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KINEMATIC RANGING in an AIR-AIR MISSILE

by

R.J. LaSpisa and F.D. Powell

Report Number 9500-820254

October, 1972

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Final Report on Naval Air Systems Command

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It is shown that acceleration of the target in the radial direction, parallel to the Line of Sight, precludes successful estimation by the classical techniques such as adaptive parameter identification, Kalman filter estimation, or the various minimum-variance estimation methods. A nonlinear estimator is described which estimates the target acceleration and velocity components and can yield accurate range and range-rate estimates if correctly initialized at launch. This estimator is able to take advantage of the many inequalities which constrain the maneuvering of an air-air target. Errors and performance of this estimator are demonstrated by computer simulations.
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1. INTRODUCTION

Proper use of range and range-rate data, or the equivalent "time-to-go," can improve the performance of an air-air missile. At the same time, instruments which directly measure range and/or range-rate, such as radars, are expensive. The purpose of this study was to determine whether adaptive parameter identification techniques could enable estimation of range and range-rate from less expensive sensors which are used for example in infrared seekers.

1.1 Background

The trajectory of an air-air homing missile may be regarded as the response of a closed loop to an external command. In this case, the external command is the trajectory of the target, which must be assumed to be maneuvering either to conduct its attack or to attempt to evade the missile. The closed loop consists of the geometry which yields the inertial Line of Sight (LOS) rotation-rate, the navigation or guidance computer which forms commands to the missile as a function of the LOS rate, and the missile's dynamic response to that command, which closes the loop. The inertial LOS rate is, as viewed from the missile, a dynamic function of the target and missile accelerations perpendicular to the LOS, and the range and range-rate.
If signals of the LOS rate and the missile acceleration perpendicular to the LOS are available measurements, it might be possible to deduce the dynamics of the geometry, which may be regarded as a "plant" within the closed loop of the entire system.

As stated, the purpose of this research study was to explore means of estimating the missile-target range and range-rate by applying adaptive parameter identification techniques to this "plant" or geometry.

This approach to estimation of range and range-rate uses the missile dynamics as a probe with which to examine the geometry or kinematics of the system. It is therefore called "Dynamic Ranging," or "Kinematic Ranging."

Figure 1.1, right, shows the closed loop from target acceleration through the kinematics and dynamics to the missile response, in the upper portion. The lower part of the diagram shows, in a symbolic form, the parameter estimation computer whose ultimate outputs are the target range and range-rate. Consider the "box" containing the geometry and kinematics: its output, the LOS rate, is a measurable signal, while the two inputs are the target and the missile accelerations perpendicular to the LOS. But while the missile acceleration is an available signal, the target acceleration is not; estimation of range and range-rate
therefore must depend significantly on the characteristics of the target acceleration perpendicular to the LOS. It will be seen that the target acceleration parallel to the LOS also has a strong effect on the estimation problem.

Fig. 1.1
System Configuration
1.2 Organization of the Report

The principal results of this study and the conclusions are gathered in Section 2. The problem is described and mathematically posed in Section 3. A nonlinear range and range-rate estimator is described in Section 4, together with an error and performance analysis. Several linear estimators are described in Section 5, together with comments on the conditions for which the problem is mathematically observable. A technological forecast is presented as Section 6. References are gathered in Section 7. Appendix A contains some details of the derivation of the nonlinear estimator, and Appendix B contains tabulations of the time-varying coefficients of its error propagation equation for a variety of typical trajectories.
2. SUMMARY AND CONCLUSIONS

A summary of the findings of this study is presented. The principal conclusions are stated and discussed.

2.1 Summary

Methods are explored for estimating Range and Range-Rate from the geometry and kinematics of the air-air homing-missile combat situation rather than from direct measurements. The methods reported herein use the various signals available from the missile autopilot, plus the inertial rotation rate of the Line of Sight provided by the seeker.

It is shown that acceleration of the target in the radial direction, parallel to the LOS, precludes successful estimation by the classical techniques such as adaptive parameter identification, Kalman filter estimation, or the various minimum-variance estimation methods, unless the sensor noise levels are very low and the geometry of the combat is favorable. A nonlinear estimator is described which estimates the target's aspect angle, its velocity and acceleration components plus turn-rate and roll-rate, and can yield accurate range and range-rate estimates if correctly initialized at launch. This estimator is able to take advantage of the many inequalities which constrain the maneuvering of an air-air target. Errors and performance of this estimator are demonstrated by computer simulations.
2.2 Conclusions

1. Target trajectories which have significant acceleration parallel to the line of sight occur in most combat geometries except head-on and tail-chase configurations. If the target has significant acceleration parallel to the line of sight, then range and range-rate are mathematically unobservable from the kinematics by linear methods.

In practice this means that range and range-rate can be estimated only in favorable geometries and while active maneuvering of the missile occurs and that the sensor noise levels must be so low that very short averaging times in the estimating filter will yield acceptable accuracy. Filter averaging times of the order of one-quarter second to one second yield range-rate errors as large as 30 to 50% when the target has 3 to 6g acceleration parallel to the LOS.

2. Two approaches are available to resolve these difficulties. A method which uses linear principles combines the kinematic techniques with stadimetric methods. Stadimetric methods require that the seeker be able to measure some function of the target's area or angular size and rely on the assumption that the target is of constant size. Using stadimetric data relieves but does not eliminate the difficulties noted in conclusion (1), above. A second method consists of a nonlinear estimator, discussed below.
3. A nonlinear estimator which estimates the target velocity vector components and acceleration components relative to the line of sight and also estimates the target turn-rate, and its rate of change, can be devised. As a stand-alone unit, this estimator requires initialization of range and range-rate. In combination with other equipment it can be used as a filter to exclude impossible target accelerations due to seeker noise or noise originating in some other method of ranging. Its principle of operation is based on the constraints on the target maneuverability dynamics, and, in particular, on the fact that the target airspeed cannot be significantly changed during an air-air missile engagement.
3. **THE MATHEMATICAL MODEL OF KINEMATIC RANGING**

The study is restricted to the Maneuver plane.

The key assumption guiding and restricting this study is that the seeker in the missile provides the line-of-sight (LOS) rate only; it does not yield any indication of target-size, or shape or changes thereof, as television or mosaic-type infrared seekers can, nor does it provide any range or range-rate information directly, as a radar does, nor even yield the variations of intensity of signal from the target. The simplest type of seeker, simply pointing at the target and carrying a rate gyro to sense the LOS rate, is assumed. On the other hand, the missile is assumed to have a pair of accelerometers in the maneuver plane, so that its accelerations may be resolved about the line of sight. Similarly, it is assumed that missile air-speed, or its estimate, can be provided if needed and that no data are transferred from the launch-airplane to the missile after the missile has been launched. The two areas which describe the problem are:

(a) The geometry and the equations which define the kinematic relations from which we hope to estimate Range and Range-Rate, and

(b) The constraints on the target.
These areas are discussed in detail below. The problem which we attack is estimation of Range and Range-Rate from the kinematics and dynamics of the combat.

3.1 Kinematics

Under these circumstances, a polar coordinate geometry with the moving origin located at the missile is the logical coordinate system. Figure 3.1 shows the geometry.

![Figure 3.1 - Geometry](image)

We first derive expressions for the velocity components parallel and perpendicular to the LOS. Similar expressions for the accelerations follow.

The instantaneous linear components of velocity parallel to the Line-of-Sight (LOS) yield the polar coordinate component,

\[
\dot{r} = V_T \cos (\gamma_T - \sigma) - V_M \cos (\gamma_M - \sigma)
\]
while the components perpendicular to the LOS are
\[ \mathbf{r} \cdot \mathbf{a} = V_T \sin (\gamma_T - \sigma) - V_M \sin (\gamma_M - \sigma). \]  
(3.2)

The angle $\xi_T$ may be called "The Target Aspect Angle" as it represents the attitude of the target velocity vector relative to the LOS, and is closely related to the target body attitude with respect to the LOS.

As missile acceleration components are assumed to be measured, we differentiate (3.1) and (3.2) so that the acceleration terms will become explicitly evident, yielding
\[ \mathbf{\ddot{r}} = \left[ \dot{V}_T \cos (\gamma_T - \sigma) - V_T \dot{\gamma}_T \sin (\gamma_T - \sigma) \right] - \left[ \dot{V}_M \cos (\gamma_M - \sigma) - V_M \dot{\gamma}_M \sin (\gamma_M - \sigma) \right] \]
\[ - \dot{V}_M \dot{\gamma}_M \sin (\gamma_M - \sigma) + \dot{\sigma} \left[ V_T \sin (\gamma_T - \sigma) - V_M \sin (\gamma_M - \sigma) \right]. \]
(3.3)

and
\[ \mathbf{\ddot{r}} + \mathbf{\ddot{r}} = \left[ \dot{V}_T \sin (\gamma_T - \sigma) + V_T \dot{\gamma}_T \cos (\gamma_T - \sigma) \right] - \left[ \dot{V}_M \sin (\gamma_M - \sigma) + V_M \dot{\gamma}_M \cos (\gamma_M - \sigma) \right] - \dot{\sigma} \left[ V_T \cos (\gamma_T - \sigma) - V_M \cos (\gamma_M - \sigma) \right]. \]
(3.4)

The components of acceleration of target and missile parallel to the instantaneous line-of-sight are the first two terms of the right of (3.3), respectively.
Therefore define

\[ a_{T_r} = \dot{V}_T \cos (\gamma_T - \sigma) - V_T \dot{\gamma}_T \sin (\gamma_T - \sigma) \]  

(3.5)

\[ a_{M_r} = \dot{V}_M \cos (\gamma_M - \sigma) - V_M \dot{\gamma}_M \sin (\gamma_M - \sigma). \]  

(3.6)

The third term on the right of (3.3) represents the apparent acceleration due to the LOS rotation-rate.

Now multiply (3.2) by \( \dot{\sigma} \) and substitute \((r \dot{\sigma}^2)\) for the third term on the right of (3.3); with this substitution and the definitions (3.5) and (3.6), (3.3) yields

\[ \ddot{r} - r \dot{\sigma}^2 - a_{T_r} + a_{M_r} = 0. \]  

(3.7)

Equation (3.4) is similarly simplified.

As before, define the components of target and missile accelerations across the instantaneous line-of-sight by

\[ a_{T_\sigma} = \ddot{V}_T \sin (\gamma_T - \sigma) + V_T \dot{\gamma}_T \cos (\gamma_T - \sigma) \]  

(3.8)

\[ a_{M_\sigma} = \ddot{V}_M \sin (\gamma_M - \sigma) + V_M \dot{\gamma}_M \cos (\gamma_M - \sigma) \]  

(3.9)

and substitute these for the first and second terms on the right of (3.4). Again, the third term on the right of (3.4) is the apparent acceleration due to the LOS rotation-rate. Now multiply (3.1) by \( \dot{\sigma} \) and substitute
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(\ddot{r} \hat{o}) for the third term on the right of (3.4), yielding the acceleration equation

\[ \ddot{r} + 2 \dot{r} \hat{o} - a_T + a_M = 0. \]  (3.10)

If we use the notation

\[ \xi_T = \gamma_T - \sigma \]
\[ \xi_M = \gamma_M - \sigma \]  (3.11)

then the velocity equations (3.1) and (3.2) may be expressed as

\[ \dot{r} - VT \cos \xi_T + VM \cos \xi_M = 0 \]  (3.12)
\[ r \dot{o} - \gamma_T \sin \xi_T + VM \sin \xi_M = 0 \]  (3.13)

and, with the more compact notation

\[ V_M \hat{r} = VM \cos \xi_M, \quad V_T \hat{r} = VT \cos \xi_T, \]
\[ V_M \hat{o} = VM \sin \xi_M, \quad V_T \hat{o} = VT \sin \xi_T, \]  (3.14)

for the components of target and missile velocity parallel and perpendicular to the LOS, we have the velocity equations in the form

\[ \dot{r} - V_T \hat{r} + V_M \hat{r} = 0 \]  (3.15)
\[ r \dot{o} - V_T \hat{o} + V_M \hat{o} = 0. \]  (3.16)
Equations (3.1) and (3.2) or their equivalents (3.12) and (3.13) or (3.15) and (3.16), together with (3.7) and (3.10) form the key elements of the mathematical model of the kinematics. The assumed noise levels, bias, initialization errors, etc., which determine the errors of the proposed solution, are gathered in Section 4.

The term \( r^2 \) in (3.7) deserves a brief discussion. This term, a component of the range-acceleration, is due purely to the geometry. Its physical significance can easily be perceived. Assume you, \((V)\), are standing motionless at the coordinates \( X = 0, Y = Y_0 \) where \( |Y_0| \) is small, so that \( V_{H_T} = V_{H_O} = a_{H_T} = a_{H_O} = 0 \). A vehicle, \((T)\), is approaching at constant velocity on the \( X\)-axis; \( \dot{X} \) may be positive or negative but \( \dot{r} \) is negative as the vehicle is approaching. The target is not accelerating, so that \( a_{T_T} = a_{T_O} = 0 \). At the start of the problem, \( \dot{r} = -|\dot{X}| \), while at the end \( \dot{r} = +|\dot{X}| \); Range-Rate, \( \dot{r} \), has changed from \(-|\dot{X}|\) to \(+|\dot{X}|\); the polar-coordinate acceleration signal which expresses this change is \( r^2 \). If the offset \( Y_0 \) is very small, the change from \(-|\dot{X}|\) to \(+|\dot{X}|\) can occur very suddenly, implying large values of \( \dot{r} \). In an air-air combat situation, it is therefore possible for very large range-accelerations to occur, due to the geometry alone, even though neither missile nor target maneuvers.
3.2 Constraints on the Target

The constraints under which the target airplane operates form an essential part of the system. An airplane can increase or decrease its airspeed by changing its thrust or by use of the earth gravitation vector component; in either case or in combination the longitudinal acceleration increment is not more than \(1g\). An airplane can also change its airspeed by increasing its drag; this can be accomplished deliberately by use of speed brakes or involuntarily as a result of the induced drag due to the lift resulting from the airplane's evasive maneuvers. But the target of an air-air missile attack must try to maintain airspeed to conserve maneuverability. We therefore consider that

\[
|\dot{V}_T| \leq 1g
\]

is a plausible constraint for deliberate airspeed changes, and we neglect the involuntary changes in this study.

The target turn-rate, \(\dot{\gamma}_T\), is governed by several inequality constraints. At high airspeed, \(|V_T \dot{\gamma}_T|\) is limited by the ability of the pilot (or structure) to withstand g-loads, equal to \(V_T \dot{\gamma}_T\), while at low airspeed the turn-rate is limited by aero-dynamic stall related to bank angle \(\phi_T\). In addition, the target pilot and aircraft roll-dynamics establish a minimum time in which a maneuver can be performed or changed by the target so that \(|\ddot{\gamma}_T|\), or \(|\dot{\phi}_T|\), is limited.

The high-speed condition provides the acceleration constraint, due to pilot or structural strength limits,
Further, turning is achieved by bank-angle. Taking the pilot's characteristics into account, the bank angle, \( \phi_T \), cannot be changed from hard-turn in one direction to the other in less than 1 second. This yields the constraint

\[
\left| \dot{\phi}_T \right| \leq 2 \left| \phi_T \right|_L.
\]

Figure 3.2 shows acceleration limit, turn-rate limit, and the bank angle for a coordinated turn, as functions of \( V/V_S \) where \( V_S \) is the stall speed for a fighter in combat configuration and \( V_S = 200 \) ft/sec. is assumed as a nominal value. The relationships for a coordinated turn are

- **Acceleration Limit**
  \[
  N_L = \begin{cases} 
  (V_T/V_S)^2, & V_T \leq 2.45 V_S \\
  6.0, & V_T > 2.45 V_S 
  \end{cases} \text{ g's}
  \]

- **Bank Angle**
  \[
  \left| \phi_T \right|_L = \cos^{-1} \left( \frac{1}{N_L} \right).
  \]

- **Roll Rate**
  \[
  |\dot{\phi}_T|_L = 2 \phi_T = 2 \cos^{-1} \left( \frac{1}{N_L} \right)
  \]

- **Turn-Rate Limit**
  \[
  |\dot{\gamma}_T|_L = \frac{(g/V_S)}{(V/V_S)} \sqrt{N_L^2 - 1} = \frac{g/V_S}{V/V_S} \tan |\phi_T|_L.
  \]

- **Turn-Acceleration Limit**
  \[
  |\ddot{\gamma}_T|_L = 2 |\dot{\gamma}_T|_L
  \]

- **Airspeed change**
  \[
  |V_T|_L \leq 1.0 \text{ g}.
  \]
Equations 3.17 and Figure 3.2 thus represent the statement of the constraints which form part of the mathematical model. In an air-air combat situation, it will usually be the case that the target will be flying at a relatively high speed, so that the left of Fig. 3.2, representing the condition $V_T \leq 2.45 V_S$, may be disregarded, and only a relatively narrow range of $V_T/V_S$ can be realistically expected.

Figure 3.2. Bank-Angle and the Limits of Target Turn-Rate and Acceleration versus $V/V_S$ ($V_S = V_{STALL} = 200$ fps)
4. **A NONLINEAR ESTIMATOR**

A derivation of a nonlinear range/range-rate estimation filter is presented below.

This filter has a format such that it is very easy to impose and take advantage of the physical constraints which limit the maneuverability of an airborne target. It therefore could be appropriately used alone if initialized or preset at launch with range and range-rate; thereafter it will correctly keep track of those variables. Alternately it may be used as a nonlinear filter in conjunction with other ranging methods, whether direct or indirect. The analysis below assumes that this estimator will be initialized or preset at launch. The error analysis and performance of Sections 4.2 and 4.3 show the effects of errors in initializing as well as the effects of various other error sources.

This nonlinear filter has as its physical origin, and therefore motivation, the hypotheses that (a) the target cannot easily change airspeed, and (b) range-acceleration is not trivial and must be estimated.

**4.1 Theoretical Basis**

The equations which present the components of velocity parallel and perpendicular to the LCS are, respectively, from (3.15) and (3.16),

\[
\begin{align*}
\dot{r} &= \frac{V_{Tr}}{r} - \frac{V_{Mr}}{r} \\
\dot{r}^* &= \frac{V_{Ts}}{r} - \frac{V_{Ms}}{r},
\end{align*}
\]  

(4.1)  

(4.2)
where the subscripts \((T)\) and \((M)\) refer to target and missile, respectively, while the subscripts \((r)\) and \((\sigma)\) refer to the components parallel to the LOS and perpendicular thereto. Thus, for example, the target velocity component perpendicular to the LOS is \(V_T^M\).

Similarly, the equations which present the components of acceleration parallel and perpendicular to the LOS are, respectively, from (3.7) and (3.10),

\[
\ddot{r} - r^2 \dot{\sigma} + a_{T_r} + a_{M_r} = 0 \quad (4.3)
\]

\[
\dot{\sigma} + 2\dot{r} \dot{\sigma} + a_{T_{\sigma}} + a_{M_{\sigma}} = 0, \quad (4.4)
\]

where the components use the same subscript code described above, so the target acceleration across the LOS is \(a_{T_{\sigma}}\). Consider Eqn. (4.3); we can measure \(\dot{r}\) directly from the seeker and can therefore compute \(r^2\), as initial values of \(r\) and \(\dot{r}\) are assumed given. We can also directly sense the missile acceleration component \(a_{M_r}\). If we could also estimate \(a_{T_r}\), we could compute \(\dot{r}\) and then by integration continue to estimate \(\dot{r}\) and \(r\). This would enable us to close a computation loop and keep the estimation process going. It will be shown that we can estimate \(a_{T_r}\) by using the assumption that target airspeed is quasi-constant during the brief interval of an air-air encounter. This is significant, as linear estimators uniformly fail to operate successfully whenever \(a_{T_r}\) is not negligible, as will be seen in Section 5.
Let us solve Eqns. (4.1) and (4.2) for the target velocity components \( V_{Tr} \) and \( V_{T\sigma} \). We know that the total velocity, or airspeed, \( V_T \), of the target can be determined from these components by the relationship

\[
V_{Tr}^2 + V_{T\sigma}^2 = V_T^2 \tag{4.5}
\]

due to the definitions \( V_{Tr} = V_T \cos \xi_T \) and \( V_{T\sigma} = V_T \sin \xi_T \).

An air-air combat target cannot easily change its airspeed \( V_T \). Since \( V_T \) is nearly constant, \( \dot{V}_T \) is nearly zero, so that differentiation of (4.5) yields

\[
V_{Tr} a_{Tr} + V_{T\sigma} a_{T\sigma} = V_T \dot{V}_T \approx 0 \tag{4.6}
\]

where \( a_{Tr} \) and \( a_{T\sigma} \) are the components of target acceleration parallel and perpendicular to the LOS. The considerable algebraic effort required to show that this differentiation is valid is presented in Appendix A.

We may solve (4.6) for \( a_{Tr} \), so that we may estimate

\[
\hat{a}_{Tr} = - \frac{V_{T\sigma} a_{T\sigma}}{V_{Tr}} \tag{4.7}
\]

We now substitute for \( a_{T\sigma} \), \( V_{T\sigma} \), and \( V_{Tr} \) from (4.1), (4.2) and (4.4) so that (4.7) becomes

\[
\hat{a}_{Tr} = - \frac{(r \ddot{o} + V_{M\sigma})(r \ddot{o} + 2\dot{r} \dot{o} + a_{M\sigma})}{(\dot{r} + V_{Mr})} \tag{4.8}
\]
All quantities on the right of (4.8) may be measured or estimated in the missile, so that we can compute $\hat{a}_T$. But we also have, from Section 3, Eqn. (3.8)

$$a_T = \dot{V}_T \sin \xi_T + V_T \dot{\gamma}_T \cos \xi_T$$

and if $\dot{V}_T \approx 0$, then

$$a_T \approx V_T \dot{\gamma}_T \cos \xi_T = V_T \frac{\dot{\gamma}_T}{\cos \xi_T}$$

We may therefore compute $\hat{a}_T$ in a different way, as

$$\hat{a}_T = - (r \dot{v} + V_{M_\sigma}) \frac{V_T \dot{\gamma}_T \cos \xi_T}{\dot{V}_T \cos \xi_T} = - (r \dot{v} + V_{M_\sigma}) \dot{\gamma}_T = - V_{T_\sigma} \dot{\gamma}_T$$

where $\dot{\gamma}_T$ is the estimate of $\ddot{\gamma}_T$, the target turn-rate.

Eqns. (4.8) and (4.10) present different forms of the same result; either may be used as appropriate to the available physical constraints. In particular, the relationship

$$\dot{\gamma}_T = \frac{a_{T_\sigma}}{V_{T_\sigma}}$$

may be used to estimate $\dot{\gamma}_T$.

If we substitute (4.11) into (4.7) we have

$$\hat{a}_T = -V_{T_\sigma} \dot{\gamma}_T.$$

This is an obvious result: $V_{T_\sigma}$ is the target velocity across the LOS, and $\dot{\gamma}_T$ is its turn-rate, therefore the product is the acceleration of the target in the range.
direction. Its magnitude is at a maximum when the target is flying perpendicular to the LOS and vanishes in a head-on or tail-chase combat configuration.

Equation (4.11) is a very convenient result, for it enables implementation of the constraints, described in Section 3, on the target maneuver capability:

1. The estimates $\hat{a}_{T_r}$ and $\hat{a}_{T_g}$ may be limited to $6g$ magnitude.

2. Since $\dot{\gamma}_T$ may be estimated as $\dot{\gamma}_T = \frac{\hat{a}_{T_g}}{V_{T_r}}$,

   it is possible to limit the quotient to a realistic turn-rate which may be either a nominal constant or a function of the target's estimated airspeed $\hat{V}_T$.

3. Further, if $\dot{\gamma}_T$ is formed in a rate-limited net, it is possible to impose a physically motivated constraint on $\dot{\gamma}_T$, to represent the finite time required by the target to change its turn-rate, e.g., a right turn to a left turn.

These limit-properties are of great value, for they make it possible to estimate $\dot{\gamma}_T$. Without these limits, the division in (4.11) would fail due to a "division by zero" whenever the target velocity vector is nearly perpendicular to the
LOS. This condition can occur in a dogfight situation. Further, the limit on $\hat{\gamma}_T$ noted above in (3) enables exclusion of noise which implies impossible roll-rates, just as the limits (1 & 2) preclude undue sensitivity to noise implying impossible turn-rates and accelerations.

The result of the analysis above is the estimator

$$\hat{r} = \hat{r} \cdot (r - a_{\alpha} - \hat{V}_T \hat{\gamma}_T).$$

Fig. 4.1 shows an estimator configuration, which requires 6 multiplications for its instrumentation in the given format. The locations and character of the limits which impose the physical constraints are shown. The adaptive algorithm which determines $\hat{\gamma}_T$ is shown as part of the overall diagram. It is assumed that missile acceleration components are sensed with respect to the seeker centerline; it is also assumed that the missile velocity components with respect to the seeker centerline are available. This net can be formed in a variety of equivalent ways; the configuration shown in Fig. 4.1 is representative but not unique.
A brief discussion of Fig. 4.1 follows. The block at the extreme upper left shows the process of inserting the initial values of range and range-rate at launch. The three blocks immediately below show the various signals used in this estimator, gathered by their several sources. Block A shows the two integrations relating \( \hat{\mathbf{r}} \) to \( \mathbf{r} \) and \( \mathbf{\dot{r}} \). Block B shows the formation of \( \hat{a}_{T_0} \) from the relationship

\[
\hat{a}_{T_0} = \hat{\mathbf{r}} + 2\hat{\mathbf{r}} \cdot \mathbf{a}_{M_0} . \tag{4.14}
\]

Blocks C and D show the formation of \( \hat{V}_{T_r} \) and \( \hat{V}_{T_0} \), respectively, from the relationships

\[
\hat{V}_{T_r} = \mathbf{r} + V_{M_r} \tag{4.15}
\]

\[
\hat{V}_{T_0} = \mathbf{r} \cdot \mathbf{a} + V_{M_0} . \tag{4.16}
\]

Block E shows the formation of \( \mathbf{\hat{a}} \) as

\[
\mathbf{\hat{a}} = \mathbf{\hat{r}} \cdot \mathbf{a}^2 - a_{M_r} + \hat{a}_{T_r} . \tag{4.17}
\]

We have stated that the term \( \hat{a}_{T_r} \) is estimated as

\[
\hat{a}_{T_r} = -\hat{V}_{T_0} \cdot \mathbf{\gamma}_T . \tag{4.18}
\]
and that this may be limited to 6g, or some similar limit representative of the abilities of the target airplane and pilot. This limited product is contained within Block F.

We now consider estimation of $\hat{\gamma}_T$, noting that if $\dot{V}_T \equiv 0$, then

$$\dot{\gamma}_T = \frac{a_T}{V_T}.$$

We do not have the true values of $a_T$ and $V_T$, but only their estimates. The procedure for estimating $\dot{\gamma}_T$ is to form an error

$$\epsilon = \hat{a}_T - \hat{\gamma}_T \hat{V}_T, \quad (4.19)$$

as shown in Block G. A least mean magnitude algorithm is

$$\hat{\gamma}_T = k \hat{V}_T \text{sgn}(\epsilon), \quad (4.20)$$

where $\text{sgn}(\epsilon)$ is defined as

$$\text{sgn}(\epsilon) = \begin{cases} +1, & \epsilon > 0 \\ -1, & \epsilon < 0 \end{cases}$$

i.e., "the sign of $\epsilon$". One simple way to limit $\hat{\gamma}_T$
to the values permitted by the constraint relationship is to replace $\hat{V}_T$ by $\text{sgn}(\hat{V}_T)$, and to choose $k$ equal to the constraint limit. It is essential that the algorithm contain at least the sign of $\hat{V}_T$ and the sign of $\epsilon$ if it is to be stable for $\hat{\gamma}_T > 0$ and also for $\hat{\gamma}_T < 0$. With these comments, the $\hat{\gamma}_T$ algorithm is now

$$\hat{\gamma}_T = k \text{sgn}(\hat{V}_T) \text{sgn}(\epsilon). \quad (4.21)$$

The least mean magnitude algorithm is preferred to the least mean square as it yields more rapid solution for this class of problem, and is less complex to instrument. In some extreme trajectories the target is flying nearly perpendicular to the LOS and $\hat{V}_T$ may be quite small. To preclude instability of the $\hat{\gamma}_T$ algorithm it is useful to disable integration of $\hat{\gamma}_T$ when $|\hat{V}_T|$ is less than some minimum value, such as $60'/\text{sec}$. Assume a target airspeed of $V_T = 1000'/\text{sec}$. This limit is $6\%$ of the airspeed, implying that the angle $\xi_T$ between the LOS and the target velocity vector is in the range $86^\circ < |\xi_T| < 94^\circ$.

With this restriction, algorithm (4.21) becomes

$$\hat{\gamma}_T = \begin{cases} k \text{sgn}(\hat{V}_T) \text{sgn}(\epsilon), & |\hat{V}_T| \geq V_T^{\min} \\
0, & \text{otherwise} \end{cases} \quad (4.22)$$

An alternate form of (4.22) which contains the entire algorithm in one expression is
\[ \hat{\gamma}_T = k \text{ sgn}(\hat{V}_{T_r}) \text{ sgn}(\varepsilon) \left[ \frac{1 + \text{ sgn}\left(\frac{\left|\hat{V}_{T_r}\right| - V_{T_{r_{\text{min}}}}}{2}\right)}{2} \right]. \]

(4.22a)

We found that \( k = 0.4 \) and \( \hat{V}_{T_{r_{\text{min}}}} = 60 \text{ ft/sec} \) are satisfactory values for the missile and target which we assumed. Eqn. (4.22) is instrumented in Block H.

The integration of \( \hat{\gamma}_T \) to form \( \gamma_T \) and the limitation of \( \hat{\gamma}_T \) to the constrained range appear in Block K. The expression in the square brackets \([\ldots]\) in (4.22a) introduces the deadspace in the algorithm and is visible in Block H. The nonlinearity in the feedback path in Block K is an analog representation of a saturation limiter which imposes the constraint \( \hat{\gamma}_{T_{L}} \) on turn-rate estimate \( \hat{\gamma}_T \). The net of Fig. 4.1 can be mechanized in analog or digital formats with equal ease. The choice of an analog format for this Figure is arbitrary.
Initializing Data at Launch from Carrier

Guidance Signals from Seeker

Missile Velocity Components from Sensor or Estimator

Missile Accel. Components from Autopilot
Figure 4.1. Range and Range-Rate Estimator Configuration
4.2 Error Analysis

The true value of the range acceleration is

$$\ddot{r} = \dot{r}^2 - a_{Mr} + a_{Tr}.$$  

and we have shown that

$$a_{Tr} = \dot{V}_T \cos \xi_T - V_T \dot{V}_T \sin \xi_T = (V_T \dot{V}_T - V_T \dot{a}_T) / V_{Tr}$$

where

$$V_{Tr} = V_T \cos \xi_T = \dot{r} + V_M \cos \xi_M$$

$$V_T = V_T \sin \xi_T = \dot{r} + V_M \sin \xi_M$$

and

$$a_{Tg} = \dot{V}_T \sin \xi_T + V_T \dot{V}_T \cos \xi_T = \ddot{r} + 2 \ddot{r} + a_{Mg}.$$  

Disregarding the effects of target velocity-change $\dot{V}_T$, the algorithm estimates $a_{Tr}$ as

$$\hat{a}_{Tr} = - \frac{(\hat{\dot{r}} + \hat{V}_M \sin \hat{\xi}_M) (\ddot{r} + \dot{V}_T \sin \xi_T)}{\hat{r} + V_M \cos \xi_M}.$$  \hspace{1cm} (4.23)

imposing limits on $|\hat{a}_{Tr}|$, $|\dot{V}_T|$, and $|\ddot{V}_T|$ in the process.

We consider in this section the several sources of error and their significance. The principal sources of error are discussed below.

1. $\dot{V}_T$. The target may change its airspeed, although relatively slowly.

2. $\hat{r}_o$, $\ddot{r}_o$. Range and range-rate may be initialized with errors.
(3) $V_M$. The missile airspeed is required. It may be sensed or may be provided adaptively, but with an error in either case.

(4) $\xi_M$. The missile angle of attack must be estimated to form $\hat{\xi}_M = -\alpha - \lambda$ in order to resolve the missile velocity into the components $V_M \sin \xi_M$ and $V_M \cos \xi_M$. Errors enter as incorrect estimates of $\lambda$ and $\alpha$, as discussed in the simulation results. The angle $\lambda$ is the angle from the missile centerline to the seeker centerline.

(5) $\dot{\sigma}$. The LOS rate may have random noise errors plus a bias.

(6) $a_{Mr}, a_{Mo}$. The accelerometers sensing missile acceleration parallel and perpendicular, respectively, to the LOS may have bias errors and random noise.

We define $x=r-\hat{r}$, $\dot{x}=r-\hat{r}$, so that $x$ is the range error, and $V_1 = \hat{V}_1 + N_1$ where

$V_1 = a_{Mr}$

$V_2 = 0$; $N_2 = \hat{V}_T$; shows $\hat{V}_T \neq 0$

$V_3 = V_M$

$V_4 = \dot{\sigma}$

$V_5 = \xi_M$

$V_6 = \ddot{\sigma}$

$V_7 = a_{Mo}$
Thus, \( N_1 \) represents the deviation from the actual value of \( V_i \), i.e., the additive noise. The error \( x(t) \) is then governed by the equation

\[
\dot{x} + a_1(t)x + a_2(t)x = \sum_{i=1}^{7} b_i(t)N_i
\]  

(4.24)

where the coefficients \( a_i, b_i \) are defined as:

\[
a_1 = 2\dot{\gamma} \tan \xi_T - \dot{\gamma}_T \tan \xi_T
\]

\[
a_2 = -\dot{\gamma}^2 + \dot{\gamma}_T \dot{\gamma} + \dot{\gamma} \tan \xi_T
\]

\[
b_1 = -1
\]

\[
b_2 = \sec \xi_T
\]

\[
b_3 = -\dot{\gamma}_T \sec \xi_T \sin (\gamma_M - \gamma_T)
\]

\[
b_4 = -r\dot{\gamma}_T - 2\dot{r} \tan \xi_T + 2r\dot{\gamma}
\]

\[
b_5 = -V_M \dot{\gamma} \sec \xi_T \cos (\gamma_M - \gamma_T)
\]

\[
b_6 = -r \tan \xi_T
\]

\[
b_7 = -\tan \xi_T.
\]

The coefficients of \( x \) and \( \dot{x} \) are functions of time. The trajectory of the errors of range estimate, \( x \), and range-rate estimate, \( \dot{x} \), may be found by solving this linear differential equation with time-varying coefficients, subject to the initial conditions \( x_0 \) and \( \dot{x}_0 \), which are the assumed initialization errors of range and range rate. As (4.24) is linear, superposition is valid so that it may be solved for the individual terms. The coefficients \( a_i \) and \( b_i \) are tabulated in Appendix B for a number of typical trajectories.
4.3 Performance

The air-air missile homing situation was programmed on the digital computer. The simulation used parameters from a typical short-range highly maneuverable air-air missile. The guidance method used for the simulation was the conventional proportional guidance:

\[ a_c = 5 V_M \dot{\alpha}, \]

where \( \dot{\alpha} \) is the LOS angular inertial rate and \( a_c \) is the acceleration command across the LOS. The transfer function describing the missile characteristics was

\[
\frac{a(s)}{a_c(s)} = \frac{-53s^2 + 106.6s + 27000}{s^3 + 72s^2 + 2160s + 27000}
\]

where \( a \) is the cross-body acceleration. This corresponds to a simplified and linearized second order model for the missile airframe combined with a first order model for actuator lag. The estimates of range and range-rate were thus not used in the guidance loop as they might be in an actual missile system. The missile was started with a specified velocity with the airframe state variables at rest at the beginning of the trajectory. The geometry of the combat situation is shown in Fig. 4.2.

A discretization step size of 0.005 seconds was found to result in negligible error in the numerical output of the digital program as a substitute for the actual continuous model airframe being analyzed.
Figure 4-2. Definition of Positive Geometric Angles
Several different geometrical situations were simulated, corresponding to different angular orientations and bank angle of the target. Missile velocity was Mach 2.0 or 2144 ft/sec at 10,000 ft. altitude, and target velocity 1,000 ft/sec. The missile airspeed was assumed constant; this does not restrict the generality of the results.

The simulations which follow are arranged in ten groups, each corresponding to a different starting situation. The angle $\psi_T$ is defined as the bank angle of the target. For example, $80^\circ$ represents a hard left turn of approximately 5.7 g's lateral acceleration. The angle $\psi_T$ represents the target's initial heading, measured positive counter-clockwise from a reference heading parallel to the attacking missile's initial centerline, which is the x-axis.

Most of the ten groupings of plots have six separate graphs, arranged in pairs on facing pages.

The first graph is a view looking down at the two dimensional chase. In each case the target is intercepted as a result of the homing navigation. This also shows initial physical orientation of the target. Connecting the missile and target at corresponding successive instants of time are dashed lines representing the position of the line-of-sight (LOS) as the chase evolves.

The second graph shows the actual values of range, range-rate, and range-acceleration as functions of time,
from \( t = 0 \) to \( t = \text{time-of-intercept} \). These variables represent the true values of the parameters which are estimated.

Graphs 3 and 4 in the group plot the errors in the range estimate due to various error sources. The plot lettered (a) has no error sources and represents the good estimates of range and range-rate obtainable with this method. The reason (a) is not identically zero arises from the inaccuracies of the algorithm and computation; it is intended as a control in analyzing the other graphs. The other plots show the effects of various errors. For example, in plot (b) range is initialized incorrectly at 99% of its actual value. A listing of the sources of error is as follows:

(a) No errors, except those inherent to the algorithm itself.
(b) Range initialized with 1% error.
(c) Range-rate initialized with 1% error.
(d) The signal \( \dot{\psi} \) is erroneously biased by the amount 0.5 deg/sec.
(e) The signal \( G \), the lateral accelerometer output, is biased by 3 ft/sec\(^2\).
(f) The signal \( H \), the longitudinal accelerometer output, is biased by 3 ft/sec\(^2\).
(g) The seeker angle relative to the missile centerline, \( \lambda \), is biased by 0.1 degrees.
(h) The error in measuring missile velocity \( V_M \), is taken as 1% or 21.44 ft/sec.
The error in estimating angle of attack, \( \alpha \), was taken to be a bias of 1 degree.

In this case the target velocity is increased with an acceleration of \( 1g \), thus illustrating the effect of violating the assumption \( \dot{V}_T = 0 \).

In all cases, only the particular source of error mentioned was included, i.e., the other error sources were removed.

Graphs 3 shows the errors from sources (a) through (e) above. Graph 4 shows error sources (f) through (j).

In the case of Figure 4.6, an additional source of error is analyzed. Figure 4.6.4 shows the effect of additive noise in the LOS rate as used by the kinematic ranging algorithm. The corrupted LOS rate was also fed to the guidance loop.

Graphs 5 and 6 of each group show the corresponding errors in range-rate instead of the error in range. Separation of cases (a) through (j) is the same as above in graphs 3 and 4.

Figure 4.12 is also an exception to the format specified. This figure displays the geometry when the target changes its maneuver during the flight of the missile. The target starts with a turn to the left, changing to a turn to the opposite direction of equal magnitude.

Figure 4.12.2 shows both the error in range and range-rate from the estimation algorithm. This error represents the numerical accuracy of the algorithm, and is not due to other sources.
FIG. 4.3.1 DYNAMIC RANGING SIMULATION WITH
TARGET INITIAL CONDITIONS. \( t = 90 \), \( \phi = -80 \), \( X = 10000 \).
\( Y = 1000 \). PLOTS OF VARIABLES DEFINING GEOMETRY
Graph 1, the upper graph to the left, shows the plan view of the geometry of the missile chase. The target has an initial range of slightly more than 10,000 ft., and initially is traveling across the LOS. The target airplane undergoes a 5.7g constant right turn which is indicated by the bank angle $\phi_T = -80^\circ$.

Graph 2, bottom left, plots actual range, range-rate, and range-acceleration. One may note the decrease in magnitude of range-rate as the chase changes from a side attack to a tail chase.

Graphs 3 and 4, top and bottom immediate right on the facing foldout page, show the errors in the range estimate due to various error sources for the geometry described above.

Curve (a) represents no errors except those due to the computation process and shows the ability to track the target when perfect signals are used in the algorithm. Maximum error for range is approximately 5 ft. Curves (b) thru (j), on Graphs 3 and 4 represent the effects of the error sources specified in Section 4.3.

Graphs 5 and 6, on the far right of the facing page display the errors in range-rate from the error sources. Curves (a) thru (j) again correspond to the errors mentioned in Section 4.3. Curve (a) shows a maximum error of 3 ft/sec.

Curves (j) on Graphs 4 and 6 represent the largest errors, and appear reasonable since the basic assumption of the algorithm ($\dot{V}_T=0$) is violated. Curve (j) thus shows the error due to target airspeed changes, reading a maximum range-error of 350 feet.

For convenience, the labels a...j in the curves imply:

- a. no error sources
- b. 1% error in $\hat{r}$ (0)
- c. 1% error in $\hat{r}$ (0)
- d. 0.5° bias in $\dot{a}$
- e. 0.1 g bias in $a_M$
- f. 0.1 g bias in $a_M$
- g. 0.1° bias in seeker gimbal angle
- h. 20 ft/s bias in $V_M$
- i. 1° error in $\hat{a}$
- j. $\dot{V}_T = 1$ g
FIG. 4.3.2 DYNAMIC RANGING SIMULATION WITH TARGET INITIAL CONDITIONS. $\phi = 90^\circ$, $\phi = -80^\circ$, $X_T = 10000$, $Y_T = 1000$. ERRORS IN RANGE ESTIMATE FROM VARIOUS SOURCES.
FIG. 4.3.3 DYNAMIC RANGING SIMULATION WITH TARGET INITIAL CONDITIONS. \( \theta = 90 \), \( \phi = -80 \), \( \dot{Y}_T = 10000 \), \( \dot{Y}_T = 1000 \). ERRORS IN RANGE RATE ESTIMATE FROM VARIOUS SOURCES.
FIG. 4.4.1 DYNAMIC RANGING SIMULATION WITH
TARGET INITIAL CONDITIONS. $\phi = 90$, $\theta = 0$, $X_T = 0.0000$, $Y_T = 1000$. PLOTS OF VARIABLES DEFINING GEOMETRY
This grouping of curves shows a non-maneuvering target, with initial heading $90^\circ$ relative to the attacking missile's centerline. In this situation, the LOS rate approaches zero after the missile response to the guidance. As a result, one can say in general that errors in the estimates will tend to grow large after LOS rate gets small. This is somewhat apparent to the curves on the right. It is also noted that curves (a) are not small compared to the others, as was observed in Figure 3. This is explained by the same reason, namely, that when neither the target nor missile maneuver, the estimation of range and range rate is difficult and more sensitive to computation error. The algorithm exhibits moderate sensitivity to initial range error and to accelerometer biases.

a. no error sources
b. 1% error in $\hat{r} (0)$
c. 1% error in $\hat{r} (0)$
d. $0.5^\circ$ bias in $\dot{\alpha}$
e. $0.1$ g bias in $a_{M_0}$
f. $0.1$ g bias in $a_{M_r}$
g. $0.1^\circ$ bias in seeker gimbal angle
h. 20 ft/s bias in $\hat{V}_M$
i. $1^\circ$ error in $\hat{\alpha}$
j. $\hat{V}_T = 1$ g
FIG. 4.4.2 DYNAMIC RANGING SIMULATION WITH TARGET INITIAL CONDITIONS. \( t = 90 \), \( \phi = 0 \), \( x_T = 10000 \), \( y_T = 1000 \). ERRORS IN RANGE ESTIMATE FROM VARIOUS SOURCES.
FIG. 4.4.3 DYNAMIC RANGING SIMULATION WITH TARGET INITIAL CONDITIONS. \( t = 90 \), \( \dot{r} = 0 \), \( x_T = 10000 \), \( y_T = 1000 \). ERRORS IN RANGE RATE ESTIMATE FROM VARIOUS SOURCES.
FIG. 4.4.1 DYNAMIC RANGING SIMULATION WITH TARGET INITIAL CONDITIONS. \( \phi = 90 \), \( \psi = 0 \), \( \omega t = 0000 \), \( \omega y = 0000 \). PLOTS OF VARIABLES DEFINING GEOMETRY
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FIGURE 4.5

Figure 4.5 shows the target with an initial heading of 90° relative to the missile centerline, and a bank angle of 80°, implying a hard turn to the left. With this configuration, the aspect angle $\xi_T$ is close to 90° at all times, in fact it passes through 90°. The error analysis equation predicts that the estimates would be extremely sensitive in this case. However, the nonlinearity in sensing the sign of $V_T$, which is not considered in the error analysis equation, prevents the estimates from becoming sensitive, and in fact the errors from all sources are only slight, this can be observed on the curves to the right.

a. no error sources  

b. 1% error in $\hat{r}(0)$  
c. 1% error in $\hat{\theta}(0)$  
d. $0.5^o$ bias in $\hat{\phi}$  
e. $0.1^o$ bias in $a_M$  
f. $0.1^o$ bias in $a_{MR}$  
g. $0.1^o$ bias in seeker gimbal angle  
h. 20 ft/s bias in $\hat{V}_M$  
i. $1^o$ error in $\hat{\alpha}$  
j. $\hat{V}_T = 1$ g

Without the dead space in Block H of Fig. 4.1, the estimator would be unstable, see Fig. 4.5.4 below.
FIG. 4.5.2 DYNAMIC RANGING SIMULATION WITH
IMPLICIT INITIAL CONDITIONS. $\gamma = 90, \theta = 80, \chi = 10000,$
$\psi = 1000. ERRORS IN RANGE ESTIMATE FROM VARIOUS
SOURCES.

FIG. 4.5.3 DYNAMIC RANGING SIMULATION
TARGET INITIAL CONDITIONS
$\gamma = 1000. ERRORS IN A
Fig. 4.5.3 Dynamic ranging simulation with target initial conditions. $t = 90$, $\theta = 80$, $x_T = 10000$, $y_T = 1000$. Errors in range rate estimate from various sources.
FIG. 4.6.1 DYNAMIC RANGING SIMULATION WITH
TARGET INITIAL CONDITIONS. \( \phi = 0 \), \( \phi = 80 \), \( x_t = 10000 \),
\( r_t = 1000 \). PLOTS OF VARIABLES DEFINING GEOMETRY
This shows the target with an initial relative heading of 0°, and a bank angle of 80°. The largest error appears to occur for case (j), target airspeed change, in agreement with previously explained simulations. The other significant error source, (d), is LOS rate bias.

FIGURE 4.6.4 below, shows the errors when noise is added to the LOS rate, instead of the fixed bias present in curve (d). This random noise was constructed to have zero mean and standard deviation equal to 5.0/Range; thus the noise becomes quite large as range decreases.

a. no error sources  

b. 1% error in \( \hat{r} (0) \)  
c. 1% error in \( \hat{r} (0) \)  
d. 0.5° bias in \( \dot{\alpha} \)  
e. 0.1 g bias in \( a_M \)  
f. 0.1 g bias in \( a_{\dot{M}} \)  
g. 0.1° bias in seeker gimbal angle  
h. 20 ft/s bias in \( V_M \)  
i. 1° error in \( \alpha \)  
j. \( \dot{V}_T = 1 \) g

FIG. 4.6.4 DYNAMIC RANGING SIMULATION WITH TARGET INITIAL CONDITIONS, \( * = 0, \phi = 80 \), \( XT = 10000 \), \( YT = 1000 \). ERRORS IN ESTIMATE WITH NOISE PRESENT IN LOS RATE.
4.5.1 Dynamic ranging simulation with initial conditions $\theta = 80^\circ$, $x_t = 10000$. Errors in range estimate from various sources.
FIG. 4.6.3 DYNAMIC RANGING SIMULATION WITH TARGET INITIAL CONDITIONS. \( t=0 \), \( \theta = 80 \), \( XT = 10000 \), \( YT = 1000 \). ERRORS IN RANGE RATE ESTIMATE FROM VARIOUS SOURCES.
FIG. 4.7.1 DYNAMIC RANGING SIMULATION WITH
TARGET INITIAL CONDITIONS, \( \mathbf{\theta} = 180, \mathbf{\phi} = 80, \mathbf{Xt} = 10000 \),
\( \mathbf{yt} = 1000 \). PLOTS OF VARIABLES DEFINING GEOMETRY.
This geometry displays a head-on situation where the target and missile are approaching each other with maximum closure rate. The target is turning left. Error curve (a) remains small. Curve (b) is worthy of note, as it demonstrates that an error in initial range estimate need not lead to divergence of the range-rate estimate. Curve (c) shows readily that an error in range-rate leads to an increasing error in range. Curve (j) again shows the largest errors, due to target airspeed change.

a. no error sources  
b. 1% error in \( \hat{r} \ (0) \)  
c. 1% error in \( \hat{r} \ (0) \)  
d. 0.5° bias in \( \dot{\alpha} \)  
e. 0.1 g bias in \( a_{M_0} \)  
f. 0.1 g bias in \( a_{M_T} \)  
g. 0.1° bias in seeker gimbal angle  
h. 20 ft/s bias in \( \hat{V}_H \)  
i. 1° error in \( \hat{\alpha} \)  
j. \( \dot{V}_T = 1 \ g \)
FIG. 4.7.2 DYNAMIC RANGING SIMULATION WITH TARGET INITIAL CONDITIONS. \( \theta = 180 \), \( \phi = 80 \), \( xt = 10000 \), \( \eta = 1000 \). ERRORS IN RANGE ESTIMATE FROM VARIOUS SOURCES.
FIG. 4.7.3 DYNAMIC RANGING SIMULATION WITH TARGET INITIAL CONDITIONS. $\theta = 180^\circ$, $\phi = 80^\circ$, $X_T = 10000$, $Y_T = 1000$. ERRORS IN RANGE RATE ESTIMATE FROM VARIOUS SOURCES.
FIG. 4.8.1 DYNAMIC RANGING SIMULATION WITH TARGET INITIAL CONDITIONS. $\gamma = \lambda 0$, $\phi = 0$, $X T = 20000$, $Y T = 1000$. PLOTS OF VARIABLES DEFINING GEOMETRY.
The figure shows a head-on situation, with the target initially at slightly more than 20,000 feet range, flying straight (no turn rate). Range-rate is quite constant, and range acceleration is negligible and barely discernible from the time axis. Errors in the estimates are very small except for a few cases. Curve (c) again shows that a constant range-rate error leads to an increasing range error. Again, the range error in curve (b) does not lead to a range-rate error. Curves (j) show the largest errors.

a. no error sources
b. 1% error in \( \hat{r} (0) \)
c. 1% error in \( \hat{r} (0) \)
d. 0.5° bias in \( \hat{\alpha} \)
e. 0.1 g bias in \( a_M \)
f. 0.1 g bias in \( a_M \)
g. 0.1° bias in seeker gimbal angle
h. 20 ft/s bias in \( \hat{V}_M \)
i. 1° error in \( \hat{\alpha} \)
j. \( \hat{V}_T = 1 \) g
FIG. 4.8.2 DYNAMIC RANGING SIMULATION WITH \( x_t = 1000 \). ERRORS IN RANGE ESTIMATE FROM VARIOUS SOURCES.

TARGET INITIAL CONDITIONS: \( +180, \phi = 0^\circ \)

\( X = 20000 \).
FIG. 4.18.3 DYNAMIC RANGING SIMULATION WITH TARGET INITIAL CONDITIONS. $\theta = 180$, $\phi = 30$, $XT = 20000$, $YT = 1000$. ERRORS IN RANGE RATE ESTIMATES FROM VARIOUS SOURCES.
Fig. 4.9.1 Dynamic ranging simulation with target initial conditions. \( \gamma = 0 \), \( \Phi = -80 \), \( XT = 10000 \), \( YT = 1000 \). Plots of variables defining geometry.
This figure has target heading $0^\circ$, and bank angle $-80^\circ$.
Thus pursuit starts as a tail chase, and ends with a rear-side attack. Target initial range is slightly more than 10,000 ft.
The estimate errors generally follow the same pattern established in previous trajectories, with the largest error sources being bias (d) and target airspeed change (j).

a. no error sources  
f. $0.1 \, \text{g}$ bias in $a_M$
b. $1\%$ error in $\hat{r}$ (0)  
g. $0.1^\circ$ bias in seeker gimbal angle
c. $1\%$ error in $\dot{\hat{r}}$ (0)  
h. $20 \, \text{ft/s}$ bias in $\hat{V}_M$
d. $0.5^\circ$ bias in $\hat{\dot{r}}$  
i. $1^\circ$ error in $\hat{\alpha}$
e. $0.1 \, \text{g}$ bias in $a_{M_g}$  
j. $\hat{V}_T = 1 \, \text{g}$
FIG. 4.9.2 DYNAMIC RANGING SIMULATION WITH TARGET INITIAL CONDITIONS, $\phi = 0, \phi = -80, XT = 10000$
$YT = 1000$. ERRORS IN RANGE ESTIMATE FROM VARIOUS SOURCES
FIG. 4. O'D. DYNAMIC RANGING SIMULATION WITH TARGET INITIAL CONDITIONS. $\tau = 0$, $\phi = 90$, $X_t = 10000$, $Y_t = 1000$. ERRORS IN RANGE RATE ESTIMATE FROM VARIOUS SOURCES.
FIG. 4.1 UNTYPICAL SIMULATION WITH
INITIAL CONDITIONS: + - 80, Y: -80, X: 10000.

TARGET PLANS OF VARIABLES DEFINING GEOMETRY.
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FIGURE 4.10

This trajectory displays a head-on trajectory with initial range around 10,000 ft. The target is executing a hard right turn. The estimate errors are somewhat related to the errors portrayed in Figure 4.7, where the turn was in the opposite direction.

a. no error sources
b. 1% error in $r(0)$
c. 1% error in $\dot{r}(0)$
d. $0.5^\circ$ bias in $\dot{\alpha}$
e. $0.1 g$ bias in $a_{h_d}$
f. $0.1 g$ bias in $a_{h_p}$
g. $0.1^\circ$ bias in seeker gimbal angle
h. 20 ft/s bias in $\hat{V}_M$
i. $1^\circ$ error in $\hat{\alpha}$
j. $\dot{V}_T = 1 g$
FIG. 4. 102 DYNAMIC RANGING SIMULATION WITH
INTERCOURSE: (a) = 180°, (b) = 0°. ERRORS IN RANGING ESTIMATE FROM VARIOUS
DEFINITION SOURCES.

ER

-210.00 -140.00 -70.00 0.00 70.00

-130.00 -65.00 0.00 65.00 130.00

TIME (SEC)

4.00 3.00 2.00 1.00 0.00

1.00

ERDOT

-150.00 -100.00 -50.00 0.00 50.00 -60.00 -40.00 -20.00 0.00 20.00

TARGET:

Y1 = 1000
Fig. 4.10.3 Dynamic Ranging Simulation with Target Initial Conditions. \( \phi = 180 \), \( \psi = 80 \), \( X = 10000 \), \( Y = 1000 \). Errors in range rate estimate from various sources.
slight maneuver established after result zero, have anticipated that p is even

a. 

b. 

c. 

d. 

e. 

FIG. 4.11.1 DYNAMIC RANGING SIMULATION WITH
TARGET INITIAL CONDITIONS, \( \phi = 90^\circ, \dot{\phi} = 0 \), \( XT = 20000 \),
\( YI = 1000 \). PLOTS OF VARIABLES DEFINING GEOMETRY.
FIGURE 4.11

Figure 4.11 portrays a side attack. Initial range is slightly more than 20,000 ft., and the target is not maneuvering. As is evident, the guidance system has established an intercept course at around 3 seconds, and after this point, missile maneuvering is negligible. This results in the signals \( \dot{r} \) and consequently \( \ddot{\gamma} \) approaching zero, and thus the main signals being used by the algorithm have little significance. Thus after 3 seconds it can be anticipated that if the algorithm is not set up properly at that point, then serious errors can and will result. This is evident in the curves to the right.

a. no errc. sources  
b. 1\% error in \( \dot{r} (0) \)  
c. 1\% error in \( \ddot{r} (0) \)  
d. 0.5\(^\circ\) bias in \( \dot{\gamma} \)  
e. 0.1 g bias in \( a_{M_r} \)  
f. 0.1 g bias in \( a_{M_r} \)  
g. 0.1\(^\circ\) in seeker gimbal angle  
h. 20 ft/s bias in \( \dot{V}_T \)  
i. 1\(^\circ\) error in \( \dot{\alpha} \)  
j. \( \dot{V}_T = 1 \) g
Fig. 4.11.2 Dynamic Range Simulation with Target Initial Conditions. \( t = 90, \theta = 0 \), \( XT = 20000 \), \( YT = 1000 \). Errors in range estimate from various sources.
FIG. 4.11.3 DYNAMIC RANGING SIMULATION WITH TARGET INITIAL CONDITIONS, $t = 90$, $\phi = 0$, $XT = 20000$, $YT = 1000$. ERRORS IN RANGE RATE ESTIMATE FROM VARIOUS SOURCES.
FIG. 4.121 DYNAMIC RANGING SIMULATION WITH TARGET INITIAL CONDITIONS. $\phi = 0, \gamma = 80 \Rightarrow 80$, $xt = 10000$, $yt = 1000$. PLOTS OF VARIABLES DEFINING GEOMETRY DUE TO CHANGE IN TARGET MANEUVER.
FIGURE 4.12

This figure displays a situation where the target performs a rapid change in acceleration by changing its turn from left to right. This trajectory is identical to that of Fig. 4.6 for the first three seconds; at t = 3, the target bank angle was changed from 80° left turn to 80° right turn, requiring one second to do so; the bank angle is 80° right turn for t>4. The algorithm, which estimates range must therefore re-adapt to accommodate the change in target turn rate. The bank angle was changed in a linear fashion; however as acceleration is related to \( \tan \phi \), the acceleration of the target changed in a nonlinear fashion. This is evident on inspection of the \( \dot{\mathbf{r}} \) trace, on the left, at \( 3 \leq t \leq 4 \) seconds. As the acceleration changed in a way which violates the estimator's assumptions, an error is formed while the maneuver change occurs. But there are many ways in which an airplane can change its maneuver, and any one is as likely as another. The estimator properties lie in the mid-range of the likely types of maneuver change methods. The curves on the facing page are the errors in the range and range-rate estimates with range error on the upper graph, and range-rate error on the lower. No additional curves representing sources of error were made.

The trace at the lower right shows the error of range-rate estimate. Notice that this error jumps from 4 ft/sec to 6 ft/sec at the time of impact at the end of the run. This change is due to imperfect tracking of the term \( \ddot{r}^2 \) since \( \dot{r} \) is extremely large at impact while \( \dot{r} \) is not exactly equal to \( r \); in this connection see the discussion of the physical meaning of the term \( \ddot{r}^2 \) on page 13. Notice also that the error of tracking the changing maneuver during \( 3 \leq t \leq 4 \) seconds causes the range-rate estimate error to increase from 1 ft./sec at \( t = 3 \) to 4 ft./sec at \( t = 4 \); the increase of error of 3 ft./sec is considered to be minimal and demonstrates the surprising power of this nonlinear estimator.
FIG. 4. Dynamic ranging simulation with target initial conditions. \( \phi = 0 \), \( \psi = 80 \), \( \beta = 80 \), \( x_t = 10000 \), \( y_t = 1000 \). Errors in range and range rate estimate due to change in target maneuver.
LINEAR ESTIMATORS

Linear estimators may, for the purpose of this discussion, be divided into two classes, depending on whether a gradient technique is used to achieve convergence. We term "adaptive" those methods which use a gradient technique in some form. A matrix-inverse problem is always implicit in the formation of an "adaptive" or gradient-seeking solution. In the "matrix-inverse" technique, that inverse problem appears explicitly and must be treated explicitly.

The adaptive methods lead to simpler instrumentations than the matrix-inverse methods and were therefore considered first. The adaptive methods failed, whether with simple single-error nets, or with multiple-error nets or Kalman filters, principally due to target acceleration effects. The mode of failure was that the estimated range-rate was as much as 30% to 50% in error at the end of the major maneuvers of the missile. The study then turned to matrix-inverse methods; these failed to converge for some encounter geometries and converged to erroneous solutions for other geometries. The modes of failure were large errors of range-rate estimate or a problem of ill-conditioning which precluded convergence. We therefore tentatively concluded that fundamental questions of mathematical observability were involved and then verified that hypothesis.
This section presents a brief outline of the adaptive methods, and of the matrix-inverse methods, with comments on the physical reasons for their deficiencies, followed by a discussion on mathematical observability, as applied to this situation. The section concludes with some comments on the possibilities available by combining the kinematic ranging linear estimator with other linear devices, in particular with a stadimetric ranging method.

5.1 Adaptive Estimators

Adaptive identifiers, as a class, share the use of the gradient of the index of performance as the guide to improving the parameter estimates from one instant to the next. The two principal well-known methods of forming parameter identifying nets are known as the response-error and equation error methods. The response-error method is known to be relatively slow, and relatively insensitive to noise, as compared to the equation-error method. The air-air intercept problem clearly requires rapid solution; efforts were therefore concentrated on the equation-error method.

A brief analytical outline of the method is followed by discussion of the representatives of the various adaptive systems.

Assume that the output of the plant whose parameters are to be identified is designated y, and that a set of signals v,
consisting of various functions of the plant's input and output, is available. If the true values of the unknown parameters are represented by the vector $x^*$ then the plant output takes the form

$$ y = v^T x^*. $$

(5.1)

Given the same set of signals, $v$, and a vector of estimates of the parameter values, $x$, we may form an estimate of the plant output

$$ \hat{y} = v^T x. $$

(5.2)

Then we may form an error as

$$ y - \hat{y} = y - v^T x = e, $$

(5.3)

and the error will become identically zero if and only if $x = x^*$. The common and simplest gradient algorithm for adjusting the parameter estimates, $x$, minimizing $J = e^2$, is

$$ \dot{x} = +k ve $$

(5.4)

which on substituting $e$ from (5.3) yields

$$ (I + k vv^T) x = k vy. $$

The gain $k$ must be positive definite; it may be a scalar or a matrix. If the value of $k$ is chosen small enough to ensure averaging, and if the parameter $x^*$ is constant, then the steady state is reached when

$$ vv^T x = vy $$

with solution

$$ x = (vv^T)^{-1} vy. $$

(5.5)
The covariance matrix \( v v^T \) is implicit in all adaptive techniques as they use the algorithm (5.4). The convergence properties and noise sensitivity of the solution are determined by the properties of that matrix.

Matrix-inverse methods, to be discussed in Section 5.2, find the solution vector \( x \) directly from (5.5). The covariance matrix \( v v^T \) is obviously involved explicitly.

If the parameters \( x \) are known a priori to be variable or to be related, the known properties may advantageously be used in a modified algorithm. Assume that it is known that \( \dot{x}_1^* = x_1^* \) and \( \dot{x}_2^* = bt \), or in general, \( \dot{x}^* = Ax^* + B(t) \), then an algorithm equivalent to (5.4) is

\[
\dot{x} = Ax + B(t) + k v e,
\]

and the term \( k v e \) is required only to correct for the unknown errors in the parameter estimates.

Our initial approach was to use the identity

\[
\hat{r} \hat{\sigma} + \hat{\kappa} \sigma = \hat{y}_T - \hat{y}_M
\]  

(5.7)

as a starting point for an adaptive algorithm. This equation results from the inertial space geometry.

\[
\begin{array}{c}
\text{M} \\
\text{T} \\
\end{array}
\]
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which shows
\[ r \sin \sigma = y_T - y_M \]

For small \( \sigma \), this equation has the approximate form
\[ r \sigma = y_T - y_M \]

with first derivative (5.7). Taking current estimates for range and range rate, which in general are different from the actual values, the equation error is
\[ x_1 v_1 + x_2 v_2 + x_3 v_3 + \dot{y}_M = e_1 \]

(5.8)

where \( x_1 = \hat{r}_1, x_2 = \dot{r}_1, x_3 = \ddot{y}_T, v_1 = \dot{\gamma}, v_2 = \sigma, v_3 = -1 \) and \( \dot{x}_1 = x_2 \),

or \( \dot{x} = AX, A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \),

where \( \dot{\gamma}, \sigma \) and \( \ddot{y}_M \) are known. The index of error is defined as
\[ J = \mathbf{e}^T \mathbf{e}_1. \]

The algorithm for updating the estimate \( \mathbf{x} = (\hat{r}, \dot{\hat{r}}, \ddot{y}_T)^T \) is
\[ \dot{x} = Ax - k v e_1, \]

which is then readily implemented for a suitable choice of \( k > 0 \). Using this method assumes that the variables \( \hat{r} \) and \( \ddot{y}_T \) are constant. But in an actual dogfight situation, both target and missile are accelerating, and the assumption is too optimistic. This approach therefore failed and was abandoned.
A second approach, consistent with the mathematical model of Section 3, was taken. In this development, the frame of reference is polar coordinates centered at the missile, with the instantaneous radius vector parallel to the line of sight. The error equation which holds in this case is

\[ r \ddot{\varphi} + 2 r \dot{\varphi} \dot{\varphi} - \dot{a}_T + a_M = e_2, \]

and the error index is then chosen as \( J = \frac{e_2^T}{e_2} \).

Again, let \( x_1 = \dot{r}, x_2 = \dot{\varphi}, x_3 = a_T \) and let \( v_1 = \dot{\varphi}, v_2 = \ddot{\varphi}, v_3 = -1 \). Then that error equation can be put into the form

\[ v^T x + a_M = e_2. \] (5.9)

The complete statement of the components of acceleration is from, Section 3,

Accelerations
perpendicular \( r \ddot{\varphi} + 2 r \dot{\varphi} \dot{\varphi} - \dot{a}_T + a_M = 0 \)
to the LOS

\[ \text{Accelerations} \]
parallel \( \ddot{r} - r \dot{\varphi}^2 - a_T + a_M = 0. \)
to the LOS

\[ \text{(5.10)} \]

\[ \text{(5.11)} \]

The first of these may be used to form the "observation equation", or, in adaptive terminology, the equation error.
But the second expression provides a statement of the value of \( \hat{r} \) which has previously been lacking. Let us define \( x_1 = \hat{r} \), \( x_2 = \hat{r} \), \( x_3 = \dot{a}_T \), and \( x_4 = \ddot{a}_T \). We can directly sense the LOS angular rate and acceleration and the unit gains \( v_3 = -1 \), \( v_4 = +1 \), and also the missile accelerations \( a_M \) and \( a_M \). Then the equation error to be minimized is again in the form

\[
v^T x + a_M = e_3
\]

and the algorithm which minimizes \( J = \frac{e_3^2}{3} \), is

\[
\dot{x} = Ax + B(t) - kv e_3
\]

where

\[
A = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
B = \begin{pmatrix}
0 \\
-a_M \\
0 \\
0
\end{pmatrix}
\]

The matrix \( A \) expresses the differential relationship between range and range-rate on the first line; it expresses the definition of \( \hat{r} \) from (5.11) on the second line, and on the third and last lines expresses the assumption that the target accelerations with respect to the LOS are quasi-constant.

The term \( r_0^2 \) in (5.11) can easily be as large as 100 ft/sec\(^2\), e.g., when in the initial geometry the target at a range of 10,000 feet has 1,000 ft/sec velocity and is flying perpendicular to the LOS. This is not negligible.
The kinematic ranging estimation method expressed by equations (5.13) and (5.14) was examined for a variety of geometrical conditions.

The simple adaptive schemes following (5.13) and (5.14), with the adaption-rate parameter $k$ as a scalar constant, were quite unsatisfactory as there was no single constant value of $k$ which worked well for all combat geometries or throughout any single trajectory. The difficulty is that a large value of $k$ is required to find the correct values of $\hat{r}$ and $\hat{p}$ at the start of the trajectory, while a low value is required to prevent undue sensitivity to noise in later parts of the trajectory. Further, the rate parameter $k$ should ideally be proportional to $\left(\frac{1}{\sigma}\right)$ so as to accommodate both to large and small maneuver geometries. The gain $k$ must thus vary with time and also with geometry to achieve satisfactory results.

It was therefore necessary to replace the scalar adaption-rate parameter $k$ by a Kalman Filter type matrix, $K$, where the values of $K$ are time-varying, according to the well-known rule

$$\dot{K} = -K v N^{-1} v^T K$$

(5.15)

where $N$ is the noise covariance. Even so, while this approach would generate good estimates of $\hat{r}$ and $\hat{p}$ for
some trajectories, it failed when the trajectory was such that \( a_{Tr} \neq 0 \).

In the geometrical conditions in which this Kalman-filter type algorithm works well, i.e., when the target acceleration along the LOS, \( a_{Tr} \), is negligible, a significant simplification is possible. Consider (5.11) and "transform-differentiate-integrate"6 by premultiplying by

\[
\frac{T_i s}{T_i s + 1}, \quad i = 1, 2, \ldots, T_i \neq T_j, \text{ yielding, if } r \approx 0,
\]

\[
\frac{T_i s}{T_i s + 1} (r \dot{a} + 2 \ddot{a} - a_{T_0} + a_{M_0}) = r \left( \frac{T_i s}{T_i s + 1} \ddot{a} \right) + \dot{r} \left( \frac{2 T_i s}{T_i s + 1} \ddot{a} + \frac{T_i}{(T_i s + 1)^2} \right) \ddot{a} - \left( \frac{T_i s}{T_i s + 1} \right) a_{T_0} + \left( \frac{T_i s}{T_i s + 1} \right) a_{M_0}.
\]

For these geometries, it is a reasonably good assumption that \( a_{T_0} \) is quasi-constant. It therefore follows that

\[
\frac{T_i s}{T_i s + 1} a_{T_0} \approx 0 \text{ is valid. The result is that the estimating variable } x_3 = \hat{a}_{T_0} \text{ may be omitted, and there is one less unknown parameter. Equation (5.16) may be put into the equation error form}
\]

\[
v_i T x + y_i = e_{i1},
\]
where \( x = (\hat{r}, \dot{\hat{r}})^T \),

\[
v_1 = \left\{ \left( \frac{T_1 s}{T_1 s + 1} \right) \dot{\sigma}, \left( \frac{2T_1 s}{T_1 s + 1} \right) \dot{\sigma}, \left( \frac{2T_1 s}{(T_1 s + 1)^2} \right) \dot{\sigma} \right\}^T
\]

and \( y_1 = \frac{T_1 s}{T_1 s + 1} a_m \). \hspace{1cm} (5.17)

If this transform-integration, (5.16), is instrumented for several different values of \( T_i \), a vector of errors \( (e_{i1}, e_{i2}, \ldots) \) may be generated. This is one of the several multiple-error techniques. This technique did not work in an adaptive approach although it was better than single-error techniques. The solution usually involved an ill-conditioned covariance matrix and was therefore noise-sensitive.

When averaging over a sufficiently large interval was used, the unknown acceleration \( a_r \) caused \( \hat{r} \) to be as much as 30% to 50% in error in 1 to 2 seconds.

The difficulty is that if the filter time-constants \( T_i \) are small, then the several vectors \( v_i \) are almost parallel resulting in an ill-conditioning problem and a poorly defined solution. On the other hand, if \( T_i \) are large enough to avoid ill-conditioning then the invalid assumption that

\( \ddot{r} \approx \ddot{a}_r = 0 \) caused excessive error in the range-rate estimate.
The difficulty common to all the methods is that whenever the averaging time must exceed 1/4 to 1/2 second, to enable the vectors which carry the information to be linearly independent, then it is essential to take into account the target acceleration along the LOS. On the other hand an averaging time of 1/2 second or less is not sufficient as the LOS does not rotate sufficiently in that interval to permit determining a solution in the presence of noise, and also to correct the initial estimates. For these reasons, the various adaptive gradient-seeking methods of parameter estimation were abandoned.

5.2 Matrix-Inverse Methods

Several schemes were derived from the classical point of view of least mean square estimation. The general theory of this method can be explained quite briefly, as follows. Consider the linear homogeneous vector differential system

$$\dot{x} = A(t)x.$$  \hspace{1cm} (5.18)

Assume the observed output of the system is the vector

$$\mathbf{y} = H(t)x + \mathbf{v}$$ \hspace{1cm} (5.19)

where $H(t)$ is known and $\mathbf{v}$ represents additive random noise. It is desired to estimate $\mathbf{x}$ knowing only $A(t)$, $\mathbf{y}(t)$ and the statistics of $\mathbf{v}$. 

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Knowing \( A(t) \), generate the state transition matrix, \( \Phi \), for the system so that the state at any time can be written as a function of the state at another time. Thus,

\[
\mathbf{x}(t+\tau) = \Phi(t+\tau,t) \mathbf{x}(t).
\]  

(5.20)

Many estimates can be constructed. The one which is usually considered the most desirable, however, has the property that it is an "unbiased" estimate of \( \mathbf{x} \) and furthermore, it minimizes the variance of the expected error in the estimate. This estimate is found by computing

\[
\hat{\mathbf{x}} = (T_n^T T_n)^{-1} T_n T_n^T y(n)
\]

(5.21)

where

\[
Y(n) = \begin{bmatrix}
y(t_n) \\
y(t_{n-1}) \\
\vdots \\
y(t_1)
\end{bmatrix}
\]

(5.22)

\[
T_n \triangleq \begin{bmatrix}
H(t_{n-L}) \Phi(t_{n-L}, t_n) \\
\vdots \\
H(t_{n-1}) \Phi(t_{n-1}, t_n) \\
H(t_n)
\end{bmatrix}
\]

(5.23)
The matrix \( T_n \) is thus a "continuously updated observation matrix" and the vector \( Y(n) \) is a "series of output observations". The expression (5.21) is valid provided the noise \( v \) is independent from computing instant to instant and has zero mean. The construction of the matrix \( T_n \) is often a formidable task as it requires keeping in storage a large amount of past data \( H(t_i) \) so that it can be updated by the state transition matrix \( \Phi(t_i, t_j) \) which is at times also difficult to obtain.

Apply this theory to, typically, the reduced dimensionality problem defined by (5.16) and (5.17); assume that \( \dot{r} \) is constant, thus \( x_2 \) is constant after convergence. But \( x_1 = r \) and is not constant. Now we have, at some specific instant, and with only one value of \( i = 1 \),

\[
v_1(t_1)x_1 + v_2(t_1)x_2 + y_1 = e_5. \tag{5.24}
\]

At the next instant we have

\[
v_1(t_2)x_1 + v_2(t_2)x_2 + y_2 = e_5. \tag{5.25}
\]

But while the value of \( x_2 \) is ideally the same at these two instants, the value of \( x_1 \) changes continuously, for \( x_1 = r \), and \( r \) is the integral of \( \dot{r} \). We may modify (5.24) so that \( x_1 \) is the same for the two instants. If \( \Delta t \)
is the time between samples, we may rewrite (5.24) as
\[ v_1(t_1) \left( x_1(t_2) + x_2 \Delta t \right) + v_2(t_1) x_2 + y_1 = e_5, \]
which may be rearranged as
\[ v_1(x_1) x_1(t_2) + (v_2(t_1) + v_1(t_1) \Delta t) x_2 + y_1 = e_5. \]

Now Eqns. (5.25) and (5.26) represent two measurements referred to the same instant, and may be solved jointly. More generally, a larger sequence of such instants may be gathered in the more general form of (5.23).

This method exhibited several modes of failure. Whenever any two of:

the missile velocity vector
the target velocity vector
the LOS

become colinear, or \( \dot{\phi} = 0 \), the matrix \( (T_n T_n) \) became ill-conditioned and indeterminate, yielding inaccurate and noise-sensitive solutions. In geometries when this mode of failure did not occur, this algorithm has a different mode of failure. If the duration of the stored memory exceeds 1/2 second, the neglected target acceleration parallel to the LOS, \( a_{T_r} \), causes the estimate of \( \dot{r} \) to be 30% to 50% in error within 1 to 2 seconds, as the assumption that \( \dot{r} = 0 \) is too incorrect. On the other hand, a shorter duration memory results in high noise sensitivity. But these algorithms work
well (Range-Rate errors of 1%) if $a_T = 0$.

In any case the computational complexity and memory storage requirements of this algorithm are excessive. An alternate was therefore examined. If

$$r = r_0 + \int \hat{r} \, dt$$

(5.27)

and $x_1 = \hat{r}_0$, $x_2 = \hat{r}$, etc., then a much simpler algorithm results. At a number of successive instants we have

$$\begin{pmatrix} v_1(1) & v_2(1) \\ v_1(2) & v_2(2) + v_1 \Delta t \\ \vdots & \vdots \\ \vdots & \vdots \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

(5.28)

and both $x_1$ and $x_2$ may be assumed constant. If the nth row of the left of (5.28) is

$$v_1(n) v_2(n) + \Delta t v_1(n-1)$$

then the matrix $(T_n^T T_n)$ may be recursively formed as

$$(T_n^T T_n) = (T_{n-1}^T T_{n-1}) + v_n v_n^T$$

(5.29)
Although sufficiently simple to be feasible for an air-air missile, this approach to range estimation also failed for the same reasons:

(a) Ill-conditioned matrix for some geometries,

(b) High noise-sensitivity for short averaging times,

(c) Large errors in range-rate estimates for longer averaging times due to the invalid assumption that $a_T = 0$.

When failure-mode (a) did not occur, and $a_T = 0$ was valid, the algorithm yields estimate errors of the order of 1%.

No attempt at estimating $a_T$, target acceleration parallel to the LOS, by linear methods was successful. Any algorithm which neglected $a_T \neq 0$ when it was significant operated unsuccessfully; either excessive noise sensitivity became a problem, or the neglected acceleration caused excessive errors in the estimation of Range-Rate, $\dot{r}$.

In summary, except for encounter geometries which yield ill-conditioned matrices, the matrix-inverse algorithms work well with long-filtering times which eliminate noise effects if the assumption that $a_T = 0$ is valid. If that
assumption is invalid, the matrix-inverse schemes fail due to the bias in range-rate estimate caused by target acceleration, or due to noise.

5.3 Observability

The failure of these various adaptive and matrix-inverse parameter identification methods, whether continuous or discrete, is fundamentally due to the existence of target acceleration in the radial direction, parallel to the line of flight. That acceleration component cannot be estimated by linear techniques because it is "unobservable" in the mathematical sense. This section relates the concept of "observability" to this application and problem.

The state of a linear system with constant coefficients in the dynamics and constant coefficients in the output relation can be estimated from knowledge of the output provided the system is mathematically observable. For this discussion, we view the quantities of range, range-rate, etc., as states which are to be determined and consider the signals of LOS-rate and missile acceleration as time-varying parameters. In state-variable notation a plant may be described as

\[
\begin{align*}
\text{Plant Dynamics:} & \quad \dot{x} = Ax + Bu \\
\text{Output:} & \quad y = Hx + v
\end{align*}
\]

where \( v \) is zero-mean Gaussian white noise, and \( x \) is an \( N \)-component vector.

(5.30)
The observability of the system is specified by a relationship between the A matrix, specifying the dynamics, and the H matrix, relating the state to the output. If the order of the system is \( N \), then the matrix

\[
Q_N = \begin{bmatrix} H^T & A^T H^T & (A^T)^2 H^T & \ldots & (A^T)^{N-1} H^T \end{bmatrix}
\]

must have full rank for the system to be observable. Let us now apply this to one of the possible formulations of the problem. For the system definition, use the acceleration-component equations from Section 2

accelerations parallel to the LOS

\[
\ddot{r} - r \dot{\sigma}^2 - a_{Tr} + a_{M_T} = 0
\]

(5.32)

Perpendicular to the LOS

\[
\ddot{r} \dot{\sigma} + 2 \dot{r} \dot{\sigma} - a_{T\sigma} + a_{M_{\sigma}} = 0.
\]

(5.33)

In order to cast this into the standard form of (5.30), define state variables

\[
\begin{align*}
    x_1 &= \dot{r} \\
    x_2 &= \ddot{r} \\
    x_3 &= a_{T_T} \\
    x_4 &= a_{T_{\sigma}}
\end{align*}
\]

(5.34)
i.e. the variables $x$ are the estimates of the unknown parameters. The last two lines of (5.34) express the fact that the two components of target acceleration must be estimated and are assumed to be unknown constants.

The state variable expression of the relationships among the several variables may be found from (5.32) and (5.34) as

$$
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4
\end{pmatrix} =
\begin{pmatrix}
0 & 1 & 0 & 0 \\
\sigma^2 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{pmatrix} +
\begin{pmatrix}
0 \\
-a_{mr} \\
0 \\
0
\end{pmatrix}
$$

(5.35)

The first row of (5.35) expresses the differential equation relating range and range-rate estimates. The second row consists of (5.34) while the third and fourth rows express the assumption that the target's unknown accelerations $a_{T_r}$ and $a_{T\sigma}$ are constant. From (5.35), the matrix $A$ is

$$
A_n =
\begin{pmatrix}
0 & 1 & 0 & 0 \\
\sigma^2 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
$$

(5.36)

and $N=4$. 

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Equation (5.33) yields the output relationship

\[
\begin{pmatrix}
-\dot{\theta} \\
-2\ddot{\theta} \\
0 \\
1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{pmatrix}
= \hat{a}_{M \sigma};
\]

(5.37)

comparison of the estimate, \( \hat{a}_{M \sigma} \), with the measured acceleration \( a_{M \sigma} \) yields an error whose minimization will presumably yield optimal estimates of the range, \( x_1 \), and range-rate, \( x_2 \). From (5.37),

\[
H = \begin{pmatrix}
-\dot{\theta} \\
-2\ddot{\theta} \\
0 \\
1
\end{pmatrix}.
\]

(5.38)

Combining (5.36) with (5.38) in the format of (5.35) yields the observability matrix

\[
0_4 = \begin{pmatrix}
-\dot{\theta} & -2\ddot{\theta} & -\dot{\theta} & -2\ddot{\theta} \\
2\ddot{\theta} & -\dot{\theta} & -2\ddot{\theta} & -\dot{\theta} & -\dddot{\theta} \\
0 & -2\ddot{\theta} & -\dot{\theta} & -2\dddot{\theta} \\
1 & 0 & 0 & 0
\end{pmatrix}.
\]

(5.39)

The fourth column of (5.39) is exactly equal to \( \dot{\theta}^2 \) times the second column, and the rank is less than the order. This system is unobservable; experiment confirmed this result.
In realistic combat trajectories it is generally erroneous to assume that \( a_{Tr} \), target acceleration in the range-direction, is zero. If, however, that assumption is valid for some situation, then we can simplify (5.35) by omitting the third row and column and similarly simplify \( H \), so that if \( a_{Tr} \equiv 0 \), a priori, then

\[
\begin{align*}
A_{31} &= \begin{pmatrix} 0 & 1 & 0 \\ \dot{\sigma} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
H_{31} &= \begin{pmatrix} -\dot{\sigma} & -2\dot{\sigma} & 0 \end{pmatrix}.
\end{align*}
\]

and

\[
\begin{align*}
0_{31} &= \begin{pmatrix} -\ddot{\sigma} & -2\ddot{\sigma} & -\dot{\sigma}^2 \\ -2\dot{\sigma} & -\dot{\sigma} & -2\dot{\sigma}^3 \\ 1 & 0 & 0 \end{pmatrix}.
\end{align*}
\]

In this case rank equals order and the system is observable if \( \dot{\sigma} \neq 0 \) and \( \ddot{\sigma} \neq 0 \). This result was experimentally confirmed.

Returning to (5.35), assume that the target acceleration perpendicular to the LOS, \( a_{T\sigma} \), is identically zero, i.e., \( x_4 = a_{T\sigma} \equiv 0 \), a priori, but the other component, \( x_3 = a_{Tr} \), is a non-zero constant. Then, making the appropriate changes in \( A \) and \( H \), we have

\[
\begin{align*}
A_{32} &= \begin{pmatrix} 0 & 1 & 0 \\ \dot{\sigma} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
H_{32} &= \begin{pmatrix} -\dot{\sigma} & -2\dot{\sigma} \end{pmatrix}.
\end{align*}
\]
In this case the third column equals $\dot{\sigma}^2$ times the first; therefore the system is unobservable.

The result may now be stated: If the target has significant acceleration parallel to the line of sight, $(a_T \neq 0)$, it is not possible to estimate range and range-rate; the system is unobservable. If the target acceleration $a_T$ is known to be zero, a priori; then the system is observable, and it may be reduced to a problem in the three states, $x_1 = \dot{r}$, $x_2 = \ddot{r}$ and $x_4 = \dot{a}_T$. In this case, Equation (5.32) can be exactly instrumented. If $x_4 = \dot{a}_T = \text{constant}$ is a reasonable assumption, it is possible to reduce the problem from three states to two by differentiating the output expression (5.37) and eliminating $x_4 = \dot{a}_T$, as in (5.16).

It is, however, an important restriction that the theorems on observability do not apply perfectly to systems whose coefficients are time-varying. In our case the system dynamics are constant, while the observation matrix is not. Intuitively, it seemed possible that if
the coefficients of the output equation vary sufficiently in the observation interval, then a result might be obtained which is of value. Many variations of the above development, using different models of the system dynamics and output equations were therefore examined to pursue this question. The results were disappointing whenever $a_T \neq 0$, or $\dot{\theta}$ was small. The system is unobservable, despite the uncertainty of the theory on this question, whenever $a_T \neq 0$. 
5.4 Other Linear Systems

It is advantageous to unify the results of the above discussions on kinematic linear parameter identification systems in order to form a basis for considering other possible linear systems. We therefore re-present the problem from a geometrical viewpoint and then consider possible extensions beyond the scope of this contract.

Consider the equation of acceleration components perpendicular to the LOS, from the kinematics,

\[ \ddot{r} + 2r \dot{\omega} - a_{T_0} + a_{m_0} = 0. \]  

(5.41)

Assume that the range acceleration, \( \ddot{r} \), is zero, and may therefore be neglected. Assume further that \( a_{T_0} \) is quasi-constant, i.e., cannot change significantly in less than 1/2 second. Then differentiation of (5.41) with a high pass filter such as \( \frac{T_1}{T_1 s + 1} \), where \( T_1 \) is of the order of 1/2 second or less, reduces (5.41) to

\[ r \ddot{v}_1 + \dot{r} \ddot{v}_2 + v_0 = 0. \]  

(5.42)

where \( \ddot{v}_1 = \left( \frac{T_1 s}{T_1 s + 1} \right) \ddot{\omega}; \) \( \ddot{v}_2 = \left( \frac{2T_1 \ddot{\omega}}{T_1 s + 1} + \frac{T_1 \dot{\omega}}{(T_1 s + 1)^2} \right); \) \( v_0 = \left( \frac{T_1 s}{T_1 s + 1} \right) a_{m_0}. \)

If in (5.42) we replace \( r \) and \( \dot{r} \) by the estimates \( \hat{r} \) and \( \hat{\dot{r}} \), (5.42) may be expressed as \( \hat{r} \ddot{v}_1 + \hat{\dot{r}} \ddot{v}_2 + v_0 = e_6; \) then the solution for \( \hat{r} \) and \( \hat{\dot{r}} \) lies on a locus such that \( e_6 = 0. \) In this
case (5.43) forms a straight line, as shown in the sketch below for some particular instant.

![Diagram showing geometric view of equation error]

**Fig. 5.1**  
Geometrical View of Equation Error.

The entire problem of estimation is based on the necessity of finding another line in the \((\hat{r}, \dot{r})\) space:

(a) The multiple-error adaptive process, by using a variety of values of \(T_1\) generates a number of other lines which are, more or less, linearly independent. This approach may be categorized as achieving distribution in the \(T_1\)-lag space at one instant of true time.

(b) The various single-error adaptive nets and the Kalman net take advantage of the fact that, at some other instant than that shown in Fig. 5.1, the line of zero error in the \((\hat{r}, \dot{r})\) space is
differently oriented, if $\dot{c} \neq 0$, and is therefore linearly independent. These approaches thus use the idea of distribution in true time, $t$, rather than in the $T_1$ space.

(c) The matrix inverse schemes gather all lines over the computing interval, thus achieving distribution in time, and then find the solution.

A physically different approach to finding a second linearly independent line in the $(\hat{r}, \hat{\theta})$ space is obviously preferable if possible. Stadimetric ranging provides such an approach. Seekers which enable measurement of some function of the target size, or area, such as mosaic IR, TV, area, or correlation, have this capability. In a single plane we have the geometrical relationship

$$r\theta = D$$

(5.44)

where $r$ Range

$\theta$ Angle Subtended by Target, assumed a small angle

$D$ Target Size

as shown in the sketch below.

![Diagram](image-url)
Assume that $D$ is constant; this assumption is fair if $D$ is the diameter of the least circle which contains the target as viewed from the seeker in both the maneuver and cross maneuver planes. From (5.44), by differentiating,

$$r \dot{\theta} + \dot{r} \theta = 0$$

and

$$\ddot{r} \theta + \dot{r} \dot{\theta} = e_\gamma.$$  \hfill (5.45)

The choices of $\hat{r}$ and $\hat{\theta}$ for which $e_\gamma = 0$ lie on a straight line which passes through the origin of the $(\hat{r}, \hat{\theta})$ space at every instant. Combining this with the kinematic ranging sketch, Fig. 5.1, yields the sketch below.

Fig. 5.2
Combining Kinematic and Stadimetric Ranging.
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If we solve (5.44) and (5.45 simultaneously, we get one equation for \( \hat{r} \) and one for \( \hat{\dot{r}} \). These are independent and their joint solution may therefore be determined. It is usually better to solve the two equations in a decoupled format as the solution can always be accomplished more rapidly, more simply and more accurately. A Least Magnitude algorithm may be used. They are analytically valid in such cases and are attractive for they estimate a moving parameter with smaller lag than quadratic algorithms, and thereby decrease the effect of target accelerations.

However, this argument rests on the rather uncertain base of the assumptions. One assumption is that the target cross-track acceleration \( a_{r_c} \) is quasi-constant, i.e., changes negligibly within two time-constants of the high-pass filter \( \frac{T_i^8}{T_i^8 + 1} \), in (5.42). The other assumption is that \( \hat{r} \approx 0 \), or at least the unknown portion \( a_{r_T} \) thereof. In fact, in any maneuver of the target, \( a_{r_c} \) and \( a_{r_T} \) change continuously and can change rapidly. This scheme is therefore less powerful than appears at first.

However, stadimetric data can be valuable in reducing the difficulty of range estimation. Equation (5.41) states one relationship between three of the unknowns in the kinematic ranging problem. Stadimetric data provides another relationship, so that the effective total number of unknowns may be reduced by one. The possibility therefore exists that linear estimation of range and range rate could be accomplished with acceptably small error, due to the unobservability of \( a_{r_T} \), by combining kinematic and stadimetric methods.
It should be observed that stadimetric ranging has an advantage over kinematic ranging. In stadimetry, the subtended angle $\theta$ is positive-definite, and its rate $\dot{\theta}$ is zero if and only if $\dot{r} = 0$; this almost never occurs. On the other hand, in kinematic ranging, the objective of a proportional navigation homing system is to keep $\dot{\phi} = 0$ at all times; consequently an ill-conditioned covariance matrix must result at least occasionally.

This observation leads to describing the possible weakness in the idea of combining kinematic and stadimetric ranging principles. In both schemes, express $\hat{r}$ as a function of $\hat{r}$, and assume $\dot{r} \approx 0$. Then, using (5.42) and (5.45)

\[
\text{Kinematic:} \quad \frac{\hat{r}}{r} = \frac{v_0}{v_2} - \frac{v_1}{v_2} \times \hat{r}
\]

\[
\text{Stadimetric:} \quad \frac{\hat{r}}{r} = -\left(\frac{\dot{\theta}}{\theta}\right) \times \hat{r}
\]

under the simplifying assumption that $a_{\tau_0} = 0$.

As previously observed for the stadimetric method, in (5.47) $\dot{\theta}$ is always positive, and $\dot{\theta}$ is almost always positive. Neglecting the rare and anomalous case of $\dot{\theta} \leq 0$, implying increasing range, $\dot{\theta}/\theta$ is positive and the solution for $\hat{r}$ as a function of $\hat{r}$ passes through the origin with a time-varying negative slope which is a quotient of non-zero finite quantities. In the kinematic approach (5.46), the
coefficient of \( \hat{r} \) will normally have a negative value but is a quotient in which the numerator and denominator rapidly approach small values and ultimately zero, in favorable geometries, and start and remain at small values in unfavorable geometries. The slope of (5.46) and the joint solution of (5.46) and (5.47), therefore are increasingly sensitive to instrument and computation noise as the maneuver becomes small. In geometric terms, the two lines in the \((r, \hat{r})\) space of Figure 5.2 become parallel as the maneuver becomes small.

In consequence, it is evident that the stadimetric/linear-kinematic method outlined above could be improved by use of the nonlinear kinematic method described in Section 4.
6. **TECHNOLOGY ASSESSMENT**

It appears that linear methods of kinematic ranging can work effectively only for favorable combat geometries and if the sensor noise levels are quite low. This implies high accuracy sensors, and frequent ineffectivity.

The nonlinear filter described in Section 4 estimates the target velocity-vector components and therefore can be used to deduce the target velocity and aspect angle. The filter also estimates the target's acceleration components, the turn-rate, and its derivative. It therefore estimates the entire specification of the target dynamics, and can be used to estimate the target future trajectory and time to go. This may be useful not only to the air-air missile terminal-control guidance problem but also to a variety of terminal fire control systems in aircraft or helicopters instead of missiles. This nonlinear filter can also be used to eliminate impossible target accelerations from data provided by other tracking systems such as radar or laser rangers.

Many of the newer seekers of TV or IR types provide some indication of the target size or shape. Target size data can be used as a partial basis for a ranging system, as outlined in Section 5. In addition, information on the target shape or its aspect-angle can be used to provide clues enabling nonlinear filtering. For example, if the target is viewed from a head-on or tail-chase aspect it is clearly
difficult for the target to accelerate parallel to the Line of Sight; it can accelerate across the line of sight with relative ease.

Linear stadimetric ranging cannot by itself estimate range and range-rate, but only their ratio. However, combining linear stadimetry and kinematic schemes has the effect of reducing the number of unknown variables to be determined simultaneously. The parameter identification problem may in fact be reduced to a problem in 2-space, without using the poor assumptions that $\dot{r}$, $a_T^r$, and $a_T^g$ are quasi-constant.

In this situation, two or (preferably) more independent measurement instants enable an unique solution. The problem of updating the hyperplane of Figs. 5.1 or 5.2 to another instant is then reduced. It is not eliminated, therefore the solution is not rigorously observable; however, the combination may be useful.

In a review of the present technology and an estimate of the trends and potentials, the interaction of cost with systems technology is perhaps most important in the air-air missile problem. These points stand out:

a) Seekers of all kinds are very expensive; a small radar seeker is very costly in weight, space and dollars. A TV seeker may be less expensive in each sense, but is not cheap. The various sophisticated IR seekers are cheaper but are still quite costly compared to the simplest IR seekers.

(b) Sensors such as rate gyros are an order of magnitude less costly than simple seekers.
c) The introduction of medium or large scale integrated digital and analog chips has at once improved the reliability of the computation process and reduced the cost by orders of magnitude.

Taken together, these elements suggest that cost/effectivity considerations lead to improving missile system performance by combining a wide range of the inexpensive sensors with a relatively sophisticated computation capability. This may enable use of a relatively unsophisticated seeker of relatively low cost.

The nonlinear range/range rate estimator shown in Fig. 4.1 requires a considerable computation capability, and requires input signals from a pair of accelerometers. It also requires the missile airspeed components which may be obtained from relatively inexpensive sensors or may be estimated from accelerometer and rate-gyro data. But the least sophisticated seeker is sufficient. Our approach was to maximize the information which could be extracted from an inexpensive seeker by combining several inexpensive sensors with a relatively sophisticated yet inexpensive computer which will soon be available.
7. REFERENCES


The Nonlinear Estimator; Derivation and Instrumentation

It is possible, due to the particular form of the equations and constraints of the Mathematical Model of the system, to develop a nonlinear range and range-rate identifier. This type of identifier is discussed in this section.

A.1 Analysis

The key elements of the Mathematical Model are quoted from Section 3:

Velocity component equations parallel to the Line-of-Sight:

\[ \dot{r} - V_T \cos (\gamma_T - \sigma) + V_M \cos (\gamma_M - \sigma) = 0, \]

or

\[ \dot{r} - \frac{V_T}{r} + \frac{V_M}{r} = 0. \]  

(A.1)

Velocity component equations perpendicular to the LOS

\[ \dot{r} \cdot \sigma - V_T \sin (\gamma_T - \sigma) + V_M \sin (\gamma_M - \sigma) = 0, \]

or

\[ \dot{r} \cdot \sigma - \frac{V_T}{r} + \frac{V_M}{r} = 0. \]  

(A.2)
We also demonstrated that the acceleration component equations parallel and perpendicular to the LOS are, respectively:

\[ \ddot{r} - \ddot{r}^2 - a_{Tr} + a_{Mr} = 0 \]  
(A.3)

and

\[ \ddot{r} + 2\dot{r}\dot{\phi} - a_{T\phi} + a_{M\phi} = 0. \]  
(A.4)

The target and missile velocity and acceleration components parallel and perpendicular to the LOS are:

**Velocities:**

Parallel: \[ V_{Tr} = V_T \cos \xi_T \]
\[ V_{Mr} = V_M \cos \xi_M \]

Perpendicular: \[ V_{T\phi} = V_T \sin \xi_T \]
\[ V_{M\phi} = V_M \sin \xi_M. \]  
(A.5)

**Accelerations:**

Parallel: \[ a_{Tr} = \dot{V}_T \cos \xi_T - V_T \dot{\gamma}_T \sin \xi_T \]
\[ a_{Mr} = \dot{V}_M \cos \xi_M - V_M \dot{\gamma}_M \sin \xi_T \]

Perpendicular: \[ a_{T\phi} = \dot{V}_T \sin \xi_T + V_T \dot{\gamma}_T \cos \xi_T \]  
(A.6)
\[ a_{M\phi} = \dot{V}_T \sin \xi_T + V_M \dot{\gamma}_M \cos \xi_T \]

where
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Angles:  \[ \xi_T = \gamma_T - \sigma \]
\[ \xi_M = \gamma_M - \sigma. \]

The principle of this approach is simply to solve A.3 for \( \ddot{r} \) as
\[ \ddot{r} = r \dot{\sigma}^2 + a_M - a_T. \]  \hspace{1cm} (A.8)

Integration of \( \ddot{r} \) will yield \( \dot{r} \) and \( r \) if the terms on the right of (A.8) are available.

The term \( r \dot{\sigma}^2 \) can be computed easily while \( a_M \), the missile acceleration parallel to the LOS, can be measured directly. The problem therefore is now to estimate \( a_T \) in some way. It is assumed that initial values of \( \dot{r} \) and \( r \) are provided.

It was shown that the target acceleration in the range direction, \( a_T \), can be determined from available data. The procedure is first to calculate \( V_T \), the target airspeed, by use of (A.1) and (A.2) as
\[ (\ddot{r} + V_M \cos \xi_M)^2 + (r \dot{\sigma} + V_M \sin \xi_M)^2 = V_T^2 \cos^2 \xi_T + V_T^2 \sin^2 \xi_T = V_{T \sigma}^2 + V_{T \sigma}^2 = V_T^2. \]  \hspace{1cm} (A.9)

It was stated in Section (4.1) that this expression may be differentiated to yield \( V_T a_T + V_T a_T = V_T \dot{V}_T \).
This is demonstrated below. Therefore, let us differentiate (A.9) with respect to time, yielding
\[ 2(\dddot{r} + V_M \cos \xi_M)[\dddot{r} + (\dot{V}_M \cos \xi_M - V_M \dot{V}_M \sin \xi_M) + V_M \dot{\sigma} \sin \xi_M] + (r \dddot{\sigma} + V_M \sin \xi_M)[r \dddot{\sigma} + r \dddot{\sigma} + (\dot{V}_M \sin \xi_M + V_M \dot{V}_M \cos \xi_M) - V_M \dot{\sigma} \cos \xi_M] = 2 V_T \dot{V}_T. \]
Substitute from (A.5) the simplifying notation for the acceleration components of the missile, $a_{N_T}$ and $a_{M_G}$, parallel and perpendicular to the LOS, respectively; these accelerations are gathered in parenthesis within the same brackets in the expression above [( )].

This yields

\[
(\ddot{r} + V_M \cos \xi_M)(\dot{r} + a_{N_T} + V_M \dot{\sigma} \sin \xi_M) + (r \dot{\sigma} + V_M \sin \xi_M)(\dot{\sigma} + \ddot{r} + a_{M_G} - V_M \dot{\sigma} \cos \xi_M) = V_M \dot{V}_T. \tag{A.10}
\]

Now solve (A.2) for $V_M \sin \xi_M$, multiply by $\dot{\sigma}$ and replace the term $(V_M \dot{\sigma} \sin \xi_M)$ on the first line of (A.10) by the term thus formed. Similarly, solve (A.1) for $V_M \cos \xi_M$, multiply by $\dot{\sigma}$ and eliminate the term $(V_M \dot{\sigma} \cos \xi_M)$ from the second line of (A.10). These substitutions yield

\[
(\ddot{r} + V_M \cos \xi_M)(\dot{r} - r \dot{\sigma}^2 + a_{N_T} + \dot{\sigma} V_T \sin \xi_T) + (r \dot{\sigma} + V_M \sin \xi_M)(\dot{\sigma} + 2r \dot{\sigma} + a_{M_G} - \dot{\sigma} V_T \cos \xi_T) = V_T \dot{V}_T. \tag{A.11}
\]

Keeping the right side of this expression, gather the coefficients of $V_T$ on the left; this reduces to

\[
(\ddot{r} + V_M \cos \xi_M)(\dot{r} - r \dot{\sigma}^2 + a_{N_T}) + (r \dot{\sigma} + V_M \sin \xi_M) + (r \ddot{\sigma} + 2r \dot{\sigma} + a_{M_G}) \dot{\sigma} ((\ddot{r} + V_M \cos \xi_M)(V_T \sin \xi_T) - (r \dot{\sigma} + V_M \sin \xi_M)(V_T \cos \xi_T)) = V_T \dot{V}_T. \tag{A.11}
\]
The term in curly braces \{ \} in (A.11) is identically zero. This follows from (A.1), \( \dot{r} + V_M \cos \xi_M = V_T \cos \xi_T \), while from (A.2), \( \dot{r} + V_M \sin \xi_M = V_T \sin \xi_T \), so that the brace has the value \( V_T \cos \xi_T (V_T \sin \xi_T - V_T \sin \xi_T (V_T \cos \xi_T)) = 0 \).

As a consequence, (A.11) takes the simpler form

\[
(r + V_M \cos \xi_M)(\ddot{r} - \ddot{r}^2 + a_M) + (r + V_M \sin \xi_M)
\]

\[
(r \dddot{r} + 2r \ddot{r} + a_M r) = V_T \dot{V}_T. \tag{A.12}
\]

By use of (A.1) through (A.6), this may be expressed in the intuitively satisfying form \( V_{T_{T_{r}}} a_{T_{r}} + V_{T_{T_{c}}} a_{T_{c}} = V_T \dot{V}_T \), (A.13) which is the form used in the discussion in Section 4, where we use the simplifying notation, from (A.5),

\[
V_{T_{r}} = V_T \cos \xi_T
\]

\[
V_{T_{c}} = V_T \sin \xi_T
\]

to express the target velocity component's parallel and perpendicular to the LOS.

The exact solution for \( a_{T_{r}} \) is thus

\[
a_{T_{r}} = V_T \dot{V}_T - V_{T_{T_{c}}} a_{T_{c}} \over V_{T_{r}}. \tag{A.14}
\]

and

\[
\ddot{r} = -a_M \dot{r} + V_T \dot{V}_T - V_{T_{T_{c}}} a_{T_{c}} \over V_{T_{r}}, \tag{A.15}
\]

while \( a_{T_{c}} V_{T_{r}} = \dot{V}_T \) if \( \dot{V}_T = 0 \).
Usually $V_T$ is small and can be neglected, and all other terms on the right of (A.15) can be estimated.

The mechanization of this algorithm is discussed in Section (A.2), following.

A.2 Mechanization of the Signals for the Ideal Algorithm

The range acceleration is given by (A.15); it may be integrated to yield $\dot{r}$ and $r$. The term $r\sigma^2$ can then be formed, while the missile acceleration component, $a_M$, can be directly sensed.

The mechanization of (A.15) requires that missile airspeed $V_M$, the missile acceleration components $a_M$ and $a_M$, and the angle $\xi_M$ be available as signals or be estimated. It will now be shown that this can be accomplished.

(a) Acceleration Components

The accelerometer components, $a_M$ and $a_M$, are a special case; two easy solutions are available:

1. Mount the accelerometers on the seeker; then one yields $a_M$ and the other yields $a_M$ directly.

2. Mount the accelerometers on the missile body; then their signals are $a_M$, $a_M$, fore-aft and cross-body, respectively; and resolution of the signals from the accelerometer through
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the "look-angle," \( \lambda \), yields

\[
\begin{bmatrix}
a_M^r \\
\alpha \\
a_M^y
\end{bmatrix} = \begin{bmatrix}
\cos \lambda & -\sin \lambda \\
\sin \lambda & \cos \lambda
\end{bmatrix} \begin{bmatrix}
a_M \\
a_M^x \\
a_M^y
\end{bmatrix}.
\]

(A.16)

As the "look-angle," \( \lambda \), between the missile centerline and the LOS can easily be made physically available in a seeker gimbal, the resolution of the acceleration components is easily accomplished.

(b) Velocity Estimation

If the missile body pitch rate, \( q \), or equivalent, is an available signal, and further, if the cross-body acceleration of the missile, \( a_{M_y} \), is also available as a signal, it is relatively easy to estimate missile velocity, \( V_M \).

The pertinent equations of motion of the missile are,

\[
\dot{\alpha} = q - \frac{a_{M_y}}{V_M} \tag{A.17}
\]

\[
a_{M_y} = Na + G\delta
\]

where
- \( \alpha \) angle of attack
- \( a_{M_y} \) cross-body acceleration
- \( \delta \) control displacement
- \( q \) body pitch rate
- \( V_M \) airspeed
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\[ N, G \] cross-body accelerations per radian, due to angle of attack and control displacements, respectively.

Assuming quasi-constant conditions, the transfer function defining acceleration is

\[ a_{M_y} = \frac{N q + G s \delta}{s + N/V_M} \]  \hspace{1cm} (A.16)

The acceleration due to control displacement may usually be neglected so that

\[ a_{M_y} \approx \frac{N q}{s + N/V_M} = V_M \left( \frac{N/V_M}{s + N/V_M} \right) q \]

where \( T = \frac{V_M}{V_M} \). Then \( V_M \) may be estimated by the algorithms

\[ \hat{V}_M = k a_{M_y} \left( a_{M_y} - \hat{V}_M \right) \]  \hspace{1cm} (A.20)

or

\[ \hat{V}_M = k ( |a_{M_y}| - \hat{V}_M |q| ) \]  \hspace{1cm} (A.21)

where \( k > 0 \) has the appropriate value.

Somewhat superior algorithms which will lag less are achieved by blissing the algorithms given by the missile longitudinal acceleration component \( a_{M_x} \).
Algorithm (A.21) is obviously simple to instrument, requiring only one multiplication.

(c) Estimation of the Angle \( \xi_M \)

The angle \( \xi_M \) is defined as \( \xi_M = \gamma_M - \sigma \).

But as \( \gamma_M = \theta_M - \alpha \), where \( \theta_M \) is the centerline attitude, then \( \xi_M = \theta_M - \alpha - \sigma \), and as the angle from missile centerline to the seeker LOS is \( \lambda = \sigma - \theta_M \)

we have \( \xi_M = -\lambda - \alpha \) \hspace{1cm} (A.22)

where \( \alpha \) is the angle of attack between the missile centerline and the velocity vector. But this can easily be estimated as

\[
\hat{\alpha} \approx \frac{q}{T + s + 1} \hspace{1cm} (A.23)
\]

where \( T \) was defined above.

The missile velocity components \( V_{Mr} \) and \( V_{Ms} \) can then be computed as \( V_{Mr} = V_M \cos \xi_M \) and \( V_{Ms} = V_M \sin \xi_M \).
APPENDIX B

Error Analysis Coefficients

The numerical data presented in this appendix represent the time varying coefficients of the error equation. As discussed in Chapter 4, the error equation is of the form

\[ \ddot{x} + a_1 x + a_2 x = \sum_{i=1}^{7} b_i N_i, \]

and the data here represent the coefficients \( a_1 \) and \( b_i \) tabulated for every 0.25 seconds. There are six tables, representing a selection of six choices of initial geometry configuration. They are ordered, and correspond respectively to Figures 4.3 through 4.8 inclusive; however, each graph is labelled to show the initial geometry. These data permit more general stochastic analyses of the effects on the estimates due to errors in the several signals which are used in the algorithm. The coefficients \( a_1 \) and \( a_2 \) are:

\[
\begin{align*}
    a_1 &= 2\dot{\theta} \tan \xi_T - \gamma_T \tan \xi_T, \quad \text{and} \\
    a_2 &= -\dot{\theta}^2 + \gamma_T \dot{\theta} + \ddot{\theta} \tan \xi_T.
\end{align*}
\]

The coefficients \( b_1 \) and error sources \( N_i \) are:

\[
\begin{align*}
    b_1 &= -1 \quad \text{\( N_1 \) \( a_M \), accel/LOS error} \\
    b_2 &= \sec \xi_T \quad \text{\( N_2 \) \( \dot{V}_T \), target airspeed change} \\
    b_3 &= -\gamma_T \sec \xi_T \sin (\gamma_M - \gamma_T) \quad \text{\( N_3 \) \( \dot{V}_M \), missile airspeed error}
\end{align*}
\]
b_4 = -\gamma_T \tan \xi_T + 2r^2 \tan \xi_T + 2r \dot{\gamma}  
N_4 \hat{\delta}, \text{LOS rate error}

b_5 = -v_M \gamma_T \sec \xi_T \cos (\gamma_M - \gamma_T)  
N_5 \hat{\xi}_M, \text{error of estimating } \hat{\delta}

b_6 = -r \tan \xi_T  
N_6 \dot{\alpha} \text{ LOS acceleration error}

b_7 = -\tan \xi_T  
N_7 a_{M \delta}, \text{ accel. } \perp \text{ LOS, error}

All angles are in radian measure unless specifically stated otherwise.

The data of this appendix are presented in a conventional "exponential format" for digital output data. Thus the tabulated item 0.422E 01 is interpreted as (0.422)10^1 = 4.22, while -0.244E-01 is interpreted as (-0.244)10^{-1} = -0.0244.
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**TABLE B-1**

Error Equation Coefficients $a_i$ and $b_i$ as functions of the time for the trajectory of Fig. 4.3 with initial conditions $\psi = 90^\circ$, $\phi = 0^\circ$, $x_T = 10,000$, $y_T = 1,000$.
TABLE B-2
Error Equation Coefficients $a_1$ and $b_1$ as functions of the time for the trajectory of Fig. 4.4 with initial conditions $\psi=90^\circ$, $\phi=0^\circ$, $x_T=10,000$, $y_T=1,000$.

$$a_1 = 2\dot{\phi} \tan \xi_T - \dot{y}_T \tan \xi_T$$
$$a_2 = -\dot{\phi}^2 + \dot{y}_T \ddot{\phi} + \ddot{\phi} \tan \xi_T$$

$$b_1 = -1$$
$$b_2 = \sec \xi_T$$
$$b_3 = -\dot{y}_T \sec \xi_T \sin (\gamma_M - \gamma_T)$$
$$b_4 = -r \dot{\phi} - 2\ddot{\phi} \tan \xi_T + 2r \ddot{\phi}$$
$$b_5 = -\dot{y}_T \dot{\phi} \sec \xi_T \cos (\gamma_M - \gamma_T)$$
$$b_6 = -r \tan \xi_T$$
$$b_7 = -\tan \xi_T$$

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<td>0.418E+01</td>
<td>0.0</td>
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<td>C.870E-01</td>
<td>-C.10E+01</td>
<td>0.409E+01</td>
<td>0.0</td>
<td>0.138E+05</td>
<td>0.0</td>
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<td>0.0</td>
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<td>0.956E+01</td>
<td>C.666E-01</td>
<td>-C.10E+01</td>
<td>0.404E+01</td>
<td>0.0</td>
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<td>0.0</td>
<td>-C.211E+05</td>
<td>0.0</td>
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<td>C.516E+00</td>
<td>-C.10E+01</td>
<td>0.400E+01</td>
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<td>0.393E+01</td>
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<td>0.394E+01</td>
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<td>0.394E+01</td>
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<td>0.126E+05</td>
<td>0.0</td>
<td>-C.154E+04</td>
<td>0.0</td>
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</table>

N
- $A_{\phi}$, accel. LOS error
- $N_2$, target airspeed change
- $N_3$, missile airspeed error
- $N_4$, $\dot{\phi}$, LOS rate error
- $N_5$, $\dot{\phi}_M$, error of estimating $\dot{\phi}$
- $N_6$, $\ddot{\phi}_M$, LOS acceleration error
- $N_7$, $a_{\phi}$, accel. perpendicular to LOS, error
TABLE B-3

Error Equation Coefficients a_1 and b_1 as functions of the time for the trajectory of Fig. 4.5 with initial conditions \( \psi = 90^\circ \), \( \phi = 80^\circ \), \( x_T = 10,000 \), \( y_T = 1,000 \).

\[
a_1 = 2 \dot{\gamma}_T \tan \xi_T - \dot{\gamma}_T \tan \xi_T
\]

\[
a_2 = -\sigma^2 + \dot{\gamma}_T \dot{\sigma} + \sigma \tan \xi_T
\]

\[
b_1 = -1
\]

\[
b_2 = \sec \xi_T
\]

\[
b_3 = -\dot{\gamma}_T \sec \xi_T \sin (\gamma_M - \gamma_T)
\]

\[
b_4 = -\ddot{\gamma}_T - 2 \ddot{\gamma} \tan \xi_T + 2 \dot{\sigma}
\]

\[
b_5 = -V_M \gamma_T \sec \xi_T \cos (\gamma_M - \gamma_T)
\]

\[
b_6 = -r \tan \xi_T
\]

\[
b_7 = -\tan \xi_T
\]

\[
N_1 a_M', \text{ accel LOS error}
\]

\[
N_2 \hat{V}_T, \text{ target airspeed change}
\]

\[
N_3 \hat{V}_M, \text{ missile airspeed error}
\]

\[
N_4 \dot{\sigma}, \text{ LOS rate error}
\]

\[
N_5 \dot{\xi}_M, \text{ error of estimating } \dot{\xi}
\]

\[
N_6 \ddot{\sigma}, \text{ LOS acceleration error}
\]

\[
N_7 a_{M_{\perp}}, \text{ accel } \perp \text{ LOS error}
\]
The matrix $T_n$ is thus a "continuously updated observation matrix" and the vector $Y(n)$ is a "series of output observations". The expression (5.21) is valid provided the noise $v$ is independent from computing instant to instant and has zero mean. The construction of the matrix $T_n$ is often a formidable task as it requires keeping in storage a large amount of past data $H(t_i)$ so that it can be updated by the state transition matrix $\Phi(t_i, t_j)$ which is at times also difficult to obtain.

Apply this theory to, typically, the reduced dimensionality problem defined by (5.16) and (5.17); assume that $\dot{r}$ is constant, thus $x_2$ is constant after convergence. But $x_1 = r$ and is not constant. Now we have, at some specific instant, and with only one value of $i = 1$,

$$ v_1(t_i)x_1 + v_2(t_i)x_2 + y_1 = e_i. $$

(5.24)

At the next instant we have

$$ v_1(t_2)x_1 + v_2(t_2)x_2 + y_2 = e_5. $$

(5.25)

But while the value of $x_2$ is ideally the same at these two instants, the value of $x_1$ changes continuously, for $x_1 = r$, and $r$ is the integral of $\dot{r}$. We may modify (5.24) so that $x_1$ is the same for the two instants. If $\Delta t$
is the time between samples, we may rewrite (5.24) as
\[ v_1(t_1) (x_1(t_1) + x_2 \Delta t) + v_2(t_1) x_2 + y_1 = e_5, \]
which may be rearranged as
\[ v_1(x_1) x_1(t_2) + (v_2(t_1) + v_1(t_1) \Delta t) x_2 + y_1 = e_5. \]

Now Eqns. (5.25) and (5.26) represent two measurements referred to the same instant, and may be solved jointly.

More generally, a larger sequence of such instants may be gathered in the more general form of (5.23).

This method exhibited several modes of failure. Whenever any two of:

- the missile velocity vector
- the target velocity vector
- the LOS

become colinear, or \( \cdot = 0 \), the matrix \((T_n^T T_n)\) became ill-conditioned or indeterminate, yielding inaccurate and noise-sensitive solutions. In geometries when this mode of failure does not occur, this algorithm has a different mode of failure. If the duration of the stored memory exceeds 1/2 second, the neglected target acceleration parallel to the LOS, \( a_T \), causes the estimate of \( \hat{r} \) to be 30% to 50% in error within 1 to 2 seconds, as the assumption that \( \dot{r} = 0 \) is too incorrect. On the other hand, a shorter duration memory results in high noise sensitivity. But these algorithms work
well (Range-Rate errors of 1%) if \( a_T r = 0 \).

In any case the computational complexity and memory storage requirements of this algorithm are excessive. An alternate was therefore examined. If

\[
\hat{r} = \hat{r}_0 + \int \hat{r} \, dt
\]

and \( x_1 = \hat{r}_0, x_2 = \hat{r}, \) etc.,

then a much simpler algorithm results. At a number of successive instants we have

\[
\begin{bmatrix}
v_1(1) & v_2(1) \\
v_1(2) & v_2(2) + v_1 \Delta t \\
\vdots & \vdots \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\end{bmatrix}
= \begin{bmatrix}
y_1 \\
y_2 \\
\end{bmatrix}
\]

and both \( x_1 \) and \( x_2 \) may be assumed constant. If the nth row of the left of (5.28) is

\[
v_1(n) \quad v_2(n) + \Delta t \quad v_1(n-1)
\]

then the matrix \((T_n^T T_n)\) may be recursively formed as

\[
(T_n^T T_n) = (T_{n-1}^T T_{n-1}) + v_n v_n^T
\]

(5.29)
Although sufficiently simple to be feasible for an air-air missile, this approach to range estimation also failed for the same reasons:

(a) Ill-conditioned matrix for some geometries,

(b) High noise-sensitivity for short averaging times,

(c) Large errors in range-rate estimates for longer averaging times due to the invalid assumption that $a_{T_r} = 0$.

When failure-mode (a) did not occur, and $a_{T_r}$ was valid, the algorithm yields estimate errors of the order of 1%.

No attempt at estimating $a_{T_r}$, target acceleration parallel to the LOS, by linear methods was successful. Any algorithm which neglected $a_{T_r} \neq 0$ when it was significant operated unsuccessfully; either excessive noise sensitivity became a problem, or the neglected acceleration caused excessive errors in the estimation of Range-Rate, $\hat{r}$.

In summary, except for encounter geometries which yield ill-conditioned matrices, the matrix-inverse algorithms work well with long-filtering times which eliminate noise effects if the assumption that $a_{T_r} = 0$ is valid. If that
assumption is invalid, the matrix-inverse schemes fail due to the bias in range-rate estimate caused by target acceleration, or due to noise.

5.3 Observability

The failure of these various adaptive and matrix-inverse parameter identification methods, whether continuous or discrete, is fundamentally due to the existence of target acceleration in the radial direction, parallel to the line of flight. That acceleration component cannot be estimated by linear techniques because it is "unobservable" in the mathematical sense. This section relates the concept of "observability" to this application and problem.

The state of a linear system with constant coefficients in the dynamics and constant coefficients in the output relation can be estimated from knowledge of the output provided the system is mathematically observable. For this discussion, we view the quantities of range, range-rate, etc., as states which are to be determined and consider the signals of LOS-rate and missile acceleration as time-varying parameters. In state-variable notation a plant may be described as

\[
\begin{align*}
\text{Plant Dynamics:} & \quad \dot{x} = Ax + Bu \\
\text{Output:} & \quad y = Hx + v
\end{align*}
\]

where \(v\) is zero-mean Gaussian white noise, and \(x\) is an \(N\)-component vector.
The observability of the system is specified by a relationship between the A matrix, specifying the dynamics, and the H matrix, relating the state to the output. If the order of the system is N, then the matrix

\[ O_N = \begin{bmatrix} H^T & A^T H^T & (A^T)^2 H^T & \ldots & (A^T)^{N-1} H^T \end{bmatrix} \]

must have full rank for the system to be observable. Let us now apply this to one of the possible formulations of the problem. For the system definition, use the acceleration-component equations from Section 2:

accelerations parallel to the \( r \hat{\varpi} - r \dot{\varpi}^2 - a_r + a_{\varpi r} = 0 \)

LOS \hspace{1cm} (5.32)

Perpendicular to \( r \ddot{\varpi} + 2 \dot{r} \dot{\varpi} - a_{\varpi r} + a_{\varpi \varpi} = 0 \)
the LOS \hspace{1cm} (5.33)

In order to cast this into the standard form of (5.30), define state variables

\[ \begin{align*}
    x_1 &= \hat{r} \\
    x_2 &= \dot{r} \\
    x_3 &= \dot{\varpi} \\
    x_4 &= a_{\varpi r} \\
\end{align*} \]

(5.34)
i.e. the variables $x$ are the estimates of the unknown parameters. The last two lines of (5.34) express the fact that the two components of target acceleration must be estimated and are assumed to be unknown constants.

The state variable expression of the relationships among the several variables may be found from (5.32) and (5.34) as

$$
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4
\end{pmatrix}
=
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{pmatrix}
+
\begin{pmatrix}
0 \\
a_{m_r} \\
0 \\
0
\end{pmatrix}
$$

(5.35)

The first row of (5.35) expresses the differential equation relating range and range-rate estimates. The second row consists of (5.34) while the third and fourth rows express the assumption that the target's unknown accelerations $a_{T_r}$ and $a_{T_o}$ are constant. From (5.35), the matrix $A$ is

$$
A_n =
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
$$

(5.36)

and $N=4$.  

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Equation (5.33) yields the output relationship

\[
\begin{pmatrix}
-\ddot{\sigma} & -2\dot{\sigma} & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4
\end{pmatrix} = \dot{a}_M;
\]

(5.37)

comparison of the estimate, \( \hat{a}_M \), with the measured acceleration \( a_M \) yields an error whose minimization will presumably yield optimal estimates of the range, \( x_1 \), and range-rate, \( x_2 \). From (5.37),

\[
H = \begin{pmatrix}
-\ddot{\sigma} & -2\dot{\sigma} & 0 & 1
\end{pmatrix}.
\]

(5.38)

Combining (5.36) with (5.38) in the format of (5.35) yields the observability matrix

\[
O_4 = \begin{pmatrix}
-\ddot{\sigma} & -2\dot{\sigma}^3 & -\dot{\sigma}^2 & -2\dot{\sigma}^5 \\
2\dot{\sigma} & -\ddot{\sigma} & -2\dot{\sigma}^3 & -\dot{\sigma}^2 y \\
0 & -2\dot{\sigma} & -\ddot{\sigma} & -2\dot{\sigma}^3 \\
1 & 0 & 0 & 0
\end{pmatrix}.
\]

(5.39)

The fourth column of (5.39) is exactly equal to \( \ddot{\sigma}^2 \) times the second column, and the rank is less than the order. This system is unobservable; experiment confirmed this result.
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In realistic combat trajectories it is generally erroneous to assume that \( \mathbf{a}_T \), target acceleration in the range-direction, is zero. If, however, that assumption is valid for some situation, then we can simplify (5.35) by omitting the third row and column and similarly simplify \( \mathbf{H} \), so that if \( \mathbf{a}_T \equiv 0 \), a priori, then

\[
\mathbf{A}_{32} = \begin{pmatrix}
0 & 1 & 0 \\
\ddot{\sigma} & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad \mathbf{H}_{31} = (-\ddot{\sigma} & -2\dot{\sigma} & 1)
\]

and

\[
\mathbf{C}_{31} = \begin{pmatrix}
-\ddot{\sigma} & -2\dot{\sigma} & -\ddot{\sigma} & \sigma^2 \\
-2\ddot{\sigma} & -\ddot{\sigma} & -2\dot{\sigma} & \sigma^3 \\
1 & 0 & 0
\end{pmatrix}
\]

In this case rank equals order and the system is observable if \( \ddot{\sigma} \neq 0 \) and \( \dot{\sigma} \neq 0 \). This result was experimentally confirmed.

Returning to (5.35), assume that the target acceleration perpendicular to the LOS, \( \mathbf{a}_T \), is identically zero, i.e., \( x_4 = \hat{\mathbf{a}}_T \equiv 0 \), a priori, but the other component, \( x_3 = \hat{\mathbf{a}}_T \), is a non-zero constant. Then, making the appropriate changes in \( \mathbf{A} \) and \( \mathbf{H} \), we have

\[
\mathbf{A}_{32} = \begin{pmatrix}
0 & 1 & 0 \\
\ddot{\sigma} & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}, \quad \mathbf{H}_{32} = (-\ddot{\sigma} & -2\dot{\sigma} & 0)
\]

...
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and

\[
O_{32} = \begin{pmatrix}
  -\dot{\delta} & -\delta^3 & -\delta^2 \dot{\delta} \\
  -2\dot{\delta} & -\delta & -2\delta^3 \\
  0 & -2\delta & 0
\end{pmatrix}
\]  

(5.41)

In this case the third column equals \(\dot{\delta}^2\) times the first; therefore the system is unobservable.

The result may now be stated: If the target has significant acceleration parallel to the line of sight, \((a_T \neq 0)\), it is not possible to estimate range and range-rate; the system is unobservable. If the target acceleration \(a_T\) is known to be zero, a priori; then the system is observable, and it may be reduced to a problem in the three states, \(x_1 = \dot{r}\), \(x_2 = \ddot{r}\) and \(x_4 = \hat{a}_{T0}\). In this case, Equation (5.32) can be exactly instrumented. If \(x_4 = \hat{a}_{T0} = \) constant is a reasonable assumption, it is possible to reduce the problem from three states to two by differentiating the output expression (5.37) and eliminating \(x_4 = \hat{a}_{T0}\), as in (5.16).

It is, however, an important restriction that the theorems on observability do not apply perfectly to systems whose coefficients are time-varying. In our case the system dynamics are constant, while the observation matrix is not. Intuitively, it seemed possible that if
the coefficients of the output equation vary sufficiently in the observation interval, then a result might be obtained which is of value. Many variations of the above development, using different models of the system dynamics and output equations were therefore examined to pursue this question. The results were disappointing whenever $a_T \neq 0$, or $\dot{c}$ was small. The system is unobservable, despite the uncertainty of the theory on this question, whenever $a_T \neq 0$. 
5.4 Other Linear Systems

It is advantageous to unify the results of the above discussions on kinematic linear parameter identification systems in order to form a basis for considering other possible linear systems. We therefore re-present the problem from a geometrical viewpoint and then consider possible extensions beyond the scope of this contract.

Consider the equation of acceleration components perpendicular to the LOS, from the kinematics,

\[ r \ddot{c} + 2r \dot{\theta} - a_{m_0} + a_{n_0} = 0. \]  
\[(5.41)\]

Assume that the range acceleration, \( \ddot{r} \), is zero, and may therefore be neglected. Assume further that \( a_{m_0} \) is quasi-constant, i.e., cannot change significantly in less than 1/2 second. Then differentiation of (5.41) with a high pass filter such as \( T_1 \frac{T_1}{T_1 s+1} \), where \( T_1 \) is of the order of 1/2 second or less, reduces (5.41) to

\[ r \dot{v}_1 + \dot{r}v_2 + v_3 = 0 \]  
\[(5.42)\]

where \( v_1 = \left( \frac{T_1 s}{T_1 s + 1} \right) \bar{c} \); \( v_2 = \left( \frac{2T_1 s^2}{T_1 s + 1} + \frac{T_1 \bar{v}}{(T_1 s + 1)^2} \right) \); \( v_3 = \left( \frac{T_1 s}{T_1 s + 1} \right) a_{n_0} \).

If in (5.42) we replace \( r \) and \( \dot{r} \) by the estimates \( \hat{r} \) and \( \hat{\dot{r}} \), (5.42) may be expressed as \( \dot{\hat{r}}v_1 + \hat{\dot{r}}v_2 + v_3 = \epsilon^c \); then the solution for \( \hat{r} \) and \( \hat{\dot{r}} \) lies on a locus such that \( \epsilon^c = 0 \). In this
case (5.43) forms a straight line, as shown in the
sketch below for some particular instant.

\[ \frac{v_0}{v_2} \]

\[ \text{slope} = - \frac{v_1}{v_2} \]

\[ e_0 = 0 \]

**Fig. 5.1**
Geometrical View of Equation Error.

The entire problem of estimation is based on the necessity of finding another line in the \((\hat{r}, \hat{r})\) space:

(a) The multiple-error adaptive process, by using a variety of values of \(T_1\) generates a number of other lines which are, more or less, linearly independent. This approach may be categorized as achieving distribution in the \(T_1\)-lag space at one instant of true time.

(b) The various single-error adaptive nets and the Kalman net take advantage of the fact that, at some other instant than that shown in Fig. 5.1, the line of zero error in the \((\hat{r}, \hat{r})\) space is
differently oriented, if \( \theta \neq 0 \), and is therefore linearly independent. These approaches thus use the idea of distribution in true time, \( t \), rather than in the \( T_1 \) space.

(c) The matrix inverse schemes gather all lines over the computing interval, thus achieving distribution in time, and then find the solution.

A physically different approach to finding a second linearly independent line in the \((\mathbf{r}, \mathbf{v})\) space is obviously preferable if possible. Stadiometric ranging provides such an approach. Seekers which enable measurement of some function of the target size, or area, such as mosaic IR, TV, area, or correlation, have this capability. In a single plane we have the geometrical relationship

\[
\mathbf{r} \theta = \mathbf{D} \tag{5.44}
\]

where
- \( \mathbf{r} \) Range
- \( \theta \) Angle Subtended by Target, assumed a small angle
- \( \mathbf{D} \) Target Size

as shown in the sketch below.
Assume that \( D \) is constant; this assumption is fair if \( D \) is the diameter of the least circle which contains the target as viewed from the seeker in both the maneuver and cross maneuver planes. From (5.44), by differentiating,

\[
\mathbf{r}_t + \dot{\mathbf{r}} \cdot \theta = 0
\]

and

\[
\dot{\mathbf{r}}_t + \dot{\mathbf{r}} \cdot \theta = e_7.
\]

The choices of \( \mathbf{r} \) and \( \dot{\mathbf{r}} \) for which \( e_7 = 0 \) lie on a straight line which passes through the origin of the \( (\mathbf{r}, \dot{\mathbf{r}}) \) space at every instant. Combining this with the kinematic ranging sketch, Fig. 5.1, yields the sketch below.

**Fig. 5.2**
Combining Kinematic and Stadimetric Ranging.
If we solve (5.44) and (5.45 simultaneously, we get one equation for \( \dot{r} \) and one for \( \ddot{r} \). These are independent and their joint solution may therefore be determined. It is usually better to solve the two equations in a decoupled format as the solution can always be accomplished more rapidly, more simply and more accurately. A Least Magnitude algorithm may be used. They are analytically valid in such cases and are attractive for they estimate a moving parameter with smaller lag than quadratic algorithms, and thereby decrease the effect of target accelerations.

However, this argument rests on the rather uncertain base of the assumptions. One assumption is that the target cross-track acceleration (\( a_{Tg} \)) is quasi-constant, i.e., changes negligibly within two time-constants of the high-pass filter \( T_1 s \), in (5.42). The other assumption is that \( \dot{r} \approx C \), or at least the unknown portion (\( a_{Tr} \)) thereof. In fact, in any maneuver of the target, \( a_{Tg} \) and \( a_{Tr} \) can change continuously and can change rapidly. This scheme is therefore less powerful than appears at first.

However, standimetric data can be valuable in reducing the difficulty of range estimation. Equation (5.41) states one relationship between three of the unknowns in the kinematic ranging problem. Standimetric data provides another relationship, so that the effective total number of unknowns may be reduced by one. The possibility therefore exists that linear estimation of range and range rate could be accomplished with acceptably small error, due to the unobservability of \( a_{Tr} \), by combining kinematic and standimetric methods.
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It should be observed that stadimetric ranging has an advantage over kinematic ranging. In stadimetry, the subtended angle \( \theta \) is positive-definite, and its rate \( \dot{\theta} \) is zero if and only if \( \hat{r} = 0 \); this almost never occurs. On the other hand, in kinematic ranging, the objective of a proportional navigation homing system is to keep \( \dot{\theta} = 0 \) at all times; consequently an ill-conditioned covariance matrix must result at least occasionally.

This observation leads to describing the possible weakness in the idea of combining kinematic and stadimetric ranging principles. In both schemes, express \( \hat{r} \) as a function of \( \hat{r} \), and assume \( \dot{\theta} \geq 0 \). Then, using (5.42) and (5.45)

**Kinematic:**

\[
\hat{r} = \frac{v_0}{v_2} - \frac{v_1}{v_2} \hat{r}
\]

(5.46)

**Stadimetric:**

\[
\hat{r} = -\left(\frac{\dot{\theta}}{\theta}\right) \hat{r}
\]

(5.47)

under the simplifying assumption that \( a_{T_0} = 0 \).

As previously observed for the stadimetric method, in (5.47) \( \dot{\theta} \) is always positive, and \( \theta \) is almost always positive. Neglecting the rare and anomalous case of \( \dot{\theta} \leq 0 \), implying increasing range, \( \dot{\theta}/\theta \) is positive and the solution for \( \hat{r} \) as a function of \( \hat{r} \) passes through the origin with a time-varying negative slope which is a quotient of non-zero finite quantities. In the kinematic approach (5.46), the
coefficient of \( \hat{r} \) will normally have a negative value but is a quotient in which the numerator and denominator rapidly approach small values and ultimately zero, in favorable geometries, and start and remain at small values in unfavorable geometries. The slope of (5.46) and the joint solution of (5.46) and (5.47), therefore are increasingly sensitive to instrument and computation noise as the maneuver becomes small. In geometric terms, the two lines in the \((r, \hat{r})\) space of Figure 5.2 become parallel as the maneuver becomes small.

In consequence, it is evident that the stadimetric/linear-kinematic method outlined above could be improved by use of the nonlinear kinematic method described in Section 4.
TECHNOLOGY ASSESSMENT

It appears that linear methods of kinematic ranging can work effectively only for favorable combat geometries and if the sensor noise levels are quite low. This implies high accuracy sensors, and frequent ineffectivity.

The nonlinear filter described in Section 4 estimates the target velocity-vector components and therefore can be used to deduce the target velocity and aspect angle. The filter also estimates the target's acceleration components, the turn-rate, and its derivative. It therefore estimates the entire specification of the target dynamics, and can be used to estimate the target future trajectory, and time to go. This may be useful not only to the air-air missile terminal-control guidance problem but also to a variety of terminal fire control systems in aircraft or helicopters instead of missiles. This nonlinear filter can also be used to eliminate impossible target accelerations from data provided by other tracking systems such as radar or laser ranglers.

Many of the newer seekers of TV or IR types provide some indication of the target size or shape. Target size data can be used as a partial basis for a ranging system, as outlined in Section 5. In addition, information on the target shape or its aspect-angle can be used to provide clues enabling nonlinear filtering. For example, if the target is viewed from a head-on or tail-chase aspect it is clearly
difficult for the target to accelerate parallel to the Line of Sight; it can accelerate across the line of sight with relative ease.

Linear stadimetric ranging cannot by itself estimate range and range-rate, but only their ratio. However, combining linear stadimetry and kinematic schemes has the effect of reducing the number of unknown variables to be determined simultaneously. The parameter identification problem may in fact be reduced to a problem in 2-space, without using the poor assumptions that $\dot{r}$, $a_T$, and $a_T^*$ are quasi-constant.

In this situation, two or (preferably) more independent measurement instants enable an unique solution. The problem of updating the hyperplane of Figs. 5.1 or 5.2 to another instant is then reduced. It is not eliminated, therefore the solution is not rigorously observable; however, the combination may be useful.

In a review of the present technology and an estimate of the trends and potentials, the interaction of cost with systems technology is perhaps most important in the air-air missile problem. These points stand out:

a) Seekers of all kinds are very expensive; a small radar seeker is very costly in weight, space and dollars. A TV seeker may be less expensive in each sense, but is not cheap. The various sophisticated IR seekers are cheaper but are still quite costly compared to the simplest IR seekers.

b) Sensors such as rate gyros are an order of magnitude less costly than simple seekers.
c) The introduction of medium or large scale integrated digital and analog chips has at once improved the reliability of the computation process and reduced the cost by orders of magnitude.

Taken together, these elements suggest that cost/effectivity considerations lead to improving missile system performance by combining a wide range of the inexpensive sensors with a relatively sophisticated computation capability. This may enable use of a relatively unsophisticated seeker of relatively low cost.

The nonlinear range/range rate estimator shown in Fig. 4.1 requires a considerable computation capability, and requires input signals from a pair of accelerometers. It also requires the missile airspeed components which may be obtained from relatively inexpensive sensors or may be estimated from accelerometer and rate-gyro data. But the least sophisticated seeker is sufficient. Our approach was to maximize the information which could be extracted from an inexpensive seeker by combining several inexpensive sensors with a relatively sophisticated yet inexpensive computer which will soon be available.
7. REFERENCES


APPENDICES

APPENDIX A

The Nonlinear Estimator; Derivation and Instrumentation

It is possible, due to the particular form of the equations and constraints of the Mathematical Model of the system, to develop a nonlinear range and range-rate identifier. This type of identifier is discussed in this section.

A.1 Analysis

The key elements of the Mathematical Model are quoted from Section 3:

Velocity component equations parallel to the Line-of-Sight:

\[ \dot{r} - V_T \cos (\gamma_T - \sigma) + V_M \cos (\gamma_M - \sigma) = 0, \]

or

\[ \dot{r} - V_T \dot{r} + V_M \dot{r} = 0. \]  \hspace{1cm} (A.1)

Velocity component equations perpendicular to the LOS

\[ r \dot{\sigma} - V_T \sin (\gamma_T - \sigma) + V_M \sin (\gamma_M - \sigma) = 0, \]

or

\[ r \dot{\sigma} - V_T \dot{\sigma} + V_M \dot{\sigma} = 0. \]  \hspace{1cm} (A.2)
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We also demonstrated that the acceleration component equations parallel and perpendicular to the LOS are, respectively:

\[ \ddot{r} - \ddot{r}c^2 - a_T + a_M = 0 \]  \hspace{1cm} (A.3)

and

\[ r\ddot{\theta} + 2\dot{r}\dot{\theta} - a_T + a_M = 0. \]  \hspace{1cm} (A.4)

The target and missile velocity and acceleration components parallel and perpendicular to the LOS are:

**Velocities:**

Parallel: \( V_{T_T} = V_T \cos \xi_T \)

\( V_{M_T} = V_M \cos \xi_M \)

Perpendicular: \( V_{T_\theta} = V_T \sin \xi_T \)

\( V_{M_\theta} = V_M \sin \xi_M. \) \hspace{1cm} (A.5)

**Accelerations:**

Parallel: \( a_{T_T} = \dot{V}_T \cos \xi_T - V_T \dot{\gamma}_T \sin \xi_T \)

\( a_{M_T} = \dot{V}_M \cos \xi_M - V_M \dot{\gamma}_M \sin \xi_T \)

Perpendicular: \( a_{T_\theta} = \dot{V}_T \sin \xi_T + V_T \dot{\gamma}_T \cos \xi_T \)  \hspace{1cm} (A.6)

\( a_{M_\theta} = \dot{V}_T \sin \xi_T + V_M \dot{\gamma}_M \cos \xi_T \)

where
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Angles: \[ \xi_T = \gamma_T - \sigma \]  
\[ \xi_M = \gamma_M - \sigma. \]  \hspace{1cm} \text{(A.7)}

The principle of this approach is simply to solve \( A.3 \) for \( \mathbf{r} \) as

\[ \ddot{\mathbf{r}} = r \dot{\sigma}^2 + a_{rT} - a_{rT}. \]  \hspace{1cm} \text{(A.8)}

Integration of \( \ddot{\mathbf{r}} \) will yield \( \dot{\mathbf{r}} \) and \( \mathbf{r} \) if the terms on the right of (A.8) are available.

The term \( r \dot{\sigma}^2 \) can be computed easily while \( a_{rT} \), the missile acceleration parallel to the LOS, can be measured directly. The problem therefore is now to estimate \( a_{rT} \) in some way. It is assumed that initial values of \( \dot{\mathbf{r}} \) and \( \mathbf{r} \) are provided.

It was shown that the target acceleration in the range direction, \( a_{rT} \), can be determined from available data. The procedure is first to calculate \( V_T \), the target airspeed, by use of (A.1) and (A.2) as

\[ (\ddot{\mathbf{r}} + V_r \cos \xi_M)^2 + (r \ddot{\sigma} + V_r \sin \xi_M)^2 = \]

\[ V_T^2 \cos^2 \xi_T + V_T^2 \sin^2 \xi_T = V_{T,r}^2 + V_{T,\sigma}^2 = V_T^2. \]  \hspace{1cm} \text{(A.9)}

It was stated in Section (4.1) that this expression may be differentiated to yield \( V_T, a_{rT} + V_r, a_{rT} = V_T, \ddot{\mathbf{r}} \); this is demonstrated below. Therefore, let us differentiate (A.9) with respect to time, yielding

\[ 2((\ddot{\mathbf{r}} + V_r \cos \xi_M)[\ddot{\mathbf{r}} + (V_r \cos \xi_M - V_r \sin \xi_M)] + V_r \dot{\mathbf{r}} \sin \xi_M)
\]

\[ + (r \ddot{\mathbf{r}} + V_r \sin \xi_M)[r \dot{\mathbf{r}} + \dot{\mathbf{r}}] + (V_r \sin \xi_M + V_r \dot{\mathbf{r}} \cos \xi_M) -
\]

\[ V_M \dot{\mathbf{r}} \sin \xi_M]) = 2 V_T \ddot{\mathbf{r}}. \]  

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Substitute from (A.5) the simplifying notation for the acceleration components of the missile, $a_{br}$ and $a_{bc}$, parallel and perpendicular to the LOS, respectively; these accelerations are gathered in parenthesis within the same brackets in the expression above [( )].

This yields

$$(\dot{r} + V_p \cos \xi)(\ddot{r} + a_{br} + V_p \dot{\phi} \sin \xi) +$$

$$(r \ddot{\phi} + V_p \sin \xi)(r \ddot{\phi} + \dot{V}_T + a_{bc} - V_p \dot{\phi} \cos \xi) = 0.$$

Now solve (A.2) for $V_p \sin \xi$, multiply by $\dot{\phi}$ and replace the term $(V_p \dot{\phi} \sin \xi)$ on the first line of (A.10) by the term thus formed. Similarly, solve (A.1) for $V_p \cos \xi$, multiply by $\dot{\phi}$ and eliminate the term $(V_p \dot{\phi} \cos \xi)$ from the second line of (A.10). These substitutions yield

$$(\dot{r} + V_p \cos \xi)(\ddot{r} - r \dot{\phi}^2 + a_{br} + \dot{V}_T \sin \xi) +$$

$$(r \ddot{\phi} + V_p \sin \xi)(r \ddot{\phi} + 2r \dot{\phi} + a_{bc} - 3V_T \cos \xi) = 0.$$

Keeping the right side of this expression, gather the coefficients of $V_T$ on the left; this reduces to

$$(\dot{r} + V_p \cos \xi)(\ddot{r} - r \dot{\phi}^2 + a_{br}) + (r \ddot{\phi} + V_p \sin \xi)$$

$$(r \ddot{\phi} + V_p \sin \xi)(r \ddot{\phi} + 2r \dot{\phi} + a_{bc}) - 3V_T \cos \xi = V_T \dot{\phi}.$$

(A.11)
The term in curly braces \{ \} in (A.11) is identically zero. This follows from (A.1), \( r + V_M \cos \xi_M = V_T \cos \xi_T \), while from (A.2), \( r \dot{\theta} + V_M \sin \xi_M = V_T \sin \xi_T \), so that the trace has the value \( (V_T \cos \xi_T \sin \xi_T - V_T \sin \xi_T \cos \xi_T) = 0 \).

As a consequence, (A.11) takes the simpler form
\[
(\ddot{r} + V_r \cos \xi_M)(\ddot{r} - r \ddot{\theta}^2 + a_{\theta r}^2) + (r \ddot{\theta} + V_r \sin \xi_M)
\]
\[
(r \ddot{\theta}^2 + 2r \ddot{\theta} + a_{\theta r}^2) = V_T \dot{V}_T.
\]

By use of (A.1) through (A.6), this may be expressed in the intuitively satisfying form \( V_T a_T r + V_T a_T \sigma = V_T \dot{V}_T \), (A.13) which is the form used in the discussion in Section 4, where we use the simplifying notation, from (A.5),
\[
V_T r = V_T \cos \xi_T
\]
\[
V_T \sigma = V_T \sin \xi_T
\]
to express the target velocity component parallel and perpendicular to the LOS.

The exact solution for \( a_T \) is thus
\[
a_T r = \frac{V_T \dot{V}_T - V_T a_T \sigma}{V_T r} \]
\]
\[
(\text{A.14})
\]

and
\[
\ddot{r} = r \ddot{\theta}^2 - \dot{\theta}^2 + \frac{V_T \dot{V}_T - V_T a_T \sigma}{V_T r} \]
\]
\[
(\text{A.15})
\]

while
\[
\frac{a_T v}{V_T} = V_T \text{ if } \dot{V}_T = 0.
\]
Usually $V_T$ is small and can be neglected, and all other terms on the right of (A.15) can be estimated.

The mechanization of this algorithm is discussed in Section (A.2), following.

A.2 Mechanization of the Signals for the Ideal Algorithm

The range acceleration is given by (A.15); it may be integrated to yield $\dot{r}$ and $r$. The term $r\dot{r}^2$ can then be formed, while the missile acceleration component, $a_{Mr}$, can be directly sensed.

The mechanization of (A.15) requires that missile airspeed $V_M$, the missile acceleration components $a_{Mr}$ and $a_{Mc}$, and the angle $\xi_M$ be available as signals or be estimated. It will now be shown that this can be accomplished.

(a) Acceleration Components

The accelerometer components, $a_{Mr}$ and $a_{Mc}$, are a special case; two easy solutions are available:

(1) Mount the accelerometers on the seeker; then one yields $a_{Mr}$ and the other yields $a_{Mc}$ directly.

(2) Mount the accelerometers on the missile body; then their signals are $a_{Mx}$ and $a_{My}$, fore-aft and cross-body, respectively; and resolution of the signals from the accelerometer through
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the "look-angle," \( \lambda \), yields

\[
\begin{bmatrix}
\bar{a}_{x} \\
\bar{a}_{y}
\end{bmatrix}
= \begin{bmatrix}
\cos \lambda & -\sin \lambda \\
\sin \lambda & \cos \lambda
\end{bmatrix}
\begin{bmatrix}
\bar{a}_{x} \\
\bar{a}_{y}
\end{bmatrix}.
\]

As the "look-angle," \( \lambda \), between the missile centerline and the LOS can easily be made physically available in a seeker gimbal, the resolution of the acceleration components is easily accomplished.

(b) Velocity Estimation

If the missile body pitch rate, \( q \), or equivalent, is an available signal, and further, if the cross-body acceleration of the missile, \( a_{y} \), is also available as a signal, it is relatively easy to estimate missile velocity, \( V_{M} \).

The pertinent equations of motion of the missile are,

\[
\dot{\alpha} = q - \frac{a_{y}}{V_{M}}
\]

\[
a_{y} = 11a + 65
\]

where \( \alpha \) angle of attack

- \( a_{y} \) cross-body acceleration
- \( \delta \) control displacement
- \( q \) body pitch rate
- \( V_{M} \) airspeed
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$N, G$ cross-body accelerations per radian, due to angle of attack and control displacements, respectively. 

Assuming quasi-constant conditions, the transfer function defining acceleration is

$$a_{py} = \frac{N_y + G \phi}{s + \frac{1}{V_{\infty}}}.$$  \hspace{1cm} (A.18)

The acceleration due to control displacement may usually be neglected so that

$$a_{py} = \frac{N_y}{s + \frac{1}{V_{\infty}}} = \frac{V_{\infty}}{s + \frac{1}{V_{\infty}}} \phi$$

$$= \frac{V_{\infty}}{T s + 1}$$ \hspace{1cm} (A.19)

where $T = \frac{1}{V_{\infty}}$. Thus $V_{\infty}$ may be estimated by the algorithms

$$\dot{V}_{\infty} = k a_{py} (a_{py} - V_{\infty})$$ \hspace{1cm} (A.20)

or

$$\dot{V}_{\infty} = k (|a_{py}| - V_{\infty} |\phi|)$$ \hspace{1cm} (A.21)

where $k > 0$ has the appropriate value.

Somewhat superior algorithms which will lag less are achieved by biasing the algorithm given by the missile longitudinal acceleration component $a_{py}$. 
Algorithm (A.21) is obviously simple to instrument, requiring only one multiplication.

(c) Estimation of the Angle $\xi_M$

The angle $\xi_M$ is defined as $\xi_M = \gamma_M - \sigma$.

But as $\gamma_M = \theta_M - \alpha$, where $\theta_M$ is the centerline attitude, then $\xi_M = \theta_M - \alpha - \sigma$, and as the angle from missile centerline to the seeker LOS is $\lambda = \sigma - \theta_M$,

we have $\xi_M = -\lambda - \alpha$ \hspace{1cm} (A.22)

where $\alpha$ is the angle of attack between the missile centerline and the velocity vector. But this can easily be estimated as $\hat{\alpha} = \frac{\alpha}{T_s + 1}$ \hspace{1cm} (A.23)

where $T$ was defined above.

The missile velocity components $V_{M_F}$ and $V_{M_\sigma}$ can then be computed as $V_{M_F} = V_M \cos \xi_M$ and $V_{M_\sigma} = V_M \sin \xi_M$. 

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APPENDIX B

Error Analysis Coefficients

The numerical data presented in this appendix represent the time varying coefficients of the error equation. As discussed in Chapter 4, the error equation is of the form

$$\ddot{x} + a_1 x + a_2 x = \sum_{i=1}^{7} b_i N_i,$$

and the data here represent the coefficients $a_1$ and $b_i$ tabulated for every 0.25 seconds. There are six tables, representing a selection of six choices of initial geometry configuration. They are ordered, and correspond respectively to Figures 4.3 through 4.8 inclusive; however, each graph is labelled to show the initial geometry. These data permit more general stochastic analyses of the effects on the estimates due to errors in the signals which are used in the algorithm. The coefficients $a_1$ and $a_2$ are:

$$a_1 = 2\dot{\alpha} \tan \xi_T - \dot{\gamma}_T \tan \xi_T,$$
$$a_2 = -\dot{\gamma}_T + \dot{\gamma}_T \dot{\xi} + \ddot{\alpha} \tan \xi_T.$$

The coefficients $b_i$ and error sources $N_i$ are:

$$b_1 = -1, \quad N_1 = \alpha_M, \text{ accel\LOS error}$$
$$b_2 = \sec \xi_T, \quad N_2 = \dot{V}_T, \text{ target airspeed change}$$
$$b_3 = -\dot{\gamma}_T \sec \xi_T \sin (\gamma_M - \gamma_T), \quad N_3 = \dot{V}_M, \text{ missile airspeed error}$$
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\[ b_4 = -r \dot{\gamma}_T - 2r \tan \xi_T + 2r \dot{\omega} \quad N_4 \quad \dot{\omega}, \text{LOS rate error} \]
\[ b_5 = -V_M \dot{\gamma}_T \sec \xi_T \cos (\gamma_M - \gamma_T) \quad N_5 \quad \dot{\gamma}_M, \text{error of estimating } \dot{\gamma}_M \]
\[ b_6 = -r \tan \xi_T \quad N_6 \quad \dot{\alpha}, \text{LOS acceleration error} \]
\[ b_7 = -\tan \xi_T \quad N_7 \quad a_{\gamma}, \text{accel. } \perp \text{LOS, error} \]

All angles are in radian measure unless specifically stated otherwise.

The data of this appendix are presented in a conventional "exponential format" for digital output data. Thus the tabulated item 0.422E 01 is interpreted as \((0.422)10^{+1} = 4.22\), while \(-0.244E-01\) is interpreted as \((-0.244)10^{-1} = -0.0244\).
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**Table 8-1**

Error Equation Coefficients \(a_i\) and \(b_i\) as functions of the time for the trajectory of Fig. 1.3 with initial conditions \(\phi = 0^\circ\), \(\psi = 0^\circ\), \(x_0 = 10,000\), \(y_0 = 1,000\).
TABLE B-2
Error Equation Coefficients $a_1$ and $b_1$ as functions of the time for the trajectory of Fig. 4.4 with initial conditions $\psi = 90^\circ$, $\phi = 0^\circ$, $x_T = 10,000$, $y_T = 1,000$.  

$$a_1 = 2\dot{\psi} \sec \psi \tan \xi_T - \dot{\psi} \tan \xi_T$$

$$a_2 = -\ddot{\psi}^2 + \dot{\psi} \dot{\psi} + \delta \tan \xi_T$$

$$b_1 = -1$$

$$b_2 = \sec \xi_T$$

$$b_3 = -\dot{\psi} \sec \xi_T \sin (\tau_M - \tau_T)$$

$$b_4 = -\dot{\tau}_M \tan \xi_T + 2\dot{\psi}^2$$

$$b_5 = -\dot{\psi}_M \dot{\tau}_M \sec \xi_T \cos (\tau_M - \tau_T)$$

$$b_6 = -r \tan \xi_T$$

$$b_7 = -\tan \xi_T$$

$N_1 a_{m_T}$, accel. - LOS error

$N_2 \dot{V}_T$, target airspeed change

$N_3 \dot{V}_M$, missile airspeed error

$N_4 \dot{\psi}$, LOS rate error

$N_5 \delta$, error of estimation

$N_6 \delta$, LOS acceleration error

$N_7 a_{m_T}$, accel. + LOS error
TABLE B-3

Error Equation Coefficients $a_i$ and $b_i$ as functions of the time for the trajectory of Fig. 4.5 with initial conditions $\psi=90^\circ$, $\phi = 80^\circ$, $x_T = 10,000$, $y_T = 1,000$.

\[ a_1 = 2\dot{\psi} \tan \xi_T - \dot{y}_T \tan \xi_T \]
\[ a_2 = -\dot{\psi}^2 + \dot{y}_T \dot{\psi} + \dot{\theta} \tan \xi_T \]
\[ b_1 = -1 \]
\[ b_2 = \sec \xi_T \]
\[ b_3 = -\dot{y}_T \sec \xi_T \sin (y_M-y_T) \]
\[ b_4 = -r\dot{y}_T - 2\dot{r} \tan \xi_T + 2r \dot{\phi} \]
\[ b_5 = -V_M \dot{y}_T \sec \xi_T \cos (y_M-y_T) \]
\[ b_6 = -r \tan \xi_T \]
\[ b_7 = -\tan \xi_T \]
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**TABLE B-8**

Error Equation Coefficients: $a_1$ and $b_1$ as functions of the time for the trajectory of Fig. 4.6 with initial conditions $\psi = 0^\circ$, $\phi = 60^\circ$, $x_T = 10,000$; $y_T = 1,000$. 

162
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**TABLE P-5**

Error Equation Coefficients $a_I$ and $b_I$ as functions of the time for the trajectory of Fig. 4.7 with initial conditions $\psi = 180^\circ$, $\phi = 90^\circ$, $x_T = 10,000$, $y_T = 1,000$.

- $a_1 = 2\phi \tan \xi_T - \dot{\xi}_T \tan \xi_T$
- $a_2 = \dot{\phi}^2 + \dot{\phi} \dot{\xi}_T + \dot{\phi} \tan \xi_T$
- $b_1 = -1$
- $b_2 = \sec \xi_T$
- $b_3 = -\dot{\xi}_T \sec \xi_T \sin (\gamma_M - \gamma_T)$
- $b_4 = -\dot{\gamma}_T - 2\dot{\gamma} \tan \xi_T + 2\dot{\phi}$
- $b_5 = -\dot{\gamma}_M \dot{\gamma}_T \sec \xi_T \cos (\gamma_M - \gamma_T)$
- $b_6 = -r \tan \xi_T$
- $b_7 = -\tan \xi_T$
- $N_1 \dot{a}_r$, accel/LOS error
- $N_2 \dot{V}_T$, target airspeed change
- $N_3 \dot{V}_M$, missile airspeed error
- $N_4 \dot{\phi}$, LOS rate error
- $N_5 \dot{\xi}_M$, error of estimating
- $N_6 \dot{\phi}_T$, LOS acceleration error
- $N_7 \dot{a}_r$, accel. $\perp$ LOS, error
TABLE B-6

Error Equation Coefficients \(a_1\) and \(b_1\) as functions of the time for the trajectory of Fig. 4.8 with initial conditions \(x_0 = 180^\circ\), \(\phi = 0^\circ\), \(x_T = 20,000\), \(v_T = 1,000\).
Kinematic Ranging in an Air-Air Missile

Abstract

Methods are explored for estimating Range and Range-Rate from the geometry and kinematics of the air-air homing-missile combat situation rather than from direct measurements. The methods reported herein use the various signals available from the missile autopilot, plus the inertial rotation-rate of the Line of Sight provided by the seeker.

It is shown that acceleration of the target in the radial direction, parallel to the Line of Sight, precludes successful estimation by the classical techniques such as adaptive parameter identification, Kalman filter estimation, or the various minimum-variance estimation methods. A nonlinear estimator is described which estimates the target acceleration and velocity components and can yield accurate range and range-rate estimates if correctly initialized at launch. This estimator is able to take advantage of the many inequalities which constrain the maneuvering of an air-air target. Errors and performance of this estimator are demonstrated by computer simulations.
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