SOME ASYMPTOTIC RESULTS FOR OCCUPANCY PROBLEMS

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Consider a situation in which balls are falling into $N$ cells with arbitrary probabilities. Limit distributions for the number of empty cells are considered when $N \to \infty$ and the number of balls $n \to \infty$ so that $n/N \to \infty$. Limit distributions for the number of balls to achieve exactly $b$ empty cells are obtained when $N \to \infty$ for $b$ fixed or $b \to \infty$ so that $b/N \to 0$. 
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Lars Holst

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

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ABSTRACT

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SOME ASYMPTOTIC RESULTS FOR OCCUPANCY PROBLEMS

Lars Holst

1. Introduction.

Suppose that balls are thrown independently of each other into $N$ cells, so that each ball has the probability $p_k$ of falling into the $k$th cell, $p_1 + \ldots + p_N = 1$. Let $Y_n$ denote the number of empty cells after $n$ throws and let $T_b$ denote the throw on which for the first time exactly $b$ cells remain empty, $0 < b < N$. The symmetrical case $p_1 = \ldots = p_N = 1/N$ is discussed in e.g. Feller (1968), see occupancy or waiting time problems.

Depending on how $b$, $n$, $N \to \infty$, different asymptotic distributions for $Y_n$ and $T_b$ can be obtained, see e.g. Holst (1971) and for the symmetric case see e.g. Samuel-Cahn (1974). In this paper some remaining problems are investigated for the nonsymmetrical case.

To give precise meanings of the limits obtained, double sequences e.g. $(p_{kN})_N$, $(Y_{nN})_N$ are considered. But in order to simplify the notation the extra index $N$ will usually be omitted.

2. A bounded number of empty cells.

The following limit theorem for $Y_n$, the number of empty cells after $n$ throws, was proved by Sevastyanov (1972).

Theorem 1. If the $p$'s are such that

\[ \max_{1 \leq k \leq N} (1 - p_k)^n \to 0 \]

and

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\begin{equation}
E(Y_n) = \sum_{k=1}^{N} (1 - p_k)^n \to m < \infty,
\end{equation}

then
\begin{equation}
P(Y_n = y) \to m^y \cdot e^{-m/y},
\end{equation}
or equivalently
\begin{equation}
Y_n \Rightarrow Po(m), \text{ when } N \to \infty.
\end{equation}

Remark. When the \( p \)'s are equal an expression for \( P(Y_n = y) \) can be obtained from which (2.3) can be derived by elementary methods, see e.g. Feller (1968). In this case (2.1) and (2.2) are replaced by
\begin{equation}
N \cdot \exp(-n/N) \to m < \infty
\end{equation}
or
\begin{equation}
n/N - \log N \to - \log m \to -\infty.
\end{equation}

For \( T_b \), the number of balls until \( b \) empty cells remain, the limit distribution is given by:

**Theorem 2.** If \( b \) is a fixed integer and for some fixed numbers \( C \) and \( D \),

\begin{equation}
0 < C \leq Np_k \leq D < \infty, \text{ for all } k \text{ and } N,
\end{equation}

then, when \( N \to \infty \),
\begin{equation}
\sum_{k=1}^{N} (1 - p_k)^T_b \to \frac{1}{2} \chi^2(2(b+1)),
\end{equation}
and
\begin{equation}
\sum_{k=1}^{N} \exp(-T_b p_k) \to \frac{1}{2} \chi^2(2(b+1)).
\end{equation}

Before proving the theorem the following functions are considered:
\begin{equation}
f(t) = f_N(t) = \sum_{k=1}^{N} (1-p_k)^t, \quad t > 0,
\end{equation}
and
Lemma 1. If Condition (2.7) is satisfied, $y > 0$ is a fixed number, and $t = t_N = t(y)$ is defined by the equation
\begin{equation}
(2.12) \quad f(t) = y ,
\end{equation}
then
\begin{equation}
(2.13) \quad 0 < C \leq \lim \inf \frac{N \log N}{t_N} \leq \lim \sup \frac{N \log N}{t_N} < D < \infty
\end{equation}
and when $N \to \infty$
\begin{equation}
(2.14) \quad f([t]) \to y ,
\end{equation}
\begin{equation}
(2.15) \quad \max_{1 \leq k \leq N} (1 - p_k)[t] \to 0 ,
\end{equation}
\begin{equation}
(2.16) \quad g(t) \text{ and } g([t]) \to y .
\end{equation}

where $[t]$ denotes the integer part of $t$.

Lemma 2. If $f$ is replaced by $g$ and $g$ by $f$ in Lemma 1, then the same conclusions hold.

Proof of Lemma 1. From Condition (2.7), it follows that
\begin{equation}
(2.17) \quad y = \sum_{k=1}^{N} (1 - p_k)^t \geq N \cdot (1 - D/N)^t .
\end{equation}
Hence for $\varepsilon > 0$ and $N$ sufficiently large
\begin{equation}
(2.18) \quad \log y \geq \log N - t \cdot (D+\varepsilon)/N
\end{equation}
and therefore
\begin{equation}
(2.19) \quad D+\varepsilon = (D+\varepsilon) \lim_{N \to \infty} \frac{1}{N} \left(1/(1-\log y/\log N)\right) \geq \lim \sup \frac{N \log N}{t_N} ,
\end{equation}
which proves the right inequality of (2.13).

To prove the left inequality of (2.13) the following estimate follows from (2.7):
\begin{equation}
(2.20) \quad y = \sum_{l}^{N} (1-p_k)^t \leq N \cdot (1 - C/N)^t ,
\end{equation}

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or

\[(2.21)\]

\[
\log y \leq \log N - t \log(1 - C/N) \leq \log N - t C/N .
\]

From this it follows that

\[(2.22)\]

\[
C = C \lim (1 - \log y / \log N)^{-1} \leq \lim inf N \log N / t_N .
\]

To prove (2.14) we observe that

\[(2.23)\]

\[
(1 - p_k)^{t-1} \geq (1 - p_k)^{[t]} \geq (1 - p_k)^t ,
\]

and using (2.7)

\[(2.24)\]

\[
(1 - D/N)^{-1} \sum_1^N (1 - p_k)^t \geq \sum_1^N (1 - p_k)^{[t]} \geq \sum_1^N (1 - p_k)^t ,
\]

or from (2.12)

\[(2.25)\]

\[
(1 - D/N)^{-1} y \geq f([t]) \geq y .
\]

From which (2.14) follows.

Combining (2.7) and (2.13) give for some \( K > 0 \) and \( N \) sufficiently large that

\[(2.26)\]

\[
\max (1 - p_k)^{[t]} \leq (1 - C/N)^{[t]} \leq (1 - C/N)^1 \rightharpoonup 0, N \rightarrow \infty
\]

which proves (2.15).

Using (2.7) and (2.13) it follows that for some constant \( K \)

\[(2.27)\]

\[
|1 - e^{-tp_k^t} / (1 - p_k)^t| \leq K \cdot \log N / N ,
\]

and therefore

\[(2.28)\]

\[
|f(t) - g(t)| \leq \sum_1^N (1 - p_k)^t \cdot |1 - e^{-tp_k^t} / (1 - p_k)^t| \leq K \sum_1^N (1 - p_k)^t \log N / N = K y \log N / N \rightharpoonup 0 ,
\]

which proves (2.16).
Proof of Lemma 2. The proof is essentially the same as that for Lemma 1. ■

Proof of Theorem 2. From the definitions it follows that

\[(2.29) \quad Y_n \leq b \iff T_b \leq n,\]

and therefore

\[(2.30) \quad P(Y_n \leq b) = P(T_b \leq n) = \frac{P(f(T_b) \geq f(n))}{z}.\]

Let \( y > 0 \) be fixed and define \( n = [t] \) with \( t = t(y) \) as in Lemma 1. According to Lemma 1 the assumptions of Theorem 1 are satisfied. Hence

\[(2.31) \quad P(f(T_b) \geq y) = P(Y_n \leq b) \to P(Y \leq b),\]

where \( Y \) is \( P_c(y) \). Furthermore it is well-known that

\[(2.32) \quad P(Y \leq b) = P\left(\frac{1}{2} \chi^2(2(b+1)) \geq y\right).\]

(2.31) and (2.32) prove (2.8). Using Lemma 2, the assertion (2.9) follows. ■

Remark. When the \( p \)'s are equal the theorem can be written

\[(2.33) \quad N \cdot (1 - 1/N) T_b \Rightarrow \frac{1}{2} \chi^2(2(b+1)),\]

and therefore

\[(2.34) \quad T_b/N - \log N \Rightarrow \log \left(\frac{1}{2} \chi^2(2(b+1))\right).\]

This result was found by Baum and Billingsley (1965) using complicated calculations. Using the result in Feller (1968) and the method of proof of Theorem 2, (2.33) and (2.34) follows. A consequence of (2.34) is

\[(2.35) \quad T_b/N \log N \to 1, \text{ in probability, as } N \to \infty.\]

Now (2.35) will be generalized. First introduce the distribution function

\[(2.36) \quad H_N(x) = \# \{ p_k : N p_k \leq x \}/N.\]
Lemma 3. If \( t = t_N = t(y) \) is defined by

\[
(2.37) \quad g(t) = g_N(t_N) = y > 0 ,
\]

and there exists a distribution function \( H(x) \) on \([C, D]\) such that

\[
(2.38) \quad H_N(x) \rightarrow H(x) , \quad N \rightarrow \infty ,
\]

and

\[
(2.39) \quad 0 < C = \inf \{x ; H(x) > 0\} ,
\]

then for \( 1/C > \varepsilon > 0 \), when \( N \rightarrow \infty \),

\[
(2.40) \quad g_N((\varepsilon + 1/C)(N \log N)) \rightarrow 0 ,
\]

and

\[
(2.41) \quad g_N((-\varepsilon + 1/C)(N \log N)) \rightarrow +\infty .
\]

Proof. From the definitions it follows that

\[
(2.42) \quad 0 < y = g_N(t_N) = N \cdot \int_C^D \exp(-t_N x/N)dH_N(x) =
\]

\[
= \int_C^D \exp((-1/C + 1/C)(N \log N) \log N) dH_N(x) .
\]

Consider

\[
(2.43) \quad g_N((\varepsilon + 1/C) N \log N) = \int_C^D \exp((1-x(1+\varepsilon C)/C) N \log N) dH_N(x) .
\]

Now for \( C < x < D \) it is true that \( 1 - x(1+\varepsilon C)/C < 0 \) and therefore the exponent in (2.43) is negative so the integral tends to 0 when \( N \rightarrow \infty \), which proves (2.40).

For proving (2.41) consider

\[
(2.44) \quad g_N((-\varepsilon + 1/C) N \log N) = \int_C^D \exp((1-x(1-\varepsilon C)/C) N \log N) dH_N(x) .
\]

For \( C < x < C/(1-C\varepsilon) \) the exponent is positive and as the integrand is positive

(2.44) could be estimated by

\[
(2.45) \quad \int_C^{C/(1-C\varepsilon)} \exp((1 - x(1-\varepsilon C)/C) N \log N) dH_N(x) \rightarrow +\infty
\]

by Condition (2.39).
Corollary to Theorem 2. If the Conditions (2.38) and (2.39) are satisfied then

\[ T_b / N \log N \rightarrow 1/C, \text{ in probability, } N \rightarrow \infty. \]

Proof. Let \( \varepsilon_1 > 0 \) and \( \varepsilon_2 > 0 \) be given. Take a \( \delta > 0 \) so that

\[ P( \frac{1}{2} X^2 (2(b+1)) < \delta) < \varepsilon_2/2. \]

For \( N \) sufficiently large it follows from Theorem 2 that

\[ P(g_N(T_b) < \delta) < \varepsilon_2/2 \]

and from Lemma 3 that

\[ g_N((\varepsilon_1 + 1/C)(N \log N)) < \delta. \]

Hence

\[ P(T_b / N \log N > \varepsilon_1 + 1/C) = P(g_N(T_b) < g_N((\varepsilon_1 + 1/C)(N \log N)) < P(g_N(T_b) < \delta) < \varepsilon_2/2. \]

In a similar way it is proven that

\[ P(T_b / N \log N < -\varepsilon_1 + 1/C) < \varepsilon_2/2. \]

Hence for \( N \) sufficiently large

\[ P(\vert T_b / N \log N - 1/C \vert > \varepsilon_1) < \varepsilon_2. \]

Thus the assertion is proved.

3. A small fraction of empty cells.

As above, \( Y_n \) denotes the number of empty cells after \( n \) throws.

Theorem 3. If

\[ 0 < C \leq Np_k \leq D < \infty, \text{ for all } k \text{ and } N, \]

\[ n/N \rightarrow \infty, \]

and

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(3.3) \[ f(n) = E(Y_n) = \sum_{k=1}^{N} (1 - p_k)^n \to +\infty, \]
then, when \( n \to \infty \),

(3.4) \[ (Y_n - f(n))/(f(n))^{1/2} \xrightarrow{\text{D}} N(0,1), \]
and

(3.5) \[ (Y_n - g(n))/(g(n))^{1/2} \xrightarrow{\text{D}} N(0,1), \]

where

(3.6) \[ g(n) = \sum_{k=1}^{N} \exp(-np_k). \]

**Proof.** Using (3.1) and (3.3) it follows that

(3.7) \[ \sum_{k=1}^{N} (1 - p_k)^n \leq N \cdot (1 - C/N)^n \to +\infty, \]
hence

(3.8) \[ n/N \log N = O(1). \]

Using (3.1), (3.2), and (3.8) give

(3.9) \[ |f(n) - g(n)| \leq \sum_{k=1}^{N} \exp(-np_k) \cdot \left| \exp(n \log (1-p_k) + np_k) - 1 \right| \leq \sum_{k=1}^{N} \exp(-np_k) \cdot K \cdot n/N^2 \leq K \cdot (n/N) \cdot \exp(-C n/N) \to 0. \]

Hence it is sufficient to prove (3.5). This will be established using convergence of characteristic functions.

In Holst (1971) p. 1672 the characteristic function of \( Y_n \) is given by

(3.10) \[ E(\exp(\im t Y_n)) = (n! / 2\pi i N^n) \cdot \int \left( e^{Nz}/z^{n+1} \right) \prod_{k=1}^{N} \left( (1 + (e^{it} - 1)\exp(-np_k z))dz \right), \]

\( |z| = n/N \)

Using Stirling's formula and changing to polar coordinates it follows that
\begin{align*}
(3.11) \quad E(\exp(it(Y_n - \mu)/\sigma)) &= (1 + o(1)).
\end{align*}

\begin{align*}
&\cdot \int_{-\pi}^{\pi} (n/2\pi)^{\frac{1}{2}} \cdot \exp(n(e^{i\theta} - 1 - i\theta)) \\
&\cdot \frac{N}{1} \prod (\exp(-it e^{-n\sigma^2}/\sigma) \cdot (1 + (e^{it/\sigma} - 1)\exp(-n\sigma^2 e^{i\theta})))d\theta \\
&= (1 + o(1)) \cdot \int_{-\pi}^{\pi} h_n(\theta, t)d\theta,
\end{align*}

where

\begin{align*}
(3.12) \quad \mu &= \sigma^2 = g(n) = \sum_{k=1}^{N} \exp(-n\sigma^2), \quad \sigma > 0.
\end{align*}

The integral will be studied by the same method as in Holst (1971).

Take $0 < a < 1/6$ and split the interval $-\pi \leq \theta \leq \pi$ into

\begin{align*}
(3.13) \quad A &= \{ \theta ; \ a \leq |\theta| \leq \pi \}, \\
(3.14) \quad B &= \{ \theta ; \ n^{a-\frac{1}{2}} \leq |\theta| < a \},
\end{align*}

and

\begin{align*}
(3.15) \quad C &= \{ \theta ; |\theta| < n^{a-\frac{1}{2}} \}.
\end{align*}

From Lemmas 4-6 below it follows that

\begin{align*}
(3.16) \quad E(\exp(it(Y_n - \mu)/\sigma)) &= (1 + o(1)).
\end{align*}

\begin{align*}
(\int_{A} h_n + \int_{B} h_n + \int_{C} h_n) \to 0 + 0 + \exp(-t^2/2), \quad n \to \infty.
\end{align*}

By the continuity theorem for characteristic functions assertion (3.5) is proved, and thus the theorem.

With the same conditions as in Theorem 3 the following lemmas hold.
Lemma 4. For every fixed real number $t$

\[(3.17) \quad \int_a^b h_n(\theta, t) d\theta \to 0, \quad n \to \infty.\]

Proof. As $n/N \to \infty$ and $\sigma \to \infty$ it follows that

\[(3.18) \quad \left| \int_a^b \right| \leq K_1 \cdot n^{\frac{1}{2}} e^{-n} \int_a^b \left| \exp(np_k e^{i\theta}) + e^{it/\sigma - 1} \right| d\theta \]

\[\leq K_2 n^{\frac{1}{2}} e^{-n} \int_a^b \left( \exp(np_k \cos \alpha) + o(1) \right) \]

\[\leq K_2 n^{\frac{1}{2}} e^{-n} N e^{n \cos \alpha} \to 0.\]

Lemma 5. For every fixed real number $t$

\[(3.19) \quad \int_{B^+} h_n(\theta, t) d\theta \to 0, \quad n \to \infty.\]

Proof. From the assumptions, it follows that there exist positive numbers $K_3 - K_9$ such that

\[(3.20) \quad \left| \int_{B^-} \right| \leq K_3 \cdot n^{\frac{1}{2}} e^{-n} \int_{B^-} \left( \exp(np_k \cos \theta) + o(1/\sigma) \right) d\theta \]

\[\leq K_4 n^{\frac{1}{2}} e^{-n} \int_{B^-} \exp(np_k \cos \alpha^{-\frac{1}{2}}) \cdot \left( 1 + K_5 \cdot \exp(-K_6 n/N) / \sigma \right) \]

\[\leq K_7 n^{\frac{1}{2}} e^{-n} \exp(n (1 - K_8 n^{2a - 1})) \]

\[\leq \exp(-K_9 n^{2a}) \to 0, \quad n \to \infty.\]

Lemma 6. For every fixed real number $t$,

\[(3.21) \quad \int_c^d h_n(\theta, t) d\theta \to \exp(-t^2/2), \quad n \to \infty.\]

Proof. Expanding in series gives

\[(3.22) \quad \log h_n(\theta, t) = -n \theta^2/2 + o(1) \]

\[+ \sum_{l=1}^{N} \left( \log (1 + \exp(-np_k e^{i\theta}) (e^{it/\sigma - 1}) - \exp(-np_k \sqrt{\sigma}) + \frac{1}{2} \log(n/2\pi). \right) \]
Now, when \( n \to \infty \),

\[
(3.23) \quad \sum_{j=1}^{N} \left| \exp(-2np_k e^{i\theta})(e^{it/\sigma} - 1)^2 \right|
\]

\[= o(1) \cdot \sum_{j=1}^{N} \exp(-np_k) / \sigma^2 = o(1),\]

and therefore

\[
(3.24) \quad \sum_{j=1}^{N} (\log (1 + \ldots) - \ldots)
\]

\[= \sum_{j=1}^{N} (\exp(-np_k e^{i\theta})(e^{it/\sigma} - 1) - it \exp(-np_k) / \sigma + o(1)).\]

Furthermore, using (3.8), (3.9) and the assumptions, it follows that

\[
(3.25) \quad \sum_{j=1}^{N} \exp(-np_k e^{i\theta}) / \sigma^2 \to 1,
\]

and therefore (3.24) can be written

\[
(3.26) \quad \sum_{j=1}^{N} (\ldots) = \sum_{j=1}^{N} (\exp(-np_k e^{i\theta})(it/\sigma - t^2/2\sigma^2)
\]

\[- it \exp(-np_k) / \sigma + o(1)
\]

\[= it \sum_{j=1}^{N} (\exp(-np_k e^{i\theta} - 1) - 1) \exp(-np_k) / \sigma
\]

\[- t^2/2 + o(1).\]

Now, when \( n \to \infty \),

\[
(3.27) \quad \sum_{j=1}^{N} (np_k)^2 \theta^2 \exp(-np_k) / \sigma \leq
\]

\[\leq K_1 (n/N)^2 n^{2a-1} N^2 \exp(-K_2 n/N) \to 0.\]

From this it follows that

\[
(3.28) \quad \sum_{j=1}^{N} (\ldots) = \theta t \sum_{j=1}^{N} np_k \exp(-np_k) / \sigma - t^2/2 + o(1).
\]
Hence for $\theta$ in $C$,

$$\log h_n(\theta, t) - \frac{1}{2} \log(2\pi/n) = -n\theta^2/2 + \theta t \sum_{k=1}^{N} np_k \exp(-np_k)/\sigma$$

$$- t^2/2 + o(1) = -(n^2\theta - t \sum_{k=1}^{N} \frac{1}{2} np_k \exp(-np_k)/\sigma)^2/2$$

$$- t^2(1 - \sum_{k=1}^{N} \frac{1}{2} np_k \exp(-np_k)/\sigma)^2/2 + o(1) .$$

Now, when $n \to \infty$,

$$\sum_{k=1}^{N} \frac{1}{2} np_k \exp(-np_k)/\sigma \leq K_3 n^{\frac{1}{2}} N^{-1} \cdot N^\frac{1}{2} \cdot \exp(-K_4 n/N) \to 0 .$$

Thus with $\psi = n^\frac{1}{2} \theta$ the integral (3.21) can be written

$$\int h_n = \int_{C} h_n \left|\psi\right| \leq n^a (2\pi)^{-\frac{1}{2}}$$

$$\cdot \exp(-(\psi - o(1))^2/2 - t^2/2 + o(1)) \, d\psi ,$$

which converges to $\exp(-t^2/2)$ when $n \to \infty$.

4. The waiting time for a small fraction.

As above let $T_b$ denote the number of balls thrown until exactly $b = b_N$ cells remain empty. Let $t_b$ be the unique solution of the equation

$$b = g(t_b) = \sum_{k=1}^{N} \exp(-t_b p_k) .$$

Theorem 4. If, when $N \to \infty$,

$$b_N \to +\infty ,$$

$$b_N/N \to 0 ,$$

and

$$C < C \leq Np_k \leq D < \infty ,$$

then

$$b_N^{\frac{1}{2}} (T_b - t_b) \sum_{k=1}^{N} p_k \exp(-t_b p_k) \Rightarrow N(0,1) .$$
Proof. From the assumptions it follows that
\begin{equation}
C \frac{b}{N} \leq \Delta = \sum_{k=1}^{N} p_k \exp(-t_b p_k) \leq D \frac{b}{N}.
\end{equation}

Thus for \( N \) sufficiently large
\begin{equation}
0 < C \leq \Delta \cdot \frac{N}{b} \leq D < \infty.
\end{equation}

As in the proof of Theorem 2 the following relation holds
\begin{equation}
P((T_b - t_b) \Delta^{1/2} \leq x) = P(Y_n \leq b),
\end{equation}

where
\begin{equation}
n = \lfloor t_b + x \frac{b^{1/2}}{\Delta} \rfloor.
\end{equation}

It is seen that
\begin{equation}
g(n) (1 + o(1)) = g(t_b + x \frac{b^{1/2}}{\Delta})
= \sum \exp(-t_b p_k) \cdot (1 - x p_k \frac{b^{1/2}}{\Delta} + O(1/L),
= b - x \cdot b^{1/2} + O(1),
\end{equation}

and thus
\begin{equation}
g(n) \to +\infty,
\end{equation}

and from (3.9) it follows that
\begin{equation}
f(n) \to +\infty.
\end{equation}

Furthermore,
\begin{equation}
b = g(t_b) \geq N \exp(-D t_b/N),
\end{equation}

implying that
\begin{equation}
t_b /N \to +\infty,
\end{equation}

and therefore
\begin{equation}
n/N \to +\infty.
\end{equation}
Hence the assumptions of Theorem 3 are fulfilled and (4.8) and (4.10) give

\[(4.16) \quad P(T_n - t_n) \Delta / b^{\frac{1}{2}} \leq x) = P(Y_n \leq b) = \]

\[= \Phi \left( (b - g(n)) / (g(n))^{\frac{1}{2}} \right) + o(1) = \]

\[= \Phi \left( (x b^{\frac{1}{2}} + O(1)) / (b(1 + o(1)))^{\frac{1}{2}} \right) + o(1) \rightarrow \Phi(x), \]

where \(\Phi(x)\) is the standardized normal distribution function. This proves the theorem.

References


