A SOLUTION FOR LAMINAR FLOW PAST A ROTATING CYLINDER IN CROSSFLOW

Kevin S. Fansler, et al

Ballistic Research Laboratories
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USA BALLISTIC RESEARCH LABORATORIES
ABERDEEN PROVING GROUND, MARYLAND
A SOLUTION FOR LAMINAR FLOW PAST A ROTATING CYLINDER IN CROSSFLOW

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UNCLASSIFIED

Fluid Dynamics
Boundary Layer Theory
Wakes
Moving-Wall Boundary Layers

Two-Dimensional Fluid Flow
Conformal Mapping
Inviscid-Flow Models

Two-dimensional subcritical flow past a rotating cylinder has been theoretically treated to obtain agreement with the boundary-layer calculations. This study combined a source-wake bound-vortex flow model with a moving-wall boundary-layer calculation to force the final inviscid-flow model to be consistent with boundary-layer theory. Consistency was obtained by an iterative process whereby the separation points of the inviscid-flow model converged toward the separation points found by boundary-layer calculations. The boundary-layer is calculated using the integral-momentum and the integral-energy.
equations where the family of moving-wall similarity boundary-layer solutions provide relationships between some parameters of the equations.

To obtain the inviscid flow model, flows in two planes were considered. The inviscid flow in the untransformed plane results from the superposition onto a uniform flow of a doublet, two sources located on the downstream side of a cylinder, their image sinks located along the axis of the cylinder, and a point vortex located in the center. The resulting complex-potential field was conformally mapped onto the physical plane by a Zhukovskii transformation. The free streamlines for the flow in the physical plane were used to simulate the free-shear layers in the near wake in order to approximate the velocity distribution on that part of the cylinder with attached flow. Other conditions were then imposed on the inviscid-flow model to take into account the vorticity transport into the wake and to allow for the possibility of boundary-layer theory to agree with the inviscid-flow model near separation.

Results were obtained for ratios of cylinder peripheral velocity to free-stream velocity from zero to 0.32. Lift coefficients were calculated; particularly good agreement with experiment was obtained for ratios equal to or less than 0.15. The calculated drag coefficients varied at most by 20% from those observed experimentally.
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I. INTRODUCTION

In this study, the lift and drag coefficients are calculated as a function of the ratio of the cylinder's tangential velocity to the velocity of the free stream flow. Here the boundary-layer is laminar to separation with the free shear layers near the separation points also being laminar. According to Swanson\(^1\), the nondimensionalized wall velocity (peripheral velocity of cylinder divided by freestream velocity) must always be less than 0.5 or the boundary layer will not be laminar to separation. From Wu's\(^2\) review article on wakes and cavities, laminar flow for the shear layers near the separation points requires flows with Reynolds numbers less than \(5 \times 10^5\). Yet the Reynolds number must be high enough for the boundary-layer approximation to be valid; that is, the boundary-layer is very thin and the pressure immediately outside the boundary-layer is impressed through the boundary layer thickness.

Rotating a body of revolution so that its axis of rotation is at an angle with its direction of relative fluid velocity not only affects the drag but also introduces a new force. This force, called the Magnus force, is directed perpendicular to the plane in which the rotational axis and direction of translation lie. G. Magnus\(^3\) correctly attributed this force to the pressure field produced by the inviscid-velocity distribution about the rotating body in the airstream. He did this, and set the tone for subsequent experiments, by considering the apparently simpler two-dimensional problem of the rotating cylinder in crossflow. Using this model, he established that the force was directed toward the side of the cylinder moving with the direction of flow.

Lord Rayleigh\(^4\) was the first to construct a mathematical model of the flow field by assuming an ideal fluid; a potential vortex combined with a doublet in a uniform flow induces a circulation and pro-

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duces a side force according to Zhukovskii's theorem. However, no mathematical method could yet be used to couple the rotation of the cylinder with the fluid to produce the circulation. A missing element was found when Prandtl introduced the concept of the boundary-layer in 1904, thus providing a better understanding of the rotating cylinder problem. Prandtl and Tietjens also showed that separation was delayed on the upper part of the cylinder wall moving with the flow while separation was hastened on the underside of the cylinder. However, no successful theoretical solutions were found for quantifying the lift and drag forces.

Reid measured the lift and drag coefficients of the rotating cylinder for rotational velocities up to $u_w = 3.4$, where $u_w$ is the tangential velocity of the surface of the cylinder divided by the free stream velocity. A maximum lift to drag ratio of 7.8 was observed, and it was noted that the drag decreased somewhat with increasing velocity of rotation. A. Thorn and his associates made comprehensive studies involving the variations of the Magnus force with Reynolds number, surface condition, end conditions and other factors. Pressure field data were also obtained with the lift and

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drag coefficients being deduced from these. More recently, Swanson has investigated the rotating cylinder and has presented experimental results for a wider range of rotation rates and Reynolds numbers than were previously available. In his careful study, three-dimensional effects due to finite length of the cylinder were minimized and pressure probe results for the boundary-layer were obtained in addition to lift and drag.

Figure 1, obtained from Swanson's work, illustrates some features of the flow in the boundary layer for a wall velocity to free-stream velocity ratio \( \left( u_w \right) \) of one. Caution must be used in applying some features of this boundary-layer flow to the problem under investigation since Swanson concluded that the boundary layer on the underside of the cylinder for this rotation rate was very probably turbulent before separation. The stagnation point location corresponding to this velocity ratio is nearly the same as the location for the nonrotating cylinder. On the top of the cylinder, at an angle of about 120° from the stagnation point, the boundary layer thickens rapidly and separation appears to occur with a velocity profile satisfying the Moore-Rott-Sears (unpublished) criterion of \( \frac{2u}{\nu} = 0 \) at a coincident point. A boundary layer then appears and grows on the cylinder in the wake starting at the upper separation point. The profile corresponding to the apparent lower separation point does not obey the Moore-Rott-Sears criterion for separation; from the form of the profiles, it is not obvious what the correct criterion would be. But, as noted before, the boundary layer is probably turbulent in this region.

Griffiths and Ma have investigated the Magnus force phenomena particularly with regard to the negative Magnus force and its relationship to the Reynolds number. The negative force is thought to occur because the side of the cylinder going against the flow will have a higher local relative Reynolds number for corresponding points than the other side. Here the relative Reynolds number is defined as


Figure 1. Boundary-Layer Profiles on a Rotating Cylinder
\( \mathbf{u}_o \times (\mathbf{l} \pm \mathbf{\omega} \times \mathbf{u}_o) / \nu \), where the minus sign corresponds to the upper side and the plus sign refers to the lower side. If the Reynolds number for the flow is in a certain range, transition to turbulence will occur on the upstream moving wall thus delaying separation and causing the circulation strength to be of opposite sign. The present theoretical investigation, however, is not concerned with the turbulent boundary layer and this interesting aspect of Magnus forces will not be discussed further. Nevertheless, Griffiths and Ma's results would seem suspect in a quantitative sense because their lift and drag forces do not agree for common regions of the Reynolds number with those of Thom and Swanson's. In fact, their drag coefficient for the non-rotating cylinder is some 30% lower than found by other investigators. Consequently, their results will not be used for comparison purposes in this study.

The most successful attempt to theoretically treat the Magnus force (for any rotation rate) is M. B. Glauert's\(^{17}\) treatment of a cylinder whose peripheral velocity is large enough so that the streamlines around the immediate vicinity of the cylinder are closed. He then assumes that the velocity outside the boundary layer is

\[
\mathbf{u}_e = \mathbf{u}_c + 2\mathbf{u}_o \sin \mathbf{x},
\]

where \( \mathbf{u}_o \) is the free stream speed, \( \mathbf{u}_c \) is the circulatory component of the fluid velocity and \( \mathbf{x} \) is the angular displacement from the line through the center of the cylinder aligned with the direction of the unperturbed flow. He further assumes a parameter perturbation expansion together with an expansion in terms of \( \exp(i\mathbf{x}) \) and substitutes the expansion into the boundary layer equations. He solves the equations and obtains

\[
\frac{\Gamma}{2\pi a u_w} = 1 - 3 \left( \frac{u_o}{u_w} \right)^2 - 3.24 \left( \frac{u_o}{u_w} \right)^4 \ldots
\]

where \( \Gamma \) is the circulation strength and \( a \) is the radius of the cylinder. This indicates that the lift force \( F_L = \rho u_o \Gamma \) is asymptotically a linear function of the peripheral velocity of the cylinder and has no upper limit. This result contradicts Prandtl's\(^{18}\) assertion that the

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upper limit for the lift coefficient is 4. Glauert is supported by Swanson's experimental results; Prandtl's upper limit is exceeded and no other upper limit was encountered.

In contrast, some other attempts to predict the lift and drag coefficients need data from experiment to obtain meaningful results. W. G. Bickley\textsuperscript{19} placed a single vortex downstream of a cylinder with circulation superimposed upon a flow around a cylinder. The location of this vortex was selected to give the experimentally obtained relationship between $C_L$ and $C_D$ for large values of $C_L$. T. Gustafson\textsuperscript{20} expanded Bickley's analysis, but distributed the vorticity on the separation streamlines extending to infinity instead of using a single concentrated vortex. The vortices traveling downstream result in net drag and lift forces on the cylinder. Gustafson was also concerned with modeling the pressure distribution upstream of separation but Bickley was not concerned with such details of the flow field.

The inviscid-flow model used in the present investigation is a specialized case of Piercy, Preston and Whitehead's\textsuperscript{21} model. Their model was used to consider the flow around other shapes such as ellipses and flat plates set at various angles to the direction of free stream flow. Two complex planes are considered: the physical plane and the untransformed plane. The inviscid flow in the untransformed plane, shown in Figure 2, results from the following superpositions onto a uniform flow: a doublet, two sources located on the downstream part of the cylinder and also rear of the cylinder, their image sinks at the center of the cylinder, and a point vortex located in the center. With this configuration, the separation streamlines emanate from the cylinder at right angles to the surface. This complex potential field is conformally mapped onto the physical plane by a Zhukovskii transformation that doubles the angle of the streamlines emanating from the cylinder at $S_1$ and $S_2$. The cylinder in the

\begin{thebibliography}{99}
\end{thebibliography}
Figure 2. Flow Past Cylinder in the Untransformed Planes (Source-Wake Bound-Vortex Model)
Physical plane together with the corresponding streamlines are shown in Figure 3. For purposes of calculating the drag and the lift forces, the flow inside the separation streamlines is ignored and the base pressure is assumed constant at the separation value. The pressure gradients near the separation points are required to be finite as is found experimentally. In this inviscid flow model, the pressure values on the separation streamlines increase asymptotically towards the freestream value as actually occurs. The separation streamlines asymptotically approach a finite width apart with distance from the cylinder. It is assumed that this inviscid flow model simulates the shear-layers near the separation points and hence produces a more realistic velocity distribution on the cylinder ahead of separation.

The general approach defined above was used for the problem of the nonrotating cylinder in crossflow by Bluston and Paulson. They employed the source-wake of Parkinson and Jandali, a specialized model of the Piercy, Preston and Whitehead model. Their inviscid-flow model in the untransformed plane had two equal sources located symmetrically about the flow axis together with their image sinks at the center of the circle to produce symmetric separation streamlines. The resultant complex potential field was transformed to the physical plane by a Zhukovskii transformation. Parkinson and Jandali needed the velocity at separation and the location of the separation point to specify their inviscid flow model. However, Bluston and Paulson obtained their inviscid-flow model by specifying the separation point and requiring that the pressure gradient be finite at separation.

The inviscid-flow model used by Bluston and Paulson was made to agree with the results of boundary-layer theory. Using the resulting velocity distribution for a trial value of the inviscid-flow separation point, they calculated the boundary layer over the surface of the cylinder until boundary layer separation was reached. A second inviscid velocity distribution was generated with the inviscid-flow separation point determined from the first boundary-layer calculation. This procedure was repeated until the location of the separation points calculated by boundary layer theory agreed approximately with the separation points of the inviscid flow model. Separation was achieved at 83° compared with about 82° for experiment; furthermore, the pressure distribution over the cylinder ahead of separation was in good agreement with experimental measurements. These results suggested the


\[ z = X_1 + iY_1 \]
\[ \bar{z} = \bar{X}_1 + i\bar{Y}_1 \]

Figure 3. Flow Past Rotating Cylinder in the Physical Planes
possibility of extending the technique to the more difficult problem of the rotating cylinder in crossflow.

Although the approach of the present problem is similar to Bluston and Paulson's approach, the problem addressed in the present report is more difficult than their problem in at least two ways. The inviscid-flow model used is complicated by the loss of symmetry (unequal source strengths located asymmetrically with respect to the free stream flow) and the addition of a point vortex. A computer had to be used to find five unknown parameters in the untransformed plane whereas, the parameters for Bluston and Paulson's problem were found analytically. The other difficulty involved calculating the boundary layer on the rotating cylinder. For the part of the cylinder where the wall is moving against the flow, there is a region in the boundary layer near the wall which has reverse flow and calculations of boundary layers with regions of reverse flow have seldom been attempted. However, an integral technique has been used to numerically compute the boundary layer for such a situation \(^{24}\) in which the integral-momentum equation and the integral-energy equation were simultaneously integrated. The moving-wall similarity solutions \(^{25}\) were used to approximate the nonsimilar velocity profiles encountered on the rotating cylinder.

II. THE INVISCID-FLOW MODEL

A. General Description

The two-dimensional, incompressible, irrotational point vortex source-wake model for a spinning cylinder is represented in the physical plane in Figure 3 where \(u_e\) is the dimensionless inviscid velocity tangential to the surface, \(u_w\) is the peripheral dimensionless velocity of the cylinder rotating in the clockwise sense, \(u_0\) is the free stream velocity that is used to nondimensionalize \(u_e\) and \(u_w\), and \(x\) is a nondimensionalized coordinate which defines an angular position on the cylinder. The cylinder and general flow field are described in two complex planes: the \(z\) complex plane having its real axis aligned with the direction of unperturbed flow and the \(z\) plane having its imaginary axis going through


the points of breakaway, $S_1$ and $S_2$. The center of the cylinder is a point on the real axis of the $\bar{z}$ plane. The $\bar{z}$ plane is used in an intermediate step of the transformation from the untransformed plane $\zeta$ to the description of the flow in the $z$ plane. At the points $S_1$ and $S_2$, the free streamlines leave the cylinder smoothly and are a finite distance apart at infinity. The stagnation point will not generally be at $x = 0$, and the flow field will normally be asymmetrical with respect to the $X_1$ axis.

To put this source-wake model in better perspective, wake flow for nonrotating cylinders with Reynolds numbers between 1500 and $10^5$ will be discussed briefly. This is a subcritical region within which the boundary layer on the cylinder is nowhere turbulent. According to Roshko and Fiszdon\textsuperscript{26}, shear layers separate from the body and envelop a region of recirculating flow called the near wake. These free or separating shear layers are thin and well defined near the cylinder; the flow outside the free shear layers is irrotational. The near wake is unstable and oscillates periodically, particularly close to the end of the near wake region; the shear layers roll up into large vortices near the closure of the near-wake and then break away creating a Karman vortex street. Behind this near wake, the flow, being vortical and turbulent, is called the turbulent far-wake. As the Reynolds number is increased, the transition to turbulence advances along the free shear layers towards the points of separation. The time-averaged pressure is found to have an almost constant value over the part of the cylinder immersed in the wake.

The source-wake bound-vortex model used in this investigation does take into account some phenomena observed. The free streamlines of the model take the place of the time-averaged free-shear layers and form boundaries of the irrotational flow external to the wake region. Also, the fluid velocities on the free streamlines decrease with distance from the cylinder towards the free stream velocity as observed in experiments. It is found, however, that streamlines continually enter the actual wake which widens downstream because of the diffusion of vorticity. Also the near wake is continuously shedding Karman vortex streets at the rear of the bubble and thus the free shear layers are unsteady over part of the near-wake region. A successful treatment to take into account the diffusion of vorticity and unsteadiness in the wake has not been developed. A flow model taking into account these features of the wake would require an extensive investigation of the wake region. Since the intention is

not to study the wake itself but rather to account for the effect of
the wake on the upstream (attached flow) pressure distribution, it is
expected that a reasonable model for the time-averaged near-wake free
shear layers would result in an improvement over their complete neglect.

To construct the flow field just described, flow in an untransformed
plane \( \zeta \) is considered as in Figure 2. The fundamental flow past the
circle is the uniform flow plus a doublet. Additionally, sources of
strength \( 2Q_1 \) and \( 2Q_2 \) are located at angles \( \gamma \) and \( \delta \) respectively from
the direction of unperturbed flow. Image sinks \(-Q_1\) and \(-Q_2\) are placed
at the origin to satisfy the boundary conditions on the circle. A point
vortex of strength, \( \Gamma \), is also placed at the origin and does not affect
the boundary conditions on the circle but does affect the positions of
\( S_1 \) and \( S_2 \) and the complex potential field. Here \( R \) is the radius of the
cylinder. Now half the distance between the two sources will be assign-
ated the value unity. Therefore, \( R \) is equal to \( \csc \alpha \).

The complex potential of the resulting flow referred to the \( \zeta \) plane
is:

\[
X(\zeta) = V_0 (\zeta + R^2/\zeta) +\{2\hat{Q}_1 [\ln (\zeta - Re^{i\gamma}) - \frac{1}{2}\ln (\zeta)] / 2\pi\} + \{2\hat{Q}_2 [\ln (\zeta - Re^{i\delta}) - \frac{1}{2}\ln (\zeta)] / 2\pi\} + (\frac{1}{2} i\Gamma/\pi) \ln \zeta \tag{1}
\]

For the \( \zeta \) plane, only the fluid dynamical properties on the circle
\( C \) will be of primary interest. The velocity on the circle \( C \) can be
obtained as shown below. From Figure 2, the directed vectors from the
sources \( 2Q_1 \) and \( 2Q_2 \) to the point \( \zeta \) can be seen to be respectively,

\[
\zeta - Re^{i\gamma} = r_1 e^{is},
\]

\[
\zeta - Re^{i\delta} = r_2 e^{it}, \tag{2}
\]

where \( s \) and \( t \) are angles between the lines drawn from the sources to a
point \( \zeta \) on the circle and the lines drawn through the sources parallel
to the real axis. Here, \( r_1 \) and \( r_2 \) are the lengths from the point \( \zeta \)
to the sources \( 2Q_1 \) and \( 2Q_2 \) respectively. Thus, using equation \( 2 \) to
rewrite the expression for the complex potential in equation \( 1 \), the
velocity potential can then be expressed as

\[
\phi = 2 V_0 \csc \alpha \cos \phi + \frac{\hat{Q}_1}{2\pi} \ln \left( \frac{r_1^2}{\csc \alpha} \right) + \frac{\hat{Q}_2}{2\pi} \ln \left( \frac{r_2^2}{\csc \alpha} \right) - \frac{\phi_\Gamma}{2\pi}, \tag{3}
\]

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Now $r_1^2$ and $r_2^2$ can be seen to be the following in terms of $\csc \tilde{\alpha}$, $\gamma$, $\delta$, and $\psi$:

$$r_1^2 = 2 (\csc^2 \tilde{\alpha}) [1 - \cos (\phi - \gamma)] ,$$

$$r_2^2 = 2 (\csc^2 \tilde{\alpha}) [1 - \cos (\phi - \delta)] .$$  (4)

The tangential velocity at the surface of the cylinder is just

$$q = - \frac{d\phi}{d\psi} \sin \tilde{\alpha}$$  (5)

Thus, substituting equation (4) into equation (3), differentiating according to equation (5), and using trigonometric identities, the following equation can be obtained:

$$q = 2 V_o \sin \phi - \frac{1}{2\pi \csc \tilde{\alpha}} \left[ \tilde{Q}_{1,2} \cot \left( \frac{\phi - \gamma}{2} \right) + \tilde{Q}_{2} \cot \left( \frac{\phi - \delta}{2} \right) \right] + \Gamma/2\pi \csc \tilde{\alpha} .$$  (6)

By using the following definitions

$$q \equiv q/V_o ,$$

$$Q_{1,2} = \tilde{Q}_{1,2} / 2\pi V_o \csc \tilde{\alpha} ,$$

$$u_c = \Gamma / 2\pi V_o \csc \tilde{\alpha} ,$$  (7)

the following equation can be obtained from equation (6):

$$q = 2 \sin \phi - [Q_{1} \cot \left( \frac{\phi - \gamma}{2} \right) + Q_{2} \cot \left( \frac{\phi - \delta}{2} \right) + u_c] .$$  (8)

Equation (8) is the expression for the dimensionless tangential velocity at a point on the surface of the cylinder in the untransformed plane $\zeta$.

The resulting complex potential and the complete circle $E$ in the $\zeta$ plane can be mapped to the physical plane $z$ by first transforming to the $\xi$ plane. The complex plane $\xi$ has its real axis aligned with the direction of the freestream flow. The complex plane $\xi$ has its real axis at an angle $-\varepsilon$ from the real axis of the $\zeta$ plane and is located such that $S_1$ and $S_2$ are symmetric about the real axis. An angle $\phi$

describing the position on the circle in the $\zeta$ plane can be described in the $\xi$ plane by the relationship

$$\sigma = \phi + \varepsilon$$

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The $\zeta$ and $\zeta$ planes will be termed the untransformed planes. Points on the $\zeta$ plane can be mapped conformally into points on the $\bar{z}$ plane by an analytic function:

$$\bar{z} = M(\zeta) .$$

(9)

The particular mapping function is a Zhukovskii transformation:

$$M(\zeta) = \zeta - \cot \alpha - \frac{1}{\zeta - \cot \alpha} .$$

(10)

Here $\cot \alpha$ is just the distance from the origin along $\xi_1$ to the straight line drawn from $S_1$ to $S_2$ in Figure 2. The points $S_1$, $S_2$, are critical points and $M'(\zeta)$ the derivative of $M$ is zero at the points. Angles of intersection in the $\zeta$ plane are doubled in the $\bar{z}$ plane by the mapping function at $S_1$ and $S_2$. Thus, the angle of intersections of the separation streamlines with the circle $E$ in the $\zeta$ plane increase from $90^\circ$ to a tangential intersection with the arc $C$. Also, the circle $E$ is mapped onto the slit $S_1A_2B_S_1$. The points $S_1$ and $S_2$ lie on the $\bar{Y}$ axis of the $\bar{z}$ plane.

The relationship between the two complex velocities in the two planes $\bar{z}$ and $\zeta$ is given by

$$W(\bar{z}) = w(\zeta) / M'(\zeta) ,$$

(11)

where

$$M'(\zeta) = 1 + 1/(\zeta - \cot \alpha)^2 .$$

(12)

From equation (11) and equation (12), it is apparent that at large distances from the flow, the complex velocities in the two planes asymptotically approach each other; and this implies that $u_0 = V_0$ and $\varepsilon = \bar{\varepsilon}$.

The transformation from the $\bar{z}$ plane to the $z$ plane is accomplished by a translation and then a rotation through an angle $\varepsilon$. The $z$ and $\bar{z}$ planes will be termed the physical planes. As mentioned before, only the dynamical quantities on the cylinder are of primary interest; hence, only the circle $E$ of Figure 2 will be mapped into the arc $C$ of Figure 3.

In the sections that follow, it will be necessary to know the tangential velocity $u_\phi$ on the arc $C$ of the cylinder in the $z$ plane in terms of the position $\phi$ on the circle $E$. The tangential velocity on the cylinder in the physical plane $z$, given the corresponding velocity in the plane $\zeta$, can be obtained if equation (11) is used specialized to the circle:
\[ u_e(x) = q(\phi)N(\phi) \tag{13} \]

when \( N(\phi) \) is defined as

\[ N(\phi) = \left| M'(\xi) \right| \tag{14} \]

The subscript \( E \) means that \( M'(\xi) \) is evaluated on the cylinder \( E \). The value \( N(\phi) \) is obtained by multiplying \( M'(\xi) \) on the circle \( E \) by its complex conjugate and using trigonometric identities to obtain squared quantities in both numerator and denominator. Then taking the square root and putting \( \sigma \) in terms of \( \phi \) and \( \epsilon \), the expression for \( N(\phi) \) becomes

\[ N(\phi) = \frac{2 \left[ \cos \bar{\alpha} - \cos (\phi + \epsilon) \right]}{1 - 2 \cos \bar{\alpha} \cos (\phi + \epsilon) \cos^2 \bar{\alpha}} \tag{15} \]

The inviscid flow velocity can then be found by using equation (13). The function for \( N(\phi) \), equation (15), is known (since \( \bar{\alpha} \) and \( \epsilon \) can be obtained from equation (33).

B. Curvature and Pressure Conditions at the Separation Points

Parkinson and Jandali\textsuperscript{23}, with their symmetric flow model, related the location and strength of the source in the untransformed plane to two parameters in the physical plane: the position of separation and the local external velocity at separation. By using the finite curvature condition at the separation point originally suggested and used by Woods\textsuperscript{27}, Bluston and Paulson\textsuperscript{22} could characterize the inviscid flow in the physical plane with one parameter: the position of separation. The number of parameters of the physical plane needed to describe the present flow model will also be reduced by imposing similar conditions. The streamline curvature condition imposed by Bluston and Paulson is applied to both separation points in the present case and thus reduces the number of descriptive parameters by two. After applying a second condition of equal pressures at the separation points, only two parameters of the physical plane will be needed to determine the five parameters in the untransformed plane. These two parameters are specified by the locations of the separation points on the rotating cylinder.

The first conditions imposed are designed to insure that streamline curvatures and pressures at the separation points are physically

meaningful and are consistent with boundary-layer theory. The curva-
tures of the free streamlines at the separation points are required to be finite. If the curvature of the separation streamline is negative
infinite, the separation streamline will be concave downwards and will cut into the rear of the cylinder. The negative infinite curvature of the separation streamline should, therefore, be forbidden since this is physically impossible. If the separation streamline has positive
infinite curvature at $S_1$, a very large adverse pressure gradient will occur upstream of separation becoming infinite at $S_1$. Clearly this is not consistent with the results of boundary layer theory since boundary-
layer separation would then occur before the assumed separation point. Thus, only the case of finite streamline curvature will be considered.

The other condition of equal pressures at the separation points results from vorticity transport considerations. In wing-airfoil
theory, no net vorticity is shed from the wing in steady flow; that is, the average flux of vorticity out of any fixed circuit around the air-
foil is zero. For the rotating cylinder, the average flux of vorticity out of any fixed circuit around the cylinder should also be zero. Now, if no vorticity were transported into the wake from the rear part of the rotating cylinder immersed in the wake, then equal amounts of vorticity (but with opposite signs) should be transported into the wake at the separation points from the upper and lower parts of the cylinder. The rate of vorticity transport downstream at a separation point is

$$\int_C u \frac{3u}{3y} dy = \frac{(u_{es}^2 - u_w^2)}{2},$$

where the subscript $s$ denotes conditions at separation. This is the result obtained considering either the lower or upper side of the cylinder. Since vorticity transport must then be the same at both separation points, this implies that the magnitudes of $u_e$ at both separation points will be equal. From Bernoulli's equation, the pressure at both separation points will be the same.

The original assumption of equal pressures at the separation points was attributed to Howarth \(^{28}\) by Piercy, Preston, and Whitehead. Although this assumption is also used in the present work, Piercy, Preston, and Whitehead \(^{21}\) took into account the possibility that vorticity is gener-

ated by the part of the cylinder that is immersed in the wake. He obtained good results for an ellipse at a nonzero angle of attack. To take this castoff vorticity into account would complicate the problem to be addressed here although it would be a suitable refinement to be considered in subsequent investigations.

C. Numerical Method of Obtaining the Model

To obtain the values of the five basic parameters appearing in equation (8) in the \( \zeta \) plane - given the two separation points in the physical plane - a system of five nonlinear equations must be solved. One might attempt to solve these equations analytically as was done for Bluston and Paulson's relatively simple model. For their model, \( u_e \) finally reduced to a trigonometric expression that involved only the untransformed-plane angles \( \phi \) and \( \tilde{a} \). An unsuccessful attempt was made to solve the present more complicated set of equations analytically; therefore, it was decided to solve for the five parameters systematically using numerical techniques.

The system of five equations is obtained from requirements imposed by separation and the previously discussed curvature and pressure conditions at separation. The first requirement is derived from the nature of the critical points, \( S_1 \) and \( S_2 \); these are the stagnation points in the untransformed plane caused by the upstream flow from the sources placed on the circle. Thus, the untransformed velocity \( q \) at the critical points is zero.

From the expression for \( q \), equation (8), the following two equations are then obtained at the critical points, \( \phi = \pm \tilde{a} - \varepsilon \), respectively:

\[
2 \sin (\tilde{a} - \varepsilon) - \left[ Q_1 \cot \left( \frac{\tilde{a} - \gamma - \varepsilon}{2} \right) + Q_2 \cot \left( \frac{\tilde{a} - \delta - \varepsilon}{2} \right) \right] + u_c = 0. \quad (17)
\]

\[
2 \sin (\tilde{a} + \varepsilon) - \left[ Q_1 \cot \left( \frac{\tilde{a} + \gamma + \varepsilon}{2} \right) + Q_2 \cot \left( \frac{\tilde{a} + \delta + \varepsilon}{2} \right) \right] - u_c = 0. \quad (18)
\]

The next requirement imposed was that the magnitude of the velocities should be equal at separation in the physical plane as required by Howarth's assumption; i.e:

\[
u_e(x_{s1}) = - u_e(x_{s2}) \quad (19)
\]

where the subscript symbols \( s1 \) and \( s2 \) indicate the upper and lower separation points respectively. The negative sign occurs in this equation because \( x \) increases on the upper side of the cylinder and decreases on the lower side of the cylinder. Now from the expression for \( u_e \) in equation (13), \( u_e \) is indeterminate at the separation points since \( q \) and \( N \) are both zero there. This indeterminancy could have
been removed if the problem had been solved analytically. Since the computer was available, an approximation to the requirement of equation (19) was obtained using numerical computation techniques. For angular positions very near to the critical points, the absolute values of the velocities in the physical plane were set equal to each other at the following two points:

\[ \pm \alpha_1 = (\bar{\alpha} + \Delta \alpha) \quad , \quad (20) \]

where \( \pm \alpha_1 \) are the values of \( \alpha \) very near to \( \pm \bar{\alpha} \) and \( \Delta \alpha \) is a small incremental angle. Now from equation (15) it is seen that

\[ N (\alpha_1 - \varepsilon) = N (-\alpha_1 - \varepsilon) \quad (21) \]

where \( \varepsilon \) is a parameter for \( N \). Hence, a condition almost equivalent to equation (19) can be imposed:

\[ q (\alpha_1 - \varepsilon) = -q (-\alpha_1 - \varepsilon) \quad . \quad (22) \]

Thus, from equation (8), the following equation will result:

\[
\begin{align*}
2 \left[ \sin (\alpha_1 - \varepsilon) - \sin (\alpha_1 + \varepsilon) \right] - Q_1 \left[ \csc \left( \frac{\alpha_1 - \gamma - \varepsilon}{2} \right) - \cot \left( \frac{\alpha_1 + \gamma + \varepsilon}{2} \right) \right] \\
- Q_2 \left[ \csc \left( \frac{\alpha_1 - \delta - \varepsilon}{2} \right) - \cot \left( \frac{\alpha_1 + \delta + \varepsilon}{2} \right) \right] + 2 u_c = 0 \quad . \quad (23)
\end{align*}
\]

The smoothness or finite curvature requirement is used to obtain two more equations. A result of the smoothness requirement is that the value of \( \frac{du_e}{dx} \) is finite at the separation point. Now from the chain rule:

\[ \frac{du_e}{dx} = \frac{du_e}{d\phi} \frac{d\phi}{dx} \quad . \quad (24) \]

But \( x \) as a function of \( \phi \) at the separation points (\( S_1, S_2 \)) are extremums since the circle \( E \) in the untransformed plane (Figure 2) is mapped onto the curved slit \( S_1 A S_2 B S \), in the physical plane (Figure 3). Therefore

\[ \frac{dx}{d\phi} \bigg|_{S_1, S_2} = 0 \quad , \]

and thus

\[ \frac{du_e}{d\phi} \bigg|_{S_1, S_2} = 0 \quad . \quad (25) \]
is a necessary condition that \( \frac{du_e}{dx} \) be finite at the critical points. But \( N(\phi) \), which is zero at the critical points, occurs in the denominator in the expression for \( \frac{du_e}{d\phi} \) as a squared term. So then in a similar manner as was done for the equal pressure condition, use points at angle \( \Delta \sigma \) away from the points \( S_1, S_2 \):

\[
\frac{du_e(\alpha_1-\varepsilon)}{d\phi} = \frac{du_e(-\alpha_1-\varepsilon)}{d\phi} = 0 \quad (26)
\]

The smoothness condition at \( \phi = \alpha_1 - \varepsilon \) becomes

\[
(1-2 \cos \bar{\alpha} \cos \alpha_1 + \cos^2 \bar{\alpha}) \left\{ \frac{2 \cos (\alpha_1 - \varepsilon)}{2 (\cos \bar{\alpha} - \cos \alpha_1)} \right\} + \frac{1}{2} \left[ Q_1 \csc^2 \left( \frac{\alpha_1-\varepsilon-\gamma}{2} \right) + Q_2 \csc^2 \left( \frac{\alpha_1-\varepsilon-\delta}{2} \right) \right] + \frac{\sin \alpha_1 (\cos^2 \bar{\alpha}-1)}{2 (\cos \bar{\alpha} - \cos \alpha_1)^2} \left\{ \frac{-2 \sin (\alpha_1-\varepsilon)}{4 \csc^2 \left( \frac{\alpha_1-\varepsilon-\gamma}{2} \right) + Q_1 \cot \left( \frac{\alpha_1-\varepsilon-\delta}{2} \right) + u_c \right\} = 0 \quad (27)
\]

The smoothness condition at \( \phi = -\alpha_1 - \varepsilon \) is given as

\[
(1-2 \cos \alpha_1 + \cos^2 \bar{\alpha}) \left\{ \frac{2 \cos (\alpha_1+\varepsilon)}{2 (\cos \bar{\alpha} - \cos \alpha_1)} \right\} + \frac{1}{2} \left[ -Q_1 \csc^2 \left( \frac{\alpha_1+\varepsilon+\gamma}{2} \right) \right] + \frac{\sin \alpha_1 (\cos^2 \bar{\alpha}-1)}{2 (\cos \bar{\alpha} - \cos \alpha_1)^2} \left\{ -2 \sin (\alpha_1+\varepsilon) \right\} + Q_1 \cot \left( \frac{\alpha_1+\varepsilon+\gamma}{2} \right) + Q_2 \cot \left( \frac{\alpha_1+\varepsilon+\delta}{2} \right) \left\{ \frac{u_c}{u_c} \right\} = 0 \quad (28)
\]

Equations (17), (18), (23), (27), and (28) constitute a nonlinear system of equations to be solved for \( Q_1, Q_2, \gamma, \delta, \) and \( u_c \). These values are determined by only two parameters in the untransformed plane, \( \bar{\alpha} \) and \( \varepsilon \). These two parameters are in turn determined by \( x_{s1} \) and \( x_{s2} \) as given by equation (33).
Equations (17), (18), (23), (27), and (28) are solved using the iterative Newton-Raphson method and the parameters are said to be found when the biggest change in any of the parameters is less than $2 \times 10^{-5}$ from one iteration to the next.

To approximate the values of the parameters that would be obtained if the pressure and curvature conditions could be applied at the separation points, the following procedure is followed. The value of $\Delta \sigma = -0.08^\circ$ is first used and the corresponding parameters are computed. The parameters for $\Delta \sigma = 0,08^\circ$ are next found and these two sets of parameters are used to find the estimated parameters for $\Delta \sigma = 0$ by linear interpolation. A question might occur as to how accurate this approximation might be, especially if the parameters changed in a nonlinear manner and rapidly with $\Delta \sigma$. Figures 4 and 5 show the variation of $Q_1$ and $\gamma$ versus $\Delta \sigma$ and both are approximately linear with $\Delta \sigma$ which is very encouraging in light of the linear interpolation used in the program. The circulation was also found to have similar linear behavior.

To numerically calculate the solution, some relationships and quantities need to be found, such as the radius of the cylinder in the $z$ plane. Using the mapping function $M(\zeta)$ described by equation (10), it is found that the position of $S_1$ in the $\bar{z}$ plane is given by $2i$. Drawing a line from the center of the circular arc $C$ in the $\bar{z}$ plane to $S_1$, the radius is seen to be

$$a = 2 \csc m_{s1}$$  \hspace{1cm} (29)

where --as mentioned before--the subscript $s1$ denotes evaluation at the upper separation point $S_1$. The length from the origin in the $\bar{z}$ plane along the real line to the arc $C$ is given by

$$M(-\csc \bar{a}) = -2 \cot \bar{a}$$  \hspace{1cm} (30)

Thus, drawing a straight line from point $A$ to $S_1$, it is seen from equation (30) that

$$\bar{a} = \left(\pi - m_{s1}\right) / 2$$  \hspace{1cm} (31)

Using equation (31), $a$ is then found to be, from equation (29):

$$a = 2 \csc 2 \bar{a}$$  \hspace{1cm} (32)

Knowing $x_{s1}$ and $x_{s2}$, the values $\bar{a}$ and $\varepsilon$, which determine the five parameters in the plane $\zeta$, can be found using Figure 3 and equation (31):

$$\varepsilon = (x_{s1} + x_{s2}) / 2$$  \hspace{1cm} (33)
Figure 4. $Q_1$ vs. $\Delta \sigma$

$x_{s1} = 90^\circ$

$x_{s2} = -79^\circ$
Figure 5. \( \gamma \) vs. \( \Delta \sigma \)

\( x_{S1} = 90^\circ \)

\( x_{S2} = -79^\circ \)
\[ a = (\pi - x_s \epsilon) / 2 \] \hspace{1cm} (33)

Since the quantities in the plane \( z \) will be calculated in terms of the angular position \( \phi \) in the \( \zeta \) plane, it is necessary to determine a relationship between \( \phi \) and \( x \). To do this, consider the directed vector \( \hat{a} \) from the center of the circular arc \( C \) in the physical planes to a point on the arc \( C \) in the physical planes. Using the mapping function given by equation (10), the expression for \( \hat{a} \) is given by the following:

\[ \hat{a} = M [\csc \theta \exp (i\sigma)] - M(0) \] \hspace{1cm} (34)

This expression can be manipulated to a form such that the imaginary part of \( \hat{a} \) can be taken. The imaginary part of \( \hat{a} \) is just a \( \sin m \). Using the imaginary part of the expression for \( \hat{a} \) from equation (34) and applying trigonometric identities, the following relationship is obtained:

\[ \sin m = \frac{\sin \sigma (1 - \cos \bar{a} \cos \sigma)}{(\sec \bar{a} + 1/\sec \bar{a}) - \cos \sigma} \] \hspace{1cm} (35)

Expressing equation (35) in terms of \( \phi \) and \( x \), the following equation is obtained:

\[ \sin(x - \epsilon) = \frac{\sin(\phi + \epsilon) [1 - \cos \bar{a} \cos(\phi + \epsilon)]}{[(\sec \bar{a} + 1/\sec \bar{a}) / 2] - \cos(\phi + \epsilon)} \] \hspace{1cm} (36)

The velocity and derivative of the velocity with respect to angle in the physical plane are calculated as these quantities appear in the integral boundary-layer equations. A subprogram was constructed to obtain these quantities found in terms of the untransformed variable \( (\phi) \) and parameters in the plane \( \zeta \). This subprogram is called at each step of the integration of the boundary-layer equations. First the relationship between \( x \) and \( \phi \) is found using equation (36). The value of \( \phi \) is found using the iterative method of Newton. Successive values of \( \phi \) found during iteration must be within \( 10^{-6} \) before the method is declared to have converged. The values of \( u_e \) and \( du_e / dx \) can then be found from equation (8), its derivative with respect to \( \phi \), equation (15), its derivative with respect to \( \phi \), and substituting the results into equations (13) and its derivative with respect to \( \phi \). For more details, the listing of the computer program is given in Appendix A.

D. Method of Obtaining Agreement Between the Flow Model and the Boundary-Layer Calculation Results

For bodies with fixed surfaces, boundary-layer calculations can approximately predict the point of separation given the external velocity distribution. In this investigation of the rotating cylinder,
the final inviscid-flow model's separation points are made to agree with the location of the separation points found by numerical calculation of the boundary layer. This flow-model can thus be made consistent with boundary-layer theory using the integral boundary-layer technique developed here.

The procedure for obtaining consistency between the flow-model and boundary layer theory is quite simple in principle. The boundary layer equations for flow over the rotating circular cylinder are solved with assumed separation points for the inviscid-flow model further back on the cylinder than they would be anticipated to actually occur. Separation points are then obtained from the results of the boundary-layer calculation and a revised inviscid flow model is then constructed using these new separation points. Again new boundary-layer separation points are found and the flow model is again modified using these new separation points. This process is repeated until the largest absolute value of the difference in separation location for the last two particular flow models is less than a value called $E_c$. The value of $E_c$ used in this work was $E_c = 0.46$ degrees.

III. FLOW-MODEL RESULTS AND DISCUSSION

The external velocity distributions consistent with the boundary-layer calculations and the corresponding lift and drag coefficients are of chief interest for comparison with experiment. The range of $u_w$ considered here ($0 < u_w < 0.32$) is chiefly limited by stability of the boundary-layer calculation method used although Swanson indicates that turbulence always appears before separation on the lower side of the cylinder when $u_w > 0.5$ for the range of Reynolds numbers investigated. He investigated the Magnus effect for $3.58 \cdot 10^4 < Re_d < 5.01 \cdot 10^5$.

A. External-Velocity Distributions

Figure 6 illustrates the first trial external-velocity distribution up to the point of separation and compares it with the converged or final velocity distribution when $u_w = 0.2$. Figures 7 and 8 present, in the adverse pressure gradient region, details of these distributions with additional distributions found in the iteration process to obtain convergence. The graphs are representative of the convergence behavior using this iterative method for all corresponding values of $u_w$ considered with one exception. For $u_w = 0.3$, various initial values of separation points were tried, but after a few iterations the boundary layer calculations would not separate up to and including the breakaway point of the inviscid-flow model. It is felt that the boundary layer needs to
Figure 6. Comparison Between First and Converged Velocity Distributions ($u_w = 0.2$)

Initial $x_{S1} = 95.1^\circ$

Initial $x_{S2} = 83.1^\circ$
Figure 7. Velocity Distribution Details - Upper Side Adverse Pressure Gradient Region

- First iteration
- Third iteration
- Last iteration
Figure 8. Velocity Distribution Details - Lower Side in Adverse Pressure Gradient Region

- $u_w = 0.2$
- Initial $x_{S1} = 95.1^\circ$
- Initial $x_{S2} = -83.1^\circ$
be calculated more accurately near separation for larger negative values of $u_w/u_e$. Granted adequate time, it is thought that consistent and more meaningful results could be achieved by improving the integral boundary-layer technique.

For a cylinder rotation rate of 0.15, the locations of separation versus the trial number in the iteration process to find a converged solution is shown in Figure 9. As mentioned before, when the absolute value $E_c$ of the differences between the positions of succeeding values of separation points on both sides of the cylinders become less than 0.46 degrees, convergence was considered to be achieved. Bluston and Paulson's value of $E_c$ was $0.5^\circ$. For $u_w = 0.15$, the value of $E_c$ decreased to $0.16$ degrees to observe the convergence behavior more closely. In Figure 9, it is seen that three more iterations were required to satisfy the tighter convergence criterion.

The convergence of the lift coefficient can also be studied in Figure 10. Here it is seen that the value of $C_L$ from circulation considerations is quite sensitive to separation location. The value of $C_L$ for $E_c = 0.16$ was about 80% of $C_L$ for $E_c = 0.46$ degrees. However, by integrating the vertical component of the pressure coefficient over the cylinder up to the point of separation, the values of $C_L$ were found to be within 10% of each other for the two values of $E_c$.

In choosing a precision criterion, one must accept a compromise between computing time and accuracy of the boundary layer calculation method. In retrospect, it appears that by using an accelerated convergence technique, a more rigorous standard for the convergence criterion could have been adopted. However, it might be better to improve the inviscid flow model in some other way; for instance, one could change the flow model to take into account the cast-off vorticity from that part of the cylinder immersed in the wake region.

Figure 11 compares some experimental velocity distributions with the results of the current analysis for the nonrotating cylinder. The curve of Fage and Falkner\textsuperscript{29} was obtained from Bluston and Paulson's paper while the curve of Petrie and Simpson\textsuperscript{30} was taken directly from


Figure 9. Location of Boundary-Layer Separation vs. Trial Number ($u_w = 0.15$)
Figure 10. Lift Coefficient vs. Trial Number ($u_w = 0.15$)
Figure 11. $u_e$ vs. $x$ - Comparison with Experiment ($u_w = 0$)

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THEORETICAL RESULTS

- **FAGE & FALKNER** ($Re_d = 10^5$)
- **PETRIE & SIMPSON** ($Re_d = 1.4 \times 10^4$)
their paper. Fage and Falkner's curve gives the better agreement with the theoretical curve, but is for a higher Reynolds number than for the other curves with the amount of turbulence in the freestream unknown. The curve of Petrie and Simpson was obtained with very little mainstream turbulence; by introducing a large amount of turbulence into the mainstream, they obtained data similar to the Fage and Falkner's results. The theoretical curve approximates the experimental curves better than the simple sine curves or series curves. Nevertheless, some sort of refinement of the inviscid-flow model might better represent the observed pressure distributions in the adverse pressure gradient region.

A check of Bluston and Paulson's velocity distribution reveals it to be nearly identical to the theoretical distribution found in the present investigation. This agreement with Bluston and Paulson further increases confidence in the procedure of finding the five parameters \((Q_1, Q_2, \gamma, \delta, u_c)\) in the untransformed plane by computer.

It has been seen that converged separation points can be found by this iterative process given certain initial trial separation points. The question naturally arises as to how the final separation points are affected by the particular choice of initial separation points. To answer this question, different sets of initial separation point values were used in the computer program. The results are tabulated as follow:

<table>
<thead>
<tr>
<th>Initial Values</th>
<th>Final Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Upper Side</td>
<td>Lower Side</td>
</tr>
<tr>
<td>(u_w)</td>
<td>(x_{s1})</td>
</tr>
<tr>
<td>0.2</td>
<td>95.1°</td>
</tr>
<tr>
<td>0.2</td>
<td>92.8°</td>
</tr>
<tr>
<td>0.1</td>
<td>94.5°</td>
</tr>
<tr>
<td>0.1</td>
<td>95.0°</td>
</tr>
</tbody>
</table>

The final values for the separation points vary at most 0.2° with these initial choices. As will be seen later, the variation in the lift and drag coefficients will also vary a small amount with initial choice of the breakaway points. This dependence of the final values on the initial values could probably be decreased further by using a more rigorous standard for convergence.
Figure 12 shows absolute values of $u_e$ plotted against the absolute value of $x$ for some different values of $u_w$. The separation points can be seen to be delayed on the upper side and advanced on the lower side in agreement with observation.

The inviscid flow fields external to the cylinder were also investigated. Figure 13 shows the streamlines in the untransformed plane for the nonrotating cylinder and Figure 14 exhibits the streamlines in the corresponding physical plane. Figure 15 and 16 gives the pattern of streamlines in the untransformed and physical plane respectively for $u_w = 0.2$. The zero stream function lines in the untransformed plane are seen to leave perpendicular to the cylinder whereas in the transformed plane these lines leave the circle tangentially as required by the transformation. It may be observed that the streamlines that form the boundary of the wake in the physical plane are very nearly parallel; the main effect of the low rotation rate is to produce a slight asymmetry in the streamlines near the separation points. The streamline pattern in the untransformed plane show greater departures from a symmetrical pattern than for the physical plane case.

The stagnation points shift to negative $x$ values as $u_w$ is increased. Below is a table giving these stagnation points for different values of $u_w$.

---

*The curves do not extend to $x = 0$ since values of $u_e$ and $du_e/dx$ are printed out only for that region being numerically calculated with the integral boundary-layer equations. The limitations of the computer code restrict integration of the boundary-layer equations on the lower side to the region where $u_w/u_e > -0.3$. Thus, the value of $|x|$ at which integration can start becomes larger with increasing $u_w$. 

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Figure 12. $|u_e|$ vs. $|x|$ -- Parameter is $u_w$.
Figure 15. Streamline Pattern Around Cylinder in Untransformed Plane

$u_w = 0.2$
TABLE II. STAGNATION POINT ON CYLINDER FOR DIFFERENT RATES OF ROTATION

<table>
<thead>
<tr>
<th>( \frac{u_w}{w} )</th>
<th>Stagnation Point Location (Degrees)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>0.1</td>
<td>-0.53</td>
</tr>
<tr>
<td>0.2</td>
<td>-0.91</td>
</tr>
<tr>
<td>0.25</td>
<td>-1.19</td>
</tr>
<tr>
<td>0.32</td>
<td>-1.76</td>
</tr>
</tbody>
</table>

This displacement direction of the stagnation point also occurs in experiments. However, no detailed comparison with experiment can be made for these small rotation rates partly because of the difficulty in measuring small changes in the stagnation point location. Below \( u_w = 0.25 \), the stagnation point is displaced linearly with \( u_w \) but the displacement for \( u_w = 0.32 \) is larger than the linear relationship would predict. The velocity distribution curves \( u_w = 0.2 \) and \( u_w = 0.25 \) do not follow the trends expected and noted in Figure 12. Comparing the upper-side velocity distributions in Figure 17 reveals that the maximum value of velocity for \( u_w = 0.15 \) is actually higher than it is for \( u_w = 0.25 \). The upper side displacement of separation increases only a minimal amount with the increase in \( u_w \). However, the lower side curve follows the trend noted for the lower values of wall velocity. The reasons for this behavior are not known.

For laminar boundary layers, the positions of separation should be a monotonic function of \( u_w \) according to experimental indications and also as expected from boundary layer theory. Figure 18 shows the computer results for the absolute values of the separation points versus \( u_w \). It appears that for values of \( u_w \leq 0.15 \), the displacements of the separation points from the nonrotating case are almost linear. However, the absolute value of the slope for the line approximating the computer results is greater for the lower side than it is for the upper side of the cylinder. In prior work, \(^{24}\) where sine-function external-velocity distributions were used with this integral method, a linear variation with \( u_w \) was also observed but extending up to values of \( u_w = 0.4 \). The slope of this line is somewhat less than for the present bound-vortex source-wake model.
Figure 17. $|u_e| \text{ vs. } |x|$-- $u_w = 0.25$ and $u_w = 0.15$
Figure 18. Location of Separation vs. Rotation Rate
B. Lift and Drag Coefficients

Approaches representing two different views can be utilized to obtain the lift coefficient. The first approach stems from the consideration that the rotating cylinder, through the action of the boundary layer, has induced in the inviscid flow a certain circulation strength around the cylinder. This inviscid flow model takes this induced circulation into account by introducing a bound vortex in the physical plane or more precisely, a point vortex of strength $\Gamma$ at the center of the circle in the untransformed plane. This bound vortex gives a lift force according to the Zhukovskii theorem. In the other approach, one is concerned with providing a flow model with velocity distributions approximating the experimental distributions. The assumption of Howarth\textsuperscript{26} is then made in which constant pressure is assumed over that part of the cylinder immersed in the wake. The lift coefficient can thus be calculated knowing the velocity distribution. No attempt will be made here to reconcile these two approaches or choose between them. Nevertheless, it might be pointed out that the latter approach is the same approach used in obtaining the drag coefficients. Calculations were made for both approaches: the bound-vortex approach and the integrated-pressure approach.

The lift coefficient for the bound-vortex approach can be found by first considering that the circulation strength and hence the lifting force of the point vortex can be shown to be invariant under the transformation between the two complex planes. The lift coefficient can then be found to be

$$C_L = 2\pi u_c \cos \alpha$$

In the second approach, the lift coefficient is obtained by first integrating the vertical component of the local force coefficient on the cylinder as calculation of the boundary-layer on the cylinder proceeds. The vertical component of the force coefficient on the cylinder surface in the wake is then calculated assuming the pressure constant in the region. Finally, these contributions are used to obtain the total lift force coefficient on the cylinder.

Using these two approaches, the lift coefficient as a function of rotation rate ($\Omega / u_0$) is shown in Figure 19. These lift coefficients values are compared with the experimental results obtained by Swanson for Reynolds numbers from $3.58 \times 10^4$ to $9.9 \times 10^4$. The integrated pressure approach gives the better agreement for most of the range of $u_w$ and the lift coefficient is linear with $u_w$ up to and including $u_w = 0.15$. For $u_w = 0.2$, two computer runs were made with different initial values of separation points; the variation in the final results are also shown in Figure 19. The variation of the lift coefficient with variation of initial values could probably be lessened by making
Figure 19. $C_L$ vs. $u_w$ - Comparison with Experiment
the convergence criterion for this iteration procedure more severe.

To obtain the drag coefficient, the horizontal component of the local force coefficient on the cylinder is integrated as the boundary layer quantities are calculated; the contribution to the drag coefficient in the wake region is found assuming that it is a region of constant pressure. The drag coefficient obtained by computer is shown together with experimental results in Figure 20. The values obtained from the model are comparable to values found experimentally, although the general trends do not agree very well. Similarly, as for the lift coefficient comparison, the value at \( u_w = 0.25 \) is in the worst agreement with the experimental value.

The values of the obtained untransformed-plane parameters corresponding to the lift and drag force results are summarized in the following table.

<table>
<thead>
<tr>
<th>( u_w )</th>
<th>( Q_1 )</th>
<th>( Q_2 )</th>
<th>( \gamma )</th>
<th>( \delta )</th>
<th>( \alpha_c )</th>
<th>( \beta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.28918</td>
<td>0.28918</td>
<td>0.3322</td>
<td>-0.3322</td>
<td>0.0</td>
<td>0.8551</td>
</tr>
<tr>
<td>0.1</td>
<td>0.18564</td>
<td>0.40316</td>
<td>0.32267</td>
<td>-0.34984</td>
<td>0.0125</td>
<td>0.85898</td>
</tr>
<tr>
<td>0.15</td>
<td>0.13037</td>
<td>0.48559</td>
<td>0.3324</td>
<td>-0.35289</td>
<td>0.0132</td>
<td>0.87119</td>
</tr>
<tr>
<td>0.2</td>
<td>0.082203</td>
<td>0.55964</td>
<td>0.3552</td>
<td>-0.3545</td>
<td>0.016</td>
<td>0.88089</td>
</tr>
<tr>
<td>0.25</td>
<td>0.035223</td>
<td>0.64684</td>
<td>0.41081</td>
<td>-0.34946</td>
<td>0.02198</td>
<td>0.89503</td>
</tr>
<tr>
<td>0.32</td>
<td>0.08099</td>
<td>0.49334</td>
<td>0.3164</td>
<td>-0.4011</td>
<td>0.0484</td>
<td>0.84293</td>
</tr>
</tbody>
</table>

Here it is seen that the angles \( \gamma \) and \( \delta \) are relatively insensitive to separation point locations. The reasons for the insensitivity are not known.

This flow model could probably be improved by taking into account the vorticity shed from that part of the cylinder immersed in the wake. This is deduced from Swanson's observation that for \( u_w = 0.2 \) and
Figure 20. $C_D$ vs. $u_w$ - Comparison with Experiment
Re_d = 4 \times 10^4$, the pressure at the upper separation point is lower than it is for the separation point where the wall is moving upstream. According to equation (16) and Bernoulli's equation, this implies that the transport of vorticity into the wake at the upper separation point is greater than for the vorticity of opposite sign transported into the wake at the lower separation point. Thus, vorticity of the same sign as for the lower cylinder surface must be cast off from the cylinder surface immersed in the wake. This feature might be taken into account by setting the velocity magnitudes of the two free streamlines equal to each other where they intersect a vertical line perhaps three or four diameters downstream of the cylinder. For an ellipse at angle of attack, Piercy, Preston and Whitehead took the cast-off vorticity in the wake region into account by imposing the equal velocity condition along two vertical lines cutting the wake a distance c and 2c from the ellipse. Here c is the length of the long axis of the cylinder. Differences in lift between calculation and observation were at most 15%, with angle of "stall" for the ellipse being predicted with good accuracy.

IV. SUMMARY AND CONCLUSIONS

An inviscid flow-with-wake model for a spinning cylinder in cross-flow was developed that is consistent with boundary-layer theory. This was accomplished by first considering flow around a cylinder in an untransformed plane with two sources on the cylinder, their image sinks at the center and a point vortex in the center of the circle. This resultant flow was transformed to the physical plane by a Zhukovskii transformation; this results in the separation streamlines at the separation points being tangent to the cylinder at the critical points of the transformation.

The condition that the pressure gradients are finite at the separation points was imposed on this elementary inviscid flow-model. The imposed pressure-gradient condition produces a more realistic inviscid flow near separation. This condition also makes possible the development of a self-consistent model; that is, the inviscid flow model takes the boundary-layer into account and is consistent with boundary-layer theory. With finite pressure gradients at separation, the number of parameters needed to describe the flow was reduced from five to three.

Another condition imposed upon the inviscid flow model, equal pressures at the separation points, results from vorticity transport considerations. The vorticity transport into the wake at the separation points was assumed to be equal with no vorticity cast off from the rear of the cylinder. After applying this condition, only two parameters of the flow in the physical plane were needed for the description of this flow. The two parameters used were the location of the separation points on the cylinder. Using these parameters, a
system of five equations, which describes these imposed conditions, was solved for the five basic parameters in the untransformed plane. These equations were solved numerically by computer.

An iteration procedure was used to obtain separation points located in agreement with boundary-layer theory. The boundary-layer was calculated using the integral technique and new separation points were obtained using a particular inviscid-flow model with assumed separation points. These new separation points found by the integral method then become the separation points for a new assumed inviscid-flow field and again the boundary layer is calculated to find a new separation point. The iteration procedure was stopped when the difference between two successive separation points was less than 0.46 degrees for both sides.

Comparison of the converged velocity distribution for the non-rotating case with that of the observed velocity distributions shows fair agreement. Comparison between the calculated velocity distributions for different values of \( u_w \) showed that separation is delayed and the maximum value of \( u_e \) increases with \( u_w \) for the upper or downstream-moving wall side of the cylinder with the exception of the velocity distributions for \( u_w = 0.2 \) and \( u_w = 0.25 \). For the lower side of the cylinder, or in other words, the side with the upstream moving wall, separation advances toward the front of the cylinder with increasing \( u_w \) without exception. The stagnation point moves downward on the cylinder with increasing \( u_w \) as is observed in experiment.

The calculated lift coefficient was compared with the observed lift coefficient. When the lift coefficient was calculated using the integrated value of the pressure over the cylinder, a linear (also obtained by experiment) relationship was obtained up to and including \( u_w = 0.15 \), and the lift coefficient increased almost linearly after \( u_w = 0.15 \). Using the circulation or bound-vortex approach, the calculated lift coefficients were somewhat lower than the preceding approach but still in good agreement with experiment up to and including \( u_w = 0.15 \). The drag coefficients for different \( u_w \) were also calculated and are at most 20% lower than the observed value.

Although the approach does have some success in predicting the Magnus force on a rotating cylinder, refinements and modifications of the present method could possibly improve the computed results significantly. Some refinements and modifications that could be applied to the present model are the following:

1. Tighter convergence criterion and use of an accelerated convergence technique,
(2) modification of model to take into account the vorticity shed into the wake from that part of the cylinder which is immersed in the wake, and

(3) use of a more complicated distribution of sources.

ACKNOWLEDGMENT

We wish to thank Dr. Raymond Sedney for making us aware of the work of Bluston and Paulson.
REFERENCES


REFERENCES (Continued)


APPENDIX A

LISTING OF COMPUTER PROGRAM TO CALCULATE FORCE COEFFICIENTS

The listing of the entire program is given here except for the data used to form the integral quantity arrays. The computer program is written in Fortran IV. The available shape factors are given in tabulated form in Reference 24. The list of principal variables for the program is given below for the main program.

Fortran Implementation

List of Principal Variables

<table>
<thead>
<tr>
<th>PROGRAM SYMBOL</th>
<th>DEFINITION</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>similarity solution type, A</td>
</tr>
<tr>
<td>AB, ABP, ABM</td>
<td>two-dimensional arrays of the pressure-gradient parameter B,</td>
</tr>
<tr>
<td>ADELT</td>
<td>angular location of lower source in the ( \zeta ) plane</td>
</tr>
<tr>
<td>AGAM</td>
<td>angular location of upper source in ( \zeta ) plane</td>
</tr>
<tr>
<td>AH, AHP, ABM</td>
<td>two-dimensional arrays of the shape factors H</td>
</tr>
<tr>
<td>AK, AKP, AKM</td>
<td>ordered one-dimensional arrays of the shape factors K</td>
</tr>
<tr>
<td>AL, ALP, ALM</td>
<td>two-dimensional arrays of the shape factors L</td>
</tr>
<tr>
<td>AKL</td>
<td>one-dimensional array of highest values of K corresponding to values ( \frac{u_w}{u_e} ) represented by the one-dimensional array AUS</td>
</tr>
<tr>
<td>AKS</td>
<td>one-dimensional array of separation values of K corresponding to the values ( \frac{u_w}{u_e} ) represented by the one-dimensional array AUS.</td>
</tr>
<tr>
<td>AT, ATP, ATM, AUS</td>
<td>two-dimensional array of ( \frac{u_w}{u_e} ) values used for interpolation purposes.</td>
</tr>
</tbody>
</table>
PROGRAM SYMBOL | DEFINITION
---|---
AU | one-dimensional array of values of $u_w/u_e$ corresponding to the columns of the two-dimensional arrays $AH$, $AT$, $AL$.
AUM | one-dimensional array of values of $u_w/u_e$ corresponding to the columns of the two-dimensional arrays $AHM$, $ATM$, $ALM$.
AUpp | one-dimensional array of values of $u_w/u_e$ corresponding to the columns of the two-dimensional arrays $ATP$, $AHP$, and $ALP$.
B | velocity parameter defined by $B = u_e/(u_w - u_e)$
BB | two-dimensional array of values of $B$
BETA | pressure gradient parameter, $B$
BH | two-dimensional array of values of the shape factors $H$.
BK | two-dimensional arrays of values of the shape factors $K$.
BT | two-dimensional array of values of the shape factors $T$.
BET1, BET2 | initial guessed value of the separation point on upper side and lower side respectively. Also used to store the last found separation values.
BETN1, BETN2 | most recent values of the separation points on the upper and lower sides respectively.
CIRC | nondimensionalized vortex strength.
CKF | initial value of $K$ to start the integration of the boundary-layer equations.
COEFD | drag coefficient.
COEFL | lift coefficient calculated from circulation considerations.
PROGRAM SYMBOL | DEFINITION
---|---
COEFLF | lift coefficient calculated from pressure considerations.
CPC6EF | local pressure coefficient.
CUI | starting value of velocity ratio, this determines the starting value of x along the cylinder.
DELTA | the non-dimensionalized displacement thickness.
DERIVX | subroutine which computes the derivatives and stores them in DX.
DMAX | absolute value of the maximum step-size to be used.
DNXT | step size to be used for next step.
DPST | actual step size used with the step just completed.
DUX | derivative of boundary-layer edge velocity with respect to position.
DX | one dimensional array for storing the derivative obtained by the subroutine for evaluating the derivative.
DERIVX | a subroutine which computes the derivatives and stores them in DX.
EPCON | value used as a test to see if convergence has been achieved.
ER | amount of maximum error for each step. If the error is larger than ER, the step size is reduced.
F11 | variable characterizing the skin friction.
KUTMER | variable step Runge-Kutta integration subroutine.
N | number of equations to be evaluated in DERIV subroutine.
PROGRAM SYMBOL | DEFINITION
--- | ---
NBP | number of elements in the arrays AUS, AKS, and AKL.
NEQ | number of equations to be solved simultaneously for the basic parameters in the plane. An input to the subroutine VELPAR.
NFIRST | values which denotes whether the upper or lower half of the cylinder is being considered.
NIT | number of iterations required to obtain UI and DUX.
NPAR | a number value equal to NEQ plus one.
PRINX | print subroutine.
PS | print step value. PS determines the frequency of reference to subroutine PRINX.
Q1 | value that is one-half of the upper dimensionless source strength.
Q2 | value that is one-half of the lower dimensionless source strength.
THATA | the non-dimensionalized momentum loss thickness.
THETA1 | initial value of the non-dimensional momentum thickness.
TERMX | termination subroutine.
UI | local external velocity on cylinder.
UW | tangential velocity of rotating cylinder non-dimensionalized by the free-stream velocity.
VELOC | subroutine to calculate the edge velocity and its derivative given a position on the cylinder.
<table>
<thead>
<tr>
<th>PROGRAM SYMBOL</th>
<th>DEFINITION</th>
</tr>
</thead>
<tbody>
<tr>
<td>VELPAR</td>
<td>subroutine used to find the basic parameters of the ( \zeta ) plane. It calls other subroutines in order to effect this mission.</td>
</tr>
<tr>
<td>W</td>
<td>one-dimensional array of length 2N. Not used here but is dimensioned in main program.</td>
</tr>
<tr>
<td>X</td>
<td>one-dimensional array of length N, which contains the variables found by numerical integration.</td>
</tr>
<tr>
<td>XINTC</td>
<td>integrated value of ( G'(z) )^3. Needed to compute the energy loss thickness and used to obtain the shape factor values.</td>
</tr>
<tr>
<td>XINTSQ</td>
<td>integrated value of ( G''(z) )^2. Used in obtaining some shape factor values.</td>
</tr>
<tr>
<td>XNEW</td>
<td>an array containing the most recently found values of the basic parameters of the ( \zeta ) plane.</td>
</tr>
<tr>
<td>XOLD</td>
<td>initial guessed values of the basic parameters of the ( \zeta ) plane.</td>
</tr>
<tr>
<td>XPV</td>
<td>print variable</td>
</tr>
<tr>
<td>XTC</td>
<td>one-dimensional array of length NTS. KUTMER will return to main program if any element of XTC changes sign.</td>
</tr>
<tr>
<td>XZ</td>
<td>position on circle in ( \zeta ) plane.</td>
</tr>
<tr>
<td>X1</td>
<td>position on cylinder in ( z ) plane.</td>
</tr>
<tr>
<td>X1D</td>
<td>the value of X1 in degrees.</td>
</tr>
<tr>
<td>X11NC</td>
<td>integration step size used to find force coefficients before KUTMER subprogram is used.</td>
</tr>
</tbody>
</table>
PROGRAM TO CALCULATE THE 2-D BOUNDARY LAYER ON A ROTATING CYLINDER WITH A SOURCE-WAKE AND A POINT VORTEX SO THAT THE SEPARATION POINTS OF THE FLOW MODEL AGREES APPROXIMATELY WITH THE RESULTS OF THE BOUNDARY LAYER CALCULATIONS. THE INTEGRAL PROPERTIES AND INTEGRAL ENERGY EQUATIONS ARE USED TOGETHER WITH THE MOVING WALL SIMILARITY FAMILY OF VELOCITY PROFILES TO CALCULATE THE BOUNDARY LAYER.

DIMENSION X(5), DX(5), AKS(20), AK(20), AU(20), XTC(4)
DIMENSION AU(17), BK(20, 25), BM(20, 25), BT(20, 25), BL(20, 25)
CALK(160), AHP(17, 160), AM(17, 160), ALP(17, 160), AUP(17), ABP(17, 160),
CALK(20, 25), AHP(17, 135), AM(17, 135), ALP(17, 135), AUP(17), ABP(17, 135),
CANN(75), AHN(17, 75), AMH(17, 75), ALH(17, 75), AUP(17), ABP(17, 75),

COMMON UW, CK, CU, CT, CH, CL, N1, N2, NBP, DCRF, CB, TAU, THETA, NIP, CN2, NIM, N2H, NA, ALPH, ALPINC, EPS, Q1, Q2, AGAM, ADEL, CTIC, CBET, AU, AK, A-
C, AT, AL, AKS, AK, AUP, AKP, AHP, ATP, ALP, AE, AB, AM, AU, X, Z, DUXC, UIC, CM, NFRS, NINTY, NITCO, PI

CONSTANT INPUT DATA

CATALAS(I), I=1,267,-3,-25,-2195,-17647,-111,-526,0.0
C, 0.476.1,273,373,5,6,667,714,75,78,857,889,9
CatalaS(I), I=1,267,1.753,1.71,1.67,1.631,1.581,1.543,
Cl, 1.493,1.479,1.459.1.438.1.418,1.403,1.381,1.358,1.332,
Cl, 1.468,1.438,1.407,1.377,1.347,1.318,1.289,1.262,1.239,
CatalaS(I), I=1,267,9900,12,100,7,100,7,100,7,1000,7,1000,
Cl0, 10000,13900,
CatalaS(33), Cl/CU2, -0.30, THETA1 /3.93, THETA1 /2.87, CK1 /3.48, CK2 /1.75
CB/CBP/101

101 FORMAT(/, 23H INTEGRATION STARTED AT ,F9.4,4H RAD, F10.3,8H DEGREES)
PI = DACOS(-1.0)
DCERF = 1000/PI
ASSIGN 10 TO NN

5 K = 0
J = 0
I = 0

C RECA DATA OBTAINED FROM THE SIMILARITY SOLUTIONS.

6 RECA(5, 70C) B, A, B, THETA, DELTA, NINTSQ, NINTC
I = I + 1
7 IFK.EQ.0 GO TO 9
IF (B.EQ.BX) CC TO 10

C THE M ARRAYS COUNT THE NUMBER OF DATA ENTERED FOR A CERTAIN VELOCITY

M(J) = 1 - 1

C IF A EQUALS 0, THE SET OF DATA OBTAINED GOES TO THE ARRAY SUBROUTINE TO BE FORMED INTO ARRAYS

IF (A.EQ.0.0) GO TO 20
J = J + 1
8 EX = 1
9 AUPP(J) = 1. + 1 / B
10 K = K + 1

C INTEGRAL VALUES ARE CALCULATED FROM DATA

THESTR = XINTC + 3 * B * THETA - B * B * DELTA

EB(J, I) = B * CELTA/THETA
ET(J, I) = -THETA * EXP / B
AH(J, I) = THETA * XINTSQ / B / B / B
BE(J, I) = THESTR / B / THETA
EB(J, I) = BETA

19 GO TO 6
20 L = 20
20 L = 20
1 N = 25
1 LJ = J
WRITE(6, 705) M, K
GO TO NN(601, 602, 603)
601 KPP = 150
602 K = 620 I = LJ
620 AUPP(I) = AUPP(I)
N1P = LJ
N2P = KPP
KC = - 10
EXTERNAL VELOCITY
CALL ARRAY(BB, BK, BH, BT, BL, L, N, M, AKP, AHP, ATP, ALP, K, KPP, LJ, ABP, AC)
GO TO 500
WRITE(6,707) AUP
WRITE(6,705) (AKP(I), AHP(J), J=LJ), I=1, KPP)
WRITE(6,704) (AKP(I), ATP(J), J=LJ), I=1, KPP)
WRITE(6,707) AUP
WRITE(6,704) (AKP(I), ALP(J), J=LJ), I=1, KPP)
WRITE(6,707) AUP
WRITE(6,704) (AKP(I), ABP(J), J=LJ), I=1, KPP)
500 CONTINUE
ASSIGN 502 TO NN
GO TO 5
602 KP = 135
605 AU(I) = 1, LJ
N1 = LJ
N2 = KP
KC = -10

INTEGRAL VALUES ARRAY FORMED WHERE EXT. VEL. IS GREATER THAN WALL VELOCITY
CALL ARRAY(BB, BK, BH, BT, BL, L, N, M, AK, AH, AT, AL, K, KPP, LJ, AB, AC)
GO TO 501
WRITE(6,707) AUP
WRITE(6,704) (AK(I), AH(J), J=LJ), I=1, KPP)
WRITE(6,704) (AK(I), AT(J), J=LJ), I=1, KPP)
WRITE(6,707) AUP
WRITE(6,704) (AK(I), AL(J), J=LJ), I=1, KPP)
WRITE(6,707) AUP
WRITE(6,704) (AK(I), AB(J), J=LJ), I=1, KPP)
501 CONTINUE
ASSIGN 503 TO NN
GO TO 5
603 KPM = 75
610 AU(I) = AUPP(I)
N1P = LJ
N2M = KPM
NC = -10

INTEGRAL VALUES ARRAY FOR REVERSE FLOW
CALL ARRAY(BB, BK, BH, BT, BL, L, N, M, AKM, AHM, ATM, ALM, K, KPP, LJ, ABP, AC)
GO TO 502
WRITE(6,709) AUM
WRITE(6,708) (AKM(I), AHM(J), J=LJ), I=1, KPP)
WRITE(6,708) (ATM(I), ALM(J), J=LJ), I=1, KPP)
WRITE(6,709) AUM
WRITE(6,708) (AKM(I), ALM(J), J=LJ), I=1, KPP)
WRITE(6,709) AUM
WRITE(6,708) (AKM(I), ABM(J), J=LJ), I=1, KPP)
502 CONTINUE

CONVERGENCE IS ACHIEVED IF CONDITION IS MET.
30 REAC(15,700) UW, BET1, BET2, XOLD
WRITE(6,52) UW, BET1, BET2, XOLD
IF(UW .GT. 10.0) GO TO 300
LCRIT = 0
NEP = 5
NPAR = 6
EPCON = 8.0E-3
CO = 1.5
32 XNEM(I) = XOLD(I)
GO TO 34
35 LCRIT = 0
IF(ABS(BET1-BETN1) .LT. EPCON .AND. ABS(BET2-BETN2) .LT. EPCON) LCRIT = 1
THE NEW SEPARATION POINTS OF THE FLOW MODEL ARE GIVEN BY THE VALUES OBTAINED FROM THE BOUNDARY-LAYER CALCULATIONS.

BET1=BETN1

INCREASE ANGLE OF THE LOWER SEPARATION POINT IF CONDITION IS MET.

ABS(BET2-BETN2)<1.1E-3) GO TO 33
BET2=BETN2
GO TO 34

33 BET2 = BETN2 - EPCON

CONDITIONS FOR INTEGRATION OF BOUNDARY LAYER FOR THE UPPER PART OF THE CYLINDER.

34 NFIRST = 1
CNXT = 1.0E-6
CPST = 5.0E-6

USE PREVIOUS VALUES OF FLOW MODEL PARAMETERS AS GUESSES FOR THE FLOW MODEL PARAMETERS TO BE FOUND.

36 XOLD(I) = XNEW(I)

CALL SUBROUTINE TO FIND THE FLOW MODEL PARAMETER VALUES GIVEN THE SEPARATION POINTS IN THE PHYSICAL PLANE AS INITIALLY GUESSED OR CALCULATED FROM THE INTEGRAL BOUNDARY-LAYER EQUATIONS.

CALL VELPAR(BET1,BET2,XOLD,XNEW,NEQ,NPAR)
C1 = XNEW(1)
C2 = XNEW(2)
AGAM = XNEW(3)
ADELT = XNEW(4)
CIRC = XNEW(5)
X1C = 0.0
X1 = X1D/DEGRF
X1 = PI - 0.001*EPS
IF(LCRIT.EQ.1) GO TO 95

CALL SUBROUTINE TO CALCULATE THE EDGE VELOCITY AND ITS DERIVATIVE WITH RESPECT TO ANGLE GIVEN A POSITION ON THE CYLINDER.

CALL VELOC(U1,DUX,X1,X2,NIT)

CLNT IS THE INTEGRAL (STARTING FROM X=0) OF THE PART OF THE LOCAL PRESSURE COEFFICIENT CORRESPONDING TO THE LOCAL HORIZONTAL FORCE.

CLNT = CSIN(X1)-UI*UI*X1/3.0+UI*UI*X1*X1*X1/10.0

CLNT IS THE INTEGRAL (STARTING FROM X=G) OF THE PART OF THE LOCAL PRESSURE COEFFICIENT CORRESPONDING TO THE LOCAL VERTICAL FORCE.

CLNT = CCOS(X1)+UI*UI*X1*X1*X1/4.0-1.0

40 X1IND = 0.1
X1C = X1C + X1IND
CPCOEF = 1.0-UI*UI
X1INC = X1INC/DEGRF
CLNT = CLNT+X1INC*(-CPCOEF)*SIN(X1)
CDINT = CCINT+X1INC*CPCOEF*CCS(X1)
X1 = X1C/DEGRF
CALL VELOC(U1,DUX,X1,X2,NIT)
X2LAST = X2
CU = U1/X1
CB = UI/(UI-UI)

IF THE CYLINDER IS NOT ROTATING, A DIFFERENT SET OF INITIAL CONDITIONS ARE REQUIRED FOR THE INTEGRAL B-L EQUATIONS.

IF(U1*U1*U1<1.1E-8) GOTO 50

CONDITIONS USED IN ORDER TO STEP AWAY FROM THE STAGNATION POINT TO A POSITION WHERE THE B-L EQUATIONS CAN BE INTEGRATED.

IF(CU*GT.CU1) GOTO 40
CK = CK1
THETA=THETA1
NA = 2
GO TO 55

50 CK = 1.62574
THETA = 0.29234

68
C INPUT TO KUTMER

55 CONTINUE

CONDITIONS USED DURING INTEGRATION OF THE BOUNDARY LAYER FOR BOTH
THE UPPER AND LOWER PART OF THE CYLINDER.

60 X(1) = X
X(2) = THETA * T - ETA
X(3) = CK * CK * X(2)
X(4) = CLINT
X(5) = CCINT

70 CMAX = 2.0E-3
ER = 5.0E-6
NINT = 1.0
PS = 1.0

CALL SUBROUTINE TO INTEGRATE BOUNDARY-LAYER EQUATIONS

CALL KUTMER(CNXT, CPST, DMAX, 5, X, DX, DERIVX, ER, 1, W, 0, PS, XPV, PRINX,
CXTY, 4, NT, TERM)

GO TO 75, 76, 77, 78, 79

75 WRITE(6, 110) NT, COM
IF(NFRST.EQ.1) GO TO 73
BETN2 = X(1) - 0.05
GO TO 35

73 BETN1 = X(1) + 0.01
GO TO 85

77 WRITE(6, 111)
GO TO 35

78 WRITE(6, 112)
GO TO 30

76 WRITE(6, 123) X(1), X(4), X(5)
IF(NFRST.EQ.1) GO TO 85
IF(NFRST.EQ.2) BETN2 = X(1)
GO TO 35

COMPUTE INPUT FOR LOWER PART OF CYLINDER. SEE COMMENTS FOR INPUT
TO UPPER PART OF THE CYLINDER.

85 NFRST = 2

BETN1 = X(1)
CLUDPR = X(4)
CLUDPR = X(5)

XIC = X10/0EGRF
X2 = -PI/0.01 * EPS
CALL VELOCIUfCUX, UX, UXZ, NIT1
DEL = -DCOS(X1) - UI*UI*X1/3.0 - UI*UI*X1*X1*X1/10.0
CDINT = CSIN(X1) - UI*UI*X17.0 + UI*UI*X1*X1*X1/10.0

88 X1C = X1C - X1INC
CPCEOF = 1.0 - UI*UI
X1INC = -X1INC/CEGRF

CALL VELOCIUfCUX, UX, UXZ, NIT1
DEL = -DCOS(X1) - UI*UI*X1/3.0 - UI*UI*X1*X1*X1/10.0

THE CONVERGENCE CRITERION FOR THE SEPARATION POINTS IS SATISFIED.
THE COEFFICIENTS OF LIFT AND DRAG ARE CALCULATED AND WRITTEN OUT.

55 COEFL = 2.0 * PI * CIRC * COS(ALPH)
WRITE(6, 710) EPCGN
COEFLF = (CLUDPR - X(4) - (1.0 - UICOM * UICOM) * (COS(BETN1) - COS(BETN2))

COEFD = (CLUDPR - X(5) - (1.0 - UICOM * UICOM) * (SIN(BETN2) - SIN(BETN1))) / 2.0
WRITE(6, 720) COEFL
WRITE(6, 721) COEFL
WRITE(6, 725) COEFD

69
GO TO 30

52 FORMAT(I1, 3OH SPIN RATE AND INITIAL GUESSES /I1X, 2HUN, 6X, 
11X, 2HUN, 6X, 
C1,6SPE, ANGLE1, 5X, 10HSEP, ANGLE2, 8X, 7HSOURCE1, 8X, 7HSOURCE2, 
C5X, 10HSORC ANGL1, 5X, 10HSORC ANGL2, 3X, 12HCIRC, VELMC, /BF15.4) 
110 FORMAT (59H ERROR RETURN...ITERTIONS IN VELOC SUBROUTINE HAS EXCE 
CEEC, 110) 
111 FORMAT (3!, K EXCEEDED UPPER LIMIT) 
112 FORMAT (3!, K EXCEEDED LOWER LIMIT) 
123 FORMAT (3!, SEPARATED AT , FI2.4, 6H RADIANS , /31H LIFT COEFFICIENT C 
CONTRIBUTION, , F12.5, /31H DRAG COEFFICIENT CONTRIBUTION, , 
50 STOP 

C ********************************************************************* 
C SUBROUTINE TO CALCULATE THE DERIVATIVES FOR DIFF. EQUATIONS. 
C SUBROUTINE DERIVX(X, DX) 
CIPNSION X5, DX5) 
CIMENSION AN(17), AKS(20), AK(12G), AUS(20), AK(135), AN(17, 135), 
CAKP(17, 160), AP(17, 160), ALP(17, 160), AUP(17, 160), AT(17, 135), AL(17, 135), AB(17, 135), ALM(14, 75), 
CAKM(75), A(14, 75), ATM(14, 75), ALM(14, 14) 
CIPMONE MAIN) 
XZ = X ZLAST 
CALL VELOC(U1, DUX, X1, XZ, RIT) 
CU = UW/UI 
CB = UI/(UW-UI) 
CQ = SQRT(CU) 
C = SQRT(CU) 
40 GO TO 4 

C SELECT SET OF ARRAYS FOR PARTICULAR VELOCITY RATIO AND OBTAIN 
C SPOE, CRIT FOR VALUE OF K KNOWN. 
C 
GO TO 1 

1 CH = TINTICU*CK ; AN ; AKM ; N2M ; AHP ; NIP ; 2 ; 2) 
CL = TINTICU*CK ; AN ; AK ; N2M ; AHP ; NIP ; 2 ; 2) 
CO TO 4 

2 CH = TINTICU*CK ; AN ; AKM ; N2M ; AHP ; NIP ; 2 ; 2) 
CL = TINTICU*CK ; AN ; AK ; N2M ; AHP ; NIP ; 2 ; 2) 
CO TO 4 

3 CH = TINTICU*CK ; AN ; AKM ; N2M ; AHP ; NIP ; 2 ; 2) 
CL = TINTICU*CK ; AN ; AK ; N2M ; AHP ; NIP ; 2 ; 2) 
CO TO 4 

4 EX(1) = 1.0 

C INTEGRAL MOMENTUM EQUATION 
C 
CX(2) = (4.*CT-DUX*X(2)*<CH*2.C)*2.0)/UI 

C INTEGRAL ENERGY EQUATION 
C 
CX(3) = 18.*CK*(CL+CUI*CT)-6.*X(3)*DUX)/UI 
CPCOEF = 1.0- UC/UI 
CX(4) = -CPCOEF*Sin(X1) 
CX(5) = CPCOEF*Cos(X1) 
X2A = ABS(X(2)) 
TTHETA = SQRT(X2A)*ABS(CL)/CL 
TAX = UI/CT/THETA 
UXCCM = UI
GO TO (5, 6, 7), NINTY
5 XZLAST = XZ
RETURN
6 XZLAST = XZ - C * 0.02
RETURN
7 XZLAST = XZ + 0.01
RETURN
END

C*********************************************************
SUBROUTINE TERM(X2DX,XTC,XVP)
C*********************************************************
C SUBROUTINE FOR TERMINATING THE INTEGRATION
C*********************************************************
DIMENSION XU(17), AKS(20), AK1(20), AUS(20), AK(135), AH(17), 135),
CAK(160), AHP(17), 160), AUP(17), 160), ATP(17), 160), ALP(17), 160),
CAT(17), 135), AL(17), 135), AB(17), 135), ABM(14), 75),
CAK(75), AHP(14), 75), AUP(14), 75), ALP(14), 75), ACM(14), 75),
COMMON(USE MAIN)
C*********************************************************
VALUE OF NA IS SET TO SELECT RIGHT ARRAYS OF INTEGRAL VALUES.
NA = 1
IF(CB.GT.0.0) NA = 2
IF(CB.LT.-99999) NA = 3
C OBTAIN SEPARATION VALUE(CKS) OF K FOR VELOCITY RATIO KNOWN.
CALL DVCINT(CU,CKS,AUS,AKS,ABP,3)
C IF ANY ELEMENT OF XTC CHANGES SIGN, INTEGRATION IS STOPPED.
XT(1) = 29.6 - FLOAT(NITCOM)
1 XTC(1) = C - CXT(1)
XT(1) = BET2 = -X(1) + C, .001
XT(1) = 1.0
XPV = X(1) * DEGRF
RETURN
ENC
C*********************************************************
SUBROUTINE PRINX(X,DX)
C*********************************************************
DIMENSION XI(5), CX(5)
DIMENSION AU(17), AKS(20), AK1(20), AUS(20), AK(135), AH(17), 135),
CAK(160), AHP(17), 160), AUP(17), 160), ATP(17), 160), ALP(17), 160),
CAT(17), 135), AL(17), 135), AB(17), 135), ABM(14), 75),
CAK(75), AHP(14), 75), AUP(14), 75), ALP(14), 75), ACM(14), 75),
COMMON(USE MAIN)
PBX = CLAST
XBX = XLAST
XCL = X(1)*DEGRF
CPCOEF = CX(4)/SIN(X(1))
EDELTA = CI*TETA
WRITE(6,1) XID, TETA, CI, CU, DELTA, TAU, CI, UCOMP, DX, X(4), X(5),
CCPCOEF
1 FORMAT(12F10.5)
RETURN
ENC
C*********************************************************
SUBROUTINE VELOCITUX(U1,DUX,X,XZ,NIT)
C*********************************************************
DIMENSION AU(17), AKS(20), AK1(20), AUS(20), AK(135), AH(17), 135),
CAK(160), AHP(17), 160), AUP(17), 160), ATP(17), 160), ALP(17), 160),
CAT(17), 135), AL(17), 135), AB(17), 135), ABM(14), 75),
CAK(75), AHP(14), 75), AUP(14), 75), ALP(14), 75), ACM(14), 75),
COMMON(USE MAIN)
NIT=0
PREC = 1.0E-6
CALP = DCOS(ALPH)
SIN = 0.5*(CALP+1.0)/CALP)
RETURN
ENC

71
ITERATIVE METHOD OF NEWTON TO DETERMINE CORRESPONDING ANGLE IN UNTRANSFORMED PLANE.

XZG = XZ
CSX = DCOS(XZG + EPS)

THE CORRECT RELATION IS GIVEN BETWEEN CORRESPONDING POINTS IN THE TWO PLANES WHEN FX IS ZERO.

FX = A1*COS(CALP*2.0*COSX*COSX-1.0)+1.0*COSX*COSX**3
C(A1-COSX)**2

NIT = NIT + 1
IF(NIT.EQ.30) GO TO 2
IF(ABS(XZ-XZGL).GT.PRECIGOTC 1

SINXZ = ESIN(XZ+EPS)
COSX = DCOS(XZ+EPS)
GF = SF*ESINXZ-Q1*COTAN(0.5*(XZ-AGAM))-Q2*COTAN(0.5*(XZ-ADEL))
C*IRC
CS = 1.0-2.0*CALP*COSX*CALP*CALP
C*GFS
FP = 2.0*C(CALP-COSX)

THE EDGE VELOCITY FOR THE GIVEN POINT ON THE CYLINDER:

UI = G/H
GP = (2.0*COSX+5.0Q1/DSINXZ+5.0*(XZ-AGAM))**2+5.0Q2/DSINXZ
C(AZ-ACELT)**2
GS = (2.0*GFS)

GS = 1.0-2.0*CALP*COSX*CALP*CALP
C

CSX = DCOSX+COSX*CALP

THE SPATIAL DERIVATIVE CF THE VELOCITY AT THE EDGE OF THE B-L.

UX = (UI*P-3.0P+1.0)*DCOS(X-EPSE)/H/H/YDEN
NITCOM = NIT
RETURN
END

************ SUBROUTINE VELPAR(BET1,BET2,XCLD,XNEW,NEQ,NPAR)************

SUBROUTINE TO CALCULATE THE PARAMETERS FOR THE FLOW MODEL GIVEN THE SEPARATION POINTS IN THE PHYSICAL PLANE. THIS SUBROUTINE USES OTHER SUBROUTINES TO OBTAIN THESE PARAMETERS BY A NEWTON-RAPHSON SCHEME FOR SOLVING A SYSTEM OF NONLINEAR EQUATIONS.

COMMON(XNEWB(5),XNEW(NEQ),XNEQ(I5)),X(5)),XOLD(NEQ),XNEQ(NEQ),X(5)

DIMENSION AU(I7),AKS(20),AKS(20),AKS(135),AM(17,135),
CAM(17,135),AKS(17,135),AKS(17,135),AKS(17,135),AKS(17,135),AKS(17,135),AKS(17,135),AKS(17,135),AKS(17,135),AKS(17,135)

COMMON XNEWB(5)

EXTERNAL JACOP
PI = DACOS(-1.0)
EPS = (BET1 + BET2)*0.5
AMAX = NEQ
N = NEQ
M = NPAR
L = N
ITMAX = 1000
EPS1 = 1.0E-7
EPS3 = 1.0E-5
S1 = 1.0
S2 = 1.0
EPS2 = 1.0E-2
CO 10 I = 1,N
CO 5 J = 1,M
CO 011 J = 0.0
CO 011 J = 0.0
10 CONTINUE

ALPH = 0.5*(PI-BET1+EPS)

ALPH = 0.5*(PI-BET1+EPS)

ALPH = 0.5*(PI-BET1+EPS)

ALPH = 0.5*(PI-BET1+EPS)
YN = 0.1
20  YN=0.5*YN
YN = YN
25  AIAC = PR*YNN/100.0
ALPINC = ALPH - AIAC

CALL SUBROUTINE THAT SOLVES A SYSTEM OF NONLINEAR EQUATIONS.

CALL NLSEQ(JACOB,P,OD,A,NMAX,IR,JN,M,L,ITMAX,SI,DS,EF,PSEL,3,1000)
IF(II,IEC.M,E.I=0) WRITE(6,105) EPS2
IF(I=IRR1.LE.EC,0R.I=ERR2.LE.0) IJK = 1
WRITE(6,107) ((XNEW(I),I=1,N),YNN)
IF(YNN.LG.6.0E-2.AND.YNN.LG.1.0) GO TO 50
CO 30  I=1,5
30  XNEW(I)=XNEW(I)
IF(YNN.LT.0.C) GO TO 20
YN = -YNN

FUNCTION COTAN(X)
COTAN = COT(X)
RETURN
FUNCTION CGCOTAN(X)
COTAN = COT(X)
RETURN
FUNCTION ACOS(X)
ACOS = ACOS(X)
RETURN

END

END

CALL SUBROUTINE JACCBIQ(NMAX,XNEW,A,NMAX,AMX,A)

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A(1,5) = -1.0
SAME = DSIN(ALPH- EPS)
A(1,6) = (XNEW(1)*CADMIN3+XNEW(2)*CADMIN4-2.0*SAME+XNEW(5))
CAPEP3 = CCOTAN(0.5*(ALPH+ EPS+XNEW(3)))
A(2,1) = CAPEP3
CAPEP4 = CCOTAN(0.5*(ALPH+ EPS+XNEW(4)))
A(2,2) = CAPEP4
SAME = DSIN(0.5*(ALPH+ EPS+XNEW(3)))
A(2,3) = -0.5*XNEW(1)/SAME3/CAPEP3
SAME = DSIN(0.5*(ALPH+ EPS+XNEW(4)))
A(2,4) = -0.5*XNEW(2)/SAME3/CAPEP4
A(2,5) = 0
SAME = DSIN(ALPH+ EPS)
A(2,6) = (XNEW(1)*CAPEP3+XNEW(2)*CAPEP4-2.0*SAME+XNEW(5))
ACM3 = CCOTAN(0.5*(ALPH+ EPS+XNEW(3)))
ACPEP3 = CCOTAN(0.5*(ALPH+ EPS+XNEW(3)))
A(3,1) = -ACM3*CAPEP3
ACPEP4 = CCOTAN(0.5*(ALPH+ EPS+XNEW(4)))
SCPEP3 = SIN(0.5*(ALPH+ EPS+XNEW(3)))
A(3,2) = SCPEP3*ACPEP4
SCPEP4 = SIN(0.5*(ALPH+ EPS+XNEW(4)))
SCPM3 = SIN(0.5*(ALPH+ EPS+XNEW(3)))
A(3,3) = -0.5*XNEW(1)*(1.0/SCPM3**2+1.0/SCPEP3**2)
SCPEP4 = SIN(0.5*(ALPH+ EPS+XNEW(4)))
SCPM4 = SIN(0.5*(ALPH+ EPS+XNEW(4)))
A(3,4) = -0.5*XNEW(2)*(1.0/SCPM4**2+1.0/SCPEP4**2)
A(3,5) = 0
SCAPE = DSIN(ALPH+ EPS)
SCAME = DSIN(ALPH+ EPS)
A(3,6) = (2.0*(SCAME-SCAPE)-XNEW(1)*(ACM3-ACPEP3)-XNEW(2)*(ACM4-ACPEP4))
C = ACM4
ACM3 = CCOS(0.5*(ALPH+ EPS+XNEW(3)))
A(4,1) = ACM3
ACM4 = CCOS(0.5*(ALPH+ EPS+XNEW(4)))
A(4,2) = ACM4
A(4,3) = ACM3/SCPM3/SCPM4**3-D*XNEW(1)/SCPM3**2/2.0
SCPEP3 = CCOS(0.5*(ALPH+ EPS+XNEW(3)))
A(4,4) = -SCPEP3/ACPEP4
SCPEP4 = CCOS(0.5*(ALPH+ EPS+XNEW(4)))
A(4,5) = -SCPEP4/ACPEP4
DELM3 = 0
CACHEM3 = CCOS(ALPH+EPS)
ABC(1) = -0.0*(2.0*ACM3+0.5*(XNEW(1)*SCPM3**2+XNEW(2)/SCPM4**2)
A(5,1) = ACM3
A(5,2) = ACM3
A(5,3) = ACM3/SCPM3/SCPM4**3-D*XNEW(1)/SCPM3**2/2.0
SCPEP3 = CCOS(0.5*(ALPH+ EPS+XNEW(3)))
A(5,4) = -SCPEP3/ACPEP4
SCPEP4 = CCOS(0.5*(ALPH+ EPS+XNEW(4)))
A(5,5) = -SCPEP4/ACPEP4
RETURN
END
SUBROUTINE ARRAY
DIMENSION AKX(200L),AD(3CC),C(KJ<25L),CKJ2(25L),CKJ3(25L)
CJ(25L),CTJ(25L),CLJ(25L),CKJ3(25L),CUJ(25L),AJS(25L)
1 = 0
C PLACE ALL THE K VALUES INTO A 1-D ARRAY.
CO 10 J = J+1
C CALL SORTXY(CKJ,J,MJ)
C CONSTRUCT ARRAY OF DIFFERENCES BETWEEN SUCCEEDING K VALUES.
CADS(1) = 0.0
C
PLACE SMALLER ARRAYS INTO LARGER 1-D ARRAYS OF K VALUES.

SURF LARGE ARRAYS SO THAT K IS IN INCREASING ORDER

CALL SORTXY(AKX, AD, K)

REDUCE THE NUMBER OF NON-ZERO ELEMENTS IN THE ARRAYBY REQUIRING ONLY THAT AN ELEMENT K EXISTS IN THE CLOSED INTERVAL BETWEEN SUCCEEDING K'S WHERE THE VELOCITY RATIO IS HELD CONSTANT.

J = 2
AK(1) = AKX(1)
CO 20 I = 2; K
IF(AK(I) = LT.1.E-6) GO TO 15
AKX = AKX(I) - AD(I)
IF(AKX .EQ. AK(J-1)) GO TO 15
IF(AKX .GT. AK(J-1)) GO TO 13
GO TO 20
13 AK(J) = AKX(I-1)
GO TO 19
15 AK(J) = AKX(I)
19 J = J + 1
GO TO 20

ENLARGE NO. OF NON-ZERO ELEMENTS TO INCLUDE SOME LARGER VALUES OF K.

XINCJ = (AK(J-1) - AK(J-2))/100.
RI = 1.0
JP1 = J-1
CO 39 I = J, KP
AK(I) = AK(J-1) + RI * XINCJ
RI = RI * 1.0
CONTINUE

FORM 2-D ARRAYS.

CO 70 J = 1, L J
PMJ = MJ
37 CO 40 I = 1, MJ
CKJ(1) = CKJ(I, J)
CKJ(I) = CKJ(I, J)
CKJ(I) = CKJ(I, J)
CHJ(I) = CHJ(I, J)
CTJ(I) = CTJ(I, J)
CBJ(I) = BBJ(I, J)
CLJ(I) = CLJ(I, J)
CALL SORTXY(CKJ, CHJ, MJ)
CALL SORTXY(CKJ, CTJ, MJ)
CALL SORTXY(CKJ, BBJ, MJ)
ASSIGN 46 TO KGT
45 CO 60 I = 1, KP
GO TO KGT, (46, 60)
46 IP6 = I + 6
IF(IP6 .GT. KP) GO TO 51
IF(AK(IP6) .LT. CKJ(I)) GO TO 6C
IF(I .LT. J) GO TO 52
51 IF(AK(I) .GE. CKJ(MJ)) ASSIGN 6C TO KGT
52 AKI = AK(I)
53 CALL EISCOT(AK, AK, CKJ, CHJ, CTJ, BBJ, CBJ, ABJ, MJ, O, P, Q, AHI)
CALL EISCOT(AK, AK, CKJ, CHJ, CTJ, W, MJ, O, AHI)
CALL EISCOT(AK, AK, CKJ, CHJ, CTJ, W, MJ, C, AHI)
AHI(J, J) = AHI
AT(J, J) = ATI
AL(J, J) = ALI
AB(J, J) = ABI
CONTINUE
70 CONTINUE
SUBROUTINE DISCOT (XA, ZA, TABX, TABY, TABZ, NC, NY, NZ, ANS)

*** DOCUMENT DATE 08-01-68  SUBROUTINE REVISED 08-01-68  ***

THE DIMENSIONS IN THIS SUBROUTINE ARE ONLY DUMMY DIMENSIONS.

CALL DISSER (XA, TABX(1), 1, NY, IDX, NN)
CALL LAGRAN (XA, TABX(NN), TABY(NN), NNN, ANS)
GOTO 70

CALL DISSER (ZARG, TABZ(1), 1, NZ, IDZ, NPZ)
NX = NY/NZ
NPZ = NPZ*IDZ
I = 1
IF (IMS) 30, 30, 40
CALL DISSER (XA, TABX(1), 1, NX, IDX, NPX(1))
CO 35 JJ = NPZ, NPZL
NPY(I) = (IJ-1)*NX+NPX(1)
NPX(I) = NPX(1)
35 I = I+1
GOTO 50

CO 45 JJ = NPZ, NPZL
IS = JJ-1, NX+NPX(1)
CALL DISSER (XA, TABX(1), IS, NX, IDX, NPX(I))
NPY(I) = NPX(I)
45 I = I+1

50 CO 55 LL = 1, IP1Z
NLOC = NPX(LL)
NLOCY = NPX(LL)

55 CALL LAGRAN (XA, TABX(NLOC), TABY(NLOCY), IP1Z, YY(1), NPY(1), IP1Z, ANS)
CALL LAGRAN (ZARG, TABZ(NPZ), YY(1), IP1Z, ANS)
RETURN

END

SUBROUTINE DISCOT (XA, TAB, I, NX, ID, NPX)

DIMENSION TAB(I)
NPT = ID+1
NPB = NPT/2
NP = NPT-NPB
IF (NX-NPT) 10, 5, 10
5 NPX = I
RETURN

10 NLOW = I*NPB
NUP = I*NX-(NPB+1)
CO 15 II = NLOW, NUP
NLOC = II
IF (TAB(II) > XA) 15, 20, 20
C SUBROUTINE LAGRAN (XA,X,Y,N,ANS)
C DIMENSION X(2),Y(2)
C SUM=0.0
C CONTINUE
C PROC=Y(I)
C PROC=PROC*B
C CONTINUE
C SUBROUTINE NSIMEQ(JACOBI,P,OD,A,NMAX,IR,JCN,M,L,ITMAX,SI,DS,SD,
C EPS1,EPS2,EPS3,Q,XLD,XNEW,X,IERR1,IERR2)
C THIS SUBROUTINE COMPUTES A ROOT OF THE SET OF SIMULTANEOUS
C NONLINEAR ALGEBRAIC EQUATIONS USING THE NEWTON-RAPHSON METHOD OR
C THE PARAMETER PERTURBATION METHOD.
C DIMENSION P(NMAX,M),OD(NMAX,M),A(NMAX,M),IR(N) JCN,M L XLD XNEW X
PROGRM INITIALIZATION.

C

IF (L.EQ.0) S=51
C

C TRANSFER THE OLD ROOT TO THE NEW ARRAY. IF L = 0, TRANSFER THE
C PARAMETERS AND TRANSFER THEM TO THE Q ARRAY. IF L = 1, COMPUTE A NEW SET OF
C PARAMETERS AND TRANSFER THEM TO THE Q ARRAY.
C

C CO 2 J=1,M
C CO 4 J=1,M
C CO 6 J=1,M
C CO 8 J=1,M

12 CO 1=1.N
XNEW(I)=OLD(I)
CO 2 J=1,M
IF (L.EQ.1) GO TO 12
CO 4 J=1,M
CO 6 J=1,M
GO TO 2

C CALL SUBROUTINE JACOBI TO COMPUTE THE COEFFICIENTS OF THE JACOBIAN
C MATRIX AND STORE THEM IN THE A ARRAY.
C CALL JACOBI (Q,NMAX,XNEW,N,M,A)
C
C CALL SUBROUTINE LSIMEQ TO COMUTE THE SOLUTION OF A SET OF
C SIMULTANEOUS LINEAR ALGEBRAIC EQUATIONS USING THE GAUSS-JORDAN
C REDUCTION SCHEME WITH THE MAXIMUM PIVOT CRITERION. THE CONTENTS
C OF THE X ARRAY ARE THE CHANGES WHICH MUST BE MADE IN THE
C XNEW ARRAY.
C CALL LSIMEQ (A,NMAX,IR,JC,N,EPST,X,IERR1)
C
C TEST TO DETERMINE IF THE JACOBIAN MATRIX IS SINGULAR OR
C ILLCONDITIONED.
C

C IF IERR1.LT.0 GO TO 11
C
C TEST TO DETERMINE IF ALL THE CHANGES IN THE XNEW ARRAY ARE
C LESS THAN EPS3.
C

C CO 3 I=1,N
IF (.NOT.ABS(X(I)).LT.EPS3) GO TO 4

3 CONTINUE
GO TO 6

C CONVERGENCE HAS NOT BEEN ACHIEVED. UPDATE THE XNEW ARRAY AND
C REPEAT THE ITERATION.
C

C CO 5 I=1,N
XNEW(I)=XNEW(I)+X(I)

C MAXIMUM NUMBER OF ITERATIONS HAS BEEN EXCEEDED. IF THE PARAMETER
C PERTURBATION METHOD IS BEING USED, HALVE THE STEP SIZE AND
C DETERMINE IF IT IS LESS THAN EPS2.
C

C IF (L.EQ.0) GO TO 10
S=S+DS
IF (DS.LT.EPS2) GO TO 10
GO TO 8

C CONVERGENCE HAS BEEN ACHIEVED. IF THE PARAMETER PERTURBATION IS
C BEING USEC, TEST TO DETERMINE IF THE STEPPING PROCESS IS
C COMPLETE. IF IT IS OR IF THE NEWTON RAPHSON METHOD WAS USEC,
C RETURN WITH THE SOLUTION IN THE XNEW ARRAY.
C

C 6 IF (L.EQ.0) GO TO 9
IF (.NOT.S.LT.SF) GO TO 9

C TRANSFER THE NEW ROOT TO THE XOLD ARRAY, INCREASE THE STEP SIZE BY
C THE CURRENT VALUE AND RETURN TO COMPUTE NEW G PARAMETERS.
C

C CO 7 I=1,M
2 KOLC(I)=XNEW(I)

C CO 8 S=S+DS
GO TO 1

C RETURN...SUCCESSFUL SOLUTION.
C

C 9 IERR2=1
RETURN
ERROR RETURN... IF THE NEWTON-RAPHSON METHOD HAS BEEN USED, THE MAXIMUM NUMBER OF ITERATIONS HAS BEEN EXCEEDED. IF THE PARAMETER PERTURBATION METHOD HAS BEEN USED, THE STEP SIZE IF LESS THAN EPS2

10 IERR2=-1
11 RETURN

**SUBROUTINE LSIMEQ (A, NMX, IR, JC, N, EPS1, X, IERR1)**

THE SUBROUTINE COMPUTES THE SOLUTION OF A SET OF SIMULTANEOUS LINEAR ALGEBRAIC EQUATIONS USING THE GAUSS-JORDAN REDUCTION SCHEME WITH THE MAXIMUM PIVOT CRITERION.

**DIMENSION** (A(NMX,6), IR(N), JC(N), X(N))

**BEGIN THE ELIMINATION PROCEDURE.**

NPI=N+1
KM1=K-1

**BEGIN THE SEARCH FOR THE MAXIMUM PIVOT ELEMENT.**

BIGA=0.0
CO 7 I=1,N
CO 3 J=1,N
IF (KM1.EQ.0) GO TO 2

**CHECK ROW AND COLUMN PIVOT SUBSCRIPTS ALREADY USED.**

CO 1 JJ=1,KM1
IF (I.EQ.JC(JJ)) GO TO 3
CO 1 JJ=1,KM1
IF (J.EQ.JC(JJ)) GO TO 3
1 CONTINUE
2 BIGA=ABS(A(I,J)) GT BIGA) GC TC 3
IRK=I
JCK=J
BIGA=ABS(A(I,J))

**CHECK TO SEE IF THE MAXIMUM PIVOT ELEMENT IS GREATER THAN EPS1.**

IF (.NOT. BIGA.GT.EPS1) GO TO 4
1 IERR1=1
RETURN

**NORMALIZE THE PIVOT ELEMENT.**

4 IRK=IR(K)
JCK=JC(K)
BIGA=A(IRK,JCK)
CO 6 JJ=1,NPI
5 A(IRK,J)=A(IRK,J)/BIGA

**ELIMINATE THE NON ZERO ELEMENTS IN THE JC(K) TH COLUMN.**

CO 7 I=1,N
IF (I.EQ.IRK) GO TO 7
AJCK=A(I,J)
CO 6 JJ=1,NPI
6 A(I,J)=A(I,J)-AJCK*A(IRK,J)
7 CONTINUE

**REORDER THE SOLUTION AND TRANSFER IT TO THE X ARRAY.**

CO 8 I=1,N
IRI=IR(I)
JCI=JC(I)
8 XI(J)=XI(I)+IRI

**SUCCESSFUL RETURN.**

IERR1=1
RETURN

**SUBROUTINE OVCINT (X, FX, XT, FT, NP, ND)**

**DIMENSION** XT(NP), FT(NP, 16)
N=NC

79
31 N1=(N-1)/2  
32 N2=N/2  
33 N3=N P-1  
34 IF(NP-N)/3 .LT. 41 .AND. 41  
35 N4=N+2  
36 IF(XT(1)-XT(2))<2.88 .AND. 60  
37 CONTINUE  
38 IF(X-2.*XT(1)+XT(2))>2.0 .AND. 20  
39 IF(X-2.*XT(NP)+XT(NP-1))>441 .AND. 441 .AND. 20  
40 IF(NP-LT.10) GO TO 42  
41 N5=NP-N  
42 N6=N4-N5  
43 N7=N5/2  
44 IF(N5-GT.1) GO TO 43  
45 IF(XT(N6)<XT(N4)-XT(N5))  
46 N8=N5+1  
47 L=(N+1)/2  
48 TR=+1 (L)  
49 N6=N4-N5  
50 N7=N4-N1  
51 CONTINUE  
52 N5=NP-N  
53 N6=N6-1  
54 N7=N7+1  
55 L=L-1  
56 IF(XT(N1)-XT(N7)) <1.72 .AND. 12  
57 UN=UN*TR*UN(L)  
58 FX=FX  
59 RETURN  
60 WRITE(6,50) XT(N1),XT(NP)  
61 STOP  
62 FORMAT(*ARG NOT IN TABLE ,X,=E14.7,9H XT(1)= ,  
63 E14.7,1CH XT(NP)=,E14.7,2X,6HDVDINT)  
64 WRITE(6,11) NP,ND  
65 FORMAT(*TABLE TOO SMALL NP=,I5,6H ND=,I5,2X,6HDVDINT)  
66 STOP  
67 IF(X-2.*XT(1)+XT(2))>6 .AND. 20  
68 CONTINUE  
69 IF(X-2.*XT(NP)+XT(NP-1))>441 .AND. 441 .AND. 20  
70 WRITE(6,52) XT(1)  
71 STOP  
72 FORMAT(*CONSTANT TABLE XT(1)=,E14.7,2X,6HDVDINT)  
73 C  
74 SUBROUTINE SORXYX(X,Y,N)  
75 DIMENSION X(N),Y(N)  
76 P=M  
77 1 M=M/2  
78 IF(M.EQ.0) RETURN  
79 K=A-M+1  
80 J=1  
81 2 J=J  
82 80
FUNCTION TDINT(X,Y,XT,NX,YT,NY,FT,NXMAX,NPY,NPY)

DIMENSION XT(NX),YT(NY),FT(NXMAX,NY),F(16),G(16)

IF(NX.GT.NXMAX)GOTO 14

A=2.*XT(NX)-XT(NX-1)
B=2.*YT(NY)-YT(NY-1)
C=2.*YT(NY)-YT(NY-1)

6       I=1,NX1
7       IF(X1.GE.XT(I))GOTO 2
8       CONTINUE
9       GOTO 12

10      I=1,NX1
11      IF(Y1.GE.YT(J))GOTO 4
12      CONTINUE
13      GOTO 20

14      CALL CDINTP(V1,TDINT,YT,J,F
15             1,NPY1,l,l)
16      RETURN

17      CALL CDINTP(X1,TDINT,XT,I,F
18             1,NPX1,v
19             NPX1,l,l)
20      RETURN

21      WRITE(6,22)X1,XT(I),XT(NX1)
22      WRITE(6,23)Y1,YT(J),YT(NY1)
23      STOP

24      FORMAT(32F10.7)

25      WRITE(6,14)INX1,NPX1,NPY1,NPY1
26      STOP

27      FORMAT(32F10.7)

28      SORTXY X?,E15.7,7H XT(I),E15.7
29      SORTXY Y?,E15.7,7H YT(J),E15.7
30      SORTXY T
31      SORTXY K
32      SORTXY J
33      WRITE(6,15)INX1,NPX1,NPY1,NPY1
34      STOP

35      FORMAT(32F10.7)

36      SORTXY X?,E15.7,7H XT(I),E15.7
37      SORTXY Y?,E15.7,7H YT(J),E15.7
38      SORTXY T
39      SORTXY K
40      SORTXY J
41      WRITE(6,15)INX1,NPX1,NPY1,NPY1
42      STOP

43      FORMAT(32F10.7)

44      SORTXY X?,E15.7,7H XT(I),E15.7
45      SORTXY Y?,E15.7,7H YT(J),E15.7
46      SORTXY T
47      SORTXY K
48      SORTXY J
49      WRITE(6,15)INX1,NPX1,NPY1,NPY1
50      STOP

51      FORMAT(32F10.7)
SUBROUTINE CCINTP(XX,FX,NX,FT,NF,NPS,N0,IXI,IFI)
C DIMENSION
XT(NPS),F(NF)
NFX=NX-1
NSX=NF-NFX
NSF=NF
KP=NPS
IX=IXI
IY=IFI
N1=(NF-
N2=NFX/2
N3=NFX-N2
N13=IC5*1*10
N22=(NFX-N2)
IF(NFX-N13>0,1,1)
N4=(NFX-N21+1)*IX
N44=N4-1*IX
K22=NSX*IX
N22=K22
IF(XT(N44)>N22,1,1)
IF(XT(N44)<N22,1,1)
IF(NFX.LT.10)GOTo 10
N5=NFX-N1
N55=N5/2
N6=N4*N5
N66=(N6-1)*IX
NSX=NSX+IX
IF(XT(N66).GE.XT(NFX),1,1)
NA=N6
N44=N66
IF(NFX.GT.2)GOTo 3
IF(XT(N44)>I45,43,43)
IF(NFX-N3>44)44,44
N4=N4-1
N44=N44-IX
GOTO 42
N4=N4-1
N5=N4-N1
N20=NSX*IX
N21=(N4-1)*IX
CO 46 1=1,N
T(I)=FT(N20)
S(I)=XT(N21)
N20=N20*1*1*6
N21=N21*IX
L=(NFX-
TR = T(U
N4*N1*1
N6=N4
N7=N4+1
JU = 1
N2=N4-1
UN=1.0
IF(N.EO.1)GOTO 13
CO 12 J=1,N2
N5=N4-N1
N3=N-J
CO 9 1=1,N3
ff n = (T(I)-T(m)/(S(N8)-S<N5J)
N5=N5*1
GOTOC10,
JU
UN=UN*JX-S(N7n
JU = !
N7=N7*1
L=L-1
TR=TR*UN+T(U
FX=TR
RETURN
WRITE(6,50IX,XT(NSXI,XT(N22»,
F0RMAT(23^,
ARG.
NOT IN TABLE  X» |E1S:
7
qH
1E14.7,10F  XT<NPSI»,E14.7,2X,6HD0!NTP»
60 IF(X-7.0*XT(N5)=0.*NSX1)XT(N20)=1.20*2C
721 IF(X-7.0*XT(N5)=1.20*2C)GOTO 72
N5=NP-N
723 N5=N5/2
N6=N4+N5
N6=(N6-1)*IX+NSX
IF(XT(N6)=1.E10)GOTO 724
K4=N6
N4=N4+N6
GOTO 72
724 IF(N5-CT.2)GOTO 723
72 IF(X-XT(N4))73,73,45
73 IF(N4=N374,45,74
74 N4=N4+1
GOTO 72
80 WRITE(65,521XT(NSX))
STOP
52 FORMAT(239 CONSTANT TABLE XI(1)=E14.7,2X,6HDDINTP)

C **********************************************************************
C SUBROUTINE KUTMER(DNXT,OPST,DPAX,N,Y,YP,DERIV,ER,ME,W,ITYP,PS,
C ITTC=0.4,NCT=INTC,NST,TERM)
C DIMENSION Y(50),YP(50),W(10),TC(25)
C DIMENSION E(50),Q1(50),Q2(5C),FA(25),FB(25)
C IF(ITYP.GE.1)GOTO 7C
C ER1=.1*E-5
C EPS=.01CNXT
C I=1
C IF(ER2.E0)ER2=000001*PS
10 CALL CERIV(Y,YP)
GOTO 105,50)
20 IF(ITYP.EQ.1)RETURN
25 CALL TERM(Y,YP,TC,PV)
GOTO 120,270,1
30 CALL PRINT(Y,YP)
GOTO 150,360,35C,1
40 CO 50 I=1,NCT
IF(ITYP.GT.1)GOTO 41
ENC
FB(I)=0
GOTO 50
FA(I)=1
FB(I)=0
CONTINUE
FC=0.
PV2=AIN(T(PV/PS*SIGN(ER2,DNXT)))
PV1=PV2+PS
IF(ABS(PV2-PV1).LT.ER2)PV2=PV2-PS
K=0
70 I=6=DNXT/6.
I3=6*K6+H6
I8=.25*CNXT
I2=.2*CNXT
CI 80 I=1,N
CI(I)=Y(I)
CI(I)=YP(I)
00 Y(I)=C1(I)+H3*Q1(I)
I=2
GOTO 10
50 CO 100 I=1,N
CO Y(I)=Q1(I)+H5*Q1(I)+YP(I))
I=3
GOTO 10
110 CO 120 I=1,N
KUT(I)=3*YP(I)
120 Y(I)=Q1(I)+H8*Q2(I)+Q1(I))
I=4
GOTO 10
130 KUT(I)=40 I=1,N
I1=1(I)-G2(I)
21(I)=4*YP(I)
140 Y(I)=Q1(I)+H2*Q2(I))
GOTO 10
150 DPAX=0.
CO 190 I=1,N
J50
1. VC(I)+6*G1(I)+Q2(I)+YP(I)
2. IF(IVE,EC.G.1)(CCTC)=15G
3. IF(1170,1EC,1FC).ME
4. IF(ABS(T1).GE.1)T2=T2/ABS(T1)
5. IF(T2.GT.ERMAX)ERMAX=T2
6. GOTO 190
7. IF(T1.GT.10C)(CCTC)=17C
8. T1=T2/CASE(T1)
9. J=J+1
10. T2=(T2-1)*T2
11. T1=(T1+1)*T1
12. IF(T2.GT.T1)T2=T1
13. GOTO 170
14. CONTINUE
15. EPST=DNXT
16. IF(1.U.EC.G.1)(CCTC)=26G
17. IF(TMAX.1.EC.G.1)(CCTC)=21G
18. IF(ERMAX.GT.1)ERMAX=1
19. IF(T1.GT.1.1D.IND.1.GT.1)CCTC=19Z
20. IF(T1+1+1.T1)GOTO 195
21. continue
22. INT1=IND1-1
23. GOTO 240
24. IF(DNXT).EPST(*)=(1..Z+(T-1.1))
25. IF(ABS(DNXT).LT.ERMAX)GOTO 220
26. IF(ASSN.GT.1)CCTC=152
27. CXTN=ERMAX
28. IF(DNXT.LT.1)DNXT=OMAX
29. IF(ERMAX.LT.ERM)GOTO 24C
30. CXTC=11+11N
31. Y(I)=(11)
32. INTC=2
33. GOTO 10
34. IF(T)(I)RC
35. PTV=PV
36. CTO=26G I=1+1NTC
37. FC =11 TC(I)
38. GOTO 29
39. IF (FA11.EQ.FE11) GOTO 37C
40. IF(FC11.EC.1CC)=6C TC=28G
41. IF(ABS(1111).GE.ER2.RK.GT.11GOTO 250
42. IA(I)=0
43. GOTO 293
44. IF(A1(I)EQ.I)
45. IF(ABS(1111).GE.ER2.RK.GT.11GOTO 290
46. FB1(I)=0
47. IF(A1(I)EQ.FB11)GOTO 37C
48. CONTINUE
49. IF(FV.GT.PV1)GOTO 330
50. IF(FC.GT.PV2)GOTO 75
51. V=PV
52. FC=1
53. SH=1+1XT
54. IF(EPST*PV2-PV1)GOTO 250
55. IF(TT.C=1)
56. GOTO 70
57. PV3=PV1
58. IF(TT.C=1)GOTO 310
59. IF(TT.C=1)GOTO 350
60. IF(FC.4)CCTC=25G
61. GOTO 290
62. FC=FC+1
63. GOTO 320
64. I=I+1
65. I=I+1
66. IF(PV1=PV2)PVC=PV-PV1
67. IF(FC.EC.1)GOTO 265
68. IF(FV+1P)=PV1
69. IF(PV1=PV2.PV2>PV-PV1)
70. IF(TT.C=1)
71. GOTO 250
72. 3

<table>
<thead>
<tr>
<th>X</th>
<th>Y1</th>
<th>Y2</th>
<th>Y3</th>
<th>Y4</th>
<th>Y5</th>
<th>Y6</th>
<th>Y7</th>
<th>Y8</th>
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<td>0.15</td>
<td>1.66</td>
<td>-1.43</td>
<td>.159</td>
<td>.349</td>
<td>.302</td>
<td>-.385</td>
<td>0.04</td>
<td></td>
</tr>
<tr>
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<td>1.66</td>
<td>-1.43</td>
<td>.159</td>
<td>.349</td>
<td>.302</td>
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<td>.349</td>
<td>.302</td>
<td>-.385</td>
<td>0.04</td>
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</tr>
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<td>Symbol</td>
<td>Description</td>
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</tr>
<tr>
<td>a</td>
<td>radius of circular cylinder in physical plane</td>
<td></td>
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<td></td>
</tr>
<tr>
<td>$c_f$</td>
<td>skin friction coefficient, $(\partial u/\partial y)_{W}/(\rho u_o^2/2)$</td>
<td></td>
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<tr>
<td>d</td>
<td>diameter of rotating cylinder in physical plane</td>
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<tr>
<td>h</td>
<td>coordinate scaling function of x, $\eta = y/h(x)$</td>
<td></td>
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<td></td>
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</tr>
<tr>
<td>i</td>
<td>unit imaginary value</td>
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<td>k</td>
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<td>m</td>
<td>angle used in $\tilde{z}$ plane, see Figure 2</td>
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<td>$m_s$</td>
<td>angle of m corresponding to separation</td>
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<td>p</td>
<td>local pressure</td>
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<tr>
<td>q</td>
<td>inviscid velocity along cylinder surface in $\zeta$ plane</td>
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<td>r, $\phi$</td>
<td>polar coordinates in $\zeta$ plane</td>
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<td>u</td>
<td>nondimensionalized local velocity in direction tangential to cylinder</td>
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<td>$u_c$</td>
<td>nondimensionalized circulation velocity on circle in $\zeta$ plane</td>
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<td>inviscid velocity along cylinder surface in z plane</td>
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<td>$u_w$</td>
<td>peripheral velocity of cylinder wall</td>
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<td>$u_o$</td>
<td>free-stream velocity</td>
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<td>dimensioned form of $u_e$</td>
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<td>$\tilde{u}_e$</td>
<td>dimensioned boundary-layer velocity in direction tangential to cylinder</td>
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<tr>
<td>v</td>
<td>boundary-layer velocity in direction normal to cylinder</td>
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<td>$\tilde{v}$</td>
<td>dimensioned local velocity in direction normal to cylinder</td>
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</table>
LIST OF SYMBOLS (Continued)

$w(z)$ complex inviscid velocity in $z$ plane

$x$ nondimensionalized coordinate along surface of cylinder - $x/a$

$x$ distance along cylinder from $X$ axis

$y$ nondimensionalized coordinate normal to surface - origin at surface of cylinder in physical plane

$y$ dimensioned value of $y$

$z$ complex coordinates in physical plane $z = x + i Y$

$z'$ complex coordinate rotated by angle $-\varphi$ to the $z$ plane, $z' = \bar{x} + i \bar{Y}$

$C_D$ the drag coefficient on the cylinder

$C_L$ the lift coefficient on the cylinder

$E_c$ value of the convergence criterion

$H$ asymptotic breadth of wake

$M(z)$ conformal mapping function, maps from $z$ plane to $\bar{z}$ plane

$N(z)$ function relating the fluid velocities on the untransformed and transformed circles

$Q_1$ nondimensionalized strength of single source on upper part of cylinder

$Q_2$ nondimensionalized strength of single source on lower part of cylinder

$\bar{Q}_1$ dimensioned value of $Q_1$

$\bar{Q}_2$ dimensioned value of $Q_2$

$\bar{R}$ radius of circle in $\bar{z}$ plane, $R = \csc \bar{\varphi}$

$S_1$ upper separation point

$S_2$ lower separation point
LIST OF SYMBOLS (Continued)

\( V_0 \)  
magnitude of free-stream velocity in \( \zeta \) plane

\( W(\bar{z}) \)  
complex velocity in \( \bar{z} \) plane

\( \alpha \)  
polar angle of upper separation point in \( \zeta \) plane

\( \gamma \)  
angle of upper source from real axis in \( \zeta \) plane

\( \delta \)  
angle of lower source from real axis in \( \zeta \) plane

\( \epsilon \)  
angle that \( \bar{\zeta} \) system needs to be turned to coincide with \( \zeta \) system

\( \bar{\epsilon} \)  
angle that \( \bar{\zeta} \) system needs to be turned to coincide with \( \bar{z} \) system after a translation of \( \bar{z} \) coordinates so that the origins of the two systems coincide

\( \zeta \)  
primary untransformed complex plane

\( \bar{\zeta} \)  
untransformed complex plane rotated by an angle of \(-\epsilon\) to the \( \zeta \) plane

\( \mu \)  
viscosity

\( \nu \)  
kinematic viscosity

\( \xi_1 \)  
real coordinate for \( \zeta \) plane

\( \bar{\xi}_1 \)  
real coordinate for \( \bar{\zeta} \) plane

\( \rho \)  
mass per unit volume

\( \sigma \)  
polar angle in \( \zeta \) plane

\( \tau \)  
surface shear stress

\( \psi \)  
polar angle in \( \zeta \) plane

\( \chi(\zeta) \)  
complex-potential flow field

\( \psi_1 \)  
imaginary coordinate for \( \zeta \) plane

\( \bar{\psi}_1 \)  
imaginary coordinate for \( \bar{\zeta} \) plane

\( \Gamma \)  
circulation strength

\( \phi \)  
flow potential
LIST OF SYMBOLS (Continued)

\[\alpha\] angular increment from separation points

SUBSCRIPTS

\(e\) boundary-layer edge conditions

\(f\) final conditions, also stands for fixed wall conditions

\(i\) initial conditions

\(s\) boundary-layer separation conditions

\(w\) wall conditions

\(1\) upper side conditions

\(2\) lower side conditions

SUPERSCRIPTS

some dimensioned quantities

transformed quantities with a few exceptions