person game is a pair \((N,v)\) where \(N = \{1,2,...,n\}\) is the set of players and \(v\) is a characteristic function on \(2^N\), i.e., \(v\) assigns the real number \(v(S)\) to each subset \(S\) of \(N\) and \(v(\emptyset) = 0\). The set of imputations is

\[
A = \{x: \sum_{i \in N} x_i = v(N) \text{ and } x_i \geq v(\{i\}) \text{ for all } i \in N\}
\]

where \(x = (x_1,x_2,...,x_n)\) is a vector with real components. For any \(S \subseteq N\), let \(x(S) = \sum_{i \in S} x_i\). For any \(X \subseteq A\) and nonempty \(S \subseteq N\), define \(\text{Dom}_S X\) to be the set of all \(x \in A\) such that there


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exists a \( y \in X \) with \( y_i > x_i \) for all \( i \in S \) and with \( y(S) \leq v(S) \).

Let \( \text{Dom} \, X = \bigcup_{\emptyset \neq S \subset N} \text{Dom}_S X \). A subset \( V \) of \( A \) is a solution if \( V \cap \text{Dom} \, V = \emptyset \) and \( V \cup \text{Dom} \, V = A \). The core of a game is

\[
C = \{ x \in A : x(S) \geq v(S) \text{ for all nonempty } S \subset N \}.
\]

For any solution \( V \), \( C \subset V \) and \( V \cap \text{Dom} \, C = \emptyset \).

A characteristic function \( v \) is superadditive if \( v(S \cup T) \geq v(S) + v(T) \) whenever \( S \cap T = \emptyset \). The game below does not have a superadditive \( v \) as is assumed in the classical theory, but it is equivalent solutionwise to a game with a superadditive \( v \). (See [1, p. 68].)

3. Example. The 13 vital coalitions for our example consist of \( N = \{1,2,3,4,5,6,7,8,9,10,11,12\} \) and elements from three classes:

\[
B = \{\{1,2\}, \{3,4\}, \{5,6\}, \{7,8\}, \{9,10\}, \{11,12\}\},
\]
\[
S = \{\{1,3,6,7,9,11\}, \{1,4,5,7,9,11\}, \{2,3,5,7,9,11\}\},
\]
\[
T = \{\{1,3,8\}, \{1,5,10\}, \{3,5,12\}\}.
\]

And \( v \) is given by: \( v(N) = 6 \), \( v(S) = 1 \) for all \( S \in B \), \( v(S) = 4 \) for all \( S \in S \), \( v(S) = 1 \) for all \( S \in T \), and \( v(S) = 0 \) for all other \( S \subset N \). For this game \( A = \{x : x(N) = 6 \text{ and } x_i \geq 0 \text{ for all } i \in N\} \). Consider also the six-dimensional hypercube

\[
B = \{ x \in A : x(S) = 1 \text{ for all } S \in B \}.
\]

The core \( C \) is the intersection of \( C(S) \) and \( C(T) \) where
\[ C(S) = \{ x \in B : x(S) \geq 4 \text{ for all } S \subseteq S \}, \]
\[ C(T) = \{ x \in B : x(S) \geq 1 \text{ for all } S \in T \}. \]

\( C \) is a proper superset of the convex hull of the six vertices of \( B \) which have \( x_i = 1 \) for \( i \) equal to five of the six odd indices 1, 3, 5, 7, 9 and 11, and \( x_{i+1} = 1 \) when \( i \) is the remaining odd numbered player. Let \( \text{Dom}_B X = \bigcup_{S \subseteq B} \text{Dom}_S X \). Note that \( \text{Dom}_B C \supset B - A \), and hence any solution \( V \) for our game is a subset of \( B \).

4. Outline of Proof. First, note that any component of an \( x \in B \) has a maximum value of \( x_i = 1 \). Consequently, the following three sets are contained in any solution \( V \), i.e., they are subsets of \( \cap V \):

\[ E = \{ x \in B : x_i = x_j = 1 \text{ for } i \neq j \text{ and } \{ i, j \} \subseteq \{1, 3, 5\} \}, \]
\[ F = \{ x \in C(T) : x_p = 1 \text{ for } p = 7, 9 \text{ or } 11 \}, \]
\[ P = \{(0,1,0,1,0,1,0,1,0,1,0)\}. \]

Next, we can show that \( U V \) must be a disconnected set. Let \( G = \{ x \in B : x((7,9,11)) < 1 \}, \) \( G^0 = \{ x \in B : x((7,9,11)) < 1 \}, \) and \( P' = \{ x \in G : x_2 = x_4 = x_6 = 1 \}. \) Throughout this section the indices \( i, j \) and \( k \) represent some ordering of the distinct indices 1, 3 and 5. The subset \( H \) of \( E \) consisting of the three triangular regions

\[ H_1 = \{ x \in G : x_{i+1} = x_j = x_k = 1; x_7 + x_9 + x_{11} = 1 \} \]

is in \( \cap V \) and \( \text{Dom}_B H \supset G^0 - (E \cup P') \). The subset \( J \) of \( F \) consisting of the three triangular regions
is also in \( \bigcap V \) and \( \text{Dom}_F J \supseteq B - C(T) \supseteq P' - P \). So any \( x \in U V - P \)
either has \( x \in E \) or \( x \in B - G^0 \), i.e., \( x_i = x_j = 1 \) or \( x((7,9,11)) > 1 \). Such \( x \) are clearly disconnected from the singleton \( P \subseteq \bigcap V \).

Finally, it is necessary to demonstrate that this game does
possess at least one solution. \( V' = C \cup E \cup F \cup U P \) is in any
solution \( V \), and \( V' \) can be enlarged to a solution in two steps.
First, include the set of imputations \( L \) in \( C(T) - (V' \cup \text{Dom} V') \)
which is simultaneously maximal with respect to all three of the
relations \( "\text{Dom}_S" \) for \( S \subseteq S \). Clearly \( L \subseteq \bigcap V \). Next, pick a
particular \( S^i = \{i+1,j,k,7,9,11\} \subseteq S \) and then add in those
elements \( L^i \) in \( C(T) - (V' \cup L \cup \text{Dom}(V' \cup L)) \) which are maximal
with respect to the relation \( "\text{Dom}_{S^i}" \) and are at the same time
symmetrical in the sense that \( x_j = x_k \). It requires some detail to
describe the sets \( L \) and \( L^i \) explicitly, and to verify that the
resulting sets \( V^i = V' \cup L \cup L^i \) are solutions for our example.
These will appear elsewhere.

5. Remarks. At one time it was apparently believed that
proving the union of all solutions connected could be a major step
in showing that every game has a solution. It is now known [2]
that a solution need not exist for every game. On the other hand,
it is possible that results on disconnecting \( U V \) might be useful
in the resolution of important open questions about whether solutions
always exist for games with full-dimensional cores, with empty cores, or which are constant-sum.

REFERENCES


This report describes a twelve-person cooperative game in characteristic function form (with side payments) for which the union of all solutions (stable sets) is a disconnected set. This disproves a longstanding conjecture in the classical theory of coalitional games.