Computation of the Incomplete Gamma Function Ratios

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COMPUTATION OF THE INCOMPLETE
GAMMA FUNCTION RATIOS

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A method, made up of several algorithms, is given for computing the incomplete gamma function ratio, $P(a, x)$ and its complement $Q(a, x)$ for all real arguments $a > 0$, $x > 0$. The difficult case when $a$ and $x$ are large is treated by a modified version of Takenaga's method. The resulting computer program is efficient, yields both $P$ and $Q$ correctly to within 1 unit in the twelfth significant digit, or, at the user's option, to within one unit in the sixth or third significant digit. A FORTRAN listing of the program is included.
FOREWORD

The work covered in this report was done in the Science and Mathematics Research Group of the Warfare Analysis Department at the request of Dr. Marlin A. Thomas, Head of the Mathematical Statistics and Systems Simulation Branch.

The authors are indebted to Alfred H. Morris for carrying out a number of calculations for them using his algebraic routine, "Flap." The authors are also indebted to Michael P. Saizan for computing some of the coefficients given in Appendix C.

The date of completion was 30 January 1976.

Released by:

RALPH A. NIEMANN
Head, Warfare Analysis Department
ABSTRACT

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## DISTRIBUTION LIST
1. INTRODUCTION

The object of this report is to describe an efficient procedure for computing the incomplete gamma function ratio

\[ P(a, x) = \frac{\gamma(a, x)}{\Gamma(a)}, \quad a > 0, \quad x > 0, \quad (1) \]

and its complement

\[ Q(a, x) = \frac{\Gamma(a, x)}{\Gamma(a)}, \quad a > 0, \quad x > 0, \quad (2) \]

to within a unit in twelve, six, or three significant digits, where

\[ \gamma(a, x) = \int_0^x e^{-t} t^{a-1} \, dt, \quad (3) \]

\[ \Gamma(a, x) = \int_x^\infty e^{-t} t^{a-1} \, dt, \quad (4) \]

\[ \gamma(a, \infty) = \Gamma(a, 0) = \Gamma(a) = \int_0^\infty e^{-t} t^{a-1} \, dt \quad (5) \]

with

\[ 0 < P < 1, \quad 0 < Q = 1 \quad P < 1. \]

The computing program, in FORTRAN, designed for the CDC-6700*, uses 7 different mathematical formulations for \( P \) and \( Q \), the asymptotic expansion \( (a \to \infty) \) for the complete gamma function, \( \Gamma(a) \), and a series expansion for its reciprocal. They are all given in the next section with their domains of application. The algorithm used with each formulation is given in Section 5.

If the user desires 12 significant digits for \( P, Q, \) or both, a parameter \( \epsilon \) is set in the call to the program, to the value

\[ \epsilon = \epsilon_1 = 5 \times 10^{-13} = 5(13) \]

*The CDC-6700 is a large scale binary computer capable of one million arithmetic operations per second. It has a 60 bit binary word length of which 48 are used to express the mantissa of a number.
If, on the other hand, the user wants only 6 or 3 significant figures, then \( \epsilon \) is set, respectively, to

\[
\epsilon = \epsilon_2 = 5 \times 10^{-7} = 5(-7). \quad \epsilon = \epsilon_3 = 5 \times 10^{-4}.
\]  

(7)

Besides three available levels of accuracy for the output quantities \( P \) and \( Q \), the program also has the capability to determine, in most cases, if the input quantities \( a \) and \( x \) are contained in “zones of non-computation.” These are domains in the \( ax \)-plane for which \( P \leq \epsilon \) or \( Q \leq \epsilon \); for such \((a, x)\), \( P \) is set to zero and \( Q \) is set to one or vice versa depending on the values of \((a, x)\). Section 3 deals with the ways in which the program senses on the above inequalities.

The average computing time per case, when \( \epsilon = \epsilon_2 \) (6 significant digit accuracy in \( P \) and \( Q \)) is about 1 millisecond; for \( \epsilon = \epsilon_1 \) (12 significant digits in \( P \) and \( Q \)) the average computer time per case is about 15 to 20\% longer and for \( \epsilon = \epsilon_3 \) about 10 to 15\% shorter.

The validation of the program is described in Section 5. A complete FORTRAN IV listing is given in Appendix I.

There are many applications for the incomplete gamma function ratios such as computing the Poisson distribution, [1, p. 959]. Perhaps their most well-known use is for computing the Chi-square distributions \( P(\chi^2|\nu) \) and \( Q(\chi^2|\nu) = 1 - P(\chi^2|\nu) \), where

\[
\begin{align*}
P(\chi^2|\nu) &= \left[ 2^{\nu/2} \Gamma(\nu/2) \right]^{-1} \int_{0}^{\chi^2} e^{-u/2} u^{\nu/2 - 1} du. \\
\end{align*}
\]  

(8)

By letting \( u = 2t \) in (8), the relationship between \((\chi^2, \nu)\) and \((x, a)\) becomes evident, i.e.,

\[
\begin{align*}
P(\chi^2|\nu) &= \int_{0}^{\chi^2} e^{-t/2} t^{(\nu/2) - 1} \Gamma(\nu/2) \Gamma(1/2) = P(t/2, \chi^2/2).
\end{align*}
\]  

(9)

*More precisely, one should state that in the case of (6), \( P \) and \( Q \) are given correctly to within one unit in the twelfth significant digit, and to within one unit in the sixth or third significant digit in the case of (7).
so that

\[ a = \nu/2, \quad x = \chi^2/2. \]

The extensive use of the Chi-square distributions in statistics is sufficient to warrant a good machine program for their equivalent, the incomplete gamma function ratios. Surprisingly, to the best of our knowledge, there is only one computer program documented in the literature, [11]. In any case, the authors are not aware of a program for \( P \) and \( Q \) that has the overall speed, versatility, accuracy, and range of the present program.
2. MATHEMATICAL FORMULATIONS AND DOMAINS OF APPLICATION

In this section we summarize the mathematical formulations and state the domains on which they are used.

It is worth noting, since we require relative accuracy, that it is necessary to compute \( Q(P) \) if \( P(Q) \) is near one (and then find \( P(Q) \) from \( 1 - Q(P) \)). Consequently, the program is designed to compute \( Q(P) \) if \( a \geq 1 \) and \( a \geq x \), because in this case \( P \) cannot be near one. Similarly, if \( 1 < a < x, x > \ln 10 \), then \( Q \) is computed for it cannot be near one. If, however, \( a < 1 \) and \( x \leq a \) then \( P \) may be near one and \( Q \) should be computed, although if \( x \) is sufficiently small \( P \) will, of course, be near zero and should be computed. This behavior of \( P \) near the origin of the ax-plane results from the nonexistence of \( \lim_{x \to 0} P(a, x) \), a fact easily concluded from (10) below.

\[
x \to 0, \quad a \to 0
\]

Figure 1 shows the domains over which the various formulations are used. It and the flowchart on page 44 complement the remarks that follow. It is assumed \( P > \epsilon \) and \( Q > \epsilon \).

We proceed to specify the formulations and their domains of application.

\[
p = \frac{R(t, x)}{a} \left[ 1 + \sum_{k=1}^{x} \frac{x^k}{(a+1) \cdots (a+k)} \right], \quad [1, p. 263]
\]

(10)

\[
R(t, x) = e^{-x^t / (t/2)}
\]

(11)

\[
a < 100
\]

1) \( 1 < a, a > x \).

2) \( 2a \) is not an integer when \( x > a \)

a) \( 1 \leq a < x < \ln 10 \)

b) \( a < 1, \quad x > 1.5, \quad P < 0.90 \)

c) \( a < 1, \quad x > 1.5, \quad R > 0.101 \frac{x(t + 2 - a)/(x + 1)}{x + 1} \)

\[
a \geq 100, \quad 3a > 4x
\]

4
\[ Q(a, x) = \frac{R(a, x)}{x} \left[ 1 + \sum_{k=1}^{N} \frac{(a-1) \cdots (a-k)}{x^k} \right]. \quad [1, \text{p. 263}] \tag{12} \]

\[
\begin{align*}
a < 100 \\
1 < a < x, \quad x > 31 (e = \epsilon_1), \quad x > 17 (e = \epsilon_2), \quad x > 9.7 (e = \epsilon_3) \tag{1*}
\end{align*}
\]

\[
\begin{align*}
a \geq 100 \\
4x > 5a
\end{align*}
\]

\[ Q(k + g, x) = \frac{Q(k + g, l, x) + R(k + g, l, x)}{(k + g - l)}. \quad [1, \text{p. 262}] \tag{13} \]

\[
\begin{align*}
g &= \begin{cases} 
1 & \text{a is an integer} \\
1/2 & \text{a is not an integer; } k = 1, 2, \ldots, a - g.
\end{cases}
\end{align*}
\]

\[ Q(g, x) = \begin{cases} 
\left( \frac{2}{\sqrt{\pi}} \right) \int_{g}^{\infty} e^{-u^2} \, du = \text{erfc}(\sqrt{x}), & g = 1/2 \\
1 - e^{-g^2}, & g = 1.
\end{cases} \tag{14} \]

\[
\begin{align*}
a < 100 \\
2a \text{ is an integer, } a \leq x < 31, 17, 9.7 \text{ for } e = \epsilon_1, \epsilon_2, \epsilon_3, \text{ respectively.}
\end{align*}
\]

\[ Q(a, x) = R(a, x) \left[ \frac{1}{x + \frac{1}{a}} \right. \left. + \frac{1}{x + \frac{2}{a}} \right. \left. + \frac{2}{x + \frac{3}{a}} \right. \left. + \cdots \right] \quad [10, \text{p. 356}] \tag{15} \]

\[
\begin{align*}
a < 100 \\
1. \quad 2a \text{ is not an integer} \\
a) \quad 1 < a < x, \text{ in } 10 < x < 31, 17, 9.7 \text{ for } e = \epsilon_1, \epsilon_2, \epsilon_3, \text{ respectively.} \\
b) \quad a < 1, x > 1.5, R \leq 0.101 x(x + 2 - a)/(x + 1)
\end{align*}
\]

*See (6) and (7) for values of \(\epsilon_1, \epsilon_2, \epsilon_3\).
\[ Q(a, x) = J - (H + L) + J(H + L) - HL + JHL \]  

(16)

\[ J = -a \sum_{k=1}^{\infty} \frac{(-x)^k}{(a + k) k!} \]

\[ L = \sum_{k=1}^{\infty} \frac{(a \ln x)^k}{k!} \]

\[ H = \left\{ \begin{array}{ll}
\sum_{k=2}^{\infty} C_k a^{k-1} & \quad 0 < a < 1/2 \\
(1/a) \left| (1 - a) + \sum_{k=2}^{\infty} C_k (a - 1)^{k-1} \right| & \quad 1/2 < a < 1.
\end{array} \right. \]

The coefficients \( C_k \) are given in Appendix A, also see (25). Discussion of (16) given below and on pages 10 and 11.

\[ a < 100 \]

\[ a < 1, \quad x < 1.5, \quad P > 0.90. \]

The need for (16) arises when \( a < 1 \), since then it is not true, even though \( x < a \), that \( P \) cannot be near one. In fact, for \( a = x = 0.038 \), \( P = 0.9997 \cdots \), and for \( a = 10^{-3}, x = 10^{-4} \), \( P = 0.9997 \cdots \). Thus, in such cases it is \( Q \), rather than \( P \), which must be computed. This behavior of \( P \) near the origin of the \( ax \)-plane, as mentioned previously, can be accounted for by noting from (10) that the double limit for \( P \) as \( x \to 0, a \to 0 \) does not exist.

If \( a > 100 \) and \( 3a < 4x < 5a \), none of the above formulations suffice for computing purposes. (If \( 4x < 3a \), (10) is used and if \( 4x > 5a \) then (12) is used.) In this main area of difficulty, we have found that Takenaga's method [9], with appropriate changes, works quite well. The final formulas are given by (17), (24). The mathematical formalism is easily programmed into an accurate and relatively efficient procedure for computing \( P(a, x) \) when \( x < a - 1/3 \), or \( Q(a, x) \) when \( x > a - 1/3 \). The details for deriving (17) (24) are given in Section 4.
Let

\[ T(a, x) = \begin{cases} 
    P(a, x) & \text{if } x < a - 1/3 \\
    Q(a, x) & \text{if } x > a - 1/3.
\end{cases} \]

Then Takenaga's analysis, after some changes, reduces to

\[ T(a, x) \equiv C \left[ B_0 + \frac{1}{12a^2} B_2 - \frac{1}{18a^4} \left( B_3 - \frac{1}{16} B_4 \right) \right. \]

\[ \left. - \frac{1}{24a^6} \left( B_4 - \frac{37}{225} B_5 + \frac{1}{432} B_6 \right) \right. \]

\[ \left. + \frac{1}{30a^8} \left( B_5 - \frac{743}{3024} B_6 + \frac{49}{4320} B_7 + \frac{5}{82944} B_8 \right) \right. \]

\[ \left. \cdots \right. \]

\[ \left. + \frac{1}{a^{24}} \left( \frac{1}{78} B_{13} - \cdots - 2.34146722651062 \times 10^{-22} B_{24} \right) \right. \]

\[ + \frac{A_s}{15a^3} + \frac{i}{3a^5} \left( \frac{A_s}{7} - \frac{A_s}{60} \right) + \frac{1}{27a^7} \left( A_s - \frac{29}{140} A_s + \frac{1}{160} A_s \right) \]

\[ + \frac{1}{a^9} \left( \frac{1}{33} A_s + \frac{193}{22680} A_s + \frac{1187}{2268000} A_s - \frac{i}{155520} A_s \right) \]

\[ \cdots \]

\[ + \frac{1}{a^{25}} \left( \frac{1}{81} A_{13} - \cdots - 2.24780853745020 \times 10^{-21} A_{24} \right) \]

*The numerical coefficients in (17) and (18) were computed by Alfred H. Morris using his "Flap" algebraic routine. [5]. The complete computed sets of coefficients are given in Appendix B.*
where, with \( b = a - 1 \)

\[
C = 1 + \frac{31}{2592} b^{-2} + \frac{3413}{1399680} b^{-3} + \frac{361733}{201553920} b^{-4}
\]

\[
= \frac{113888281}{50791587840} b^{-5} + \frac{7565202533}{7836416409600} b^{-6} - \ldots 
\]

\[
a = 3(a - 1/3)^{1/2}
\]

The \( A_k \) and \( B_k \), for a given integer \( k \), have different meanings depending on whether \( x < a - 1/3 \) or \( x > a - 1/3 \). If \( x < a - 1/3 \), then

\[
A_k = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^{2k+1} e^{-z^2/2} \, dz
\]

\[
B_k = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^{2k} e^{-z^2/2} \, dz, \quad k = 0, 1, \ldots
\]

where

\[
s = a \left( \left( \frac{x}{a - 1/3} \right)^{1/3} - 1 \right) \leq 0.
\]

If \( x > a - 1/3 \), then (20) and (21) are replaced by

\[
A_k = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} z^{2k+1} e^{-z^2/2} \, dz
\]

\[
B_k = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} z^{2k} e^{-z^2/2} \, dz, \quad k = 0, 1, 2, \ldots
\]

In this case also, \( s \) is given by (22), but now it is positive. We remark that the program uses a slightly different form for (17). See (115).

The evaluation of \( R(a, x) \) (see (11)), which contains \( 1/\Gamma(a) \), is carried out in two different ways. If \( a < 30 \), then the factor \( x^a e^{-x} \) is computed in the form \( e^{-x + a} x^a \), and subsequently \( \Gamma(a) \) is evaluated as a separate factor by the recurrence relation

\[
\Gamma(j + \lambda) = (j + \lambda - 1) \Gamma(j + \lambda - 1), \quad j = 1, 2, \ldots, a - \lambda, \quad \lambda \neq 0,
\]
where \(\lambda\) denotes the fractional part of \(a\) and \(a > 1\). The procedure is initiated by computing \(\Gamma'(\lambda)\) from a series expansion for its reciprocal. The quantity \(1/\Gamma'(\lambda)\) is computed to 14 significant digits by the polynomial approximations, [13],

\[
1/\Gamma'(\lambda) = \begin{cases} 
\lambda \left[ 1 + \sum_{k=2}^{17} C_k \lambda^{k-1} \right], & 0 < \lambda < 1/2 \\
1 + \sum_{k=2}^{17} C_k (\lambda + 1)^{k-1} & 1/2 < \lambda \leq 1 
\end{cases}
\] (25)

the \(C_k\)'s are given in Appendix A. By using the first equation of (25) and \(\Gamma'(\lambda) = \Gamma'(1+y) = y \Gamma'(y)\) where \(\lambda = 1+y\), so that \(1/2 < \lambda < 1\), \(-1/2 < y < 0\), the second equation of (25) is obtained. If \(\lambda = 0\), then the above recurrence relation is started with \(j = 2\) (instead of one) and \(\Gamma'(1) = 1\).

If \(30 < a\), then the asymptotic expansion for \(\ln \Gamma(a)\), \((a \rightarrow \infty)\), [1, p. 257],

\[
\ln \Gamma(a) \approx (a - 1/2) \ln a + a + \frac{1}{2} \ln 2\pi + L(a),
\] (26)

where

\[
L(a) = \frac{1}{12a} - \frac{1}{360a^3} + \frac{1}{1260a^5} - \frac{1}{1680a^7} + \left( -\frac{\theta}{12}, 0 < \theta < 1 \right)
\] (27)

is combined with the logarithm of the numerator of (11) to give, after exponentiation,

\[
R(a, x) \equiv \frac{1}{\sqrt{2\pi}} \exp \left[ (a - x) + a \ln \frac{x}{a} - L(a) \right]
\] (28)

It is easy to show that the argument of the exponential in (28) is always negative for admissible values of \(a\) and \(x\). Hence no scaling problems can occur due to the large magnitude of \(a\) and \(x\).

We do not give detailed derivations of the above equations except for (17) – (24) which are developed in Section 4. Equations (10), (12), and (13) are easily obtained using integration by parts on (1) or (2); (14) follows directly from (2) when \(a = 1/2\) and \(t = u^2\); (15) is derived in [10]. Equation (16) follows by substituting
\[ P = \left| \frac{x^n}{\Gamma(n+1)} \right| \left| 1 + a \sum_{k=1}^{\infty} \frac{(-x)^k}{(a+k)k!} \right| \quad (\text{See [1, p. 262]}) \]  

(29)

\[ x^n = 1 + \sum_{k=1}^{\infty} \frac{(a \ln x)^k}{k!} \]

(30)

\[ \frac{1}{\Gamma'(a+1)} = \begin{cases} 
1 + \sum_{k=2}^{\infty} C_k a^{k-1}, & 0 < a < 1/2, \quad (\text{See (25)}) \\
\frac{1}{a} \left| 1 + \sum_{k=2}^{\infty} C_k (a-1)^k \right|, & 1/2 < a < 1. 
\end{cases} \]

(31)

in the expression \( 1 - P (= Q) \). The derivation of the series expansion for \( 1/\Gamma'(a) \) is sketched in [13]. The asymptotic expansion for \( 1/\Gamma'(a) \) is discussed in [6, p. 293].

Table 1 has been compiled to give some idea of the number of terms of (10), (12), or (15) that are needed for various arguments \((a, x)\) with \( \epsilon = \epsilon_1 \) and \( \epsilon = \epsilon_2 \). The third column of the table identifies the equation that was used by its number in the text. The fourth and fifth columns list the number of terms of that equation needed for \( \epsilon = \epsilon_1 \) and \( \epsilon = \epsilon_2 \) respectively. For example, \( a = 7.1, \ x = 28 \) uses 12 terms of (15) for \( \epsilon_1 \) and 7 terms of (12) for \( \epsilon_2 \). No examples are given for (13) since the number of iterations is always the integer part of \( a \). Also no data is given for (16). Since (16) is used when \( a < 1 \) and \( x < 1.5 \), it yields an efficient algorithm. In the worse case, when \( x > 1.5 \) and \( \epsilon = \epsilon_1 \), no more than 18 terms of \( J \) are needed: the evaluation of \( L \) requires less than 12 terms and in the worse case of \( a = 1/2 \), no more than 17 terms of \( H \) are used.

In the next section we show how, for small \( \epsilon > 0 \), advantage is taken of a property of \( P \) and \( Q \). Namely, if \( a \) is large, then \( x \) cannot be too far from \( a \), otherwise \( P < \epsilon \) or \( 1 - P = Q < \epsilon \) depending on whether \( a > x \), or \( x > a \) respectively. In other words, there is a very limited region of the \( a-x \)-plane for which \( \epsilon < P < 1 - \epsilon \) and for which extensive calculations, using one of the above formulations, is necessary. Inequalities will be derived which, in the case of equality, give very close approximations to the boundaries of this region. Earlier, we referred to the exterior and boundary of this region as the "zone of non-computation." The computing time for any one of the inequalities is relatively short. As a result, the average computing time per case is markedly reduced, since the average computation time for computing \( P \) and \( Q \), when \( (a, x) \) is in the "zone of computation" \( (\epsilon < P < 1 - \epsilon) \) is generally much longer than when it is not.
Table 1. Number of Terms used by Subprograms for Some (a, x)

<table>
<thead>
<tr>
<th>a</th>
<th>x</th>
<th>Eq.</th>
<th>No. of Terms $\epsilon_1$</th>
<th>No. of Terms $\epsilon_2$</th>
<th>a</th>
<th>x</th>
<th>Eq.</th>
<th>No. of Terms $\epsilon_1$</th>
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</table>

*Indicates (a, x) not in "zone of computation" (see Section 3).
3. ZONE OF COMPUTATION

The function \( P(a, x) \) (Q(a, x)) has the property that for a given \( \epsilon > 0 \), there exists a function \( \tilde{f}(a) \) such that \( P < \epsilon \) (Q < \epsilon) for all \( x < f(a) \). Of course \( f \) and \( \tilde{f} \) are also functions of \( \epsilon \). Figure 1 shows the curves \( f(a) \) and \( \tilde{f}(a) \) on two scales for \( \epsilon = \epsilon_1 \) and \( \epsilon = \epsilon_2 \). Referring to the figure, \( P \) or \( Q \) is computed using \( 10, 28 \) if \( a, x \) is contained between \( f \) and \( \tilde{f} \). We call this region the "zone of computation." If \( \epsilon \) is asserted to be outside this region by the estimates given below for \( f \) and \( \tilde{f} \), then \( P \) is set to zero or one and \( Q \) to one or zero. As can be seen from Figure 2, the "zone of computation" makes up only a small part of the plane region \( x > 0, a > 0 \). Therefore, it is advantageous to obtain close estimates for \( f \) and \( \tilde{f} \) which can be used efficiently by the program.

Two different approaches are used to find two different sets of estimates for \( f \) and \( \tilde{f} \). Associated inequalities are given which can be easily incorporated into the program to effectively decide if \( a, x \) is not between \( f \) and \( \tilde{f} \).

Let the first estimates for \( f(a) \) and \( \tilde{f}(a) \) be denoted by \( f_1 \) and \( \tilde{f}_1 \), respectively. The expression for \( \tilde{f}_1 \) is derived in [121]. It is given by

\[
\tilde{f}_1 = a \left( 1 - \frac{1}{\sqrt{\sigma_0 a}} + \sqrt{\frac{\sigma_0 a}{2}} \right), \quad a > 15,
\]

where \( \sigma \) satisfies the equation

\[
\text{erfc}(\sqrt{\sigma}) = 2 \epsilon.
\]

The constant \( \sigma \) is determined once \( \epsilon \) is specified. For example, when \( \epsilon = \epsilon_1, \sigma = 7.1306 \), when \( \epsilon = \epsilon_2, \sigma = 4.892 \), when \( \epsilon = \epsilon_3, \sigma = 3.29053 \). A set of correction factors to improve (32) has been given in [8], and they show that \( \tilde{f}_1 > \tilde{f} \). Therefore,

\[
Q(a, x) < \epsilon \quad \text{for all } x > \tilde{f}_1 > \tilde{f}.
\]

Similarly

\[
\tilde{f}_1 = a \left( 1 - \frac{1}{\sqrt{\sigma_0 a}} + \sqrt{\frac{\sigma_0 a}{2}} \right), \quad a > 15,
\]

where again \( \sigma \) can be established from [8] that \( \tilde{f}_1 < \tilde{f} \). Therefore,

\[
P(a, x) < \epsilon \quad \text{for all } x < \tilde{f}_1 < \tilde{f}.
\]
In order to use these results in the program, new functions \( \overline{F} \) and \( \underline{F} \) are defined. Using the fact that \( Q \) is a decreasing function of \( x \) and an increasing function of \( a \), we conclude from (32), for given \((a, x)\), that if

\[
\overline{F} = x - a \left( 1 - \frac{1}{q_0 a} + \frac{1}{a} \right) > 0, \quad a > 15
\]

then \( Q < \epsilon \). In the same way, we conclude from (33), that if

\[
\underline{F} = x - a \left( 1 - \frac{1}{q_0 a} - \frac{1}{a} \right) < 0, \quad a > 15
\]

then \( P < \epsilon \). Clearly the quantities \( \overline{F} \) and \( \underline{F} \) are cheap to compute, requiring only a square root and a few arithmetic operations. However, they are only useful if \( a \) is not small \((a > 15)\).

The other estimates for \( x \) and \( x \) are defined by \( \overline{x} \) and \( \underline{x} \), respectively. We first show for fixed \( a \) and \( \epsilon \), that there exists a unique positive value of \( x \), say \( x_2 \), such that

\[
R(a, x_2) = a \left( \frac{x_2}{(a+1)} \right) < \epsilon, \quad x_2 < a. \quad \text{(36)}
\]

and

\[
Q(a, x) < \epsilon \quad \text{for all } x < x_2 < a. \quad \text{(37)}
\]

Then we show that if \( \overline{x}_2 \) is determined (uniquely) from

\[
R(a, \overline{x}_2) = \overline{x}_2 \left( \frac{\overline{x}_2 + 2}{(a+1)} \right) < \epsilon, \quad 0 < a < 1, \quad \text{or } a > 2. \quad \text{(38)}
\]

then

\[
Q(a, x) < \epsilon \quad \text{for all } x > \overline{x}_2. \quad \text{(39)}
\]

For \( 1 < a < 2 \), it can be shown \( Q(a, x) > \epsilon \) for all \( x < \overline{x}_2 \), therefore, we replace (38) by

\[
R(a, \overline{x}_2) = (\overline{x}_2 + 1 - a) \epsilon, \quad 1 < a < 2, \quad \overline{x}_2 > a \quad \text{(40)}
\]

and show that (39) holds.
Let, for a given value of \(a\) and \(\epsilon\),

\[
f(x) = R(a, x) - a \left[ 1 - x/(a + 1) \right] \epsilon, \quad x < a.
\]

Then for small \(\epsilon > 0\), there exists a unique positive value of \(x\) which we identify as \(x_2\) such that \(f(x_2) = 0\), i.e., such that (36) is satisfied. Indeed, \(f(0) < 0\), \(f'(x) > 0\) for \(x < a\), and \(f(a) > 0\) for small \(\epsilon > 0\). The last inequality on \(f\) follows from

\[
f(a) = \frac{e^{-a} a^\epsilon}{\Gamma(a) - \frac{a}{a + 1} \epsilon = a \left| \frac{e^{-a} a^\epsilon}{\Gamma(a) + 1} - \frac{a}{a + 1} \right|}.
\]

If \(a \leq 1\), then \(\Gamma(a + 1) \leq 1\), and

\[
f(a) > a \left[ e^{-a} (a - \epsilon) > a \left[ e^{-1} - \epsilon \right] > 0, \quad (\epsilon < 1/e \sim 0.368).\right.
\]

If \(a > 1\), then using (26)

\[
f(a) > \frac{e^{-a} a^\epsilon}{e^{-a} a^\epsilon} \sqrt{2\pi/a} e^{a/2} > \frac{a}{2\pi} e^{a/2} - \epsilon
\]

where the inequality follows by noting that an asymptotic series with decreasing successive terms of alternating sign yields an approximation with an error less than the first neglected term and of the same sign. The right hand side of (41) for \(a > 1\) takes its minimum value at \(a = 1\), hence,

\[
f(a) > \frac{1}{\sqrt{2\pi}} e^{-1/12} - \epsilon > 0, \quad (\epsilon < 0.367).
\]

Now using (10)

\[
P(a, x) = \frac{R(a, x)}{a} \left| 1 + \sum_{k=1}^\infty \frac{x^k}{(a + 1) \cdots (a + k)} \right| \quad (42)
\]

\[
< \frac{R(a, x)}{a} \left| 1 + \sum_{k=1}^\infty \left( \frac{x}{a + 1} \right)^k \right|
\]

\[
= R(a, x) \left| a \left( 1 - \frac{x}{a + 1} \right) \right| = \xi(a, x)
\]
Consequently, since $\xi$ is an increasing function of $x$ when $x < a$, (37) must hold, i.e.,

$$P(a, x) < \xi(x, a) \leq \xi(\bar{x}_2, a) = \epsilon, \quad x < \bar{x}_2 \leq a. \quad (43)$$

The following argument for the existence and uniqueness of $\bar{x}_2$ in (40) depends on the function

$$g(x) = R(a, x) (x + 1 - a) \epsilon, \quad x > a, \quad a > 1$$

where $a$ and $\epsilon$ are fixed. The proof is similar to the one for (36). Clearly $g(\infty) = -\infty$, and it is easy to show $g'(x) < 0$ for $x > a$. The proof will be complete if it can be proved that $g(a) > 0$. We have, using (26) again with $a > 1$, that

$$g(a) = R(a, a) / (x + 1 - a) > \epsilon, \quad a > 1 > 0.$$

The expression on the right also appeared in (41). Therefore, $g(a) > 0$ provided

$$\left(1/v_\theta\right) e^{-1/2} > \epsilon, \quad \text{i.e., for } \epsilon < 0.367.$$

We proceed to show that (39) holds for (40). From (2), with $a = 1 < x$,

$$Q(a, x) = \frac{1}{\Gamma(a)} \int_x^\infty \exp \left[-t + (a - 1) \ln t\right] dt, \quad a > 1 \quad (44)$$

$$\leq \frac{1}{\Gamma(a)} \int_x^\infty \left[\frac{1 + (a - 1)/t}{1 + (a - 1)/x}\right] \exp \left[-t + (a - 1) \ln t\right] dt$$

$$= \frac{e^{-x} x^a}{\Gamma(a)(x + 1 - a)} = \frac{R(a, x)}{(x + 1 - a)}, \quad x > a - 1 \geq 0.$$

It is easy to show, by differentiation, that $R(a, x)/(x + 1 - a)$ is a decreasing function of $x$ for $x > a - 1 \geq 0$. Hence,

$$Q(a, x) \leq R(a, x)/(x + 1 - a) \leq R(a, \bar{x}_2)/(\bar{x}_2 + 1 - a) = \epsilon, \quad 0 < a < \bar{x}_2 \leq x. \quad (45)$$

Next, we establish the existence and uniqueness of $\bar{x}_2$ in (38). Consider the function

$$h(x) = [(x + 1) R/(x(x + 2 - a))] - \epsilon, \quad 0 < a < 1, \quad \text{or } a > 2 \text{ and } x > (a - 2).$$
so that

\[ h'(x) = \frac{[R/x(x + 2 - a)]}{x} + \frac{a}{x} - (x + 1)(x + 3 - a)/(x + 2 - a) \]

\[ = \frac{(R/x)}{1 + \frac{(1 - a)(2 - a)}{x(x + 2 - a)^2}} \]

Now if \( 0 < a < 1 \), then \( h(0 +) = \infty \), \( h(\infty) = -\epsilon \) and \( h'(x) < 0 \). Hence, there exists a unique positive value of \( x \), say \( \bar{x}_2 \) such that \( h(\bar{x}_2) = 0 \). Clearly this implies (38) is satisfied by \( \bar{x}_2 \) since \( x + 2 - a > 0 \).

If \( a > 2 \), then \( h(a - 2 +) = \infty \), \( h(\infty) = -\epsilon \) and \( h'(x) < 0 \). Hence, by the same arguments as above, there exists a unique positive value of \( x \), say \( \bar{x}_2 \) that satisfies (38).

For \( 0 < a < 1 \), it is shown in [41] that \( Q \leq h(x) + \epsilon \).

This is easily established here by considering

\[ M(x) = Q(a, x) + (h(x) + \epsilon) \]

and

\[ M'(x) = (1 - a)(2 - a) \frac{R(a, x)}{x(x + 2 - a)^2} \]

Then if \( 0 < a < 1 \), \( M(0 +) = -\infty \), \( M(\infty) = 0 \), \( M'(x) > 0 \). Consequently \( M(x) < 0 \) for \( 0 < a < 1 \). Similarly for \( a > 2 \), \( x > a - 2 \), \( M(a - 2 +) = -\infty \), \( M(\infty) = 0 \), \( M'(x) > 0 \) so that \( M(x) < 0 \) for \( a > 2 \), \( x > a - 2 \) (note also that \( M(x) = 0 \) for \( a = 1 \) and \( a = 2 \)). Finally, since \( h'(x) < 0 \), \( h(x) + \epsilon \) is a decreasing function of \( x \), so that for all \( x > \bar{x}_2 \),

\[ Q(a, x) \leq (x + 1) R/[x(x + 2 - a)] \leq (\bar{x}_2 + 1) R(\bar{x}_2, a)/[\bar{x}_2(\bar{x}_2 + 2 - a)] = \epsilon \quad (46) \]

This establishes (39) for \( \bar{x}_2 \) satisfying (38) and includes the values \( a = 1 \), \( a = 2 \). Incidentally, \( h(x) + \epsilon \) can be obtained as an approximation for \( Q \) from (98) with \( a = 1 \).

In [6, p. 70], the first inequality in (45) is derived in a different way. The inequalities (34) and (35) can be sharpened [8]. The same can be said for (43), (45), (3), and (46), [4]. We did not feel that sharper estimates at the expense of more computing time were justified since the above estimates were already very good as can be seen by looking at Table 2. The table also shows that (34) and (35) are better than (43) and (46) for extremely large \( a \).
Table 2. Comparison of $x_i(a), \bar{x}_i(a)$ with $x(a), \bar{x}(a), i = 1, 2$

<table>
<thead>
<tr>
<th>$x_2$</th>
<th>$x_1$</th>
<th>$x$</th>
<th>$a$</th>
<th>$P(a, x_2)$</th>
<th>$P(a, x_1)$</th>
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<td>15.010</td>
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<td>4.43 (-7)</td>
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<tr>
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<td>153.15</td>
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<tr>
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<tr>
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Table 2. Comparison of $x_i(a)$, $\tilde{x}_i(a)$ with $x(a)$, $\tilde{x}(a)$, $i = 1, 2$. (Continued)

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<th>$\tilde{x}$</th>
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<th>$Q(a, \tilde{x}_1)$</th>
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<td>0.3075</td>
<td>4.99 (-13)</td>
<td>3.10 (-18)</td>
<td>5 (-13) = $\epsilon_1$</td>
</tr>
<tr>
<td>50</td>
<td>51.67</td>
<td>49.999</td>
<td>9.46</td>
<td>4.99 (-13)</td>
<td>1.23 (-13)</td>
<td>5 (-13) = $\epsilon_1$</td>
</tr>
<tr>
<td>74.994</td>
<td>75.95</td>
<td>74.991</td>
<td>22.85</td>
<td>4.99 (-13)</td>
<td>1.39 (-13)</td>
<td>5 (-13) = $\epsilon_1$</td>
</tr>
<tr>
<td>99.985</td>
<td>100.68</td>
<td>99.980</td>
<td>38.15</td>
<td>4.98 (-13)</td>
<td>3.20 (-13)</td>
<td>5 (-13) = $\epsilon_1$</td>
</tr>
<tr>
<td>249.99</td>
<td>250.29</td>
<td>249.970</td>
<td>146.42</td>
<td>4.96 (-13)</td>
<td>4.36 (-13)</td>
<td>5 (-13) = $\epsilon_1$</td>
</tr>
<tr>
<td>499.985</td>
<td>500.15</td>
<td>499.951</td>
<td>349.58</td>
<td>4.95 (-13)</td>
<td>4.70 (-13)</td>
<td>5 (-13) = $\epsilon_1$</td>
</tr>
<tr>
<td>999.98</td>
<td>1000.06</td>
<td>999.924</td>
<td>783.43</td>
<td>4.94 (-13)</td>
<td>4.85 (-13)</td>
<td>5 (-13) = $\epsilon_1$</td>
</tr>
<tr>
<td>4999.99</td>
<td>4999.92</td>
<td>4999.831</td>
<td>4504.52</td>
<td>4.92 (-13)</td>
<td>4.96 (-13)</td>
<td>5 (-13) = $\epsilon_1$</td>
</tr>
<tr>
<td>9999.99</td>
<td>9999.83</td>
<td>9999.751</td>
<td>9295.57</td>
<td>4.91 (-13)</td>
<td>4.97 (-13)</td>
<td>5 (-13) = $\epsilon_1$</td>
</tr>
</tbody>
</table>

*(34) not applicable for $a = 2.60 (-5) = 2.60 \times 10^{-5}$.

**$\tilde{x}_2$ computed from (40) instead of (38).
As stated before, the use of (34), (35), (43), (45) and (46) contribute significantly
to the overall efficiency of the program. Moreover, the computation of $R(a, x)$ in
(43), (45) or (46) is not always wasted if these inequalities are not satisfied, because
this function is also needed for computing $P$ from (10) or $Q$ from (12) or (15).

In the next section we give a somewhat detailed discussion of Takenaga’s method,
for use when $a \geq 100$, and the changes we made to it.
4. TAKENAGA'S METHOD AND MODIFICATIONS

In this section Takenaga's method [9] for evaluating $P$ when $a$ is not small is summarized. Changes were made in his analysis in order to make it efficient for calculation. In addition, it was extended to include the direct computation of $Q$. These modifications will be given in detail. Basically, the procedure is based on expansions in terms of incomplete normal moment functions, [7], and it works best when $a$ and $x$ are close and large. Even though the method can be used for $a$ as small as 20, it is called in the program only when $a > 100$, (and $3a < 4x < 5a$). Therefore, it will be assumed in discussing the method that $a > 100$.

Let

$$t = [(z + a)/\beta]^3, \quad (47)$$

where $a$ and $\beta$ are parameters to be specified. With $z$ as a new integration variable and $a$ replaced by $b + 1$, (3) transforms to

$$\gamma(b + 1, x) = \int_r^s \frac{3}{\beta} \left(\frac{z + a}{\beta}\right)^{3h+2} \exp \left[ - \left(\frac{z + a}{\beta}\right)^3 \right] dz, \quad (48)$$

where

$$r = -a, \quad s = \beta x^{1/3} - a, \quad b = a - 1. \quad (49)$$

We can also write from (1),

$$P(a, x) = P(b + 1, x) = \int_r^s S(b, z) dz, \quad (50)$$

where

$$S(b, z) = \left\{ \frac{3}{\beta} \left(\frac{z + a}{\beta}\right)^{3b+2} \exp \left[ - \left(\frac{z + a}{\beta}\right)^3 \right] \right\} \Gamma(b + 1). \quad (51)$$

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Then
\[
\log S(b, z) \sim \log \frac{3}{\beta} + (3b + 2) \log \frac{a}{\beta} + (3b + 2) \log (1 + z/a) \tag{52}
\]

\[
- \left( \frac{a}{\beta} \right)^3 \left[ 1 + \frac{3z}{a} + 3 \left( \frac{z}{a} \right)^2 + \frac{1}{2} \frac{(z/a)^3}{(b + 1)^2} \log b - b + \frac{1}{2} \log 2\pi + \tilde{L}(b) \right],
\]

where the quantity in brackets of the last line represents the asymptotic expansion of
\[
\log \Gamma(b + 1) = \log b + \log \Gamma(b), \text{(see (26))}.
\]

More specifically,
\[
\log \Gamma(b + 1) \sim \left( b + \frac{1}{2} \right) \log b - b + \frac{1}{2} \log 2\pi + \tilde{L}(b) \tag{53}
\]

with
\[
\tilde{L}(b) = \frac{1}{12b} - \frac{1}{360b^3} + \frac{1}{1260b^5} - \frac{1}{1680b^7} + \frac{1}{1188b^9} - \cdots \tag{54}
\]

(In (27), we have denoted the first four terms of \( \tilde{L}(a) \) by \( L(a) \)). In (52), we make the substitution
\[
\log(1 + z/a) = - \sum_{k=1}^{\infty} \frac{1}{k} (-z/a)^k, \quad -1 < z/a < 1. \tag{55}
\]

This series can be used for the range of values of \((a, x)\) for which the method is to be applied. Indeed, taking
\[
a = 3(b + 2/3)^{1/2} \tag{56}
\]
\[
\beta = 3(b + 2/3)^{1/6} \tag{57}
\]

and noting from (47) that \( z \) is an increasing function of \( t \), we have from (49)
\[
-1 < z/a < (\beta/a) x^{1/3} - 1 = [x/(b + 2/3)]^{1/3} - 1 = s/a. \tag{58}
\]
In the program, we use (50) only for cases when \( x < b + 2/3 \), (a corresponding expression for \( Q(b + 1, x) \) is used when \( x > b + 2/3 \) and will be discussed later in this section). Consequently, for \( x < b + 2/3 \), (58) shows \( z/a < 0 \) so that the right hand inequality in (55) is satisfied, (it is interesting to note that, regardless of the constraint on \( x \), \( z/a < 1 \) provided \( 0 < x < 8(b + 2/3) \) which always holds for \( a > 6 \) and \( e < P < 1 - e \), where \( e_1 < e \). This follows from the results of the previous section).

The left inequality in (55), on the other hand, cannot be satisfied, because \( z \) takes the value \( -a \) when \( t = 0 \). This difficulty is circumvented in the following way. Let

\[
P(a, x) = P(a, \bar{X}) + \int_{\bar{X}}^{x} e^{-t} t^b dt/\Gamma(a),
\]

where \( a = b + 1 \), as noted above, and

\[
\bar{X} = a \left(1 - \frac{1}{9a} - \frac{\bar{y}/3\sqrt{a}}{3}\right)^{1/3}, \quad \bar{y} = 11.0.
\]

Then by the results of the previous section (see (33) with \( \bar{y} \) replaced by \( \bar{y} \)) \( P(a, \bar{X}) < 4 \times 10^{-28} \). Hence, we can take the second term on the right hand side of (59) in place of \( P(a, x) \) with an insignificant loss in relative accuracy. Under these conditions, we get from (47), that when \( t = \bar{X}, z = \bar{z} \), and

\[
\bar{z} = 9a^{1/3} \left(1 - \frac{1}{9a} - \frac{\bar{y}/3\sqrt{a}}{3}\right) - a
\]

\[
= a \left(\frac{a/(a - 1/3)}{1} \left(1 - \frac{1}{9a} - \frac{\bar{y}/3\sqrt{a}}{3}\right) - 1\right)
\]

\[
= a \left(1 + \frac{1}{9a} + \frac{1}{81a^2} + \cdots \right) \left(1 - \frac{1}{9a} - \frac{\bar{y}/3\sqrt{a}}{3}\right) - 1
\]

\[
> a \left(1 + \frac{1}{9a} + \frac{2}{81a^2}\right) \left(1 - \frac{1}{9a} - \frac{\bar{y}/3\sqrt{a}}{3}\right) - 1
\]

Thus, since in our case \( a > 100 \), we see from (61) and (58) that

\[
\frac{\bar{y}}{3\sqrt{a}} \left(1 + \frac{1}{9a} + \frac{2}{81a^2}\right) < z/a < z/a < s/a < 0.
\]

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This justifies the use of (55), since for \( \tilde{y} = 11.0 \) and \( a > 100 \)

\[
\frac{\tilde{y}}{3\sqrt{a}} \left( 1 + \frac{1}{9a} + \frac{2}{91a^2} \right) < 0.37
\]

The choices for \( \alpha \) and \( \beta \), given by (56) and (57), were shown by Takenaga, after using (55) in (52), to result in zero coefficients for the terms \( z/a, (z/a)^3, \) log \( b \), \( b \) in (52). Thus, with

\[
\log \alpha/\beta = (1/3) \log(b + 2/3) = (1/3) [\log b + \log(1 + 2/3b)]
\]

he found:

\[
\log S(b, z) = \log C - \frac{1}{2} \log 2\pi - \frac{z^2}{2} - \frac{a^2}{3} \sum_{k=4}^{\infty} (1/k) (-z/a)^k,
\]

(62)

where

\[
\log C = (b + 1/2) \log (1 + 2/3b) - 2/3 - \bar{L}(b)
\]

(63)

\[
= - (b + 1/2) \sum_{k=1}^{\infty} (1/k)(-2/3b)^k - 2/3 - \bar{L}(b)
\]

\[
= \frac{1}{36b} - \frac{1}{81b^2} + \frac{1}{360b^3} + \frac{2}{1215b^4} - \frac{691}{306180b^5} + \frac{16}{15309b^6}
\]

\[
- \frac{373}{3674160b^7} + \frac{80}{177147b^8} - \frac{44069}{38972340b^9} + \ldots
\]

Then from these results, he obtained

\[
S(b, z) = C \left| \left| \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \exp \left| \frac{a^2}{3} \sum_{k=4}^{\infty} (1/k)(-z/a)^k \right| \right| \right|
\]

(64)

*The coefficients in (63) and (65), or (68) and (80), were computed by Alfred Morris with his "Flap" Algebraic Routine, [5]. Additional coefficients are given in Appendix B. In [9] there is a misprint in (14) and another in (19) of that paper."
Now expanding the second exponential function of (64) in powers of $z/a$, one gets the power series in (65), namely

$$
\exp \left[ -\frac{a^2}{3} \sum_{k=4}^{\infty} \frac{(1/k)(-z/a)^k}{k} \right] = 1 - \frac{1}{12a^2} z^4 - \frac{1}{18a^4} \left( z^8 - \frac{1}{16} z^8 \right)$$

$$
- \frac{1}{24a^6} \left( z^8 - \frac{37}{225} z^{10} + \frac{1}{432} z^{12} \right)
$$

$$
\vdots
$$

$$
- \frac{1}{a^{24}} \left( \frac{1}{78} z^{26} - \cdots - 2.34146722651062 \times 10^{-22} z^{48} \right)
$$

$$
+ \frac{1}{15a^8} z^9 + \frac{1}{3a^5} \left( \frac{z^7}{7} - \frac{z^7}{60} \right) + \frac{1}{27a^7} \left( z^9 - \frac{29}{140} z^{11} + \frac{1}{160} z^{13} \right)
$$

$$
+ \frac{1}{a^9} \left( \frac{1}{33} z^{11} - \frac{193}{22680} z^{13} + \frac{1187}{2268000} z^{15} - \frac{1}{155520} z^{17} \right)
$$

$$
\vdots
$$

$$
+ \frac{1}{a^{25}} \left( \frac{1}{81} z^{27} - \cdots - 2.24780853745020 \times 10^{-21} z^{49} \right) + \cdots
$$

In order now to carry out the integration of (64), using (65), we need to evaluate integrals of the form

$$
\bar{B}_j = \int_{\Gamma}^S z^{2j} Z(z) \, dz, \quad \bar{A}_j = \int_{\Gamma}^S z^{2j+1} Z(z) \, dz.
$$

(66)

where $j$ is a non-negative integer (in the program $j < 24$) and

$$
Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}.
$$

In [9], the indefinite integration of (64), using (65), is carried out, and a cumbersome procedure is suggested for evaluating the constant of integration. The approach below is more direct and easier to apply.
We express the integration of (64) in inverse powers of $a$.

$$\int_{\mathbb{I}} S(h, z) \, dz = C \sum_{k=0}^{N} \tilde{d}_k \, a^{-k}. \quad (65)$$

where

$$\tilde{d}_0 = \tilde{B}_0, \quad \tilde{d}_1 = 0, \quad \tilde{d}_2 = \frac{1}{12} \tilde{B}_2, \quad \tilde{d}_3 = \frac{1}{15} \tilde{A}_2,$$

and in general for $k \geq 2$,

$$\tilde{d}_k = \begin{cases} \sum_{(k+2)/2}^{k} N_{k_j} \tilde{B}_{j} \quad & (k \text{ even}, N_{k_0} = 1) \\ \sum_{(k+1)/2}^{k} M_{k_j} \tilde{A}_{j} \quad & (k \text{ odd}) \end{cases} \quad (68)$$

The constants $N_{k_j}, M_{k_j}$ which result from the expansion in (65) are given in Appendix B. Some particular values are $N_{2,2} = \frac{1}{12}, M_{7,5} = 29/(27 \times 140)$. If $k$ is even, a typical term of the sum on the right hand side of (68), multiplied by $a^{-k}$, is given by

$$a^{-k} N_{k_j} \tilde{B}_j = N_{k_j} \, a^{-k} \int_{\mathbb{I}} z^{2j} \, dz \quad N_{k_j} \, a^{-k} \int_{\mathbb{I}} z^{2j} \, dz. \quad (69)$$

or in the case of $k$ odd,

$$a^{-k} M_{k_j} \tilde{A}_j = M_{k_j} \, a^{-k} \int_{\mathbb{I}} z^{2j+1} \, dz \quad M_{k_j} \, a^{-k} \int_{\mathbb{I}} z^{2j+1} \, dz. \quad (70)$$

We want to show that the last term on the right hand side of these identities can be dropped for $b (= a - 1) \geq 99$ with no significant loss in relative accuracy. First, we obtain an upper bound on $\hat{z}$. From (61),

$$\hat{z} = a \left( \frac{b + 1}{b + 2/3} \right)^{1/3} \left( 1 - \frac{1}{9, b + 1} \right) - \sqrt{3}\left( \frac{b + 1}{b + 2/3} \right)^{1/3} \sqrt{b + 1}.$$
Then
\[
\left( \frac{b + 1}{b + 2/3} \right)^{1/3} \left| 1 - \frac{1}{9(b + 1)} \right| = \left| 1 + \frac{1}{3(b + 2/3)} \right|^{1/3} \left| 1 - \frac{1}{9(b + 2/3)(1 + 1/3(b + 2/3))} \right|
\]
\[
= \left( 1 + \frac{3}{a^2} \right)^{1/3} \left| 1 - \frac{1}{a^2(1 + 3/a^2)} \right|
\]
\[
< \left( 1 + \frac{1}{a^2} - \frac{1}{a^4} + \frac{5}{3a^6} \right) \left( 1 - \frac{1}{a^2} + \frac{3}{a^4} - \frac{9}{a^6} + \frac{27}{a^8} \right)
\]
\[
< 1 + \frac{1}{a^4} - \frac{10}{3} \frac{1}{a^6} + \frac{40}{3} \frac{1}{a^8} < 1 + 1/a^4, \quad (a \geq 100).
\]

Therefore,
\[
\mathcal{I} < \tilde{y} \left( \frac{b + 2/3}{b + 1} \right)^{1/6} + \frac{1}{a^3} < -10.9938 \quad (b > 99, \quad \tilde{y} = 11.0).
\]

Now, letting \( v = 10.9938 \) and taking \( k = j = 0 \) in (69), we have
\[
\int_{-\infty}^{\mathcal{I}} Z \, dz \int_{\mathcal{I}}^{\infty} Z \, dz \int_{v}^{\infty} Z \, dz < 2.05 \times 10^{-28} \quad \text{(See [L, p. 972])}
\]

Consequently, the last term in (69), with \( k = j = 0 \), can be neglected, (the constant \( C \), in (67), does not affect this conclusion, since it is always close to one, for \( a \geq 100 \), as (18) or (63) shows).

New consider the absolute value of a particular term in (67) for \( k > 0 \) and even.
\[
|N_{kj}| a^{-k} \int_{-\infty}^{\mathcal{I}} z^{2i} Z \, dz < |N_{kj}| a^{-k} \int_{v}^{\infty} z^{2i} Z \, dz.
\]

Then with \( u = z^{2}/2 \) and the use of (45),
\[ |N_{kj}| \frac{a^{-k} \int_{\gamma}^{\infty} z^{2j} \, dz}{2\sqrt{\pi}} = |N_{kj}| \frac{a^{-k} \int_{\gamma}^{\infty} u^{j+1/2} \, e^{-u} \, du}{2\sqrt{\pi}} \]

\[ = |N_{kj}| \frac{a^{-k} \int_{\gamma}^{\infty} Q(j + 1/2, v^2/2) \, I(j + 1/2)}{2\sqrt{\pi}} \]

\[ < |N_{kj}| \frac{a^{-k} \int_{\gamma}^{\infty} e^{-v^2/2} \, (v^2/2)^{j+1/2}}{(v^2/2) + 1 - j + 1/2} \]

(See (45))

\[ = |N_{kj}| \frac{a^{-k} \int_{\gamma}^{\infty} e^{-v^2/2} \, (v^2/2)^{j+1/2}}{(v^2 + 1 - 2j)} \]

A similar analysis for \( k \) odd shows

\[ |M_{kj}| \frac{a^{-k} \int_{\gamma}^{\infty} z^{2j+1} \, dz}{2\sqrt{\pi}} < |M_{kj}| \frac{a^{-k} \int_{\gamma}^{\infty} e^{-v^2/2} \, (v^2/2)^{j+2}}{(v^2 - 2j)} \]

(73)

It can be shown for every case considered in the program, i.e., \( 0 \leq k \leq 24 \), \((k + 2)/2 < j < k \) (k even), \((k + 1)/2 < j < k \) (k odd) that by bounding the right hand sides of (72) and (73), the left hand sides of these inequalities are very small. For example setting \( b = 99 \) (for which \( a^{-k} \) takes its largest value), \( k = 23, j = 22 \), \( M_{23,22} \approx 3 \times 10^{-19} \) (see Appendix B), we have from (73)

\[ M_{23,22} \frac{a^{-23} \int_{\gamma}^{\infty} z^{25} \, dz}{2\sqrt{\pi}} < \frac{3 \times 10^{-19} \times 1.1 \times 10^{-34}}{76.86} \times 4.5 \times 10^{31} = \text{O}(10^{-34}). \]

For the same value of \( k \) with \( j = 12 \), \( M_{23,12} > 0.0133 \cdots \).

\[ M_{23,12} \frac{a^{-23} \int_{\gamma}^{\infty} z^{25} \, dz}{2\sqrt{\pi}} < \frac{1.4 \times 10^{-2} \times 1.1 \times 10^{-34}}{95.86} \times 6.7 = \text{O}(10^{-37}). \]

Thus, the second terms on the right hand sides of (69) and (70) will be discarded, or equivalently the integrals \( B_j \) and \( A_j \) in (66) are replaced by

\[ B_j = \int_{\gamma}^{\infty} z^{2j} \, dz \]

(74)

\[ A_j = \int_{\gamma}^{\infty} z^{2j+1} \, dz \]

(75)
These integrals are easily evaluated on a computer by using the recurrence relations

\[ B_j = (2j - 1) B_{j-1} - s^{2j-1} Z(s) \]  \hspace{1cm} (76)

\[ A_j = 2j A_{j-1} - s^{2j} Z(s) \]  \hspace{1cm} (77)

with

\[ B_0 = \int_{-\infty}^{s} Z \, dz = \frac{1}{2} \text{erfc} \left( \frac{|s|}{\sqrt{2}} \right), \quad s \leq 0 \]  \hspace{1cm} (78)

\[ A_0 = -\frac{1}{\sqrt{2\pi}} e^{-s^2/2} = -Z(s) \]  \hspace{1cm} (79)

There is no problem to compute \( B_0 \) since a very efficient and accurate error function routine is available. It is based on Cody’s minimax approximations, [2], and it yields 14 significant digits for \( \text{erf} \) or \( \text{erfc} \) depending on the magnitude of the argument. For completeness, the particular Cody approximations that we use are specified in Appendix A. A discussion of how \( s \) is computed is given on page 41.

Finally the quantity \( C \), whose logarithm is given by (63), is expressed as a power series in inverse powers of \( b \) in order to evade computing \( \log C \) from the series in (63), and subsequently \( C \) from \( \exp (\log C) \). Thus we obtain

\[ C = 1 + \frac{1}{36} b^{-1} + \frac{31}{2592} b^{-2} + \frac{3413}{1399680} b^{-3} + \frac{361733}{201553920} b^{-4} + \cdots \]  \hspace{1cm} (80)\* 

\[ \frac{113888281}{50791587840} b^{-5} + \frac{7565202533}{7836416409600} b^{-6} + \cdots \]

Takenaga’s procedure is extended to advantage by using the same approach when \( x > b + 2/3 = a \cdot 1/3 \) to compute \( Q \) instead of \( P \). As noted earlier, \( F \) in this case can be close to one so that \( Q \) must be computed directly to maintain the specified relative accuracy.

\*See footnote on page 25.
The computation of $Q$ only requires changing the limits of integration in (74) and (75) from $-\infty$ and $s$ to $s$ and $\infty$, respectively. Thus (76) – (79) are replaced by

$$B_j = (2j - 1) B_{j-1} + s^{2j-1} Z(s)$$  \hspace{1cm} (81)

$$A_j = 2j A_{j-1} + s^{2j} Z(s)$$  \hspace{1cm} (82)

$$B_0 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2} \, dt = \frac{1}{2} \text{erfc}(s/\sqrt{2})$$  \hspace{1cm} (83)

$$A_0 = \frac{1}{\sqrt{2\pi}} \cdot e^{-s^2/2} = Z(s)$$  \hspace{1cm} (84)

Certainly these changes are quite simple. We also note from (49) or (58) that $s > 0$ since $x > b + 2/3$.

In order to establish these results for $Q(b + 1, x)$, we require the integration of $S(b, z)$ in (51) from $s$ to $\infty$, with $s > 0$. Again this raises the question, when we proceed as before, concerning the validity of using the expansion in (55). In order to justify its use we must show that $|z/a| < 1$ for all $z$ in the domain of integration. Clearly this will not be the case if $z \in [s, \infty]$. We get around this problem much like before. Note that since $x > b + 2/3$, a lower bound on $z/a$ is easily found, namely

$$(z/a) > s/a = [x/(b + 2/3)]^{1/3} - 1 > 0$$

An upper bound on $z/a$ is found by considering

$$Q(a, x) = Q(a, \tilde{X}) + \int_{\tilde{X}}^{X} e^{-t^2} \, dt , \quad a = b + 1 ,$$  \hspace{1cm} (85)

where $\tilde{X} = a(1 - \frac{1}{9a + \tilde{y}/3\sqrt{a}})^3 > x > 0$. It follows from (46) and [1, p. 972] that for $\tilde{y} = 11.0$, $Q(a, \tilde{X}) < 7 \times 10^{-29}$. Thus, we can replace $Q(a, x)$ by the second integral on the right hand side of (85). We get an upper bound on $z/a$ by using $t \leq \tilde{X}$ in (47). Indeed,

$$z < a \left\{ (\tilde{y}/a) \tilde{X}^{1/3} - 1 \right\} = a \left\{ \left( \frac{b + 1}{b + 2/3} \right)^{1/3} \left( 1 - \frac{1}{9a} + \tilde{y}/3\sqrt{a} \right) - 1 \right\}$$

$$< \left( \frac{b + 2/3}{b + 1} \right)^{1/6} \tilde{y} + 1/a^3 < 10.9939 = v. \quad (\text{See } (71)).$$

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Since \( b > 99 \), we have, by noticing that \( v/a \) is a decreasing function of \( b \),

\[
0 < s/a < z/a < v/a < 0.37.
\]

This result allows us to use (55) when \( s > 0 \), \( (x > b + 2/3) \).

The algebraic procedures which yield (65) in inverse powers of \( a \), and the subsequent integration show that individual terms contain integrals of the form (after dropping \( Q(a, \bar{X}) \) in (85))

\[
\overline{B}_j = \int_s^v z^{2j} Z \, dz, \quad \overline{A}_j = \int_s^v z^{2j+1} Z \, dz, \tag{86}
\]

where

\[
Z = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.
\]

Proceeding as in the case for \( x \leq b + 2/3 \), the expressions corresponding to (69) and (70) are given by

\[
a^{-k} N_{kj} \overline{B}_j = N_{kj} a^{-k} \int_s^\infty z^{2j} Z \, dz - N_{kj} a^{-k} \int_v^\infty z^{2j} Z \, dz \tag{87}
\]

\[
a^{-k} M_{kj} \overline{A}_j = M_{kj} a^{-k} \int_s^\infty z^{2j+1} Z \, dz - M_{kj} a^{-k} \int_v^\infty z^{2j+1} Z \, dz. \tag{88}
\]

We note that the coefficients of (68), \( N_{kj} \) and \( M_{kj} \), are unchanged and that (68) holds with \( \overline{B}_j \) and \( \overline{A}_j \) given by (86). But, we have argued above (see (72), (73)) that the second term on the right of (87) and (88) can be dropped or, equivalently, the integrals \( \overline{B}_j \) and \( \overline{A}_j \) in (86) can be replaced by

\[
\overline{B}_j = \int_s^\infty z^{2j} Z \, dz, \quad \overline{A}_j = \int_s^\infty z^{2j+1} Z \, dz. \tag{89}
\]

It is interesting to note that these integrals were obtained directly by formally integrating (64) from \( s \) to \( \infty \). They are evaluated by the recurrence relations (81) and (82).

The final integrated expressions for \( P(b + 1, x) \) and \( Q(b + 1, x) \) are given by (17) and again in (115) of the next section where the computer program is discussed.
5. DESCRIPTION OF PROGRAM

In this section we summarize the equations and inequalities on which the CDC-6700 computing program is based. In addition, the general features of the program are described and a master flowchart is included at the end of this section as Figure 3.

The program requires as input 5 quantities, the two locations where two arguments a and x are stored, the two output locations for P and Q, and a location which is used to specify whether $\epsilon_1 (= 5 \times 10^{-13})$, $\epsilon_2 (= 5 \times 10^{-7})$, or $\epsilon_3 (= 5 \times 10^{-4})$ is to be used.

If $\epsilon_1$ or $\epsilon_2$ is specified, then the output quantities P and Q are given correctly to within one unit in the twelfth (sixth) significant digit. Of course, if (a, x) is outside the "zone of computation" P and Q may have no correct significant digits since they are simply given as 1 and 0 or 0 and 1. The absolute error in such cases, of course, will always be within the specified $\epsilon$.

In order to give some estimate of the average computing time per case, two grids of points in the ax-plane were used for $\epsilon = \epsilon_1$ and two others for $\epsilon = \epsilon_2$. The two grids for $\epsilon_1$ are specified first, identifying them as A and B.

<table>
<thead>
<tr>
<th>Grid A</th>
<th>$\epsilon = \epsilon_1$</th>
<th>Grid B</th>
</tr>
</thead>
<tbody>
<tr>
<td>a = 0.0001 (0.001) 0.05</td>
<td>a = 0.0001 (0.001) 0.05</td>
<td>a = 0.0001 (0.001) 0.05</td>
</tr>
<tr>
<td>x = 0 (0.1) 126.1</td>
<td>x = 0 (0.1) X (a)</td>
<td>x = 0 (0.1) X (a)</td>
</tr>
<tr>
<td>a = 0.1 (0.1) 3!</td>
<td>a = 0.1 (0.1) 3!</td>
<td>a = 0.1 (0.1) 3!</td>
</tr>
<tr>
<td>x = 0 (0.5) 1260</td>
<td>x = 0 (0.5) X (a)</td>
<td>x = 0 (0.5) X (a)</td>
</tr>
<tr>
<td>a = 31.0 (0.5) 99.5</td>
<td>a = 31.0 (0.5) 99.5</td>
<td>a = 31.0 (0.5) 99.5</td>
</tr>
<tr>
<td>x = 0 (0.5) 1260</td>
<td>x = $X_1$ (0.5) X (a)</td>
<td>x = $X_1$ (0.5) X (a)</td>
</tr>
<tr>
<td>a = 100.1 (5) 1000.1</td>
<td>a = 100.1 (5) 1000.1</td>
<td>a = 100.1 (5) 1000.1</td>
</tr>
<tr>
<td>x = 0 (10) 1260</td>
<td>x = $X_1$ (10) X (a)</td>
<td>x = $X_1$ (10) X (a)</td>
</tr>
</tbody>
</table>

$X (a)$ = minimum grid value of x, given a, such that Q $\leq$ $\epsilon$.

$X_1$ = value of x, given a, for which P(a, x) $= \epsilon$. See (33).
\[ x_1 = a \left( 1 - \frac{1}{g_2} - \frac{\bar{y}^2}{3\sqrt{a}} \right)^{\frac{1}{2}}, \quad \bar{y} = 7.1306, \quad (\bar{y} = 4.892, \epsilon = \epsilon_2). \]

All the grid points, \((a, x)\), specified are considered.

Denoting the corresponding grids for \(\epsilon = \epsilon_2\) by \(A_2\) and \(B_2\), we have

<table>
<thead>
<tr>
<th>Grid (A_2)</th>
<th>(\epsilon = \epsilon_2)</th>
<th>Grid (B_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a = 0.0001 (0.001) 0.05)</td>
<td>(a = 0.0001 (0.001) 0.05)</td>
<td>(a = 0.0001 (0.001) 0.05)</td>
</tr>
<tr>
<td>(x = 0 (0.1) 117.1)</td>
<td>(x = 0 (0.1) X (a))</td>
<td>(x = 0 (0.1) X (a))</td>
</tr>
<tr>
<td>(a = 0.1 (0.1) 20)</td>
<td>(a = 0.1 (0.1) 20)</td>
<td>(a = 0.1 (0.1) 20)</td>
</tr>
<tr>
<td>(x = 0 (0.5) 1170)</td>
<td>(x = 0 (0.5) X (a))</td>
<td>(x = 0 (0.5) X (a))</td>
</tr>
<tr>
<td>(a = 20.5 (0.5) 99.5)</td>
<td>(a = 20.5 (0.5) 99.5)</td>
<td>(a = 20.5 (0.5) 99.5)</td>
</tr>
<tr>
<td>(x = 0 (0.5) 1170)</td>
<td>(x = x_1 (0.5) X (a))</td>
<td>(x = x_1 (0.5) X (a))</td>
</tr>
<tr>
<td>(a = 100.1 (5) 1000.1)</td>
<td>(a = 100.1 (5) 1000.1)</td>
<td>(a = 100.1 (5) 1000.1)</td>
</tr>
<tr>
<td>(x = 0 (10) 1170)</td>
<td>(x = x_1 (10) X (a))</td>
<td>(x = x_1 (10) X (a))</td>
</tr>
</tbody>
</table>

The major portion of the points, \((a, x)\), in the \(A\) grids lie outside the "zone of computation," whereas almost all of the points in the \(B\) grids lie within the "zone of computation." The \(A\) grids were designed to get a representative average time per case in the situation where any point in the region \(0.0001 \leq a \leq 1000.1, 0 \leq x \leq U^*\) (where \(U^*\) denotes the maximum value on \(x\)) is equally likely to occur as input. On the other hand, the \(B\) grids were set up to obtain a representative average time per case in those situations where the input point \((a, x)\) is most likely to occur in the "zone of computation." In Table 3, the results for the various grids are summarized.

*We note for \(a = 0.0001 (0.001) 0.05\) \(U\) is taken as 126.1 for \(\epsilon = \epsilon_1\) and 117.1 for \(\epsilon = \epsilon_2\).*
Table 3. Summary of Grid Results

<table>
<thead>
<tr>
<th>EQ. GRID</th>
<th>(10)</th>
<th>(12)</th>
<th>(13)</th>
<th>(15)</th>
<th>(16)</th>
<th>(17)</th>
<th>E</th>
<th>TOTAL CASES</th>
<th>AVG. TIME (msec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>GRID A₁</td>
<td>20021</td>
<td>39089</td>
<td>1891</td>
<td>17520</td>
<td>769</td>
<td>5872</td>
<td>1131595</td>
<td>1216757</td>
<td>0.75</td>
</tr>
<tr>
<td>GRID B₁</td>
<td>19958</td>
<td>39155</td>
<td>1879</td>
<td>17512</td>
<td>769</td>
<td>5860</td>
<td>2274</td>
<td>87407</td>
<td>2.02</td>
</tr>
<tr>
<td>GRID A₂</td>
<td>12920</td>
<td>20708</td>
<td>561</td>
<td>5606</td>
<td>769</td>
<td>4020</td>
<td>876965</td>
<td>921549</td>
<td>0.68</td>
</tr>
<tr>
<td>GRID B₂</td>
<td>12788</td>
<td>20783</td>
<td>551</td>
<td>5598</td>
<td>769</td>
<td>3993</td>
<td>1687</td>
<td>46169</td>
<td>1.59</td>
</tr>
</tbody>
</table>

The numbers in parentheses refer to the equations in the text, e.g., (10) was used 20021 times in Grid A₁. The column headed E gives the number of cases that were recognized by the program to be outside the “zone of computation.” The average time per case is given in the last column in milliseconds. The much smaller computing times for the A grids are to be expected, since most of the cases were outside the “zone of computation.”

The flow of the program, depending on the input, should be easy to follow from the flow chart at the end of this section. The circled numbers are used to denote subprograms for computing P or else Q, and the number itself corresponds to the equation in Section 2 upon which the subprogram is based. For example, (10) refers to that subprogram which uses (10) for the computation of P; (12) refers to that subprogram which uses (12) to determine Q. The quantities $\bar{F}$ and $F$ are defined by (34) and (35). The term $R(a, x) = e^{-\frac{x^2}{2a}}$, (see also (11)), and $\epsilon$ refers to either $\epsilon_1$, $\epsilon_2$, or $\epsilon_3$, (see (6) and (7)).

We continue with a brief description of the algorithm used with each subprogram. It is assumed $(a, x)$ is in the “zone of computation” and that all the subprograms except (17) require $a < 100$.

If $a > 1$, then (10) is used if $a > x$. Also, (10) is used when $2a$ is not an integer provided $1 < a < x < ln 10$ or $a < 1, x < 1.5$ and $P < 0.90$, or $a < 1, x > 1.5$ and $x < \frac{\ln 10}{2}$.
R > 0.101 \ x \ (x + 2 - a)/(x + 1). Let \( s_{n-1} \) denote the \((n - 1)\)st term of the series in (10), then the \(n\)th term is found from

\[
s_n = \left( \frac{x}{a + n} \right) s_{n-1}, \quad s_0 = 1, \quad n = 1, 2, \ldots \quad (90)
\]

The summation of the series is terminated at \(n = N\) when

\[
s_N < \epsilon/2 \text{ and } x/(a + N + 1) < 2/3,
\]

which assures a truncation error, \[
\sum_{N+1}^\infty s_k < \epsilon, \text{ (see Appendix D)}.
\]

Program (1) applies when \(1 < a < x\), and \(x > 31\) if \(\epsilon = \epsilon_1\); \(x > 17\) if \(\epsilon = \epsilon_2\); \(x > 9.7\) if \(\epsilon = \epsilon_3\). The \((n + 1)\)st term of the asymptotic series (12) for \(Q\) is given by

\[
V_{n+1} = \left( \frac{a - 1 - n}{x} \right) V_n, \quad V_0 = 1, \quad n = 0, 1, \ldots \quad (91)
\]

The summation is terminated when

\[
|V_{n+1}| < \epsilon. \quad (92)
\]

Program (3) is used when \(a < x < 31\) for \(\epsilon = \epsilon_1\), \(a < x < 17\) for \(\epsilon = \epsilon_2\), or \(a < x < 9.7\) for \(\epsilon = \epsilon_3\), and if \(2a\) is an integer. The algorithm for computing \(Q\) from (13) then is given by

\[
W(k + g - 1, x) = \left( \frac{x}{k + g - 1} \right) W(k + g - 2, x) \quad (93)
\]

\[
Q(k + g, x) = Q(k + g - 1, x) + W(k + g - 1, x), \quad k = 1, 2, \ldots, a - g, \quad (94)
\]

where \(g = 1\) if \(a\) is an integer or \(g = 1/2\) if \(a\) is not an integer. Also

\[
W(k + g - 1, x) = R(k + g - 1, x)/(k + g - 1) \quad (95)
\]

\[
W(g - 1, x) = \begin{cases} (\pi x)^{-1/2} e^{-x} & g = 1/2 \\ e^{-x} & g = 1 \end{cases} \quad (96)
\]

\[
Q(g, x) = \begin{cases} \text{erfc}(\sqrt{x}) & g = 1/2 \\ e^{-x} & g = 1 \end{cases} \quad (97) \quad \text{(See (14))}.
\]
Program \( \text{Qj) } \) is based on a continued fraction expansion for \( Q \). It is used when \( 2a \) is not an integer and \( 1 < a < x \) with \( \sqrt{n} 10 < x < 31 \) for \( \epsilon = \epsilon_1 \) (\( \sqrt{n} 10 < x < 17 \) for \( \epsilon = \epsilon_2 \), or \( \sqrt{n} 10 < x < 9.7 \) for \( \epsilon = \epsilon_3 \)) or \( a < 1, x > 1.5 \) and \( R < 0.101 \frac{x(2 + x - a)}{(x + 1)} \). The algorithm for computing two successive approximates, based on (15), in each iteration follows. Let

\[
D_1 = D_2 = 1, \quad E_1 = x, \quad E_2 = (x + 1 - a), \quad n = 1, 2, \cdots.
\]

Then

\[
\begin{align*}
D_{2n+1} &= x D_{2n} + n D_{2n-1} \\
E_{2n+1} &= x E_{2n} + n E_{2n-1}
\end{align*}
\]

(98)

Increase \( n \) to \( n + 1 \)

\[
\begin{align*}
D_{2n} &= D_{2n-1} + (n - a) D_{2n-2} \\
E_{2n} &= E_{2n-1} + (n - n) E_{2n-2}
\end{align*}
\]

(99)

The procedure is stopped when

\[
\left| \frac{D_{2n}}{E_{2n}} - \frac{D_{2n-1}}{E_{2n-1}} \right| < \epsilon \left| \frac{D_{2n}}{E_{2n}} \right|.
\]

(100)

Program \( \text{(16)} \) is used when \( a < 1, x < 1.5 \) and \( P > 0.90 \). From (16), three series are evaluated, \( J, L, H \).

We have

\[
J = -a \sum_{k=1}^{\infty} \frac{(-x)^k}{(a + k) k!} = -a \sum_{k=1}^{\infty} J_k = -a \sum_{k=1}^{\infty} T_k/(a + k),
\]

(101)

then

\[
T_k = (-x/k) T_{k-1}, \quad J_k = T_k/(a + k) \quad k = 2, \cdots,
\]

(102)

\[
T_1 = -x, \quad J_1 = -\frac{x}{a + 1}.
\]

(103)
The iterations are stopped when
\[ |J_k| < -\epsilon \sum_{i=1}^{k} J_i \quad (104) \]

For \( L \), we have
\[ L = a \ln x \left[ 1 + \sum_{k=1}^{\infty} \frac{(a \ln x)^k}{(k+1)!} \right] = a \ln x \left[ 1 + \sum_{k=1}^{\infty} L_k \right] \quad (105) \]

Then
\[ L_1 = \left( \frac{a \ln x}{k+1} \right) L_{k-1}, \quad L_1 = (a \ln x)/2, \quad k = 2, \ldots \quad (106) \]

These iterations are terminated when
\[ |L_k| < \epsilon/2 \quad (107) \]

The \( H \) series,
\[
H = \begin{cases} 
\sum_{k=0}^{17} C_k a^{k-1} & 0 < a < 1/2 \\
(1/a) \left[ (1 - a) + \sum_{k=2}^{17} C_k (a - 1)^{k-1} \right] & 1/2 \leq a < 1,
\end{cases}

(108)

is terminated when \( k = 17 \) or before if \( |K_n| < (\epsilon/2) \left| \sum_{k=2}^{n} K_k \right| \), where \( K_k = C_k a^{k-1} \)

if \( 0 < a < 1/2 \) or \( K_k = C_k (a - 1)^{k-1} \) if \( 1/2 \leq a < 1 \). We note also that the \( H \) series make up part of the series that are used for evaluating \( 1/\Gamma(\lambda) \) when \( 0 < \lambda < 1 \), (see (25)). Consequently, when \( 0 < a < 1 \), \( H \) is obtained during the computation of \( 1/\Gamma(a) \) and stored. The evaluation of \( 1/\Gamma(a) \) is taken up later in this section on page 42 and in Appendix A.
Program (17) is the only one used when \(a \geq 100\), and \(3a \leq 4x \leq 5a\). The individual integrals \(A_k, B_k\) that appear in (17) are given by (20) and (21) if \(x \leq a - 1/3\) and by (23) and (24) if \(x > a - 1/3\). In case the first inequality is satisfied, \(P\) is computed, and the recurrence relations for evaluating the integrals of (21) and (20) are given by (76) and (77). In case the second inequality holds, \(Q\) is computed, and the recurrence relations for evaluating (24) and (23) are given by (81) and (82).

We write again the basic equations for (17), with (76), (77), (81), (82) slightly changed for greater efficiency in computation. Let

\[B_j = Z(s) b_j,\quad A_j = Z(s) a_j,\quad \left(Z(s) = \frac{1}{\sqrt{2\pi}} e^{-s^2/2}\right).\]  

(109)

Then

\[
\begin{align*}
&b_j = (2j - 1) b_{j-1} + s^{2j-1} \\
&a_j = 2j a_{j-1} + s^{2j}
\end{align*}
\]

(110)

replace (76) and (77) and are used when \(x \leq a - 1/3\) with

\[
\begin{align*}
&b_0 = \left\lfloor 1/2Z(s) \right\rfloor \text{erfc}(|s|/\sqrt{2}) \\
&a_0 = -1.
\end{align*}
\]

(111)

If, on the other hand, \(x > a - 1/3\), then using (109), (81) and (82) become

\[
\begin{align*}
&b_j = (2j - 1) b_{j-1} + s^{2j-1} \\
&a_j = 2j a_{j-1} + s^{2j}
\end{align*}
\]

(112)

with

\[
\begin{align*}
&b_0 = \left\lfloor 1/2Z(s) \right\rfloor \text{erfc}(s/\sqrt{2}) \\
&a_0 = 1.
\end{align*}
\]

(113)

In all cases

\[s = 3(a - 1/3)^{1/2} \left\lfloor x/(a - 1/3) \right\rfloor^{1/3} - 1\]  

(See (22)).

(114)
Thus, in terms of $a_k$, $b_k$, instead of $A_k$, $B_k$, (17) becomes

\[
T(a, x) = Z(s) C \left[ b_0 - \frac{1}{12a^2} b_2 - \frac{1}{18a^4} \left( b_3 - \frac{1}{16} b_4 \right) \right]
\]

\[
- \frac{1}{24a^6} \left( b_4 - \frac{37}{225} b_5 + \frac{1}{432} b_6 \right)
\]

\[
+ \frac{1}{30a^8} \left( b_5 - \frac{743}{3024} b_6 + \frac{49}{4320} b_7 + \frac{5}{82944} b_8 \right)
\]

\[
\vdots
\]

\[
+ \frac{1}{a^{24}} \left( b_{14} - \cdots - 2.3414 \times 10^{-22} b_{24} \right)
\]

\[
+ \frac{a_2}{15a^3} + \frac{1}{3a^5} \left( \frac{33}{7} a_4 + \frac{3413}{1399680} a_5 + \frac{36173}{201553920} a_6 \right)
\]

\[
+ \frac{1}{a^7} \left( \frac{a_6}{22680} - \frac{1187}{2268000} a_7 + \frac{1}{155520} a_8 \right)
\]

\[
\vdots
\]

\[
+ \frac{1}{a^{25}} \left( \frac{a_{24}}{81} - \cdots - 2.2478 \times 10^{-21} a_{24} \right)
\]

where, with $b = a - 1$,

\[
C = 1 + \frac{1}{36} b^1 - \frac{31}{2592} b^2 + \frac{3413}{1399680} b^3 + \frac{361733}{201553920} b^4
\]

\[
\frac{113888281}{50791587840} b^5 + \frac{7565202533}{7836416409600} b^6 \quad \cdots \quad \text{(See (18))}
\]

The program uses no more terms than are shown for (115) and (116). The numerical coefficients in these equations, and some additional ones for (116), not used in the program, are given in Appendix B.
The right hand side of (115) is computed by the following algorithm:

(a) Compute $a_0, b_0, \hat{r}_0 (= b_0)$ (use (111) for $s < 0$ or (113) for $s > 0$)

(b) Compute $1/a^2, 1/a$ ($a = 3(a - 1/3)^{1/2}$)

(c) Set $i = 2$, $\Sigma = \hat{r}_0$ ($\Sigma$ denotes accumulated sum)

(d) Compute $b_{i-1}, b_i, a_{i-1}, a_i$ (by (110) for $s < 0$ or (112) for $s > 0$)

(e) Compute $\hat{r}_i \equiv \left( \sum_{1 \leq i/2} b_i \right)/a^i, \hat{r}_{i+1} = \left( \sum_{1 \leq i/2} M_{i+1,j} a_j \right)/a^{i+1}$

(See (67) and (68)).

(f) Add $\hat{r}_i$ and $\hat{r}_{i+1}$ to accumulated sum, $\Sigma$, to obtain

$$\sum = \sum_{k=0}^{i} \hat{r}_k + \hat{r}_i + \hat{r}_{i+1} \quad (\hat{r}_i = 0)$$

(g) Is $|\hat{r}_i| < \epsilon b_0$, and $|\hat{r}_{i+1}| < \epsilon b_0$, or is $i = 24$?

(h) If no in both cases then increase $i$ by 2 and return to (d)

(i) If yes in either case, for (g), proceed to (j)

(j) Compute $Z(s) \subset \Sigma$ and exit.

The calculation of $s$ does not result in any significant increase in relative error in computing $P$ or $Q$. A simple analysis shows the relative error in $P$ or $Q$ is not greater than twice the absolute error in $s$. Nevertheless, in order to minimize the absolute error in $s$, it is evaluated by the algebraic equivalent of (114), namely

$$s = \frac{a^{1/3}}{(a - 1/3)^{1/3}} \left\{ 1 + \left( \frac{x}{a} \right)^{1/3} + \left( \frac{x}{a} \right)^{2/3} \right\},$$

where

$$b = (x - a) + 1/3.$$
If (10), (12), (15), or (16) is used the function $R(a, x)$, as defined in (11) is computed. If $a > 30$ its computation is carried out by using (28) with $L(a)$ given by (27). This eliminates any scaling problem with evaluating $R(a, x)$ for $30 < a$.

If $a < 30$, then $R$ is evaluated by

$$R(a, x) = \left( e^{-x} + a \right) \frac{\Gamma(a)}{\Gamma(1 + a)}$$  \hspace{1cm} (117)$$

where the following procedure is used to compute $\Gamma(a)$. Let $\lambda$ be the fractional part of $a$.

If $\lambda = 0$, then

$$\begin{cases} \Gamma(j + 1) = j \Gamma(j), & j = 1, 2, \ldots, a - 1, \ a > 1. \\ \Gamma(1) = 1, & a = 1. \end{cases}$$ \hspace{1cm} (118)$$

If $\lambda \neq 0$, then $1/\Gamma(\lambda)$ is found to 14 significant digits by the polynomials

$$1/\Gamma(\lambda) = \begin{cases} \lambda \left[ 1 + \sum_{k=2}^{17} C_k \lambda^{k-1} \right] & 0 < \lambda < 1/2 \\ 1 + \sum_{k=2}^{17} C_k (\lambda - 1)^{k-1} & 1/2 < \lambda < 1; \end{cases}$$ \hspace{1cm} (119)$$

the $C_k$'s are given in Appendix A with the rule for terminating these series. Then $\Gamma(\lambda)$ is computed from

$$\begin{cases} \Gamma(j + \lambda) = (j - 1 + \lambda) \Gamma(j - 1 + \lambda), & j = 1, \ldots, a - \lambda, \ a \geq 1 \\ \Gamma(1) = \Gamma(\lambda), & a < 1. \end{cases}$$ \hspace{1cm} (120)$$

For values of $a > 20$ it was observed that the asymptotic series for $\ln \Gamma(a)$, (26), yielded a faster algorithm, in spite of evaluating $\Gamma(a)$ from $e \ln \Gamma(a)$, than the one based on (118)–(120). However, the relative error was larger. In the procedure above 1.5 more significant figures were obtained in some cases, for $x = x_1$, than by using the asymptotic series.
The error function, erf(x), and its complement erfc(x) are evaluated by Cody's
minimax approximations, [2], (see Appendix A). We note that the error function, or
its complement, is needed in (11), (111), (113).

This completes the discussion of the algorithms for the computer program. A
few final comments are in order regarding the checkout of the program.

The program was extensively checked by computing P (or Q) for a large ran
g of arguments, (a, x), such that both (10) and (12) could be used for the same
arguments. This overlapping procedure was also used with (10) and (13), (10)
and (17), (10) and (15), (13) and (15), (12) and (17). The subprogram (16) was validated by comparing its results with those from a double
precision version of (10). A final check that was made compared the computation
of Q(x + 1, x) from (17), (x > 100), with an asymptotic expansion for this quantity
given by

\[ Q(x + 1, x) \approx \frac{1}{2} + \frac{1}{3 \sqrt{\pi x}} \left( 1 - \frac{23}{180 x} + \frac{23}{2016 x^2} + \cdots \right), \quad (x \to \infty). \quad (121) \]

Values of x as large as \(10^6\) were tested and the computed values of Q by (17) and
(121) met the desired relative accuracy for \(\epsilon = \epsilon_1, \epsilon = \epsilon_2\), and \(\epsilon = \epsilon_3\), i.e., correct to
within one unit in the 12th, 6th and 3rd significant digit, respectively. For complete-
ness, the method by which (121) was obtained and its leading sixteen terms are given
in Appendix C.

The program, as noted earlier, senses if for a given (a, x) P or Q is greater than the
specified \(\epsilon(\epsilon_1\ or \epsilon_2\ or \epsilon_3)\), i.e., if (a, x) is in the zone of computation. If the user
desires P or Q computed for values smaller than \(\epsilon\), i.e., when (a, x) is outside the zone
of computation, this can easily be accomplished by changing one location (ACO) in
the program, (see page E-1).

A FORTRAN listing for the entire program is given in Appendix E.
Figure 3. Master Flow Chart $a > 0 \times 0$
REFERENCES


REFERENCES (continued)


APPENDIX A

COMPUTATION OF THE ERROR FUNCTION,
AND THE RECIPROCAL OF THE GAMMA
FUNCTION OF $a$ WHEN $a < 1$
COMPUTATION OF erf(x) AND 1/′(a), a ≤ 1

The formulations for the computation of erf(x) or erfc(x), depending on x, are taken from [2]. They are based on rational Chebyshev approximations.

If |x| < 0.5

\[
\text{erf}(x) = x \sum_{k=0}^{j} p_k x^{2k} \left/ \sum_{k=0}^{j} q_k x^{2k} \right.
\]  

(A-1)

\[
p_0 = 2.42667 95523 05318 (2) \quad q_0 = 2.15058 87586 98612 (2)
\]
\[
p_1 = 2.19792 61618 29415 (1) \quad q_1 = 9.11649 05404 51490 (1)
\]
\[
p_2 = 6.99638 34886 19136 (0) \quad q_2 = 1.50827 97630 40779 (1)
\]
\[
p_3 = 3.56098 43701 81539 (2) \quad q_3 = 1.0 \quad (0)
\]

The maximum relative error < 10^{-14.65}.

If \( \frac{1}{2} \leq x \leq 4.0 \)

\[
1 - \text{erf}(x) = \text{erfc}(x) = e^{-x^2} \sum_{k=0}^{7} p_k x^k \left/ \sum_{k=0}^{7} q_k x^k \right.
\]  

(A-2)

\[
p_0 = 3.00459 26102 01616 005 (2) \quad q_0 = 3.00459 26095 69832 933 (2)
\]
\[
p_1 = 4.51918 95371 18729 422 (2) \quad q_1 = 7.90950 92532 78980 272 (2)
\]
\[
p_2 = 3.39320 81673 43436 870 (2) \quad q_2 = 9.31354 09485 06096 211 (2)
\]
\[
p_3 = 1.52989 28504 69404 039 (2) \quad q_3 = 6.38980 26446 56311 665 (2)
\]
\[
p_4 = 4.31622 27222 05673 530 (1) \quad q_4 = 2.77585 44474 39876 434 (2)
\]
\[
p_5 = 7.21175 82508 83093 659 (0) \quad q_5 = 7.70001 52935 22947 295 (1)
\]
\[
p_6 = 5.64195 51747 89739 711 (1) \quad q_6 = 1.27827 27319 62942 351 (1)
\]
\[
p_7 = 1.36864 85738 27167 067 (7) \quad q_7 = 1.00000 00000 00000 000 (0)
\]

The maximum relative error < 10^{-16.13}.

A-1
If $x > 4.0$

\[
\text{erfc}(x) = \frac{e^{-x^2}}{x} \left[ \frac{1}{\sqrt{\pi}} \sum_{k=0}^{4} p_k x^{-2k} \right]
\]

(A-3)

\[
p_0 = -2.99610 \ 70770 \ 35421 \ 74 \ (-3) \quad q_0 = 1.06209 \ 23052 \ 84679 \ 18 \ (-2)
\]
\[
p_1 = -4.94730 \ 91062 \ 32507 \ 34 \ (-2) \quad q_1 = 1.91308 \ 92610 \ 78298 \ 41 \ (-1)
\]
\[
p_2 = -2.26956 \ 59353 \ 96869 \ 30 \ (-1) \quad q_2 = 1.05167 \ 51070 \ 67932 \ 07 \ (0)
\]
\[
p_3 = -2.78661 \ 30860 \ 96477 \ 88 \ (-1) \quad q_3 = 1.98733 \ 20181 \ 71352 \ 56 \ (0)
\]
\[
p_4 = -2.23192 \ 45973 \ 41846 \ 86 \ (-2) \quad q_4 = 1.00000 \ 00000 \ 00000 \ 00 \ (0)
\]

The maximum relative error $\leq 10^{-15.61}$.

The reciprocal of the complete gamma function $\Gamma(a)$, when $a < 1$, is computed from (25), (see also (119)). The coefficients $C_k$ which occur are given in [13]. The program uses at most the first seventeen with a maximum relative error in $1/\Gamma(a)$ of less than $5 \times 10^{-14}$. We have

\[
C_1 = 1.0 \quad C_9 = -0.11651 \ 67591 \ 85907 \ (-2)
\]
\[
C_2 = 0.57721 \ 5649 \ 01533 \quad C_{10} = -0.21524 \ 16741 \ 14951 \ (-3)
\]
\[
C_3 = -0.65587 \ 80715 \ 20254 \quad C_{11} = 0.12805 \ 02823 \ 88116 \ (-3)
\]
\[
C_4 = -0.42002 \ 63503 \ 40952 \ (-1) \quad C_{12} = 0.20134 \ 85478 \ 07882 \ (-4)
\]
\[
C_5 = 0.16653 \ 86113 \ 82291 \quad C_{13} = -0.12504 \ 93482 \ 14267 \ (-5)
\]
\[
C_6 = -0.42197 \ 73455 \ 55443 \ (-1) \quad C_{14} = 0.11330 \ 27231 \ 98170 \ (-5)
\]
\[
C_7 = -0.96219 \ 71527 \ 87697 \ (-2) \quad C_{15} = -0.20563 \ 38416 \ 97761 \ (-6)
\]
\[
C_8 = 0.72189 \ 43246 \ 66310 \ (-2) \quad C_{16} = 0.61160 \ 95104 \ 48142 \ (-8)
\]
\[
C_{17} = 0.50020 \ 07644 \ 46922 \ (-8)
\]

It was stated on page 38 that $H$ was obtained during the computation for $1/\Gamma(a)$. We have for

\[
0 < a < 1/2, \quad H = \sum_{k=2}^{17} C_k a^{k-1}
\]

A-2
which is terminated when \( k = 17 \) or before if \( k = n < 17 \) with

\[
\left| C_n \ a^{n-1} \right| < \frac{1}{2} \epsilon \left| \sum_{k=2}^{n} C_k \ a^{k-1} \right|.
\]

Then

\[
1/\Gamma(a) = a \{1 + \tilde{H}\}, \quad 0 < a < 1/2.
\]

For \( 1/2 < a < 1 \), \( \tilde{H} = \sum_{k=2}^{17} C_k (a - 1)^{k-1} \), \( H = \frac{1}{a} (1 - a + \tilde{H}) \).

where \( \tilde{H} \) is terminated when \( k = 17 \) or before if \( k = n < 17 \) with

\[
\left| C_n (a - 1)^{n-1} \right| < \frac{1}{2} \epsilon \left| \sum_{k=2}^{n} C_k (a - 1)^{k-1} \right|.
\]

Then

\[
1/\Gamma(a) = 1 + \tilde{H}, \quad \frac{1}{2} < a < 1.
\]
COEFFICIENTS IN TAKENAGA'S METHOD

In this Appendix, we list the coefficients associated with the modified Takenaga procedure as used in (17) (see (17) or (115)). Also the coefficients for (18) or (80) are listed as well as those needed in (63). For (18) and (63) additional coefficients beyond those actually stored in the program are given. All of the coefficients were obtained initially in rational form by A. Morris, [5].

The factor $C$, which appears in (17) is found from $e^{\ln C}$, where the expansion of $\ln C$ in terms of $R = b$ is given by (63). The expansion of $\ln C$ to $b^{-11}$ is

$$\ln C \equiv \frac{1}{36} b \quad (b = a - 1)$$

\[
\begin{align*}
-1/81 b^2 \\
+1/360 b^3 \\
+2/1215 b^4 \\
-691/306180 b^5 \\
+16/15309 b^6 \\
-373/3674160 b^7 \\
+80/177147 b^8 \\
-44069/38972340 b^9 \\
+1792/9743085 b^{10} \\
+114953857/63836692920 b^{11}
\end{align*}
\]

From the expression (B-1), we obtain $C$ in terms of inverse powers of $b$ (see (80)) as far as the eleventh power, i.e.,...
\[ C = e^{\ln C} = 1 \quad (B-2) \]
\[ + \frac{1}{36} b \]
\[ - \frac{31}{2592} b^2 \]
\[ + \frac{3413}{1399680} b^3 \]
\[ + \frac{361733}{201553920} b^4 \]
\[ - \frac{113388281}{50791587840} b^5 \]
\[ + \frac{7565202533}{7836416409600} b^6 \]
\[ + \frac{81332300683}{1974776935219200} b^7 \]
\[ + \frac{245886906474757}{568735757343129600} b^8 \]
\[ - \frac{11336175048324863533}{10134871195854569472000} b^9 \]
\[ + \frac{30902061509879955337}{204319003308428120555520} b^{10} \]
\[ + \frac{4346156164844627985130999}{2390532338708609010499584000} b^{11} \]

We proceed to give the coefficients \( N_{2k,j}, M_{2k+1,j} \) for (17) or (115). Also see (68). We have from (115)

\[
T(a, x) = Z(s) C \left[ b_0 + \sum_{k=1}^{12} a^{2k} \left( \sum_{j=1}^{2k} N_{2k,j} b_j \right) \right] + \sum_{k=1}^{12} a^{(2k+1)} \left( \sum_{j=1}^{2k} M_{2k+1,j} a_j \right) .
\]

The \( a_j, b_j \) are defined by (109) and (89). Then listing the \( N_{2k,j} \), first

\[
\begin{array}{c|c|c}
2k = 0 & 2k = 6 \\
\hline
j = 0 & 1.00000000000000 & 0.416666666666667 D 01 \\
2k = 2 & 5 & 0.685185185185185 D 02 \\
2k = 4 & 6 & 0.96450612839506 D -04 \\
2k = 6 & 7 & 0.333333333333333 D -01 \\
2k = 8 & 8 & 0.819003527336861 D 02 \\
2k = 10 & 9 & 0.378086419753086 D 03 \\
2k = 12 & 10 & 0.20093878608230 D 05 \\
\end{array}
\]
\[ N_{2k,j} \]

(continued)

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<th>( 2k = 10 )</th>
<th>( 2k = 18 )</th>
</tr>
</thead>
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<tr>
<td>( j = 6 )</td>
<td>( j = 10 )</td>
</tr>
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<td>0.2777777777777778D 01</td>
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</tr>
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</tr>
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<td>( 2k = 12 )</td>
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</tr>
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<td>( j = 7 )</td>
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</tr>
<tr>
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<td>( j = 8 )</td>
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</tr>
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</tr>
<tr>
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\(N_{2k,j}\) (continued)

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<td>-0.167472688827663D-02</td>
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<td>15</td>
<td>0.154597809511729D-03</td>
<td>0.17791995495527D-03</td>
</tr>
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<td>-0.103838417905802D-04</td>
</tr>
<tr>
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<td>0.226017561582738D-06</td>
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</tr>
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\(M_{2k+1,j}\)

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\[ M_{2k+1,j} \]

(continued)

| \( j \) | \( 2k + 1 = 13 \) | \( 2k + 1 = 19 \) \\
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| \( j \) | \( 2k + 1 = 15 \) | \( 2k + 1 = 21 \) \\
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| \( j \) | \( 2k + 1 = 17 \) | \( 2k + 1 = 21 \) \\
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B-5
\( \text{M}_{2k+1,j} \) (continued)

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</tr>
<tr>
<td>15</td>
<td>-0.166412253953536D 03</td>
<td>16</td>
<td>-0.189106579970833D 03</td>
</tr>
<tr>
<td>16</td>
<td>0.913690962754997D 05</td>
<td>17</td>
<td>0.116720392056449D 04</td>
</tr>
<tr>
<td>17</td>
<td>0.284271767098726D 06</td>
<td>18</td>
<td>-0.422734470127948D 06</td>
</tr>
<tr>
<td>18</td>
<td>0.503434278629405D 08</td>
<td>19</td>
<td>0.913761861504362D 08</td>
</tr>
<tr>
<td>19</td>
<td>-0.491182369134031D 10</td>
<td>20</td>
<td>-0.116534009325387D 09</td>
</tr>
<tr>
<td>20</td>
<td>0.243170716076481D 12</td>
<td>21</td>
<td>0.840151056535048D 12</td>
</tr>
<tr>
<td>21</td>
<td>-0.512512677374051D 15</td>
<td>22</td>
<td>0.313185291085580D 14</td>
</tr>
<tr>
<td>22</td>
<td>0.296710726943426D 18</td>
<td>23</td>
<td>0.503843072514408D 17</td>
</tr>
<tr>
<td></td>
<td></td>
<td>24</td>
<td>0.224780853745020D 20</td>
</tr>
</tbody>
</table>
ASYMPTOTIC EXPANSION FOR Q(x + 1, x) \ (x \to \infty)

In this appendix we show how the asymptotic expansion of (121) was obtained. The form of the expansion is given by

\[
Q(x + 1, x) \cong \frac{1}{2} + \frac{1}{3} \sqrt{2 \pi} \sum_{k=0}^{M-1} e_k / x^k, \quad (x \to \infty),
\]

where

\[
e_0 = 1, \quad e_1 = -23/180, \quad e_2 = 23/2016,
\]
as shown on page 43. The \(e_k\) for \(k = 0, 1, \cdots, M = 15\) are given on page C-8 of this appendix.

We start with

\[
Q(x + 1, x) = \Gamma(x + 1, x) / \Gamma(x + 1) \quad \text{(See (2), (4), (5))}, \tag{C-2}
\]

where the asymptotic expansions for the two functions on the right hand side are given by

\[
1/\Gamma(x + 1) = 1/x \Gamma(x) = e^{-\ln x} \Gamma(x) \cong e^{\left(\sqrt{2\pi} x^{1/2} / x\right)} \sum_{k=0}^{M-1} f_k / x^k, \quad (x \to \infty), \tag{C-3}
\]

with

\[
\ln \Gamma(x) \equiv \left(x + \frac{1}{2}\right) \ln x - x + \frac{1}{2} \ln 2\pi + \sum_{m=1}^{\infty} \frac{B_{2m}}{2m(2m-1)x^{2m-1}}, \quad (x \to \infty), \tag{C-4}
\]

\((B_{2m}\), the \(2m^{th}\) Bernoulli number), \quad \text{(See [1], p. 257, 810),}

and

\[
\Gamma(x + 1, x) \cong e^{-x} x^{1/2} \sum_{k=1}^{2M} \frac{b_k}{x^{k-1/2}}, \quad (x \to \infty). \tag{C-5}
\]
The coefficients \( b_k \), \( \Gamma \left( \frac{k}{2} \right) \), and \( f_k \) are constants. The multiplication of the series in (C-3) and (C-5) to yield (C-1) was carried out by A. Morris using his "Flap" routine, [5]. In addition, he also used "Flap" before hand to determine the \( f_k \) and \( b_k \Gamma \left( \frac{k}{2} \right) \) in rational form. For completeness they are included on pages C-6 and C-7, respectively.

The \( b_k \) were computed independently by M. Saizan in decimal form and are given below.

\[
\begin{array}{c}
b_1 = 0.7071067811865475244004436210+00 \\
b_2 = 0.66666666666666666666666+00 \\
b_3 = 0.11785113019775792073347406540+00 \\
b_4 = 0.2362962962962962962962962-01 \\
b_5 = 0.32736425054932755759278350060-02 \\
b_6 = 0.1410934744268077601410347450-02 \\
b_7 = 0.10111917961412562334538223690-03 \\
b_8 = 0.3135410542179502253576627620-03 \\
b_9 = 0.2472527389373814204443595020-04 \\
b_{10} = 0.29664995371442559371228781380-04 \\
b_{11} = 0.187733147223528359070456900-04 \\
b_{12} = 0.56531048757643453773952173460-05 \\
b_{13} = 0.30356279857570629915558755250-06 \\
b_{14} = 0.65675582619137971472473326840-06 \\
b_{15} = 0.39661662515866434381091036460-06 \\
b_{16} = 0.11709055456525309149975384840-06 \\
b_{17} = 0.4618562455125904626849106600-08 \\
b_{18} = 0.14926778659329048172708068980-07 \\
b_{19} = 0.85193073079439970634256291640-08 \\
b_{20} = 0.25741666714184579016143015190-08 \\
b_{21} = 0.79680309544185471304773757500-10 \\
b_{22} = 0.34528658089603027319503611100-09 \\
b_{23} = 0.2016334255707250246858179830-09 \\
b_{24} = 0.5843946251833163681724231240-10 \\
b_{25} = 0.1484457788493658882335740850-11 \\
b_{26} = 0.8090537285365531785372399600-11 \\
b_{27} = 0.46917458313033741970282534700-11 \\
b_{28} = 0.135352575724733746228894010-11 \\
b_{29} = 0.2931694177930550846889634750-13 \\
b_{30} = 0.1914788206765601283127746780-12
\end{array}
\]

We note the first three terms of (C-5) are given in [1, p. 263].

*The symbols \( b_k \) and \( a_k \) are used in this appendix to retain the notation in [6]. They should not be confused with those in (109).
The derivation below for (C-5) is given for completeness and is based on the material given in [6, p. 85–87].

We have

$$I(x + 1, x) = \int_x^\infty t^x e^{-t} \, dt. \quad (C-6)$$

Letting $t = x(1 + u)$, (C-6) transforms to

$$I(x + 1, x) = e^{x^2} \int_0^\infty e^{-x^2u} \int_{x(1+u)}^u \, du. \quad (C-7)$$

The integral on the right in (C-7) is treated in [6, p. 87]. Its asymptotic expansion ($x \to \infty$) is obtained by a series reversion. Indeed, letting

$$V = u, \quad v_n(1 + u) = \frac{u^2}{2} + \frac{u^3}{3} + \cdots, \quad |u| < 1, \quad (C-8)$$

we note that

$$\frac{dV}{du} = 1, \quad \frac{1}{1 + u} = \frac{u}{1 + u}$$

or

$$\frac{du}{dV} = \frac{1 + u}{u}. \quad (C-9)$$

Then, assuming an expansion for $u$ of the form

$$u = \sqrt{\alpha} + \sum_{k=2}^\infty a_k V^{k/2}, \quad (a_1 = \sqrt{\alpha}), \quad (C-10)$$

we have by differentiating (C-10) and using (C-9)

$$\frac{1 + u}{\alpha} = \sum_{k=2}^\infty \frac{k}{2} a_k V^{k-1}, \quad a_1 = \sqrt{\alpha}.$$
It follows from this result that

\[ 1 + \sum_{k=1}^{\infty} a_k V^k = \sum_{k=1}^{\infty} a_k V^k \sum_{k=1}^{\infty} \frac{k}{2} a_k V^{k-1} \]  

(C-11)

Expressing the product on the right as a Cauchy product, (C-11) becomes

\[ 1 + \sum_{k=1}^{\infty} a_k V^k = \sum_{i=1}^{\infty} V^{i-1} \sum_{j=1}^{i} \frac{j}{2} a_j a_{i-j} \]  

(C-12)

which contains a linear set of equations for the \( a_k \), \( k = 0, 1, \ldots \) with \( a_0 \equiv 0 \). For example, equating coefficients of like powers of \( V \), we have

\[ V^{-1/2} : \quad 0 = \frac{1}{2} a_1 a_0 \Rightarrow a_0 = 0, \text{ since } a_1 = \sqrt{2} \]

\[ V^0 : \quad 1 = \frac{1}{2} a_1^2 + a_2 a_0 \Rightarrow a_1 = \sqrt{2} \]

\[ V^{1/2} : \quad a_1 = \frac{1}{2} a_1 a_2 + a_1 a_1 \Rightarrow a_2 = \frac{2}{3} \]

\[ \vdots \]

In general, by setting \( k = n - 1, i = n + 1 \), (C-12) can be solved for \( a_n \). We get

\[ a_{n-1} = \sum_{j=1}^{n} \frac{j}{2} a_j a_{n-j} \quad (a_0 = 0) \]

or

\[ a_n = \frac{2}{(n+1)a_1} \left[ a_{n-1} \sum_{j=2}^{n-1} \frac{j}{2} a_j a_{n+1-j} \right], \quad n = 3, 4, \ldots \]  

(C-13)

Thus,

\[ a_2 = \frac{2}{3}, \quad a_3 = \sqrt{2}/18, \quad a_4 = \frac{2}{135}, \quad a_5 = \sqrt{3}/1080, \quad a_6 = 4/8505, \]

etc.
The expression (C-13) gives the coefficients for the inverted series in (C-10). Therefore, following [6], the coefficients $b_k$ of asymptotic series (C-5) can now be determined. We have

\[
\frac{du}{dV} = \frac{d}{dV} \left( \sum_{k=1}^{\infty} a_k V^{k/2} \right) = \sum_{k=1}^{\infty} \left( \frac{k}{2} \right) a_k V^{(k-2)/2} \\
= \sum_{k=1}^{\infty} b_k V^{(k-2)/2}, \quad b_k = \frac{k}{2} a_k.
\]

The $b_k$ are the desired coefficients that appear in (C-5). Thus, e.g.,

\[
b_1 = \frac{1}{2} a_1 = 1/\sqrt{2}, \quad b_2 = a_2 = 2/3, \quad b_3 = \frac{3}{2} a_3 = \frac{\sqrt{2}}{12}, \quad b_4 = 2a_4 = -4/135.
\]

The $a_k$ are listed below for completeness.

\[
a_1 = 1.4142135623730950488016387240+01 \\
a_2 = 6.66666666666666666666666670+00 \\
a_3 = 7.85674201318386138223160402+00 \\
a_4 = 14.41418414814814814814814810+01 \\
a_5 = 130.94570021973102303719340030+02 \\
a_6 = 470.31158142269253380364491490+03 \\
a_7 = 288.91194175464463812988067690+03 \\
a_8 = 783.8526657044875563394031050+04 \\
a_9 = 549.495611642083069823295677820+05 \\
a_{10} = 5932.9990742885118742457562760+05 \\
a_{11} = 341.33299495186973546742648300+05 \\
a_{12} = 94.21841459640575268920289150+06 \\
a_{13} = 4670.196901164712294701346960+07 \\
a_{14} = 9382.226088482816389247609770+07 \\
a_{15} = 5268.22216667621911747880484620+07 \\
a_{16} = 1463.6319331578664374691981050+07 \\
a_{17} = 543.36028868343225445704831380+09 \\
a_{18} = 16585.3739255452414120076650+08 \\
a_{19} = 9284.176011362526006667638201+07 \\
a_{20} = 2574.166671418457901614361519+09 \\
a_{21} = 7588.60039086843306004546443580+11 \\
a_{22} = 31368.050733623718435632396460+10 \\
a_{23} = 1753.386309310652386856678110+10 \\
a_{24} = 4869.955209736096973477019270+11 \\
a_{25} = 11907.56623079429715086892680+12 \\
a_{26} = 6223.4902195119475272140768920+12 \\
a_{27} = 3475.36578743240145968585890+12 \\
a_{28} = 9668.041123195267258749210070+13 \\
a_{29} = 20218.580537507247219923889480+14 \\
a_{30} = 12765.294711770673522085164520+13
\]
The rational coefficients $f_n$ for (C-3) are included in the expression below, (C-14). The quantity $T = 1/\sqrt{x}$. The coefficient of $1/x^3$, $f_3$, is given by $139/51840$.

$1$

$-1/12*T**2$

$1/288*T**4$

$139/51840*T**6$

$-571/2488320*T**8$

$-163879/209018880*T**10$

$5246819/75246796800*T**12$

$534703531/902961561600*T**14$

$-4483131259/86684309913600*T**16$

$-432261921612371/514904800886784000*T**18$

$6232523202521089/86504006548979712000*T**20$

$25834629665134204969/13494625021640835072000*T**22$

$-1579029138854919086429/9716130015581401251840000*T**24$

$-746590869962651602203151/116593560186976815022080000*T**26$

$15.1513601028097903631961/279824544487443560529920000*T**28$

The coefficients $b_k \Gamma(k/2)$ for (C-5) are included in the expression below, (C-15). The quantity $T = 1/\sqrt{x}$. The coefficient of $1/x^{3/2}$, $b_3 \Gamma(2)$, is given by $-4/135$. The coefficient of $1/x^3$, $b_7 \Gamma(7/2)$, is given by $(-139/103680) \sqrt{2\pi}$. 

C-6
The coefficients $c_k$ for (C-1) are included in rational form in (C-16) below and then $c_k$, $k \geq 0$, are given in decimal form in (C-17). Again $T = 1/\sqrt{x}$. The coefficient of $1/x^{3/2}$ from (C-16) is $(23/270)/\sqrt{2\pi}$.

\[1/2\]
\[
\begin{align*}
&2^{3}2^{2}1\cdot(-1/2)\times P\times(-1/2)!+T \\
&-2^{3}2^{2}1\cdot(-1/2)\times P\times(-1/2)!+T \\
&2^{3}2^{2}1\cdot(-1/2)\times P\times(-1/2)!+T \\
&2^{5}7760\times P\times(-1/2)\times P\times(-1/2)!+T \\
&-2^{3}2^{3}7760\times P\times(-1/2)\times P\times(-1/2)!+T \\
&-50838/3\times (-1/2)!+T \\
&-128\times(-1/2)!+T \\
&67896,5/1128\times(-1/2)!+T \\
&113219871,92871,7208817790520002\times(-1/2)!+T \\
&-170646,3053/4\times (-1/2)!+T \\
&-1146352,7623741,773573213,21760602\times(-1/2)!+T \\
&6273446,4235,1233/4\times 76938,338971,70046002\times(-1/2)!+T \\
&14145,1302,4932,2333,4483,9735,26,42,25521,1062\times(-1/2)!+T \\
&-430544,0711,64,9145,1563,1472,1936516384,731432,00602\times(-1/2)!+T \\
&-9322493595,5270,274,16859,077,7531,452541,782346,03,08,0128000002\times(-1/2)!+T \\
&23349062,07795573,4403,647,1301184,1318,0,86,125586,4412800002\times(-1/2)!+T \\
\end{align*}
\]

(C-16)

Decimal form for $e_k$, $k \geq 0$.

\[
\begin{align*}
&c_0 = 1.000000 \\
&c_1 = 1.277777 \\
&c_2 = 1.140877 \\
&c_3 = 4.996149 \\
&c_4 = 1.637048 \\
&c_5 = 1.681256 \\
&c_6 = 9.015541 \\
&c_7 = 1.404808 \\
&c_8 = 1.056513 \\
&c_9 = 2.154392 \\
&c_{10} = 2.106740 \\
&c_{11} = 5.241389 \\
&c_{12} = 6.352544 \\
&c_{13} = 1.852644 \\
&c_{14} = 2.691328 \\
\end{align*}
\]

(C-17)

C-8
CRITERION FOR TERMINATING THE SERIES IN (10)

In the main text of this report an expression for \( P(a, x) \) is given by (10), i.e.,

\[
P(a, x) = \frac{e^x}{al'(a)} \left[ 1 + \frac{x}{a + 1} + \frac{x^2}{(a + 1)(a + 2)} + \cdots + s_n + T_{N+1} \right],
\]

where

\[
s_n = x^n / [(a + 1)(a + 2) \cdots (a + n)]
\]

\[
T_{N+1} = s_{n+1} + s_{n+2} + \cdots.
\]

The objective in this Appendix is to prove the statement made in Section 5 dealing with (90). Given an \( \epsilon > 0 \), if \( N \) is defined as the smallest positive integer for which \( s_N < \frac{\epsilon}{2} \), we wish to show the truncation error, \( T_{N+1} \), associated with the series in square brackets of (D-1) is less than \( \epsilon \), provided \( x/(a + N + 1) < 2/3 \). The proof requires only a few lines.

From (D-3)

\[
T_{N+1} = s_N \left( \frac{x}{a + N + 1} \right) \left[ 1 + \frac{x}{a + N + 2} + \frac{x^2}{(a + N + 2)(a + N + 3)} + \cdots \right]
\]

\[
< \frac{\epsilon}{2} \cdot \frac{2}{3} \cdot \sum_{k=0}^{\infty} (2/3)^k = \epsilon
\]

where we have used the inequalities \( x/(a + N + j) < x/(a + N + 1) < 2/3, j = 2, 3, \ldots \).
FORTRAN LISTING OF THE PROGRAM

Calling sequence to the Incomplete Gamma Function Ratio Subroutine.

CALL PAX(a, x, P, Q, IOP) where

\[ P = \int_0^x e^{-t} t^{a-1} \, dt / \int_0^\infty e^{-t} t^{a-1} \, dt \]

\[ Q = 1 - P \]

IOP = 0 for 12 significant digits of accuracy with an error no greater than 1 unit in the 12th significant digit of P and Q.

IOP = 1 for 6 significant digits of accuracy with an error no greater than 1 unit in the 6th significant digit of P and Q.

IOP = 3 for 3 significant digits of accuracy with an error no greater than 1 unit in the 3rd significant digit of P and Q.

The routine is designed in most cases to set P = 0, Q = 1 if P ≤ ε and P = 1, Q = 0 if Q ≤ ε. The quantity ε is ordinarily set to take the specified value \( \epsilon = 5 \times 10^{-13} \), \( 5 \times 10^{-7} \), or \( 5 \times 10^{-4} \) when IOP is set to 0, 1, 3, respectively. However, ε may be set internally to zero or any other positive value less than the value of ε specified by IOP. In the program ε is stored in AC1.

Restrictions: \( x > 0 \) and \( a > 0 \).

Error Returns: If either \( a < 0 \) and/or \( x < 0 \) then P is set equal to 2.
SUBROUTINE PAX (A, X, Y, ANS, QANS, IOP3)
DIMENSION VA(2C), VB(20), ZA(30), ZB(30), SA(20), SB(20)

DATA 1 P2/2.4266 79552 30532 E2 /
2 P1/2.1979 26161 32942 E1 /
3 P2/6.9963 63488 61914 E0 /
4 P3/-3.5609 44370 18154 E-2 /
5 Q2/2.15C5 88758 69861 E2 /
6 Q1/9.1164 95540 45149 E1 /
7 Q2/1.5082 79763 64778 E1 /
8 Q3/1. /
9 CO/3.0045 92610 20162 E2 /

DATA 1 C1/4.5191 89537 11673 E2 /
2 C2/3.3932 81167 34344 E2 /
3 C3/1.5298 92850 46940 E2 /
4 C4/4.31c2 22722 2.567 E1 /
5 CF/7.2117 54357 893.9 E0 /
6 C6/5.6419 55174 76974 E-1 /
7 C7/-1.3686 4857 3 82717 E-7 /
8 ALPH2/-2.9961 6787 73542 E-3 /
9 ALPH1/-4.973 99106 23251 E-2 /

DATA 1 D1/7.9095 69253 27898 E2 /
2 D2/9.3135 4.948 5261 E2 /
3 D3/4.3698 1064 E2 /
4 D4/2.7758 54447 43988 E2 /
5 D5/7.751 15243 52295 E1 /
6 D6/1.2782 72731 96294 E1 /
7 D7/1.

C CST2=SP1D344.3/DP13996 B. )
C CST3=SP1D31731/DPZ.155 3521.
C CST4=SP1D34 (113.4924/DP60791 58784 7 )
C CST5=SP1D31562 0.2533/DP78364 76419 7800. )

DATA 1 CST5/.96539 05736 46935E-3 /
2 CST4/.22462 66600 53312E-2 /
3 CST3/.17947 20737 75513E-2 /
4 CST2/.24334 14494 74166E-2 /
5 GAMMT/1.7724 53850 90552 /
6 RT2PI/2.5666 24274 61130 /
7 RPT1/1.7724 53150 96552 /

1 ALPH1/7.136/ACO/5.E-13/.
1 X01/31.1C0/.,X.3/17.0,CEP/2.3025 85092 99405/
2  ,UX1/1.3366/,UX3/2.9245/
3  ,RT2/.14142 13562 37310E+1/,ABAR/100. /

ASAV=A
INO=0
ANS=0.
IF ( IOP3.NE.0 ) GO TO 1105
ALPHA=ALPHA1
X0=X01
ACC=ACC1
UX=UX1
GO TO 1111
1105 CONTINUE
IF ( IOP3.NE.1 ) GO TO 1107
ALPHA=ALPHA3
ACC=ACC3
X0=X03
UX=UX3
GO TO 1111
1107 CONTINUE
ALPHA=ALPHA5
ACC=ACC5
X0=X05
UX=UX5
1111 CONTINUE
IF ( A.LE.C.0 .OR.X.LT.0.0 ) GO TO 1131
IF ( X.GT.C.0 ) GO TO 1151
1121 CONTINUE
GO TO 1171
1131 CONTINUE
ANS=2.
RETURN
1151 CONTINUE
IF ( A.LT.15. ) GO TO 1331
RTA=SQRT(A)
IF ( ACO.LT.ACC1 ) GO TO 1155
IF ( A.GE.X ) GO TO 1161
FP=1.-1.(/9.*AI +ALPHA//3.*RTA)
FP=-FP*FP*FP*A*X
IF ( FM.GE.0.L ) GO TO 1191
1155 CONTINUE
IF ( A.LT.ABAR ) GO TO 1331
IF ( (.75*A)*GT.X.OR.1.25*A)*LT.X ) GO TO 1331
1161 CONTINUE
IF ( X.LE.UX ) GO TO 1171
FM=1.-1.(/9.*AI -ALPHA//3.*RTA)
FM=-FM*FM*FM*A*X
IF ( FM.LT.0.L ) GO TO 1121
1171 CONTINUE
ANS=0.
QANS=1.
RETURN
1191 CONTINUE
1311 CONTINUE
ANS=1.
QANS=C.
RETURN
1331 CONTINUE
IF ( A.LE.X ) GO TO 1441
1341 CONTINUE
IF ( A.GE.30. ) GO TO 1343
ALGx=ALOG(x)
T1=-X*A*ALGx
CALL GAHC03 ( A,H,ACC,GANS )
R=EXP(T1)/GANS
GO TO 1349
1343 CONTINUE
AISO=1./(A*A)
T1=A-X*A*ALOG(X/A)+((AISO-1.3333 33333 33333 33333)*AISO*4.666 66666
1 666667)**FISQ-14C.)/(166G.*A)
R=R*EXP(T1)*.35894 22604 01433
1349 CONTINUE
IF ( A.GT.X ) GO TO 1351
1350 CONTINUE
IF ( A.LT.1. ) GO TO 1459
IF ( A.GE.X ) GO TO 3011
GO TO 3005
1351 CONTINUE
IF ( R.LE.(A*(1.-X/(A+1.1))*ACD)) GO TO 1171
GO TO 1350
1441 CONTINUE
IF ( X.GE.XC ) GO TO 1341
TWOA=2.*A
J=INT(TWOA)
T1=ABS(FLOAT(J)-TWOA)
IF ( T1.GT.0.1G0 ) GO TO 1341
I=J/2
AF=A-FLOAT(I)
IF ( AF.GT.0. ) GO TO 6011
GO TO 6071
1459 CONTINUE
IF ( X.LT.1.5 ) GO TO 1461
T1=(2.*X-1.)/(1.+A)
IF ( R.LE.(ACD*X*T1) ) GO TO 1311
IF ( R.GT.(1.01*X*T1) ) GO TO 3011
GO TO 7:11
1461 CONTINUE
INO=1
GO TO 3011
C
C THIS IS PART F

C
2011 CONTINUE
ACCD=.5*ACC
T7=A*ALGx
CEE=2.
T=T7/CEE
SUM=T
2031 CONTINUE
CEE = CEE + 1.
T = (T + T7) / CEE
SUM = SUM + T
IF (ABS(T) .GT. ACC7) GO TO 2031
T2 = T7 * (1. + SUM)
Tk = X
CEE = 1.
AJK = Tk / (A + CEE)
SUM = AJK
2071 CONTINUE
CEE = CEE + 1.
Tk = -(X * Tk) / CEE
AJK = Tk / (A + CEE)
SUM = SUM + AJK
IF (ABS(AJK) .GE. -ACC*SUM) GO TO 2071
T3 = -A*SUM
T1 = H
T4 = T1 + T2
T5 = T1 + T2
QANS = T3 - T4 + T3 * (T4 + T5) - T5
ANS = 1. - QANS
RETURN
3005 CONTINUE
IF (X .LE. CEP) GO TO 3011
IF (X .LT. X3) GO TO 7011
IF (A .LE. 2.) GO TO 3006
IF (RMX_n .LE. (AC0 + XM2 - A)) GO TO 9L11
3006 CONTINUE
IF (R .LE. (AC0 + XM2)) GO TO 1311
GO TO 9011
C
C
C
C
3011 CONTINUE
HAVACC = ACC*.5
ROVA = R/A
SNP1 = 1.
SUM = SNP1
CEE = A
3031 CONTINUE
CEE = CEE + 1.
T1 = X / CEE
SNP1 = SNP1 * T1
SUM = SUM + SNP1
IF (T1 .GT. 66666 66666 66666 66667) GO TO 3031
3041 CONTINUE
IF (SNP1 .LE. HAVACC) GO TO 3051
CEE = CEE + 1.
SNP1 = SNP1 * X / CEE
SUM = SUM + SNP1
GO TO 3041
3051 CONTINUE

1-5
ANS=ROVA*SUM
QANS=1.-ANS
IF ( INO.EQ.0.AND.ANS.GT.C.9 ) GO TO 2011
RETURN
C
C THIS IS PART E1
C
5011 CONTINUE
ASAV=A
A=A-1.
ALPHY=3.*SQRT(A+2./3.)
ALPHINV=1./ALPHY
A2THRU=A+.66666 66666 66667
DELT4=(X-1.)-.66666 66666 6667
S2=DELT4/A2THRU
Z=EXP(33333 33333 33333*ALOG(X/A2THRU))
S0=(ALPHY*SZ)/((Z+1.)*Z+1.)
XSAV=X
X=IS3*SO1/2.
RTX=ARS(XC)/RT2
T5=EXP(-X)
IF ( X.GT.0.25 ) GO TO 5361
T1=((P3*X+P2)*X+P1)*X+P0
T3=((Q3*X+Q2)*X+Q1)*X+Q0
DEL=RTX*(T1/T3)
CERF=1.-DEL
GO TO 5381
5361 CONTINUE
IF ( X.GT.16. ) GO TO 5371
T1=Q0*RTX*Q1*RTX*Q2*RTX*Q3*RTX*(Q4+RTX*(Q5+RTX*(Q6+RTX*Q7))))
1)
T3=Q0*RTX*Q1*RTX*Q2*RTX*Q3*RTX*(Q4+RTX*(Q5+RTX*Q6+RTX*Q7))))
1)
CERF=T1/T3
DEL=1.-CERF
GO TO 5381
5371 CONTINUE
T=1./X
T1=ALPHX+T*(ALPH1+T*(ALPH2+T*(ALPH3+T*ALPH4)))
T3=BE1+T*(BE1+T*(BE2+T*(BE3*T*BE4)))
CERF=X/RTX*1.5/RTPI*T1/(T3*X))
DEL=1.-CERF
5381 CONTINUE
ZA0=1.
IF ( X.LE.0.25 ) ZAO=1.5*CERF*RT2PI/T5
IF ( X.GT.16. ) ZAO=0.5*CERF*RT2PI
IF ( SC.GE.CS ) GO TO 5389
ZA0=-1.
5389 CONTINUE
X=XSAV
Z6=1./Z6
Z4=1./Z4
X=SO=SL*S
T7=1./ABS(S)
TOL=ACC*Z60

E-6
DO 5391 L=2,5
K=2*(L-1)
T7=T7/S0*SQ
Z8(L)=T7
ZB(L)=-T7
1+FLOAT(K-1.)*ZB(L-1)
ZA(L)=-T7
*SQ+FLOAT(K)*ZA(L-1)
5391 CONTINUE
VB(1)=ZB(1)
VB(2)=-.83333 33333 333333E-1*ZB(3)
VA(1)=.66666 66666 66667E-1*ZA(3)
S8(1)=VB(1)
ALPHSO=ALPINV*ALPINV
ALPHGB=ALPHSO*ALPINV
SB(2)=VB(2)*ALPHSO
SA(1)=VA(1)*ALPHCG
T15=SB(1)+SB(2)
T17=SA(1)
VB(3)=-{-2614}-.0625*Z8(5) *.55555 55555 55555E-1
VA(2)=.47619 04761 90476E-1*ZA(4) -.55555 55555 55555E-2*ZA(5)
T23=ALPHSO
I5=3
MF=0
5395 CONTINUE
I7=I5-1
T23=T23*ALPHSO
SB(15)=VB(15)*T23
SA(17)=VA(17)*T23*ALPINV
T15=T15+SB(15)
T17=T17+SA(17)
IF ( ABS(SB(15)) .LT. TOL. AND. ABS( SA(17)) .LT. TOL ) GO TO 5431
I3=2*I5
I5=I5+1
T7=T7/S0*SQ
T25=4.*FLOAT(I5)-7.
ZB(I3)=-T7+T25*ZB(I3-1)
ZA(I3)=-T7+T7*SQ+(T25+1.)*ZA(I3-1)
T7=T7*S0*SQ
ZB(I3-1)=-T7+(T25+2.)*ZB(I3)
ZA(I3-1)=-T7*SQ+(T25+3.)*ZA(I3)
MF=MF+1
GO TO ( 5397, 5399, 5431, 5433, 5435, 5437, 5439, 5441 )
5399 CONTINUE
VB(4)=(-Z8(5))-.1644 44444 4444*ZB(6)* .23148 14814 81481E-2*Z8(7)
VA(3)=(ZA(5))-.20714 20571 42857*ZA(6)* .0625*ZA(7)) *.37037 03703
1 J) *.41666 66666 66667E-1
GO TO 5395
5401 CONTINUE
VA(11) = -0.1442 37532 31844E-1*ZB(12) +0.6354 46152 81766E-2*ZB(15) +
2 =0.1562 3656 62573E-2*ZB(13) -0.1424 9273 42552E-2*ZB(15) +
3 =-0.6797 46263 92697E-5*ZB(16) -1.751 13179 57175E-6*ZB(17) +
4 =-0.6425 0.2019 26872E-1*ZB(10) -0.35605 8723 32111E-1*ZB(14) +
VB(11) = -0.1515 51719 15151E-1*ZB(12) +0.6159 47705 46944E-2*ZB(13) +
1 =-0.1516 52840 22335E-2*ZB(14) +0.3214 45695 3248E-3*ZB(15) -
5413 CONTINUE
VA(12)=-1.3333 33333 33333 E-1*ZA(13)-7.8228 7.319 4.7945E-2*ZA(14)
1 1.1642 35335 54489E-2*ZA(15)-1.6641 22559 5.3536E-3*ZA(16)+
2 9.1369 9627 4.397E-5*ZA(17)-2.8427 17670 6.726E-1*ZA(18)+
3 5.0343 42785 29415E-8*ZA(19)-4.9119 23691 3.493E-1*ZA(20)+
4 2.4317 716 76481E-12*ZA(21)-5.1251 26773 7.455E-1*ZA(22)+
5 2.9671 7.669 4.3426E-18*ZA(23)

VB(12)=-1.3338 58888 88888E-1*ZA(13)+7.9379 4.362 75139E-2*ZA(14)
1 -1.6039 76493 89170E-2*ZA(15)+1.9549 74355 11729E-3*ZA(16)-
2 7.1368 11111 28500E-5*ZA(17)+2.2351 7.7615 82735E-6*ZA(18)-
3 3.5525 6541 8641E-8*ZA(19)+2.9351 25131 22.191E-11*ZA(20)-
4 1.1351 52915 7779E-12*ZA(21)+.15756 9166 7390E-15*ZA(22)-
5 3.3171 286 1753E-19*ZA(23)

GO TO 5395

5415 CONTINUE
VA(12)=+1.12345 67901 23457E-1*ZA(1)+ -7.5962 63771 72480E-2*ZA(1)
1 5)+.17648 20355 6013E-2*ZA(1)+-.1891 65795 7.833E-3*ZA(17)+
2       1.1672 3390 5649E-4*ZA(18)-.42273 44701 27945E-6*ZA(19)+
2 9.1376 16615 4362E-8*ZA(20)-1.1653 40093 25347E-9*ZA(21)+
3 8.4115 10565 3574E-12*ZA(22)-.31118 52910 6598E-14*ZA(23)+
4 50.384 37725 1441E-17*ZA(41)+.22478 98137 45252E-2*ZA(25)+

VB(13)=-1.2520 51282 5128E-1*ZA(14)+.77823 28747 65910E-2*ZA(15)-
1 -1.1677 2643 27663E-2*ZA(14)+.17731 99504 95527E-3*ZA(17)-
2 1.0383 6419 5562E-4*ZA(18)+.34987 60792 2823E-6*ZA(19)+
3 6.8864 71270 71394E-4*ZA(21)+.7734 5530 6075E-11*ZA(22)+
4 4.7332 7376 28715E-12*ZA(22)+.13608 28591 5537E-14*ZA(23)-
5 1.8316 1846 8932E-17*ZA(24)+.23414 67226 51662E-21*ZA(25)

GO TO 5395

5431 CONTINUE
T=1./A
CST=(CST*TA*CST4)T*CST3)T*CST2)T-.11956 6765 32699E-1)*1
1 +.27777 77777 77777 E-1)*T+1.
PAP=1.*CST1*171*T13)*T5)/RT2PI
A=ASAV
IF ( SC.LT.0 ) GO TO 5441
QANS=PAPX
ANS=1.-QANS
GO TO 5451

5441 CONTINUE
ANS=PAP
QANS=1.-ANS

5451 CONTINUE
RETURN

C

THIS IS PART B1

C

6011 CONTINUE
N=0
RTX=SQRT(X)
T=EXP (-X)
RN=RT*X*GAMPS5
C THIS IS PART B1 SUPPLEMENT
C
IF (X.GT.16.) GO TO 6031
T1=60*RTX*(C1+RTX*(C2+RTX*(C3+RTX*(C4+RTX*(C5+RTX*(C6+C7*RTX))))))
1 )
T3=60*RTX*(D1+RTX*(D2+RTX*(D3+RTX*(D4+RTX*(D5+RTX*(D6+RTX*D7))))))
1 )
DEL=(T5 * T1)/T3
GO TO 6091
6031 CONTINUE
T1=ALPH0+T*(ALPH1+T*(ALPH2+T*(ALPH3+T*ALPH4)))
T3=BETC+T*(BET1+T*(BET2+T*(BET3+T*BET4)))
DEL=1. T5/RTX*(1./RTPI + T1/(T3*X))
GO TO 6091
6071 CONTINUE
N=1
RN=EXP(-X)
DEL=RN
6091 CONTINUE
PNCX=DEL
CEE=AF*FLOAT(N)-1.
6101 CONTINUE
IF (N.EQ.1.) GO TO 6111
N=N+1
CEE=CEE+1.
RN=(X*RN1)/CEE
PNCX=PNCX+RN
GO TO 61"1
6111 CONTINUE
R=X*"N
QANS=PNCX
ANS=1.-QANS
RETURN
C THIS IS PART C
C
7011 CONTINUE
A2NM1=1.
A2N=1.
B2NM1=X
B2N=X+1.-A
AN=1.
7031 CONTINUE
A2NM1=X*A2N*AN*A2NM1
B2NM1=X*B2N*AN*B2NM1
AMC=A2NM1/B2NM1
AN=AN+1.
SA2N=AN-A
A2N=A2NM1+SA2N*A2N
B2N=B2NM1+SA2N*B2N
ANG=A2N/B2N
IF (ABS(ANG-AMC) .GE. (ACC*ANG)) GO TO 7031
QANS=R*AN.
ANS=1.-QANS
RETURN

THIS IS PART D

9611 CONTINUE
SNP1=1.
SUM=SNP1
CEE=A

9031 CONTINUE
CEE=CEE-1.
SNP1=(SNP1*CEE)/X
SUM=SUM+SNP1
IF ( ABS(SNP1).GE.ACC ) GO TO 9031
QANS=(R*SUM)/X
ANS=1.-QANS
RETURN
END
SUBROUTINE GAMCO3 ( A, M, AC, T1 )
DIMENSION C(30)
DATA ( C(I), I=1,18 ) /
1  0.57721 5639 9 1533 E+0 ,
2  -65587 39715 20254 E+0 ,
3  -42002 6351 6 3952 E-1 ,
4  -16653 95113 82231 E+0 ,
5  -42197 73455 55443 E-1 ,
6  -96219 71527 87637 E-2 ,
7  -72199 41246 56310 E-2 ,
8  -11651 67591 85907 E-2 ,
9  -21524 16741 14951 E-3 ,
1  0.12605 02823 64116 E-3 ,
2  -20134 85478 07342 E-4 ,
3  -12534 93482 14627 E-5 ,
4  -13310 97331 98170 E-5 ,
5  -20563 16416 77761 E-6 ,
6  -61160 95134 4*1+2 E-8 ,
7  -50200 01644 45622 E-8 /
ACC=.5*AC
I=INT(A)
AF=A-FLOAT(I)
IF ( AF.EQ.0. ) GO TO 3091
CLE=AF-1.
IF ( AF.LT.0.5 ) CEE=AF
P+I=CEE
K=3
ANK=C(X)*CEE
SUM=ANK
3011 CONTINUE
CEE=CEE+P+I
K=K+1
ANK=C(X)*CEE
SUM=SUM+A*K
IF ( AMS(AK).LT.(ACC*A*A*SUM) ) GO TO 3031
IF ( K.EQ.18 ) GO TO 3031
GO TO 3011
3031 CONTINUE
H=SUM
IF ( AF.GE.0.5 ) H=(1.-A+SUM)/A
T1=1.+SUM
IF ( AF.LT.0.5 ) T1=AF*T1
T1=1./T1
IF ( I.EQ.0 ) RETURN
AJ=AF-1.
J=0
3051 CONTINUE
J=J+1
AJ=AJ+1.
T1=AJ*T1
IF ( J.LT.1 ) GO TO 3051
3071 CONTINUE
GO TO 3151
3091 CONTINUE
J=0

E-12
T1=1.
IF ( I.EQ.1 ) RETURN
I=I-1
3131 CONTINUE
J=J+1
T1=FLOAT(J)*T1
IF ( J.LT.I ) GO TO 3131
3151 CONTINUE
RETURN
END