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<tr>
<td>Contract or Grant Number: General Hydrodynamics Research Program</td>
</tr>
<tr>
<td>Report Date: September 1976</td>
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<td>Number of Pages: 7</td>
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Abstract:

The frequencies of the fluctuating forces resulting from interactions between a pair of contrarotating propellers are calculated. The frequencies may equal the sum of the blade frequencies of the two propellers and harmonics of these frequencies; but only certain combinations are possible, depending on the blade numbers of the propellers. A table presents the results in numerical form for various combinations of propeller blade numbers.
Frequencies of the Alternating Forces due to Interactions of Contrarotating Propellers

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and
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Introduction

Somewhat to our surprise, we have not found any published discussion of the frequencies of the fluctuating forces developed by interactions between a pair of contrarotating propellers. Although the calculation of the expected frequencies of the alternating thrust and torque is straightforward, the calculation for side forces involves sufficient subtlety to warrant documentation.

This Note is concerned with the frequencies of the fluctuating forces developed by interactions of contrarotating propellers. Although the calculation of the frequencies of the latter forces are already well known, since they are identical to those developed by a single propeller in a nonuniform inflow. This is so because the various nonuniform velocities contribute to the fluctuating forces by superposition (or at least it is so assumed in the conventional first-order linear theory). Accordingly, this Note will be concerned only with the forces resulting from interaction between the two propellers.

Symbols and Statement of the Problem

Suppose the forward propeller has \( Y \) blades and the aft propeller \( Z \) blades. The individual blades of the forward propeller are labeled with an index \( y \) going from zero to \( (Y-1) \), and those of the aft propeller with an index \( z \) going from zero to \( (Z-1) \). (The \( Y \)th and \( Z \)th blades are identical to the zeroth blades.) The angles between the upward vertical and a reference radial line on the forward \( y \)th blade is called \( \theta_y \), and the equivalent angle for the \( z \)th aft blade is called \( \phi_z \).

If the forward propeller rotates counterclockwise at \( M \) revolutions per unit time, and the aft propeller rotates clockwise at \( N \) revolutions per unit time, then the angles will vary with time as

\[
\theta_y = -2\pi Mt + 2\pi y/y/\gamma
\]

and

\[
\phi_z = 2\pi Nt + 2\pi z/Z + \phi_0
\]

The fixed angle \( \phi_0 \), in the aforementioned indicates that the aft zeroth blade makes an angle \( \phi_0 \) to the vertical at time \( t = 0 \). The omission of an equivalent fixed angle for the forward propeller indicates that the time origin is chosen so \( t = 0 \) when \( \theta_0 = 0 \).

Fluctuating forces will be developed by the aft propeller in response to nonuniformities in the velocity field induced by the forward propeller, and forces will be developed by the forward propeller because of nonuniform velocities induced by the aft propeller. These are the only forces that will be developed if the flow into the forward propeller is axisymmetric. If the flow into the forward propeller is not axisymmetric, additional fluctuating forces will be generated by each propeller in response to these preexisting nonuniformities. But the frequencies of the latter forces are already well known, since they are identical to those developed by a single propeller in a nonuniform inflow. This is so because the various nonuniform velocities contribute to the fluctuating forces by superposition (or at least it is so assumed in the conventional first-order linear theory). Accordingly, this Note will be concerned only with the forces resulting from interaction between the two propellers.

Detailed Analysis

The fluctuating side force developed by the aft propeller will be considered first. The fluctuating tangential force \( T_z \) developed by the \( z \)th aft blade is assumed to be a periodic function of the angle \( \theta_z \) between that blade and the zeroth forward blade. Since all aft blades are identical, the form of this function will be the same for each aft blade. Since all \( Y \) forward blades are identical with each other, the velocity field induced by each forward blade will be identical to the others. Thus, the velocity field and associated fluctuating forces induced by the forward propeller will have an angular periodicity of \((2\pi/2Y)\) rather than simply \(2\pi\).

Accordingly, the periodic tangential force on the \( z \)th aft blade can be expressed as a Fourier series

\[
T_z = \sum_{k=1}^{\infty} B_k \cos(kY\theta_z - \psi_z)
\]

where \( \theta_z \), the angle between the \( z \)th aft blade and the zeroth forward blade, is given by

\[
\theta_z = \phi_z - \theta_0
\]

The horizontal component of this tangential force is given by

\[
H_z = T_z \cos \phi_z
\]

The total horizontal force is obtained by summing over the \( Z \) blades, viz.

\[
H = \sum_{k=1}^{\infty} \sum_{z=0}^{Z-1} B_k \cos[kY(\phi_z - \theta_0) - \psi_z]
\]

Using the identity \( \cos a \cos b = \frac{1}{2} \cos(a + b) + \frac{1}{2} \cos(a - b) \), Eq. (6) becomes

\[
H = \frac{1}{2} \sum_{k=1}^{\infty} \sum_{z=0}^{Z-1} B_k \left[ \cos(kY + i)\phi_z - kY\theta_0 - \psi_z \right] + \cos(kY - l)\phi_z - kY\theta_0 - \psi_z]
\]

After introducing the time dependence of the blade angles \( \phi_z \), and \( \theta_0 \) from Eqs. (1) and (2) into Eq. (7), the latter may be written in symbolic form as

\[
H = \frac{1}{2} \sum_{k=1}^{\infty} \sum_{z=0}^{Z-1} B_k \left[ \Sigma + \Gamma \right]
\]
where

$$
\sum_{\gamma=0}^{\gamma=m} \cos \left[ a_z + 2\alpha (z/Z) (kY \pm 1) \right]
$$

(8b)

with

$$
a_z = (kY \pm 1) (2\pi N_2 \theta_2) + 2\pi k YM_1 - \psi_k
$$

(8c)

When written in this form the entire \( z \)-dependence of each term of the sum appears in one place in Eq. (8b).

The sums over \( z \) are of a standard form encountered in the theory of fluctuating propeller forces. Their values are given by the form

$$
\sum_{r=0}^{\infty} \cos (x + ry) = \frac{\cos(x + ry) \sin \left( \frac{1}{2} ry \right)}{\sin \left( \frac{1}{2} y \right)}
$$

(See, e.g., formula 3.61-9 in "Smithsonian Mathematical Formulae," Publication 2672 of the Smithsonian Institution, 1939.) In the present case, \( x = a_z, \ y = (2\alpha Z) (kY \pm 1), \) and \( R = (Z - 1). \) Examination of the quotient of the two sines,

$$
\frac{\sin(kY \pm 1)}{\sin(\alpha Z/kY \pm 1)}
$$

indicates that the numerator is always zero, since \( k \) and \( Y \) are integers. Accordingly, the entire sum will be zero except if the denominator is also zero, which is the case when

$$
kY \pm 1 = mZ
$$

(9)

where \( m \) is an integer. When this occurs, the quotient has the indeterminate form \((0/0)\) which can be evaluated to

$$
\frac{\sin mxZ}{\sin mZ} = Z(1 - \cos \left( \frac{\pi}{2} \right) mZ)
$$

(10)

Equation (9) indicates that side forces will be generated by interaction between the propellers only for certain values of \( k \) and \( m, \) depending on the blade numbers \( Y \) and \( Z \) of the two propellers. If Eq. (9) is satisfied with the plus sign, the sum \( \Sigma_z \) will be nonzero while \( \Sigma_y \) will be zero, and vice versa. In either case, the sum has the form

$$
\Sigma_y + \Sigma_z = Z(1 - \cos \left( \frac{\pi}{2} \right) mZ)
$$

(11)

where \( a \) is used instead of \( a_z, \) or \( a, \) since they have the same value

$$
a = mZ \left( 2\pi N_2 \theta_2 + 2\pi k YM_1 - \psi_k \right)
$$

(12)

**Final Results for Side-Force Frequencies**

The total horizontal force can now be written by inverting Eqs. (11) and (12) into Eq. (8a). The final result is

$$
H = \frac{1}{2} \sum B_k \cos \left[ 2\pi (mZ + kYM_1) t + \pi mZ \theta_2 - \psi_k \right]
$$

(13)

where \( k \) and \( m \) are integers whose values are restricted to those satisfying the relation

$$
kY \pm 1 = mZ
$$

(14)

The cyclic frequencies associated with these allowed values are given by

$$
f_{km} = kYM_1 + mZ \theta_2
$$

(15)

If Eq. (14) is satisfied by \( k = 1 \) and \( m = 1, \) then the lowest frequency will simply be the sum of the blade frequencies of the 2 propellers. This occurs whenever \( Z = Y \pm 1; \) i.e., when the two propellers differ by one blade. For higher values of \( k \) and \( m, \) the frequencies will equal the sum of the appropriate harmonics of the two blade frequencies. Table 1 lists the allowed frequencies for various blade combinations when both propellers are contrarotating at the same speed. Note that no side forces are generated when both propellers have an even number of blades (not necessarily the same number).

**Table 1. Frequencies of alternating forces for contrarotating propellers (expressed as harmonics of shaft frequencies)**

<table>
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<th>Number of blades of one propeller</th>
<th>Side forces</th>
<th>Thrust &amp; torque</th>
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<tr>
<td>3</td>
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The calculation has been carried out for the horizontal force. The results can be applied to the vertical force simply by defining the angle \( \theta_2, \) first appearing in Eq. (2), to be measured relative to the horizontal rather than the vertical; and the same for the other angles. The frequencies and magnitudes are identical for both vertical and horizontal forces; only the phase is different, the difference being an angle \( (\pi Z/2). \)

The calculation of the frequencies of the force generated by the forward propeller is also identical except that the symbols \( k, Y, \) and \( M \) are associated with the aft propeller, and \( m, Z, \) and \( N \) with the forward propeller. The frequencies of the forces are the same for both propellers. The magnitudes of the forces generated by the two propellers will generally be different, however.

**Alternating Thrust and Torque**

For completeness, the allowed frequencies for alternating thrust and torque are given. The cyclic frequencies are

$$
kYM_1 + mZ \theta_2
$$

(16)

but now \( k' \) and \( m' \) are integers which satisfy

$$
k'Y = m'Z
$$

(17)

When both propellers have the same number of blades, the lowest frequency equals the sum of the separate blade frequencies.
Both Propellers Rotating in Same Direction

The results may be applied to 2 propellers rotating in the same direction by simply using either $-M$ or $-N$ in the equations, as the case may be, instead of the positive value. A situation worthy of some discussion is that which occurs when both propellers rotate at the same speed in the same direction. This is the case of two propellers on one shaft.

Suppose the speed of both propellers is $N$ in the same direction. Then the formulas apply with $M = -N$. The frequencies of the side forces are given by Eq. (15) as

$$f_{1m} = N(kY - mZ)$$

If there is concern with side forces only, then a pair of even bladed propellers should be used, because there are no interaction side forces at all when both propellers have an even number of blades. Such a pair will have alternating thrust and torque, however. For a combination of a 4-blader forward and 6-blader aft, for example, alternating thrust and torque will occur at a frequency equal to twice the aft blade frequency plus 3 times the forward one, e.g., at 24 times shaft frequency if both are contrarotating at the same speed.

If side forces are of major concern, the worse choice is a combination where the blade numbers differ by one. Such a combination is good, however, if only alternating thrust and torque are of concern. When side forces as well as alternating thrust and torque are of concern, the best compromise may be a 5 blader forward and 7 blader aft, or perhaps a 4 blader forward and 6 blader aft.

Finally, the reader is reminded that all of the above applies only to forces resulting from interaction between the two propellers. These will be the only forces when the inflow to the forward propeller is axisymmetric. If the inflow is not axisymmetric, additional fluctuating forces will be superposed upon those discussed in this Note.
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