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Electrostatic Waves in Bounded Hot Plasmas

by

Masayuki Omoro

June 1967

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ELECTROSTATIC WAVES IN BOUNDED HOT PLASMAS

by

Masayuki Omura

SUIPR REPORT NO. 156

Technical Report
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Prepared by
Institute for Plasma Research
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ABSTRACT

This report deals with the study of electrostatic waves in bounded hot plasmas and, except for a few dc measurements, is theoretical.

The study is broadly divided into two parts. The first section investigates waves in a diode system with no dc magnetic field applied. A dc analysis of the equilibrium plasma produced within a thermionic diode is undertaken to provide a basis for the rf analysis, and expected plasma densities in the diode are computed for a variety of practical situations. Electrostatic wave resonances in the diode are predicted by using the hydrodynamic model for the nonuniform plasma. The impedance of a uniform plasma diode is obtained by using a kinetic model of the plasma. This model enables us to take into account the end plate electron absorption loss, a process similar to that which causes end plate diffusion in the Q-machine. The absorption loss is found to have a large effect on the impedance of the diode.

The second part of the report deals with the study of guided waves along a cylindrical column of Maxwellian plasma in a magnetic field. A dc study of the plasma column is first conducted. Theoretical density and current profiles are obtained and are compared with the measured results. Since an rf analysis using a self-consistent dc solution is involved, the plasma column is approximated by a uniform plasma with a sharp boundary and no drift. To obtain the rf fields in such a column, a plane-wave solution for an infinite Maxwellian plasma in an applied magnetic field is first obtained. By superimposing infinitely many plane waves, the fields within the column are constructed, and by matching the appropriate fields at the boundary, a dispersion relation is derived.

The solutions to the dispersion relation reveal the existence of a new type of unstable waves. When only the electrons are assumed to respond to the electric field, the surface waves which propagate when $\omega / \omega_p < 1$ are found to be unstable. When the ion motion is included, additional unstable surface waves are obtained whenever $\omega / \omega_i < 1$. A study of the instability shows that it is due to finite Larmor radii effects and is driven by the transverse energy of the particles.
While the "electron" surface wave instability is relatively weak and can easily be stabilized by making $\omega_{ce} > \omega_{pe}$, the "ion" surface wave instability is found to be very strong, and the stabilizing condition of $\omega_{ci} > \omega_{pi}$ is not easily attained in high density plasmas such as those in fusion machines. Thus, the unstable ion surface waves may have a serious effect on the containment of the fusion plasma.
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SYMBOLS

- \( a \) : average radius
- \( a_i \) : expansion coefficient
- \( A \) : vector potential
- \( B \) : plasma radius
- \( B_e \) : magnetic field
- \( \Delta B \) : perturbed magnetic field
- \( \Delta B_{\text{pu}} \) : spectrum of the perturbed magnetic field
- \( \Delta B_{\text{su}} \) : spectrum of the perturbed displacement field
- \( \mathcal{E} \) : electric field
- \( \mathcal{E}_u \) : perturbed electric field
- \( \mathcal{E}_{\text{pu}} \) : spectrum of the perturbed electric field
- \( \mathcal{E}_{\text{su}} \) : perturbed electric field in the mode
- \( \mathcal{E}_{\text{pu},\text{su}} \) : perturbed electric distribution function
- \( f_{\text{pu}} \) : perturbed electron distribution function
- \( f_{\text{pu},\text{su},i} \) : perturbed electron distribution function for the \( i \)th species in a plasma column
- \( F_{\text{pu}} \) : elliptic integral of the first kind
- \( \zeta \) : Hermite polynomial
- \( \zeta_{\text{p}} \) :
\( \mathbf{K} \)

Hankel function

\( \sqrt{-1} \)

unit vectors along \( x \), \( y \) and \( z \) axis

unit tensor

\( I \)

total current density

\( \mathbf{I} \)

modified Bessel function

perturbed conduction current

azimuthal current due to the \( j \)th species
tox and electron emission currents

complete elliptic integral of the first kind

wave number

components of the wave number

gyro-tensor

magnetization vector

electron mass

mass of the \( j \)th species

integer

dc electron density

ion density

electron density

unit vector along the dc magnetic field

density per unit length for the \( j \)th species

polarization vector (perturbed)

zero and first order pressure

reflection coefficient
radius
\sqrt{\frac{\delta kT}{m_1 \omega_{ci}^2}}
cathode radius
coordinate vector in \( r, \theta \) plane
time
temperature
transverse and longitudinal temperature
drift velocity
velocity in one dimension
velocity vector
electron thermal speed, \( \sqrt{kT/m} \)
thermal speed of the \( j \)th species
phase velocity
velocity of the spherical shell plasma
dc potential
coordinate vector
linear coordinates
Hilbert transform of gaussian
\( n_+(0)/n_-(0) \)
\( n_+(L)/n_-(L) \)
arrival rates of electrons, atoms and ions
\( \sqrt{\frac{2}{\omega^2 - \omega_p^2}}/\nu_o \)
\( \sqrt{1 - \alpha^2} \)
free space permittivity
relative dielectric tensor
\( \varepsilon_{ij} \) elements of the dielectric tensor

\( \zeta_n \) 
\[ \frac{(\omega-n_c^2)}{\sqrt{2} \ k_v \theta} \]

normalized dc potential, eV/\( \times T \)

\( \eta \) Boltzmann's constant, \( 1.38044 \times 10^{-23} \text{ joules/K} \)

\( \kappa \) 
\[ \lambda_D, \lambda_{De}, \lambda_{DL}, \lambda_a \] Debye lengths, \( \lambda_D = \frac{v_0}{\omega_p} \)

\( \lambda \) 
\[ \sqrt{2} k_v \theta / \omega_c \]

dummy variables

\( \xi, \xi \) dummy variables

\( \pi \) electric susceptibility tensor for the electrons

\( \pi_j \) electric susceptibility tensor for the \( j \)th species

\( \pi_{ij} \) elements of the susceptibility tensor

\( \rho, \rho' \) normalized radius

\( \rho_1 \) perturbed charge density

\( \rho(x, \xi), \sigma(|x|) \) conductivity kernels

\( \varphi \) dummy variable

\( \emptyset \) rf potential

\( \Psi(a, b; x) \) confluent hypergeometric function of the second kind

\( \mu_i, \mu_e, \mu_a \) emission rates of ions, electrons and neutrals

\( \mu_0 \) free space permeability

\( \mu(\psi) \) tensor relating \( \mathbf{B} \) to \( \mathbf{M} \)

\( \omega \) angular frequency

\( \omega_p, \omega_c, \omega_{pe} \) electron plasma frequency, \( \sqrt{n_e e^2 / m_e} \)

\( \omega_{ce} \) electron cyclotron frequency, \( e B_0 / m_c \)

\( \omega_{pi} \) ion plasma frequency \( \sqrt{n_i e^2 / m_i} \)

\( \omega_{ci} \) ion cyclotron frequency \( e B_0 / m_i \)

\( \omega_j = \mathbf{z} \omega_j \) rotational vector for the \( j \)th species of a plasma column
\[ a_H = \sqrt{\frac{v_c^2 + v_y^2}{2}} \]

\[ a_{H_1} \]

Ion hybrid frequency

\[
\frac{1}{\omega_H} = \frac{1}{\omega_{Hi}} - \frac{1}{\omega_{pi} - \omega_{ci}} + \frac{1}{\omega_{ce}}
\]
ACKNOWLEDGEMENT

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I. INTRODUCTION

The object of this study is an investigation of electrostatic waves in bounded hot plasmas. While the bulk of prior work on electrostatic waves is restricted to infinitely extended plasmas, our study emphasizes the effects due to finite plasma boundary. Originally we hoped to avoid complications due to instabilities by restricting our work to near equilibrium plasmas. Yet in the study of a perfectly Maxwellian magnetoplasma column, we discovered unstable surface waves which are due to finite Larmor radii effects.

Plasmas in perfect thermostatic equilibrium are generated inside a black body, and our work on the subject of equilibrium plasma generation and the waves in such plasmas is summarized in Section A below. A near equilibrium column of plasma is obtained in a magnetic field between electron and ion emitting end plates. Prior work on electrostatic waves in such plasma columns and our own work which lead to the discovery of unstable waves is summarized in Section B. A brief review of known instabilities in Section C places the newly discovered instability into proper perspective.

A. PLASMAS IN PERFECT THERMOSTATIC EQUILIBRIUM

Theory and interpretation of experimental data on plasmas often suffer because the deviations from equilibrium are not known exactly. A situation in which all of the constituents are in equilibrium may be found inside a black body. A study of the dc states of an equilibrium plasma bounded by thermonically emitting walls is undertaken in Chapter II. An earlier work by Nottingham considers the diode geometry consisting of two parallel infinite walls emitting only electrons. In such a system the electron density within the diode space was found to be highly nonuniform and became very low at a distance of a few Debye lengths away from the wall. We show that under equilibrium conditions a small amount of ions emitted from a thermonic emitter can neutralize the electron space charge to form a plasma in the diode space. This is illustrated by computations based on diode walls made from pure refractory metals. The effects of enhanced ion emission from the walls due to the
introduction of alkali vapor in the diode space are also examined. Some experimental measurements for a sodium plasma are given and compared with the theoretical prediction.

Electrostatic waves within the equilibrium plasma diode are studied in Chapter III. The nonuniform sheath effects are investigated by means of the hydrodynamic model of the plasma. Previously the hydrodynamic model was used by Weissglas and later by Parker to explain the Dattner resonances in a nonuniform plasma column. As opposed to these previous works on free boundary sheaths, our study is directed to the cathode sheaths bounded by an emitter.

In the second part of Chapter III we present a study of end plate loss effects by using a kinetic model of the plasma. The end plate losses are due to the same mechanism which causes end plate diffusion in a plasma column such as that produced in the Q-machine. When particles in the magnetized plasma column are absorbed by the end plate and re-emitted with different guiding centers, the net result is a diffusion of the particles away from the column. In the diode system consisting of infinite parallel planes, we are not concerned with diffusion, but the process of absorption and re-emission causes rf energy loss in the diode. An electron emitted from the wall acquires rf energy as it traverses the diode space, and the entire energy is lost as it is absorbed at the other wall. Previous work by Hall used kinetic theory in considering a plasma capacitor but assumed perfect reflection of the particles at the boundary and thereby neglected the end plate loss effects. In our study we include the absorption effect at the end plate, but because of the extreme complexity of kinetic analysis, we were able to study only the case of a uniform plasma in the diode space.

B. ELECTROSTATIC WAVES IN A PLASMA COLUMN

In a practical plasma the emitting plates are not infinite in extent. In fact, plasmas are often produced between two circular emitting plates which are separated by a distance several times greater than the plate radius. Confinement of the plasma in such a column is achieved by applying a magnetic field along the axis. A dc study of such a column of magnetized plasma is given in Chapter IV. Theoretical density and
current profiles are computed from an appropriate solution to the Boltzmann equation and a comparison is made with some experimental results. The study indicates that the plasma column may be approximated by a uniform plasma with a sharp boundary. To simplify the study of waves in the plasma column, it is essential to assume that the column is uniform.

In 1957 Gould and Trivelpiece\cite{7} used the cold plasma model to predict electrostatic wave propagation along a uniform plasma column such as the one shown in Fig. 1.1. The dispersion relations for cylindrically symmetric waves along such a column are shown in Fig. 1.2. When \( \omega_c > \omega_p \), propagation is confined within two frequency ranges, \( 0 < \omega < \omega_p \) and \( \omega_c < \omega < \sqrt{\omega_p^2 + \omega_c^2} \). Whether the plasma completely fills the waveguide or not, these passbands are unaltered as long as \( \omega_c > \omega_p \). However, when \( \omega_c < \omega_p \), filled and unfilled waveguides have different passbands. For the completely filled waveguide, the two passbands are \( 0 < \omega < \omega_c \) and \( \omega_p < \omega < \sqrt{\omega_p^2 + \omega_c^2} \). When the plasma does not fill the waveguide, the upper passband is as before \( \left( \omega_p < \omega < \sqrt{\omega_p^2 + \omega_c^2} \right) \), but the lowest order mode of the low-frequency band propagates in the band \( 0 < \omega < \sqrt{\left(\omega_p^2 + \omega_c^2\right)/2} \) as shown by the dotted line. This particular mode is a

![Fig. 1.1. Configuration of plasma in a cylindrical waveguide.](image)
Dispersion relations for waves in a cylindrical column of cold plasma are given for two cases. The diagram on the right shows dispersion relations with $\omega_c = 1.2 \omega_p$ and on the left dispersion relations with $\omega_c = 0.5 \omega_p$ are shown. The quantity $ka$ is the product of wavenumber and the waveguide radius.

A surface wave in the frequency range $\omega_c < \omega < \sqrt{(\omega_p^2 + \omega_c^2)/2}$ such that the fields are strongest near the edge of the plasma. Such a surface mode cannot be excited in a filled waveguide. In Chapter VI attention is directed toward this surface wave which becomes unstable when the temperature effects are included properly.

Agdur and Weissglas used the hydrodynamic model of the plasma to point out the temperature effects. Later Jayson and Lichtenberg used an electron plasma model with Maxwellian velocity distribution along the magnetic field but with zero transverse temperature to compute dispersion relations for waves in filled waveguides. Kuehl et al. used a fully thermal plasma model but with infinite magnetic field to compute the dispersion relation in the identical geometry. The models of Lichtenberg and Jayson and Kuehl et al. yield Landau damping of the modes, while the hydrodynamic model does not. The study of waves in fully thermal unbounded magnetoplasma has been conducted by several authors, but the waves in a magnetoplasma column have not been treated before due to the extreme computational complexity involved. This paper reports the study of waves in a magnetoplasma column and the discovery of surface wave instabilities.
In preparation for the study of electrostatic-wave propagation in a magnetized plasma column, a dielectric tensor appropriate for an infinitely extended Maxwellian plasma is obtained in Chapter V. Our dielectric tensor is simpler than the "dielectric" tensor presented by Stix.\textsuperscript{13} The simplicity of our dielectric tensor results from subdividing the plasma current into polarization and magnetization currents. A cylindrical wave solution is appropriate for a uniform column of Maxwellian plasma is synthesized from plane waves and the dispersion relation for the waves is obtained in Chapter VI. At low magnetic fields ($\omega < \omega_p$) the surface wave along the column of a Maxwellian electron plasma is shown to be unstable. The nature of the instability is studied by means of Derfler's stability criteria.\textsuperscript{14,15} The surface waves which occur near the ion cyclotron frequency when $\omega_i < \omega_p$ are also found to be unstable. Since the complete analysis is so vast, only the lowest-order cylindrically symmetric mode is numerically analyzed.

C. INSTABILITIES

Plasma instabilities can be divided into two general groups, microinstability and macroinstability. The instabilities which cannot be derived from the standard magnetohydrodynamic equations but which depend on detailed microscopic equations for the plasma are classified as microinstabilities. We are interested primarily in the microinstabilities which occur in a finite geometry.

Among the macroinstabilities in a current carrying plasma are the well-known kink and sausage instabilities which are caused by the interaction of the plasma with its self-magnetic field. Another instability which occurs in a plasma with a directed current is the Kadomtsev instability.\textsuperscript{16} The Kelvin-Helmholtz instability\textsuperscript{17} occurs in a stratified fluid in which the adjacent layers are in relative motion.

The above instabilities can be obtained from the fluid model of the plasma. The second class of instabilities, the microinstabilities, may be divided into two broad groups. The first are those instabilities which are caused by the peculiarities of the velocity distribution and may be called velocity space instabilities. The second group of instabilities depends on the plasma geometry. A simple example of the first
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Among the macroinstabilities in a current-carrying plasma are the well-known kink and sawtooth instabilities which are caused by the interaction of the plasma with its self-magnetic field. Another instability which occurs in a plasma with a directed current is the Kelvin-Helmholtz instability. The Kelvin-Helmholtz instability occurs in a stratified fluid in which the adjacent layers are in relative motion.

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type of the zero-stress surface. This type occurs when charged particles collide. Veer's square instability occurs when there is temperature anisotropy at the plasma surface. Inobuchi and Taniuti investigated the characteristics of a plasma with two temperature along the magnetic field but with non-adiabatic variation of the perpendicular direction. Their model can be viewed under the extreme temperature anisotropy.

Our interest is primarily with instabilities of the second group. Veer's rectangular square instability is independent of plasma parameters. Those in the second group require resonant wave-particle interactions between the charged particles in the instability to produce the resonant structures, rather than electron temperature. The instabilities arise because of resonant between the charged particles in the instability to produce the resonant structures, rather than electron temperature.
II. DC STATES OF HOLLOW CATHODE PLASMA

Often plasmas produced in the laboratory differ from a true thermodynamic equilibrium plasma. In such nonequilibrium plasmas the energy distribution of the constituents (electrons, ions, neutrals and radiation) deviates significantly from Boltzmann's probability law \( \exp \left( \frac{-E}{RT} \right) \). These deviations are an apparently inexhaustible source for the discovery of new instabilities, designed by nature to ultimately restore equilibrium. Theory and interpretation of experimental data frequently suffer because the deviations from equilibrium are never known exactly. A situation where all constituents are as close as possible to strict thermodynamic equilibrium may be found within a boundary consisting of walls which thermionically emit electrons, ions, and neutrals. In this chapter we will examine the dc properties of equilibrium plasmas bounded by such thermionically emitting walls. Numerical results are given for some practical situations.

A. PLASMA DIODE

In this section we investigate theoretically the plasma produced between the two infinite plane emitting walls as shown in Fig. 2.1. There is no net dc potential across the diode, and the emitters emit electrons and ions in some known number ratio \( \frac{Z}{A} = n_+ (L) / n_-(L) \), which is fixed by thermodynamic considerations of the material.

![Diagram of Emitter Diode](image-url)
In the past a great deal of work has been done on diodes consisting of thermionic emitters. The majority of these papers deal with the study of dc and oscillating states with a nonzero dc potential applied across the diode. An early work on the thermionic diode with zero net potential was given by Nottingham, who treated the problem in the absence of ions. Without the ions he found that the electron density was highly nonuniform, and if the spacing was much greater than the Debye length, the density at the midplane was very low. Eichenbaum and Hernqvist included the ion emission but studied the problem using the collisionless Boltzmann equation, never allowing the empty regions of the phase space to fill up. Figure 2.2 illustrates the model used by Eichenbaum and Hernqvist with the shaded region of the ion phase space unfilled. Such a nonequilibrium analysis is not realistic when there is no net potential across the diode. Langmuir examined an equilibrium case but with one of the planes at infinity.

In our problem, we assume a strict thermodynamic equilibrium and fill the phase space completely according to Boltzmann's probability law, and hence the scale-height law is used for the electron and ion densities. By taking the potential to be zero at the midplane \((x = 0)\), Poisson's equation becomes

\[
\frac{d^2 V}{dx^2} = -\frac{e}{\varepsilon_0} \left[ n_+(x) - n_-(x) \right]
\]

\[
= -\frac{e}{\varepsilon_0} \left[ n_+(\zeta) \exp\left[ -\frac{eV}{kT} \right] - n_-(\zeta) \exp\left[ eV\right] \right] \ . \quad (2.1)
\]
At this point normalized quantities
\[ \xi = \frac{x}{\sqrt{2} \lambda_{D0}} \]
\[ \eta = \frac{eV}{a^2} \]
and
\[ \alpha^2 = \frac{n_+(0)}{n_-(0)} \]
with the Debye length at \( x = 0 \),
\[ \lambda_{D0} = \left( \frac{\varepsilon_0 \kappa T}{n_-(0) e^2} \right)^{1/2} \]
are introduced, and Eq. (2.1) is thus reduced to
\[ \frac{d^2 \eta}{d\xi^2} = 2[\eta - \alpha^2 e^{-\eta}] \quad (2.2) \]
Let us first consider the case of \( \alpha < 1 \), the "electron" rich case. The boundary condition at \( \xi = 0 \) is
\[ \eta = 0, \quad \eta' = 0. \]
By introducing a new variable
\[ w^2 = e^\eta \quad (2.3) \]
the solution to Eq. (2.2) is found to be
\[ F(\alpha, w) = K(\alpha) - \xi \quad (2.4) \]
where \( F(\alpha, w) \) is the elliptic integral of the first kind\(^{27}\) with modulus \( \alpha \),
- 9 -
\[ F(x, u) = \int_{x}^{u} \frac{dt}{\sqrt{(1-t^2)(1-\alpha^2 t^2)}} \]  \hspace{1cm} (2.5) 

and \( K(\alpha) = F(x, 1) \) is the complete elliptic integral. Tables for these integrals are available.

By inverting Eq. (2.4), we obtain

\[ v = \text{sn}[K(\alpha) - \xi, \alpha] \]  \hspace{1cm} (2.6) 

where \( \text{sn}[K(\alpha) - \xi, \alpha] \) is the Jacobi elliptic function \( \text{sn} \) with modulus \( \alpha \) and argument \( K(\alpha) - \xi \). Hence,

\[ \tau = -2 \ln \text{sn}[K(\alpha) - \xi, \alpha] \]  \hspace{1cm} (2.7) 

Equation (2.7) enables us to calculate the potential as a function of \( z \) provided \( \alpha \) is known. Generally, \( \tau \) is a quantity we seek in terms of a similar quantity given at the emitter surface. Let

\[ \beta^2 = \frac{n_p(L)}{n_e(L)} \]

We know from the scale-height law that

\[ n_e(L) = n_e(L) \exp(-\zeta_L) \]

and

\[ n_p(L) = n_p(L) \exp(-\zeta_L) \]

Hence,

\[ \frac{n_p(L)}{n_e(L)} = \frac{n_p(L)}{n_e(L)} \exp(-\zeta_L) \]

or

\[ \beta^2 = \frac{n_p(L)}{n_e(L)} \exp(-\zeta_L) \]

\[ -10 - \]
The subscript \( L \) denotes quantities evaluated at the surface of the emitter. Since

\[
S = \frac{x}{\sqrt{2}} \frac{\mu}{\mu_D} \frac{x}{\sqrt{2}} \frac{\mu_L}{\mu_D} = \frac{x}{\sqrt{2}} \frac{x}{\sqrt{2}} \frac{\mu_L}{\mu_D} \left( \frac{\mu_L}{\mu_D} \right)^{1/2} = \frac{x}{\sqrt{2}} \frac{x}{\sqrt{2}} \frac{\mu_L}{\mu_D} \exp \left( -\frac{\mu_L}{\mu_D} \right) = \frac{x}{\sqrt{2}} \frac{x}{\sqrt{2}} \frac{\mu_L}{\mu_D} v_L,
\]

Eq. [2.4] can be rewritten as

\[
\frac{x}{\sqrt{2}} \frac{\mu_L}{\mu_D} v_L = \frac{E(\nu) - F(\nu, w)}{v_L} \tag{2.5}
\]

We evaluate Eq. [2.1] at \( x = L \) and obtain

\[
\frac{x}{\sqrt{2}} \frac{\mu_L}{\mu_D} v_L = \frac{E(\nu) - F(\nu, w)}{v_L} \tag{2.9}
\]

With Eq. [2.9] a family of curves \( \frac{x}{\sqrt{2}} \frac{\mu_L}{\mu_D} v_L \) versus \( \nu \) with \( \nu \) as a parameter, can be plotted as shown in Fig. 2.1. From the graph \( \nu \) can be determined for given values of \( \mu \) and \( \mu_L \left( \frac{\sqrt{2}}{\mu_D} \right) \) and, consequently, \( \mu, \mu_L \) and \( v, w \) can be obtained.

The following properties of the Jacobi elliptic function and the elliptic integrals are worth noting here. For \( \nu = 1 \) we have

\[
\sin \nu x = \sin x - \frac{\nu^2}{2} \cos x \sin x \cos x \ldots \tag{2.10}
\]

\[
E(\nu) = \frac{\nu^2}{2} \cos \nu \left( 1 + \frac{\nu^2}{3} \right) \ldots \tag{2.11}
\]

Thus, for \( \nu = 1 \), Eq. [2.7] becomes

\[
x = -2 \mu \left[ \sin \left( \frac{\nu}{2} - \frac{\nu}{4} \right) \right]
\]

which is the relation obtained by Nutting for the case of a pure electron cloud.
FIG. 2.3. RATIO OF ELECTRON AND ION DENSITY, 
\( \alpha^2 = \frac{n_e(0)}{n_i(0)} \), IN MID-PLANE AS A FUNCTION OF 
SEPARATION DISTANCE \( 2L \) IN UNITS OF DEBYE LENGTH 
\( \lambda_L \) FOR DIFFERENT VALUES OF \( \beta^2 = \frac{n_+(L)}{n_-(L)} \).
For $\alpha$ approaching unity we have

$$sn(u,\alpha) \approx \tanh u + \frac{1}{4} (1-\alpha^2) \sech^2 u [\sinh u \cosh u - u] \ldots \quad (2.12)$$

and

$$K(\alpha) \approx \ln \left[ \frac{4}{(1-\alpha)^{1/2}} \right] \ldots \quad (2.13)$$

For small values of $w$ or $\sqrt{\beta/\alpha}$, we also have

$$F(\alpha,w) \approx w - \frac{1}{6} w^2 (1+\alpha^2) \ldots \quad (2.14)$$

By shifting the origin to the wall,

$$\xi' = - \xi + L,$$

using Eq. (2.9) and letting the other wall go to infinity ($\alpha \rightarrow 1$), we obtain from Eq. (2.7)

$$\eta \approx - 2 \ln sn[F(1,\sqrt{\beta}) + \xi', 1]$$

$$\approx - 2 \ln sn[\sqrt{\beta} + \xi', 1]$$

$$\approx - 2 \ln [\tanh (\sqrt{\beta} + \xi')]$$

which is the result given by Langmuir for the semi-infinite plasma bounded by an emitting plane.

Thus far, we have been concerned with electron rich cases. The ion rich cases can be derived simply by re-defining the parameters $\alpha$, $\beta$, $\eta$ and $\lambda_0$ as shown in the table below.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Electron Rich</th>
<th>Ion Rich</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta^2$</td>
<td>$n_+(L)/n_-(L)$</td>
<td>$n_-(L)/n_+(L)$</td>
</tr>
<tr>
<td>$\alpha^2$</td>
<td>$n_+(0)/n_-(0)$</td>
<td>$n_-(0)/n_+(0)$</td>
</tr>
<tr>
<td>$\lambda_0^2$</td>
<td>$\kappa T \epsilon_0/(e^2 n_-(0))$</td>
<td>$\kappa T \epsilon_0/(e^2 n_+(0))$</td>
</tr>
<tr>
<td>$\eta$</td>
<td>$eV/\kappa T$</td>
<td>$-eV/\kappa T$</td>
</tr>
</tbody>
</table>
With the new definitions, the Poisson's equation becomes identical to Eq. (2.2) with $\alpha \leq 1$.

**B. NUMERICAL EXAMPLES WITH PURE METAL EMITTERS**

To give some idea of the types of densities and potential profiles which can be expected inside a "black body," we give some numerical results based on emission data of pure refractory metals. In practice the quantity $\beta^2 = n_+(L)/n_-(L)$ is not easily obtained because the data on ion emission rate from various metals are scarce. For the numerical computations we use the following empirical formula on ion emission from molybdenum emitter as given by Wright\textsuperscript{29} and the Richardson equation for electron emission. Current densities are given in amps per cm$^2$.

\begin{align*}
\ln J_+ &= 28.39 - \frac{e\varphi_+}{kT} - 0.453 \ln T + 6.22 \times 10^{-4} T, \quad (2.16a) \\
J_- &= AT^2 \exp \left( - \frac{e\varphi_-}{kT} \right) \quad (2.16b)
\end{align*}

with

- $A = 55$ amp/cm$^2$
- $\varphi_+ = 8.3$ volts
- $\varphi_- = 4.37$ volts

In the computation the emitters are assumed to be 1 cm apart. In Fig. 2.4 plots of $n_0$, $n_+(0)$, $n_-(0)$ and $\lambda_{D0}$ versus $1/T$ are given. The neutral density $n_0$ is obtained from the data provided by Honig.\textsuperscript{30} The graphs show that above 2000 $^\circ$K the space charge is practically neutralized at $x = 0$. For $T < 1800$ $^\circ$K the diode space consists almost entirely of electrons, the density is very low and the Debye length is of the order of the diode separation. Figs. 2.5 and 2.6 give potential and density profiles for $T = 2200$ $^\circ$K.

We were somewhat startled to find that the space charge neutralization requires only a small amount of ion emission from the walls. For the example shown in Fig. 2.6, the ion density at the wall is $10^6$/cc, which
FIG. 2.4. ELECTRON DENSITY, $n_e(0)$, ION DENSITY, $n_+(0)$, DEBYE LENGTH $\lambda_{DO}$ IN MID-PLANE, AND THE NEUTRAL DENSITY $n_0$ AS A FUNCTION OF $1/T$ FOR A PURE MOLYBDENUM Emitter.
corresponds to an ion current density of $10^{-9}$ amps/cm$^2$. Even with such a small ion emission rate, we can derive a neutral plasma whose density is orders of magnitude greater than that predicted by the analysis of Nottingham (ions absent) or Eichtenbaum and Hernquist (nonequilibrium analysis with unfilled regions in phase space).

The midplane electron density, ion density and neutral density versus $1/T$ for tungsten and niobium are shown in Figs. 2.7 and 2.8. For tungsten the data on the neutral density are obtained from graphs provided by Honig, and the following empirical formula as given by Smith$^{31}$ is used to compute the ion density at the tungsten emitter.
FIG. 2.6. ELECTRON DENSITY $n_e$, AND ION DENSITY $n_i$, AS FUNCTIONS OF THE DISTANCE FROM THE MID-PLANE.
FIG. 1. ELECTRON DENSITY $\rho$ AT THE MID-PLANE AND THE NEUTRAL DENSITY $n_0$ AS FUNCTION OF $z/L$ FOR A PURE TUNGSTEN CATHODE.
\[ \log_{10} \beta_0 = 0.367 \log_{10} \beta + 1.549 \cdot 10^{-7} - \frac{2.347}{2.347 - q} = 22.49 \]  

(2.17)

with \( \beta_0 = 11.9 \) units. The result for tungsten emitters is qualitatively similar to that of the molybdenum emitter. The calculation for the molybdenum emitter is based on the ion emission data obtained in the laboratory by the author. \[ \text{[Reference]} \]

The emission ion emission data obeyed the following empirical relation:

\[ \log i_i = 24.1 - \frac{10}{e^2} \log T - 2 + 2 \cdot 10^4 T \]

with \( T \) in K units. The current densities are given in amps per cm².

The neutral density for molybdenum is obtained from vapor pressure data provided by Benzaquen. \[ \text{[Reference]} \]

C. \textit{EMISSION FROM A HOLLOW CYLINDRICAL EMITTER}

We now consider the cylindrical emissor, as shown in Fig. 2.5, to see the effect of the geometry on the plasma density and the sheath potential. Under the condition

\[ \frac{2}{\pi} = \frac{x}{x_0} \cdot 1 \]

Penrose's emission is given as

\[ \frac{F(x)}{F_0} = \frac{1}{x} \frac{dF}{dx} - \frac{2}{x_0} \left( \frac{x}{x_0} \right)^{1/2} \left( \frac{x}{x_0} \right)^{1/2} \]  

(2.14)

Again the scale-height law is assumed to govern the density distribution, and hence,

\[ \frac{F(x)}{F_0} = \frac{1}{x} \frac{dF}{dx} - \frac{2}{x} \left( \frac{x}{x_0} \right)^{1/2} \exp \left[ - \frac{dF}{dF_0} \right] - \frac{2}{x_0} \left( \frac{x}{x_0} \right)^{1/2} \exp \left[ - \frac{dF_0}{dF_0} \right] \]  

(2.15)
FIG. 2.9. CYLINDRICAL EMITTER OF RADIUS $a$.

where $n_e(c)$ and $n_i(c)$ are the ion and electron densities at $r = c$.
Normalization similar to those of Section A yields

$$\frac{d^2 n}{dr^2} + \frac{1}{r} \frac{dn}{dr} = e^n - a^2 e^{-n}$$

(2.20)

where

$$s = \frac{r}{a_c} = \sqrt{\frac{e^2 n_i(c)}{c^2 T_i}}.$$

Equation (2.20) cannot be solved in terms of classical transcendental equations. An analytic solution does exist, however, when the ions are absent ($c = 0$). For this special case the solution is given by

$$n = -2 \frac{de}{dr} \left(1 - \frac{r^2}{a^2}\right).$$

(2.21)
For the general case of nonzero \( a \), one must resort to numerical integration. The following normalization is found to be convenient for numerical analysis:

\[
p' = \sqrt{a} \rho = \frac{r}{\lambda_{DC}} \sqrt{a} = \frac{r}{\lambda_0} \sqrt{a}, \quad \lambda_0 = \sqrt{\frac{\epsilon T \epsilon_0}{\varepsilon^2 n(x)}}
\]

The variable \( p' \) measures the radius in units of \( \lambda_0/\sqrt{a} \), which is a known quantity, while the variable \( \rho \) is normalized to \( \lambda_{DC} \), which is not known a priori. Poisson's equation now becomes

\[
\frac{d^2 \eta}{dp' \rho^2} + \frac{1}{\rho} \frac{d \eta}{dp'} = \frac{1}{\lambda_0} \left[ e^{-\eta} - \alpha^2 e^{-\eta} \right]
\]

or by re-arranging the right hand side we have

\[
\frac{1}{\rho} \frac{d}{dp'} \left( \rho \frac{d \eta}{dp'} \right) = \frac{1}{\lambda_0} \left[ e^{-\eta} + \alpha^2 e^{-\eta} \right]
\]

where

\[\alpha^2 = 1 - \alpha^2\]

Equation (2.22) introduces a parameter \( \alpha^2 \) which is generally a small number directly proportional to the fractional deviation from neutrality at \( \rho' = 0 \). For computational purposes, we used Eq. (2.23). For a given value of \( \alpha \), a series solution can be obtained for small values of \( \rho' \). Let

\[\eta = a_1 \rho' + a_2 \rho'^2 + a_3 \rho'^3 + a_4 \rho'^4 + \ldots\]

Then

\[
\frac{1}{\rho} \frac{d}{dp'} \left( \rho \frac{d \eta}{dp'} \right) = \frac{a_1}{\rho} + 2a_2 \rho' + 3a_3 \rho'^2 + 4a_4 \rho'^3 + \ldots
\]

21
s \sinh \gamma = a_1 e^{\gamma} + a_2 e^{-\gamma} + \left(a_3 + \frac{1}{2} a_4 \right) \gamma \ldots

and

e^{-\gamma} = 1 - a_1 e^{\gamma} + \left(a_2 - a_3 - \frac{1}{2} a_4 \right) \gamma \ldots

By inserting these relations into Eq. (2.23) and equating the coefficients of the like power terms, we get

\begin{align*}
a_{2n+1} &= 0 \quad \text{for } n = 1, 2, 3 \ldots \\
a_2 &= \frac{\gamma^2}{\alpha} \\
a_4 &= \frac{1}{12\alpha} \left[ (2 - 4) a_2 \right] \\
a_6 &= \frac{1}{36\alpha} \left[ (2 - 4) a_4 - \frac{4 a_4^2}{2} \right] \\
a_8 &= \frac{1}{90\alpha} \left[ (2 - 4) a_6 - \frac{6 a_6^2}{2} - \frac{4 a_6 a_2}{3} + 4 a_2 a_4 \right]
\end{align*}

Using this solution as the starting point, the differential equation is integrated numerically by a SUBALGOL procedure called "Adams Predictor Correction Method," available at the Stanford Computation Center. The integration procedure is repeated with different values of \( \epsilon \) until the boundary condition at the wall is satisfied. The resulting potential profiles for a molten-platelet emitter are shown with the temperature as a parameter in Fig. 2.1. The diameter of the emitters is taken to be 1 cm. The curves show that the density and potential profiles for the cylindrical system look like those of the planar system when the radius is much larger than the Debye length, \( \lambda_D \).
FIG. 2.11. POTENTIAL PROFILE INSIDE A HOLLOW CYLINDRICAL MOLTEN NICKEL EMISSOR FOR DIFFERENT TEMPERATURES.

3. CONTACT IONIZATION PLASMAS

In 1932, Langmuir and Kingston showed that, when a neutral atom strikes a hot metal whose work function is greater than the ionization potential of the atom, the atom bounces off the metal as an ion with high probability. Since Langmuir's demonstration, many papers have been written on this subject, and, in fact, many of the plasma generators designed today are based on this phenomenon. An excellent theoretical treatment of the surface phenomena involved with the so-called contact ionization process is given in a recent series of papers by Levine and Oyngusius. They derive the emission rates of the ions, electrons and atoms under thermodynamic equilibrium conditions. The emission rates $\gamma_i$, $\gamma_e$ and $\gamma_a$ are given in terms of temperature and the
fraction \( \phi \) of the metal surface covered by a mono-layer of adsorbent. Under steady state conditions, ion and atom emission rates are equated to the arrival rates of the ions and atoms \( \mu_1 \) and \( \mu_a \).

\[
\gamma_a + \gamma_1 = \mu_a + \mu_1
\]

To obtain the steady state coverage, we assume \( \gamma_1 = \mu_1 \), and the atom emission rate is equated to the atom arrival rate which is computed from the vapor pressure of the adsorbent material. In this manner, the steady state coverage \( \phi \) is determined and hence emission rates can be computed. Strictly speaking, our computations violate the thermodynamic equilibrium condition since we assume that the vapor temperature is not necessarily equal to the metal temperature. Using the resulting emission data, we compute the density and potential profiles for several different sets of conditions. Shown in Figs. 2.11 and 2.12 are the profiles for a typical case with cesium vapor and tungsten emitters. Note that the space charge is neutral in the greater portion of the diode space despite the large difference in the electron and ion densities at the wall.

![Fig. 2.11. Theoretical DC Potential Profile for a Diode with Tungsten Walls and Cesium Vapor.](image)

For the computation, the wall temperature was 1300 °K and the vapor temperature was 5 eV. The wall separation was taken to be 1 cm.
The midplane densities for sufficiently large plate separation \((L \gg \lambda_{DO})\) are computed for tantalum emitters with sodium, potassium and cesium vapors. Shown in Figs. 2.13, 2.14 and 2.15 are the midplane plasma frequency and the Debye lengths as functions of emitter temperature with the vapor temperature as a parameter. In the range of emitter temperatures covered, the Ta-Na system is electron rich, the Ta-K system is either electron or ion rich depending on the emitter temperature, and the Ta-Cs system is ion rich. It is interesting to note that even with sodium, whose ionization potential is higher than the work function of tantalum, a good workable plasma can be obtained. The probability of a sodium atom being ionized by tantalum is small, but a sufficiently high number become ionized to neutralize the electron space charge in the diode space.
FIG. 2.13. PLASMA FREQUENCY AND THE DEBYE LENGTH FOR A SODIUM PLASMA PRODUCED BETWEEN TWO TANTALUM PLATES VS PLATE TEMPERATURE. Sodium temperature is used as a parameter.

FIG. 2.14. PLASMA FREQUENCY AND THE DEBYE LENGTH FOR A POTASSIUM PLASMA PRODUCED BETWEEN TWO TANTALUM PLATES. The quantities are plotted as functions of tantalum temperature with potassium vapor temperature as a parameter.
E. DENSITY MEASUREMENTS WITH SODIUM PLASMA

Sodium plasma is produced in a diode which is described in the schematic drawings of Figs. 2.16 and 2.17. The diode consists of two circular tantalum buttons of diameter 1.5 inches separated by 1 inch. The buttons are heated by electron bombardment, and the whole tube is immersed in an oil bath to keep the tube temperature uniform and low. Provision is made also to vary the oil bath temperature. The two gridded probes are for the Landau wave measurements discussed elsewhere. The density is measured by a Langmuir probe consisting of a long wire of 15/1000 inch diameter. The probe is provided with an accurate meter which indicates the probe distance from the center line, and provisions are made to permit angular motion of the probes as well. The buttons are surrounded by heat shields which also act as guard rings. To confine the plasma in the radial direction and make the diode closely approximate an infinite system, an axial magnetic field is applied.
FIG. 2.16. SCHEMATIC DRAWING OF THE PLASMA TUBE. A = sodium vapor source; B = tantalum plates; C = gridded probe; D = heat shield (guard ring); E = bomborder filament; \( V_B \) = bomborder power supply; \( V_F \) = filament power supply.

FIG. 2.17. SCHEMATIC DRAWING OF THE PROBE SYSTEM. A = tantalum plate; B = gridded probe; C = Langmuir probe.
The plasma density is found to be highly dependent on the guard-ring potential, as was noted also by Carlin. The curves of Fig. 2.1a show the probe characteristics for various guard-ring potentials with a magnetic field of 3.5 Gs applied. Shown in Fig. 2.1b are the experimental and theoretical plasma frequencies as functions of the button temperature. The data are taken with the probe tip 2 cm from the center line, the guard ring biased positive and an axial magnetic field of 35 Gs applied. In the analysis of the data, we take that portion of the probe which is inside the cathode radius to compute the effective area of the probe. Hence, this method gives only the average density of the plasma. Nevertheless, the agreement between the theory and the experiment is good.

With the oil bath temperature at 320 °K, the neutral sodium density computed from vapor pressure data is \( n = 1 \times 10^{19} \text{ cm}^{-3} \). The plasma density with the button temperature at 3.2 °K is \( n = 2 \times 10^{17} \text{ cm}^{-3} \), which represents 40 percent ionization. Higher percentage ionization can be achieved at higher button temperatures. When a hard sphere model is used for sodium atoms, the electron neutral collision frequency is about \( 3 \times 10^5 \text{ Hz} \), while the plasma frequency is about \( 1 \times 10^9 \text{ Hz} \) for the conditions above.

From the measurements we can conclude that the sodium plasma is \( \times 10^{-4} \)-suited for low density quiescent plasma experiments. Undesirable effects, such as disappearance of the vapor source, shorting of the electrodes due to sodium condensing and corrosion, are not noticeable. We believe that this is the first sodium plasma produced by the resonance ionization process. On the basis of our computations, Berliner and Simon 

- 29 -
FIG. 2.14. PROBE CURVES FOR WITHOUT CHARGING POTENTIALS. The emitter temperature was at 200°K, and the magnetic field was 10 kGauss. Probe tip was 2.2 cm from the tube exit.

FIG. 2.15. EXPERIMENTAL AND THEORETICAL PLASMA FREQUENCIES VS BULLET TEMPERATURES. The oil bath temperature was 37°K.
FIG. 1. Experimental and theoretical plasma frequencies vs. electron temperature. The electron temperature was 3000°K.
and hence

\[ T = \frac{1}{2} \sum_{i=1}^{N} \frac{1}{t_i} \]

Similarly, we would have used in the previous chapter to arrive at the equation of the curve. The equation of curve

\[ \frac{1}{t} \quad \frac{1}{t} \quad \frac{1}{t} \]

where the curve due to the intensity of current

\[ I \]

and in for the dimensional equation, we have

\[ t \]

and the various terms for the various dimensions are

\[ t \]

Equations and equations made to make the system of equations. To

correct equations, appropriate for the use dimensional, once those are

part not to be perfect, we make further modifications. We analyze the

time and magnitudes and the parts into have the following

and modifications

\[ T \]

and

\[ \sum_{i=1}^{N} \frac{1}{t_i} \]

so that the equation with the use of dimensional equations as

required.
\[
\frac{\partial^2}{\partial t^2} \Phi = \frac{1}{c^2} \frac{\partial^2}{\partial x^2} \Phi + \frac{1}{c^2} \frac{\partial^2}{\partial y^2} \Phi + \frac{1}{c^2} \frac{\partial^2}{\partial z^2} \Phi
\]

where \( \Phi \) is a function of position and time. Assuming that all the mass is concentrated at the center in motion of the electron would be

\[
\Phi(r, t) = \frac{e}{2\pi \varepsilon_0 c^2} \int \frac{m v dr'}{2 \pi} \int \frac{r''}{r'^2} \frac{1}{r'^2} \frac{1}{r''^2} \frac{1}{r''^2} dr''
\]

and the \( m \) and \( v \) components of \( \Phi \) are obtained from \( \Phi \).

The boundary conditions that are necessary to satisfy the Maxwell's equations are obtained by solving the equations with the aid of the boundary conditions and the transmission of the electromagnetic radiations. The boundary conditions at \( r = a \) and \( r = b \) are determined from the boundary conditions on the surface of the plasma. These conditions are

\[
\begin{align*}
\frac{\partial \Phi}{\partial r} & = 0 \\
\Phi & = 0
\end{align*}
\]

By using the following relations and conditions

\(*\) Note that \( j_0 \) is the conduction current.
The text contains a mathematical expression:

\[ \frac{1}{\sum} \frac{1}{\sigma^2} = \frac{1}{\mu^2} \]

The text also contains a paragraph, but the content is not legible due to the quality of the image. It appears to discuss a topic related to statistical analysis or similar fields.
For the attenuation and amplification of the first six resonances for infinitely long tubes spaced at a distance $L$, modes are numbered and the corresponding frequencies are given in Table 1.

$$\frac{\omega}{c} = \frac{n^2}{L}$$

where $c$ is the wave-length and $n$ the wave number. At resonance we get $n = \frac{L}{2}$ where $n$ is an integer and hence in the finite case, the quantity $\frac{L}{2}$ is a constant for each resonance.
FIG. 1. PLots of \( \frac{1}{\lambda} \) versus \( \log \frac{1}{\lambda_{0}} \).

TABLE I. THE FIRST SIX RESONANCE FREQUENCIES OF THE PLASMA AS A FUNCTION OF TEMPERATURE.

The electric field is symmetrical about \( v = 0 \). The odd numbered modes are anti-symmetrical for the even numbered modes.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Mode</th>
<th>Mode</th>
<th>Mode</th>
<th>Mode</th>
<th>Mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T )</td>
<td>( \omega_{0}/2\pi )</td>
<td>( \lambda_{0} )</td>
<td>( \omega_{1}/2\pi )</td>
<td>( \lambda_{1} )</td>
<td>( \omega_{2}/2\pi )</td>
</tr>
<tr>
<td>1000</td>
<td>1.16 x 10^{-7}</td>
<td>0.68</td>
<td>2.12</td>
<td>0.68</td>
<td>3.12</td>
</tr>
<tr>
<td>1500</td>
<td>1.22 x 10^{-7}</td>
<td>0.65</td>
<td>2.17</td>
<td>0.65</td>
<td>3.20</td>
</tr>
<tr>
<td>2000</td>
<td>1.32 x 10^{-7}</td>
<td>0.62</td>
<td>2.24</td>
<td>0.62</td>
<td>3.29</td>
</tr>
<tr>
<td>2100</td>
<td>1.48 x 10^{-7}</td>
<td>0.60</td>
<td>2.28</td>
<td>0.60</td>
<td>3.32</td>
</tr>
<tr>
<td>2200</td>
<td>1.57 x 10^{-7}</td>
<td>0.58</td>
<td>2.34</td>
<td>0.58</td>
<td>3.37</td>
</tr>
<tr>
<td>2300</td>
<td>1.68 x 10^{-7}</td>
<td>0.56</td>
<td>2.40</td>
<td>0.56</td>
<td>3.43</td>
</tr>
</tbody>
</table>
1. Ion-Rich Case

In the ion-rich cases, Bethe resonances of the cathode spots are expected since the electron density near the diode walls is low. "Bethe resonances," the resonances observed on a plasma column, are due to nonuniformity in the plasma. As can be seen from Eq. (1), propagation occurs when the applied frequency is greater than the plasma frequency, and the wave is "cut off" when \( n < n_p \). If the frequency is adjusted to be above the plasma frequency at the edge of the column but below that at the center, the wave propagates inward to the point where it is cut off due to the increasing electron density and is reflected. If the total phase shift is \( 2\pi \) when the wave returns to the wall, a standing wave will be set up and resonance results. An excellent analysis of these resonances for the positive column of a discharge is given by Parker.\(^3\) Shown in Fig. 1.1 are the first electric field strengths of the three nodes for the ion-rich case corresponding to the solution shown in Figs. 2.3 and 2.11. The first resonance occurs at a frequency below the on-plane plasma frequency, and the electric field is

![Diagram of electric field strengths for the first three resonances vs distance from the anode.](image-url)

**FIG. 1.1.** RF ELECTRIC FIELD STRENGTH FOR THE FIRST THREE RESONANCES VS DISTANCE FROM THE ANODE.
concentrated in the smooth region. Clearly this resonance is like the
furthest resonance. The resonance frequency; of the second, third and
higher order modes are greater than \( \nu_{pe} \) and the fields for these
penetrate into the plasma. The higher order mode frequencies are very
close together and, hence, the modes may not be detectable experimentally.
The first and second resonances are spaced far enough apart so that the
first resonance may be detectable.

2. END PLATE LOSES

An electron acquires rf energy from the electric fields as it traverses the diode space, but the entire rf energy is lost when the electron
is absorbed by the wall. In this section, a kinetic model of the plasma is used to study this end plate loss effect.

We shall assume that the plasma is uniform within the tube. Then
the linearized Boltzmann equation for a uniform one-dimensional colli-
sionless electron plasma is given by

\[
\frac{\partial f}{\partial t} + \frac{\partial}{\partial x} \left( v f \right) - \frac{1}{2} \frac{\partial}{\partial x} \left( v^2 f \right) = \frac{e E(x,t)}{m} \nabla f
\]

By Laplace transforms in time, we obtain

\[
\frac{\partial}{\partial x} \left( \frac{1}{2} v f \right) = \frac{e E(x,t)}{m} \frac{\partial f}{\partial x}
\]

where \( p = \frac{1}{2} \). The general solution to Eq. (2.12) is given by

\[
f(x,t) = \exp \left( \frac{E(x,t)}{A} \right) \int_B \frac{e E(p,x)}{m} \int_0^t \frac{1}{2} v f(x,y, \exp \left( - \frac{E(x,y)}{A} \right)) \frac{\partial f}{\partial x} dt
\]

For the analysis, we shall use the geometry shown in Fig. 3.7. We shall
assume further that \( f_0(x) \) is a symmetric function of \( x \). In order to
make \( f_1 \) regular for \( \text{Re}(p) > 0 \), the arbitrary constant in Eq. (2.13)
must be selected so that
where throughout this section we take $v > 0$. To obtain the boundary condition, we shall assume that a certain fraction $\mathcal{R}(v)$ of the particles impinging upon the wall with velocity $v$ is specularly reflected, and the rest perfectly absorbed by the wall. In other words, at $x = -L/2$

\[
 f_1[p, -L/2, v] = \mathcal{R}(v) f_1[p, -L/2, -v]
\]

(3.16)

and at $x = L/2$

\[
 f_1[p, L/2, v] = \mathcal{R}(v) f_1[p, L/2, -v]
\]

(3.17)
The coefficient $R(p)$ can be a complex number which allows for time

delay in reflection. By applying these boundary conditions to Eqs. (3.11)
and (3.15), one evaluates $f_{1}(p, z = L/2, \pi, j)$ to obtain

$$
\begin{align*}
\left. \frac{\partial}{\partial n} \right|_{x=L/2} \mathbf{E} &= \left. \frac{1}{\varepsilon} \mathbf{E} \right|_{x=L/2} - \mathbf{J} = \mathbf{J} + \mathbf{J} \\
\left. \frac{\partial}{\partial n} \right|_{x=L/2} \mathbf{H} &= \left. \frac{1}{\mu} \mathbf{H} \right|_{x=L/2} - \mathbf{J} = \mathbf{J} + \mathbf{J} \\
\left. \frac{\partial}{\partial n} \right|_{x=L/2} \mathbf{H} &= \left. \frac{1}{\mu} \mathbf{H} \right|_{x=L/2} - \mathbf{J} = \mathbf{J} + \mathbf{J} \\
\left. \frac{\partial}{\partial n} \right|_{x=L/2} \mathbf{H} &= \left. \frac{1}{\mu} \mathbf{H} \right|_{x=L/2} - \mathbf{J} = \mathbf{J} + \mathbf{J}
\end{align*}
$$

and

$$
\begin{align*}
\left. \frac{\partial}{\partial n} \right|_{x=L/2} \mathbf{E} &= \left. \frac{1}{\varepsilon} \mathbf{E} \right|_{x=L/2} - \mathbf{J} = \mathbf{J} + \mathbf{J} \\
\left. \frac{\partial}{\partial n} \right|_{x=L/2} \mathbf{H} &= \left. \frac{1}{\mu} \mathbf{H} \right|_{x=L/2} - \mathbf{J} = \mathbf{J} + \mathbf{J} \\
\left. \frac{\partial}{\partial n} \right|_{x=L/2} \mathbf{H} &= \left. \frac{1}{\mu} \mathbf{H} \right|_{x=L/2} - \mathbf{J} = \mathbf{J} + \mathbf{J} \\
\left. \frac{\partial}{\partial n} \right|_{x=L/2} \mathbf{H} &= \left. \frac{1}{\mu} \mathbf{H} \right|_{x=L/2} - \mathbf{J} = \mathbf{J} + \mathbf{J}
\end{align*}
$$

From Maxwell's equation we have

$$
I_{1}(p) = J_{1}(p, x) + p \varepsilon_{0} E(p, x)
$$

$$
= p \varepsilon_{0} E(p, x) - \pi \int_{0}^{\pi} \left( \left. \frac{\partial}{\partial n} \mathbf{E} \right|_{x=L/2} - \mathbf{J} \right) \, dv
$$

(3.2C)

where the total current density $I_{1}$ in the one dimensional system is

known to be independent of $x$. The quantity $I_{1}$ represents the driving

current which was assumed to be zero in the study of the lossless plasma

resonance in Section IIIA. By using Eqs. (3.15) and (3.19), Eq. (3.2C)

can be expressed in the following form

$$
I_{1} = p \varepsilon_{0} E(p, x) + \int_{-L/2}^{L/2} a(|x-\xi|) E(p, \xi) \, d\xi + \int_{-L/2}^{L/2} \rho(x, \xi) E(p, \xi) \, d\xi
$$

(3.21)
The coefficient $E_0$ can be a complex number which allows for time
delay in reflection. By applying these boundary conditions to Eqs. (11) and
(12), we evaluate $|E_0|, |E_1|, \ldots$ to obtain

$$E_0 = V_0 E - \frac{E}{K^2} + \frac{E}{K^2}$$

and

$$E_1 = \frac{E}{K^2} - \frac{E}{K^2}.$$

From these, the equations on the page are derived:

$$|E_0|^2 = |E|^2 - \frac{|E|^2}{K^2} + \frac{|E|^2}{K^2}$$

and

$$|E_1|^2 = \frac{|E|^2}{K^2} - \frac{|E|^2}{K^2}.$$

where the total current density $J$ in the one-dimensional system is
assumed to be independent of $t$. The quantity $J_0$ represents the initial
current which was assumed to be zero in the study of the needless plasma
resonance as Section III. The resulting $|J|^2$ and $|J_1|^2$ can be expressed in the following form:

$$|J|^2 = |E|^2 - \frac{|E|^2}{K^2} + \frac{|E|^2}{K^2}$$

and

$$|J_1|^2 = \frac{|E|^2}{K^2} - \frac{|E|^2}{K^2}.$$
The general case for the conductive barrier for an infinite plasma.

For the finite model consider the infinite plasma conductive barrier to be modified by an additional reflection term, i.e.,

The problem is now reduced to solving the integral Eq. 3.12 for the effective field. The general solution to this equation for an arbitrary external distribution function is difficult to obtain. To get an idea of how the end plate barriers affect the plasma distributions, we shall in the following analyses assume that the plasma has spherical shell distribution of velocities as follows:

\[ f(v) = \frac{1}{\sqrt{2\pi} \sigma v^3} \exp \left( -\frac{v^2}{2\sigma^2} \right) \]

This distribution function when projected into one dimension yields

\[ f(v) = \frac{\theta(v)}{c_0} \exp \left( -\frac{v^2}{2\sigma^2} \right) \]

By identifying the mean squared velocity to the thermal speed, we obtain the following relation.
\[ \frac{\pi}{2} \cdot 2 \cdot \frac{1}{2} \int_0^1 \frac{1}{2} \int_0^1 e^{-x} \]  

Some important characteristics of the 'spherical shell' model is that it does not exhibit London damping. However, it does give results which are simple and informative. We shall further simplify the problem by assuming that the reflection coefficient \( R \) is real and independent of \( x \).  

By inserting Eq. (2.29) into Eqs. (1.22) and (1.2) \( \epsilon = \epsilon_0 \) and \( \mu = \mu_0 \) are found to be  

\[ \epsilon = -\frac{\mu}{\mu_0} \cdot \frac{2.4}{2.4} \cdot \frac{1.4}{1.4} \]  

\[ \mu = \frac{2.4}{2.4} \cdot \frac{1.4}{1.4} \]  

With these values of \( \epsilon \) and \( \mu \), we can differentiate the integral equation (1.3.21) twice with respect to \( x \) to eliminate the integrals, and obtain the following differential equation  

\[ \frac{\partial^2}{\partial x^2} \left\{ \frac{d^2 E(x)}{dx^2} \right\} = \left( \frac{\partial^2}{\partial x^2} \frac{dE(x)}{dx} \right) \]  

with the boundary conditions  

\[ E(-\frac{L}{2}) = \left. \frac{1}{1+x_0} - \frac{\mu_0}{L} \frac{1+x_0}{1+2x} \right| \frac{dE(x)}{dx} \bigg|_{x = -\frac{L}{2}} \]  

and  

\[ E(\frac{L}{2}) = \left. \frac{1}{1+x_0} - \frac{\mu_0}{L} \frac{1+x_0}{1+2x} \right| \frac{dE(x)}{dx} \bigg|_{x = \frac{L}{2}} \]
where \( \mu \) is replaced by \( \nu \). The solution to Eq. (1.2) which satisfies Eqs. (1.2) and (1.3) is given by

\[
\psi(x) = \frac{1}{\omega \mu (1 - \frac{2}{\nu})} \left[ 1 - \frac{\frac{2}{\nu} \cos \frac{x}{2}}{\cos \frac{x}{2} - \frac{1 - \frac{2}{\nu}}{1 - \frac{2}{\nu}}} \right]
\]

where \( \gamma = (\sqrt{1 - \frac{2}{\nu}})^2 \). The field consists of a constant part and a space-varying part. When \( \mu = \nu \), \( \gamma \) is real and the space-varying part varies exponentially in space.

The impedance of the shield is obtained by integrating Eq. (1.3):

\[
Z = -\frac{1}{\nu \mu} \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{\psi(x) dx}{1 + \frac{2}{\nu}}
\]

\[
= \frac{1}{\omega \mu (1 - \frac{2}{\nu})} \left[ 1 - \frac{\frac{2}{\nu} \cos \frac{x}{2}}{\cos \frac{x}{2} - \frac{1 - \frac{2}{\nu}}{1 - \frac{2}{\nu}}} \right]
\]

For the special case of perfect reflection at the walls, \( \mu = 1 \), the impedance is given by

\[
Z = \frac{L}{\omega \mu (1 - \frac{2}{\nu})} \left[ 1 - \frac{2}{\nu} \tan \frac{x}{2} \right]
\]

which is the result obtained by Hall.

Shown in Figs. 3.5 and 3.6 are the real and imaginary parts of the impedance for different values of \( R \). Note that, when \( R = 1 \), resonances occur whenever \( \nu = \frac{(2m+1)}{2} \) where \( m \) is an integer. However, with \( R < 1 \), these resonances disappear except for the lowest order one. Since the spherical shell model does not exhibit Landau damping, the latter in this case can be attributed entirely to the end plate absorption, and the "Q" of the resonance is directly related to the reflection coefficient \( R \).
Fig. 4.1. The imaginary part of the normalized mode impedance versus $\omega_p$. The parameter for the plots is the reflection coefficient $\Gamma$.

Fig. 4.2. The real plot of the normalized mode impedance versus $\omega_p$. 
It is of interest to compare the diode impedance with that of a somewhat similar system having a pair of parallel plane grids immersed in an infinite plasma. By assuming that the grids intercept field lines but not the particles, the impedance between the pair of grids can be obtained as the method described by Simmons. The separation between the grids is assumed to be equal.


\[
 Z_{\perp} = \frac{L}{\ln \frac{1}{\mu} \left( 1 - \frac{\mu}{\mu_p} \right)^2} \left[ 1 - \frac{\mu^2}{\mu_p^2} \cos \left( \frac{2\pi}{\lambda} \frac{L}{\lambda} \right) \right] \tag{3.34}
\]

where \( \mu \) is as defined previously. Shown in Fig. 4.7 is the impedance as a function of \( \mu \). The impedance of the parallel grids differs from

![Normalized Impedance for a Pair of Infinite Grids Immersed in an Infinite Plasma](image)

**FIG. 4.7.** Normalized Impedance for a Pair of Infinite Grids Immersed in an Infinite Plasma.
the diode impedance in several respects. First, the grid system exhibits resonance at the plasma frequency while the resonance for the diode is shifted above \( \omega_p \). Also, the grid impedance is infinite at \( \omega_p \), while the diode impedance has poles along the real \( \omega \) axis only when \( R = 1 \). Another difference is that the impedance of the grids is purely imaginary for frequencies below the plasma frequency while the diode impedance, except for the case of \( R = 1 \), is complex for all frequencies. The loss in the grid system is attributed to the energy carried away by the waves from the grid region to the external region.

C. CONCLUDING REMARKS

Electrostatic resonances in a non-uniform plasma diode are predicted by using the hydrodynamic model and a fictitious boundary condition. These resonances probably will be modified greatly when kinetic models are used to study the diode. The analysis of a uniform plasma diode using the spherical shell velocity distribution shows that wall absorption plays an important role in the determination of the diode impedance. In fact, when complete absorption occurs at the walls, the only resonance of significance occurs when the diode separation is exactly equal to a half-wavelength. Because of the absorption effect, resonances are not found when multiples of half-wavelength "fit" into the diode space. Thus, some of the resonances predicted by the hydrodynamic model may not be observable, especially those found in the electron-rich diode where all of the electron trajectories hit the wall. (See the phase diagram of Fig. 2.2). However, the resonances predicted for the ion-rich case may be observable since most of the electrons in this case are trapped in the diode space and do not strike the wall.

Neither the hydrodynamic model nor the spherical shell model is capable of exhibiting the Landau damping effect. An analysis employing a more realistic gaussian velocity distribution will further modify the resonances and the impedances.
IV. DC THEORY OF CYLINDRICAL HOT MAGNETOPLASMA

In Chapter II we studied the dc properties of plasma confined between infinite planes. In a practical system the infinite system can only be approximated crudely. Most of the plasma devices which use the contact ionization process consist of circular emitters which face each other coaxially but are separated by a distance many times greater than the diameter of the emitters. The radial confinement of the plasma in such a system is achieved by the axial magnetic field. In this chapter we investigate the dc properties of a cylindrical column of magnetoplasma. First, a theoretical analysis will be given followed by some dc measurement results on the cylindrical column.

A CYLINDRICALLY SYMMETRIC SOLUTION TO THE BOLTZMANN EQUATION

The Boltzmann equation for the \( p \)th species of the plasma in electric and magnetic fields is given by

\[
\frac{\partial f}{\partial t} + \frac{e}{m} \left( \mathbf{v} \cdot \mathbf{E} + \mathbf{A} \times \mathbf{v} \right) + \frac{e}{m} \mathbf{A} \cdot \mathbf{v} = \frac{2f}{f_{\text{eq}}} - \frac{2f}{f_{\text{eq}}} = \left( \frac{2f}{f_{\text{eq}}} \right)_{\text{coll.}}
\]

We seek a steady state solution \( \frac{\partial f}{\partial t} = 0 \) to the Boltzmann equation under the assumption of negligibly small collision terms. In other words,

\[
\frac{\partial f}{\partial t} = \frac{2f}{f_{\text{eq}}} - \frac{2f}{f_{\text{eq}}} = 0
\]

We shall assume further that the plasma is cylindrically symmetric \( (\theta, \phi = 0) \) and uniform in the direction of the externally applied dc magnetic field \( (\partial / \partial z = 0) \). Any function of the constants of the motion, such as the total energy, satisfies Eq. (4.2). One such solution is given by

\[
f_j(v, r) = f_0 \left[ -\frac{v^2}{2} - \frac{v^2}{2} + \left( \frac{\mathbf{v} \cdot \mathbf{A}}{\mu_0} \right) - \frac{\mathbf{v} \cdot \mathbf{A}}{\mu_0} - \frac{\mu_0}{\mathbf{v} \cdot \mathbf{A}} \right]
\]
where \( \vec{z}_j = \hat{r}_z \times \vec{a}_j \) is a vector along the magnetic field, \( \vec{r} \) is the coordinate vector perpendicular to the magnetic field, \( \vec{a}(r) \) is the vector potential, and \( V(r) \) is the scalar potential. As can be seen, Eq. (4.3) is given in terms of the total energy and the total angular momentum which are constants of motion. The electric and magnetic fields can be derived from \( V(r) \) and \( \vec{a}(r) \) by

\[
\vec{E} = -\nabla V(r) \\
\vec{B} = \nabla \times \vec{a}(r) 
\]

We shall prescribe the solution to have Maxwellian velocity distribution along the axis, and, hence,

\[
f_j(v, r) = A_j \exp \left[ \frac{m_j}{kT_j} \left( -\frac{v^2}{2} + \nabla \cdot (\vec{a}_j \times \vec{r}) + \frac{e_j}{m_j} \left( \vec{a}_j \times \vec{r} \right) - \frac{e_j}{m_j} V(r) \right) \right].
\]

(4.4)

By assuming further that the diamagnetic field produced by the plasma rotation has negligible effect on the dc magnetic field, the vector potential becomes

\[
\vec{A} = B_0 \times \vec{r}
\]

and Eq. (4.2) can be reduced to

\[
f_j(v, r) = A_j \exp \left[ \frac{m_j}{kT_j} \left( -\frac{v^2}{2} + \nabla \cdot (\vec{a}_j \times \vec{r}) + \frac{e_j}{m_j} \left( \vec{a}_j \times \vec{r} \right) - \frac{e_j}{m_j} V(r) \right) \right].
\]

(4.5)

where \( \omega_{cj} = \frac{|e_j B_0|}{m_j} \) is the cyclotron frequency of the \( j \)th species. Equation (4.5) can be rewritten as

\[
f_j(v, r) = A_j \exp \left[ \frac{m_j}{kT_j} \left( -\frac{v^2}{2} + \frac{\omega_{cj}^2 + \omega_{cj}^2}{2} \left( \frac{v^2}{2} + \frac{\omega_{cj}^2 + \omega_{cj}^2}{2} \right) - \frac{e_j}{m_j} V(r) \right) \right].
\]

(4.6)
From Eq. (4.6) we can see that the whole species rotates as a solid body with rotational frequency \( \omega_j \).

When Eq. (4.6) is specialized for electrons and ions, we have

\[
f_{oi}(\vec{v}, \vec{r}) = A_1 \exp \left[ \frac{\mathbf{i} \left( \mathbf{v} - \omega_j \times \mathbf{r} \right)^2}{2} + \left( \frac{\omega_j^2 + \omega_1 \omega_{ci}}{2} \right) r^2 - \frac{eV(r)}{m_1} \right] \tag{4.7}
\]

and

\[
f_{oe}(\vec{v}, \vec{r}) = A_e \exp \left[ \frac{\mathbf{m}}{kT_e} \left( \frac{\mathbf{v} - \omega_e \times \mathbf{r}}{2} + \left( \frac{\omega_e^2 - \omega_e \omega_{ce}}{2} \right) r^2 + \frac{eV(r)}{m_e} \right) \right] \tag{4.8}
\]

The density profiles and the current densities can be obtained by performing the appropriate integration of Eqs. (4.7) and (4.8).

\[
n_e(r) = n_e(0) \exp \left[ \frac{m_e}{kT_e} \left( \left( \frac{\omega_e^2}{2} - \omega_e \omega_{ce} \right) r^2 + \frac{e}{m_e} V(r) \right) \right], \tag{4.9}
\]

\[
n_i(r) = n_i(0) \exp \left[ \frac{m_i}{kT_i} \left( \left( \frac{\omega_i^2 + \omega_1 \omega_{ci}}{2} \right) r^2 - \frac{e}{m_i} V(r) \right) \right], \tag{4.10}
\]

\[
\mathbf{J}_e = -e(\omega_e \times \mathbf{r})n_e(r), \tag{4.11}
\]

\[
\mathbf{J}_i = e(\omega_i \times \mathbf{r})n_i(r). \tag{4.12}
\]

For the special case of complete neutrality, the simple solution yields a gaussian density profile with mean squared radius

\[
r_m^2 = \frac{2kT_1}{m_1(\omega^2_1 + \omega_1 \omega_{ci})} = \frac{2kT_e}{m_e(\omega^2_e - \omega_e \omega_{ce})} \tag{4.13}
\]

and the electrons and ions each rotate as a solid body about the axis with frequencies \( \omega_e \) and \( \omega_1 \) respectively. The following restrictions must be placed on \( \omega_e \) and \( \omega_1 \) to avoid densities which rise with increasing \( r \).
Since \( \zeta \) and \( \zeta_c \) are defined to be positive, the rotational vector \( \vec{\Omega} = \vec{e}_z \) is anti-parallel to the magnetic field and \( \vec{\Omega} \cdot \vec{B} = 0 \) is parallel to the magnetic field. The above conditions restrict \( \zeta_n^2 \) to

\[
\zeta_n^2 = \frac{e^2}{2m_n}\frac{1}{\epsilon_n^2} - \frac{e^2}{2m_n}\frac{1}{\epsilon_n^2} = -\frac{e^2}{2m_n}\frac{1}{\epsilon_n^2}
\]

By using Eq. (6.1), the rotational frequencies \( \omega_x \) and \( \omega_y \) can be obtained in terms of \( \zeta_n^2 \) as follows.

\[
\omega_x = \omega_y = \frac{e^2}{2m_n}\frac{1}{\epsilon_n^2}
\]

There are two possible rotational frequencies for a given value of \( \zeta_n^2 \) for each species.

Under the assumption of neutrality, the self-magnetic field can be computed from \( \vec{\Omega} \) and \( \vec{\Omega} \) by using Ampere's law. The magnetic field for the neutral case is given by

\[
\vec{B} = \mu_0 n \vec{J}
\]

and hence, the total axial current obtained from Eqs. (6.16) and (6.17) is

\[
\oint \vec{J} \cdot d\vec{l} = \mu_0 n \oint \vec{J} \cdot d\vec{l} = \mu_0 n \oint \vec{J} \cdot d\vec{l}
\]

From Ampere's law, the following relation is obtained for the z-component of the magnetic field

\[
B_z = e^2 \frac{x}{2\epsilon_n^2} \int_0^\infty \frac{e^2}{2m_n}\frac{1}{\epsilon_n^2} \exp\left(-\frac{e^2}{2m_n}\frac{1}{\epsilon_n^2}\right)
\]

\[
= \frac{e^2}{2m_n}\frac{1}{\epsilon_n^2} \exp\left(-\frac{e^2}{2m_n}\frac{1}{\epsilon_n^2}\right)
\]
Comparing this field with \( \mathbf{E} \), we have

\[
\frac{E_z}{E} = \frac{c^2}{\omega_B^2} \frac{\rho_i}{\rho_e} \exp\left(-\frac{r_i^2}{r_e^2}\right)
\]

where \( c \) is the speed of light. Thus, the self-magnetic field can be ignored if

\[
\frac{E_z}{E} = \frac{\rho_i}{\rho_e} \exp\left(-\frac{r_i^2}{r_e^2}\right) < 1.
\]

One way to maximize the left-hand side of (3.2) is to choose the smaller of the two possible rotational frequencies for each species given by Eqs. (3.1) and (3.2). In other words,

\[
\frac{\rho_i}{\rho_e} < 1
\]

Later we shall assume that \( \rho_i \) takes the minimum value allowed by the inequality of (3.2). For such a special case the inequality of (3.2) becomes

\[
\frac{E_z}{E} = \frac{\rho_i}{\rho_e} \exp\left(-\frac{r_i^2}{r_e^2}\right) < 1
\]

where \( \rho_i \) is the ion thermal speed. Many laboratory plasmas satisfy such an inequality.

The elementary solution obtained in this section can be used to construct more complicated solutions appropriate for a given plasma. In the following section we will consider some solutions synthesized from the elementary one.
B. SUPERPOSITION OF ROTATING CLOUDS

As example of a more complicated solution to the Boltzmann equation for the $i$th species is

$$f_i(r, v) = \int d^3 r' \exp \left[ - \frac{1}{\tau} \left( - \frac{r - r'}{2} \cdot \dot{r} - \frac{1}{2} \frac{\nabla}{} \nabla \cdot r' \cdot \dot{r} - \frac{e V(r)}{n} \right) \right]$$

which is a superposition of rotating clouds with different rotational frequencies. A valid solution is given by

$$f_i(r, v) = \int d^3 r' T \exp \left[ - \frac{1}{\tau} \left( - \frac{r - r'}{2} \cdot \dot{r} + \frac{1}{2} \frac{\nabla}{} \nabla \cdot r' \cdot \dot{r} - \frac{e V(r)}{n} \right) \right]$$

We have not found a practical system which can be described by Eqs. (4.25) and (4.25).

C. SUPERPOSITION OF DISPLACED ELEMENTARY SOLUTIONS

Under the assumption of complete neutrality, a more interesting type of solution is obtained. Such a solution is

$$f_i(r, v) = A \int d^3 \xi F(\xi) \exp \left[ \frac{1}{\tau} \left( \frac{- v - \xi}{2} \cdot \dot{r} + \frac{1}{2} \frac{\nabla}{} \nabla \cdot (r - \xi) \cdot \dot{r} + \frac{\omega^2 + \omega v \xi}{\epsilon} (r - \xi)^2 \right) \right]$$

where $\xi$ is a vector in the $r$, $\theta$ plane. Equation (4.26) is just the convolution of the elementary solution and the "displacement" function $F(\xi)$. Equation (4.26) satisfies the Boltzmann equation, but there is no way to handle the non-neutral cases in representation. The self-magnetic field will also spoil the solution. To insure neutrality the distribution function for the electrons must be of the form.
\[ f_e(\vec{r}, \vec{v}) = A \int d\xi \, F(\xi) \exp \left[ \frac{m_e}{kT_e} \left( -\frac{\vec{v} - \vec{v}_e \times (\vec{r} - \vec{\xi})}{2} \right)^2 \right] \]

\[ + \left\{ \frac{\omega^2}{2} - \frac{\omega \omega \cdot \vec{v}}{2} \right\} \left( \vec{r} - \vec{\xi} \right)^2 \] \hspace{1cm} (4.27)

where the rotational frequency \( \omega_\epsilon \) is related to \( \omega_1 \) by Eq. (4.13).

An example of \( F(\xi) \), which may be applicable to a practical situation, is

\[ F(\xi) = 1 \quad \text{for} \quad 0 < |\xi| < b_0 \]

\[ = 0 \quad \text{for} \quad |\xi| > b_0 \] \hspace{1cm} (4.28)

This is expected to give a solution which closely approximates a plasma produced between two circular diode plates of radius \( b_0 \) and confined by an axial magnetic field. For ions the expressions for the density profile and the current density become

\[ N_i(r) = \frac{2N_0}{\pi r_0^2} \int_0^{b_0} \xi \, I_0(2\xi r_0^{-2}) \exp \left[ -\frac{\xi^2 + \xi_0^2}{r_0^2} \right] d\xi \] \hspace{1cm} (4.29)

\[ J_{\theta i}(r) = \frac{1}{\pi r_0^2} \int_0^{b_0} \left\{ \xi \, I_0(2\xi r_0^{-2}) - \xi_0 I_1(2\xi r_0^{-2}) \right\} \exp \left[ -\frac{\xi^2 + \xi_0^2}{r_0^2} \right] d\xi \] \hspace{1cm} (4.30)

where \( J_{\theta i}(r) \) is the \( \theta \) component of the ion current density, \( I_0 \) and \( I_1 \) are zero- and first-order modified Bessel functions of the first kind, \( N_0 \) is the number of particles per unit length, and \( r_0 \) is as defined in Eq. (4.13). From Eqs. (4.27) and (4.28), the electron density and current profiles can be obtained and are found to be related to \( N_i(r) \) and \( J_{\theta i}(r) \) in the following manner

\[ N_e(r) = N_i(r) \] \hspace{1cm} (4.31)

\[ J_{\theta e}(r) = -\frac{\omega_e}{\omega_1} J_{\theta i}(r) \] \hspace{1cm} (4.32)
Equations (4.29) and (4.30) can be integrated readily numerically, and the results for \( \omega_i = -\omega_{ci}/2 \) are shown in Figs. 4.1, 4.2, and 4.3. The rotational frequency \( \omega_i \) is arbitrary, but the choice of \( \omega_i = -\omega_{ci}/2 \) gives a solution with best confinement (smallest mean radius). In the figures the parameter varied for the plots is

\[
r_m = \sqrt{\frac{8\pi T_i}{\sqrt{m_1 \omega_i^2/\omega_{ci}}} \quad (4.33)}
\]

The graphs show what we expect of a magnetically confined plasma as the magnetic field is varied. The density profile becomes flat-topped as the magnetic field is increased and, at infinite magnetic field, the profile becomes perfectly flat-topped with a sharp boundary. In general, the density drop at the plasma edge occurs roughly within a distance of \( 2r_m \). The current density is highest where the density gradient is highest. The ion drift velocity is of the order of the ion thermal velocity at the plasma edge. Solving Eqs. (4.21) and (4.22) for \( \omega_i \) under the condition of \( T_i = T_e \) and \( \omega_i = -\omega_{ci}/2 \), we find \( \omega_e \approx \omega_{ci}/4 = -\omega_{ci}/2 \).

![Graph](image)

**FIG. 4.1. DENSITY PROFILE.** Normalized density is plotted versus normalized radius for different values of \( r_m = \sqrt{8\pi T_i/(m_1 \omega_{ci}^2)} \).
FIG. -2. ION CURRENT DENSITY. The $\hat{z}$ component of the ion current density is normalized to $\sqrt{2} e v_{\theta i} N_0$ and plotted vs $r/r_0$.

FIG. -3. NORMALIZED ION DRIFT VELOCITY VS $r/r_0$. The drift velocity is normalized to the ion thermal velocity $v_{\theta i}$.
Thus, the electron drift velocity is also of the order of the ion drift velocity, and, hence, negligibly small when compared to the electron thermal velocity.

W. H. Cutler has derived an expression very similar to Eq. (29) but with an entirely different method. After approximation Cutler arbitrarily sets

\[ \frac{r^2}{n} = \frac{1}{n} \frac{1}{1 + c_1} \]

This value does not satisfy the inequality (1) for real \( c_1 \) and, hence, is disallowed in our theory.

D. EXPERIMENTAL MEASUREMENTS OF DC DENSITY AND DRIFT PROFILES

In the sodium plasma produced in the manner described in Chapter II, we attempted to measure the azimuthal electron drift and the density profile. For the measurement a modified Langmuir probe is used. The tip of this probe is shown in Fig. .. It can be rotated so that the dc density profile can be measured by facing the probe surface toward the buttons, while plasma rotation may be measured by facing the probe surface in the azimuthal direction. The detailed circuitry and the probe technique used to overcome various problems is discussed in Appendix A.

MOLY ROD 0.060" DIA

MOLY TUBE 0.135" OD - 0.080" ID

FIG. --, LANGMUIR PROBE
With the collecting surface facing the azimuthal direction, the probe curve is obtained first with the magnetic field directed one way and then with the magnetic field reversed. Presumably the azimuthal drift can be detected by noting the differences in the probe current for the two cases. Shown in Fig. 4.5 is a typical set of probe curves in the magnetic field. As the probe potential is increased from some negative value, the current rises exponentially and, at the plasma potential, breaks away from the exponential rise to form a "knee." We have plotted in Fig. 4.6 the current at the "knee" versus radius. There is a significant difference in the probe current when the magnetic field is reversed, indicating that an azimuthal drift is present. However, due to the lack of an appropriate probe theory, the measured result cannot be quantitatively related to the azimuthal drift. A similar measurement for ion drift is desirable, but considerable improvement on the probe circuits is required since the ion currents are of the order of the leakage current or less.

At 180 gauss, the radius \( r_m \) as defined by Eq. (4.32) is 3 cm. This value is much too high to give current density profiles such as those of Fig. 4.6. Comparison of theoretical density profile was also made with the experimental density profile of a plasma produced in a similar manner by Cutler.\(^{39}\) There again we found that invariably the experimental

![Fig. 4.5. Typical set of probe curves. The two curves are obtained by keeping the probe in the identical position but by changing the direction of the magnetic field.](image)
FIG. 4.6. AZIMUTHAL PROBE CURRENT VS RADIAL DISTANCE. The two curves are obtained with identical but oppositely directed magnetic fields. The parameters of the experimental plasma were: $T = 1800 \, ^\circ\text{K}$; $n = 3 \times 10^9$/cc; $\lambda_0 = 1.7$ mm.

density profile had a much sharper drop at the plasma edge than the theoretical profile. The discrepancy emphasizes the importance of the electric field which we have ignored in our theory. The electric field may be set up in the following manner. The primary electrons supplied within the solid cylinder, due to their small Larmor radii, have a sharper boundary than the ions. Hence, charge separation occurs and an electric field is set up which pulls the electrons outward and ions inward. Since the probe measures the electron current, it is not surprising that the profile is sharper than that predicted by a theory which assumes complete neutrality. Indeed, measurements show that there is an electric field present, though not concentrated near the plasma edge as might be expected. Shown in Fig. 4.7 is the potential profile of the column. An electric field of approximately 0.25 volts/cm is present.

For the study of waves along a plasma column undertaken in Chapter VI, we shall assume that the plasma column is uniform with a sharp
boundary, and that there is no azimuthal drift. The results of this chapter serve to point out the range of validity of the dc model used in the rf analysis.
V. WAVES IN AN INFINITE MAGNETOPLASMA

In this chapter we investigate the waves in an infinite uniform hot magnetoplasma keeping in mind our final goal of obtaining results for a finite cylindrical plasma column. We are primarily concerned with the derivation of the dielectric tensors which describe the plasma behavior. We will demonstrate a new method to separate the plasma conduction current into polarization and magnetization currents. The separation is found to simplify the analysis considerably.

A. THE DIELECTRIC TENSOR IN A MAXWELLIAN ELECTRON PLASMA

Let us consider the linearized collisionless Boltzmann equation for the electrons. For the moment we shall assume that the ions are infinitely heavy. Then the zero and first order equations are given by

\[
\left( \mathbf{v} \times \mathbf{B}_0 \right) \cdot \frac{\partial f_0}{\partial \mathbf{v}} = 0,
\]

\[
\frac{\partial f_1}{\partial t} + \mathbf{v} \cdot \frac{\partial f_1}{\partial \mathbf{x}} = - \frac{e}{m} \left[ \mathbf{v} \times \mathbf{B}_0 \right] \cdot \frac{\partial f_1}{\partial \mathbf{v}} - \frac{e}{m} \left[ \mathbf{E}_1(\mathbf{x},t) + \mathbf{v} \times \mathbf{B}_1(\mathbf{x},t) \right] \cdot \frac{\partial f_0}{\partial \mathbf{v}}
\]

where \( f_1 = f_1(t,\mathbf{x},\mathbf{v}) \). By Fourier transforming in \( \mathbf{x} \) and Laplace transforming in \( t \), Eq. (5.2) is reduced to

\[
(\omega - ik \cdot \mathbf{v})f_1 - \frac{e}{m} \left[ \mathbf{v} \times \mathbf{B}_0 \right] \cdot \frac{\partial f_1}{\partial \mathbf{v}} = \frac{e}{m} \left( \mathbf{E} + \mathbf{v} \times \mathbf{B} \right) \cdot \frac{\partial f_0}{\partial \mathbf{v}}
\]

where now \( f_1 = f_1(\omega,k,\mathbf{v}) \). We introduce the following gyro-tensor

\[
\mathbf{D} = \mathbf{n} \mathbf{n} + (I - \mathbf{n} \mathbf{n}) \cos \varphi - \mathbf{n} \times I \sin \varphi
\]

and its integral

\[
\mathbf{L}(\varphi) = \int_{0}^{\varphi} \mathbf{D}(\varphi') d\varphi'
\]

\[
= \mathbf{n} \mathbf{n} \varphi + (I - \mathbf{n} \mathbf{n}) \sin \varphi + \mathbf{n} \times I (\cos \varphi - 1)
\]
where \( \vec{n} \) is a unit vector in the direction of the magnetic field and "\( \mathbf{I} \)" is the unit tensor. The detailed derivation of the solution to Eq. (5.3) is given in Appendix B. The solution is found to be

\[
f_1(\omega, k, \vec{v}) = \frac{1}{m_0 c} \int_0^\infty \exp \left[ \frac{i \omega \varphi}{\omega_c} + \frac{1}{2} \vec{k} \cdot \vec{L} \cdot \vec{v} \right] \vec{E} \cdot \vec{D} \cdot \frac{\partial f}{\partial \vec{v}} \, d\varphi. \quad (5.6)
\]

This solution can be verified by substituting it directly into Eq. (5.3).

The above solution agrees with those given by Fried and Stix. Through the use of the gyrotensors \( \vec{D}(\varphi) \) and \( \vec{L}(\varphi) \), our solution appears in a much simpler form than that given by Stix. The current density can be obtained by the integration

\[
< \vec{\rho} \vec{v} > = -e \int \vec{f}_1(\omega, k, \vec{v}) d^2\vec{v}. \quad (5.7)
\]

In Appendix C we show that it is possible to separate the current density into polarization and magnetization currents, as suggested by Lorentz's electron theory of matter, i.e.

\[
< \vec{\rho} \vec{v} > = i \omega \vec{P} - i \vec{k} \times \vec{M} \quad (5.8)
\]

where \( \vec{D} = \epsilon_0 \vec{P} + \vec{P}_0 \) and \( \vec{B} = \mu_0 (\vec{H} + \vec{M}) \). For a Maxwellian plasma we obtain (see Appendices D and E),

\[
\vec{P} = -\frac{\epsilon_0 \omega_e^2}{\omega_c} \int_0^\infty \exp \left[ -\frac{i \omega \varphi}{\omega_c} - \frac{\omega_\varphi^2}{2 \omega_c^2} (k \cdot L)^2 \right] \vec{L}(\varphi) \cdot \vec{E} \, d\varphi \quad (5.9)
\]

and

\[
\vec{M} = -\frac{i \omega_\varphi^2}{\omega_c^3 \mu_0} \int_0^\infty \vec{\mu}(\varphi) \cdot \vec{B} \exp \left[ -\frac{i \omega \varphi}{\omega_c} - \frac{\omega_\varphi^2}{2 \omega_c^2} (k \cdot L)^2 \right] d\varphi \quad (5.10)
\]

where

\[
\vec{\mu}(\varphi) = (I - \vec{n}, \vec{n}) \varphi \sin \varphi + (\vec{n} \times I)\varphi(1 - \cos \varphi) + 2\vec{n}, \vec{n}(1 - \cos \varphi)
\]
The importance of separating the current density into the two parts is realized when we deal with boundary conditions in a finite plasma. Without the vectors $\mathbf{\Pi}$ and $\mathbf{\Pi}$, it is necessary to work with surface charges and surface currents at the boundary. Such a task is very difficult, if not impossible, for a hot plasma boundary.

From Eq. (5.3) the electric susceptibility tensor $\mathbf{\tau}$, as well as the dielectric tensor $\mathbf{\varepsilon}$, can be obtained from the following relation

$$\mathbf{D} = \varepsilon_0 \mathbf{E} = \varepsilon_0 \mathbf{E} + \mathbf{\Pi} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon_0 \mathbf{E} + \mathbf{\Pi} + \mathbf{\Pi}.$$

We shall be concerned primarily with the dielectric properties of the plasma, since we are interested in electrostatic waves in which the magnetic energy is negligibly small. The dielectric tensor can be expressed in the following form

$$\mathbf{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix}$$

with

$$\varepsilon_{yy} = \varepsilon_{xx} = 1 + \frac{\mathbf{P}}{\mathbf{E}} \int_0^\infty \sin \varphi \, e^{-\alpha} \, d\varphi$$

$$\varepsilon_{zz} = 1 + \frac{\mathbf{P}}{\mathbf{E}} \int_0^\infty \varphi \, e^{-\alpha} \, d\varphi$$

$$\varepsilon_{yx} = -\varepsilon_{xy} = \frac{\mathbf{P}}{\mathbf{E}} \int_0^\infty (1 - \cos \varphi) e^{-\alpha} \, d\varphi$$

where

$$u = \frac{k \mathbf{P}}{a_c} - \frac{k v_{\mathbf{\Pi}}^2}{a_c^2} \left( k \cdot \mathbf{\Pi} \right)^2$$

$$= \frac{k \mathbf{P}}{a_c} - \frac{k v_{\mathbf{\Pi}}^2}{2a_c^2} \varphi^2 + \frac{k v_{\mathbf{\Pi}}^2}{a_c} (1 - \cos \varphi)$$

$$- 62 -$$
and \( k \) and \( k' \) are the wave numbers perpendicular and parallel to the magnetic field respectively. In Appendix D we show how to express Eqs. (5.11) to (5.13) in terms of the Hilbert transform of the gaussian as defined by Fried and Conte. The result is

\[
\epsilon_{xx} = \epsilon_{yy} = 1 + \frac{\varepsilon' \sigma}{\varepsilon_0 \omega_c} \exp\left[-\frac{\lambda^2}{2}\right] \sum_{n=0}^{\infty} \frac{1}{n+1} (Z_{n+1} - Z_{n-1}) (5.14)
\]

\[
\varepsilon_{zx} = 1 - \frac{\varepsilon' \sigma}{\varepsilon_0 \omega_c} \exp\left[-\frac{\lambda^2}{2}\right] \sum_{n=0}^{\infty} I_n Z_n' (5.15)
\]

\[
\varepsilon_{xy} = -\varepsilon_{yx} = -\frac{\varepsilon' \sigma}{\varepsilon_0 \omega_c} \exp\left[-\frac{\lambda^2}{2}\right] \sum_{n=0}^{\infty} I_n \left( Z_n - \frac{Z_{n+1} + Z_{n-1}}{2} \right) (5.16)
\]

where \( I_n = I_n(\lambda^2/2) \) are modified Bessel functions,

\[
\zeta_n = \frac{\omega - \omega_c}{\sqrt{2} k_B v\theta},
\]

\[
\lambda = \frac{\sqrt{2} k_B v\theta}{\omega_c},
\]

and \( Z_n = Z(\zeta_n) \) is the Hilbert transform of the gaussian, commonly called the Fried function, defined by

\[
Z(\zeta_n) = \frac{1}{\sqrt{\pi}} \int_{\zeta_n}^{\infty} \frac{e^{-x^2}}{x - \zeta_n} \, dx \quad \text{Im}(\zeta_n) > 0. (5.17)
\]

Properties of this function are discussed in References 3 and 4, and a few simple relations involving \( Z(\zeta) \) are given in Appendix D.

The dielectric tensor described by Eqs. (5.11) to (5.16) is much simpler than the "dielectric" tensor derived by earlier authors and described in the book by Stix. In the representation given by Stix, the plasma conduction current is not divided into magnetization and polarization currents and hence the "dielectric" tensor is not truly the tensor.
which relates the electric field to the displacement field. Even under quasi-static approximation, our results do not agree with that given by Stix.

B. THE DIELECTRIC TENSOR IN PLASMAS WITH TEMPERATURE ANISOTROPY

The derivation of the dielectric tensors for plasmas with temperature anisotropy is given in Appendix C. Equations (5.14), (5.15), and (5.16) remain unchanged, but some of the variables must be redefined as follows

\[ \lambda^2 = \frac{2k_1^2 \nu_0^2}{\omega_c^2} \]

and

\[ \zeta_n = \frac{\omega - \omega_0}{\sqrt{2k_1^2 v_0^2}} \]

where

\[ v_{\parallel} = \frac{\nu T}{m} \]

and

\[ v_{\perp} = \frac{\nu T}{m} . \]

The limiting case of interest is \( v_{\perp} = 0 \). In this limit the dielectric tensor elements become

\[ \varepsilon_{xx} = \varepsilon_{yy} = 1 + \frac{\omega_p^2}{2\sqrt{2} \omega_c^2 k_{\parallel} v_{\parallel}} \left[ 2\left( \frac{\omega - \omega_c}{\sqrt{2} k_{\parallel} v_{\parallel}} \right) - Z \left( \frac{\omega + \omega_c}{\sqrt{2} k_{\parallel} v_{\parallel}} \right) \right] \]

\[ \varepsilon_{zz} = 1 - \frac{\omega_p^2}{2k_{\parallel}^2 v_{\parallel}^2} Z' \left( \frac{\omega}{\sqrt{2} k_{\parallel} v_{\parallel}} \right) \]

and

\[ \varepsilon_{xy} = -\varepsilon_{yx} = \frac{-1}{2\sqrt{2} \omega_c^2 k_{\parallel} v_{\parallel}} \left[ 2Z \left( \frac{\omega - \omega_c}{\sqrt{2} k_{\parallel} v_{\parallel}} \right) - Z \left( \frac{\omega + \omega_c}{\sqrt{2} k_{\parallel} v_{\parallel}} \right) \right] . \]
These agree with the results obtained by Lichtenberg and Jayson. Lichtenberg and Jayson derived the tensor by assuming that the plasma consisted of a continuum of streaming electrons with a gaussian velocity distribution along the magnetic field. They obtained the polarization tensor by integrating over the contribution from the continuum of electron streams. In Chapter VI the dispersion relation for the Lichtenberg-Jayson model is compared with that of a fully thermal plasma model.

The cold plasma dielectric tensor is recovered by letting $v_{\perp l}$ approach zero in Eqs. (5.18) to (5.20).

C. DIELECTRIC TENSOR FOR A MULTI-COMPONENT PLASMA

The polarization vector for a multi-component plasma is given by the sum of the polarization in each of the components. Thus, the dielectric tensor is given by

$$\bar{\epsilon} = I + \sum_j \bar{\pi}_j$$

where $\bar{\pi}_j$ is the electric susceptibility tensor for the $j^{th}$ species.

The elements of $\pi_j$ are given by

$$\pi_{zzj} = -\frac{\omega_{pj}^2}{\omega^2} \zeta_{0j}^2 \exp\left[-\frac{\lambda_j^2}{2}\right] \sum_{n} I_n z_n'$$

$$\pi_{xyj} = -\pi_{yxj} = -\frac{4\omega_{pj}^2 \zeta_{0j}}{\omega^2 \omega_{cj}} \exp\left[-\frac{\lambda_j^2}{2}\right] \sum_{n} I_n \left(z_n - \frac{z_{n+1} + z_{n-1}}{2}\right)$$

and

$$\pi_{xxj} = \pi_{yyj} = \frac{\omega_{pj}^2 \zeta_{0j}}{2\omega \omega_{cj}} \exp\left[-\frac{\lambda_j^2}{2}\right] \sum_{n} I_n \left(z_{n+1} - z_{n-1}\right)$$

- 65 -
Having obtained the dielectric tensors, we can now derive the dispersion relation for the plane electrostatic waves in the magnetoplasma. The distinguishing feature of electrostatic waves is that the electric field is derivable from a scalar potential $\varphi(x,t)$, and, hence, $\nabla \times \mathbf{E}(x,t) = 0$.

This approximation, commonly called the quasi-static approximation, is valid whenever the phase velocities of the waves are much less than the speed of light. Using this analysis and assuming plane wave propagation of the form $\exp[i\omega t - i\mathbf{k} \cdot \mathbf{r}]$, we obtain the following result, relating the potential and the electric field,

$$ \mathbf{E}[\omega, \mathbf{k}] = i\mathbf{k} \varphi[\omega, \mathbf{k}] \quad (5.25) $$

From the second Maxwell equation we get

$$ -66 - $$

where

$$ I_n = I_n \left( \frac{\lambda_n^2}{2} \right), $$

$$ Z_n = Z(\zeta_{nJ}), $$

$$ \zeta_{nJ} = \frac{\omega - \nu c_J}{\sqrt{2} k \parallel \nu \theta_J}, $$

$$ \lambda_J = \frac{\sqrt{2} k \parallel \nu \theta_J}{\omega c_J}. $$

D. THE QUASI-STATIC DISPERSION RELATION FOR THE INFINITE PLASMA

Having obtained the dielectric tensors, we can now derive the dispersion relation for the plane electrostatic waves in the magnetoplasma. The distinguishing feature of electrostatic waves is that the electric field is derivable from a scalar potential $\varphi(x,t)$, and, hence, $\nabla \times \mathbf{E}(x,t) = 0$.

This approximation, commonly called the quasi-static approximation, is valid whenever the phase velocities of the waves are much less than the speed of light. Using this analysis and assuming plane wave propagation of the form $\exp[i\omega t - i\mathbf{k} \cdot \mathbf{r}]$, we obtain the following result, relating the potential and the electric field,

$$ \mathbf{E}[\omega, \mathbf{k}] = i\mathbf{k} \varphi[\omega, \mathbf{k}] \quad (5.25) $$

From the second Maxwell equation we get
\[ \mathbf{D}(t, \mathbf{x}) = -i \mathbf{E} \cdot \mathbf{E}_{\mathbf{r}, \mathbf{k}} \exp(\mathbf{a}^\perp t) - \mathbf{E} \cdot \mathbf{E}_{\mathbf{r}, \mathbf{k}} \exp(\mathbf{a}^\perp t) = 0 \]

Direct substitution of Eq. (5.26) into Eq. (5.22), yields the following relation

\[ (\varepsilon_{xx} \mathbf{k}^2 + \varepsilon_{xy} \mathbf{k}^2 + \varepsilon_{xz} \mathbf{k}^2) \mathbf{L}_x \mathbf{L}_y \mathbf{L}_z = 0 \]

which in turn yields the dispersion relation

\[ \mathbf{k}^2 + \varepsilon_{xx} \mathbf{k}^2 = 0 \]

where \( \mathbf{k}^2 = k_x^2 + k_y^2 \) is the square of the wave number perpendicular to the magnetic field. As expected, the dispersion relation is independent of the angle between the perpendicular component of \( \mathbf{k} \) and the \( x \) axis. This dispersion relation, though it appears to be different, is identical to the result obtained by Stix. 12

In an electron plasma the dispersion relation can be given in terms of one integral or one series as follows

\[ \mathbf{k}^2 = \mathbf{k}^2 + \left( \frac{\mathbf{p}}{m} \right) \left[ 1 - \frac{k_z^2}{c^2} s_1 \right] = 0 \]

where

\[ s_1 = \int_0^\infty \exp \left[ - \frac{k_z^2}{c^2} t - \frac{1}{2} \mathbf{k}^2 \frac{1}{c^2} \mathbf{t}^2 \frac{1}{c^2} (1 - \cos t) \right] dt \]

\[ = \exp \left[ - \frac{k_z^2}{c^2} \right] \sum_{-\infty}^\infty \frac{k_z c}{\sqrt{\varepsilon} k z} I_n Z_n \]

and \( \mathbf{t}, I_n \) and \( Z_n \) are as defined in Section VA.
The special case of purely perpendicular propagation \((k = 0)\) has been studied by Bernstein. For this case we have from Eq. (5.28)

\[
e_{xx} = \frac{\omega}{c}
\]  

(5.30)

as the dispersion relation. By using Eq. (5.29) and the asymptotic relation for \(X_n\) given in Appendix D, we obtain the following dispersion relation for the "Bernstein" modes.

\[
1 - \frac{\omega}{c} \exp \left( -\frac{k^2}{c^2} \right) \sum_{n=0}^{\infty} \frac{ ni, (k^2/c)}{\omega^2 - \frac{n^2}{c^2}} = 0 .
\]  

(5.31)

This dispersion predicts resonances near the harmonics of the cyclotron frequency. Several papers on the experimental observations of such resonances have been published. \(^{47,49}\) A novel method of summing the series of Eq. (5.31) in terms of Whittaker's cylinder function has been given by Derringer. \(^{49}\) A computer program based on this result is used in obtaining the electric susceptibility tensor for the ions.

E. CYLINDRICAL WAVES

In this section we illustrate a method of synthesizing cylindrical waves from the plane waves. The rf potential for a plane wave may be expressed as follows

\[
\Phi(x, z) = \Phi(z, x) e^{-i k \cdot z} .
\]  

(5.32)

From the potential the electric field and the displacement vectors can be obtained

\[
\vec{E}(x, z) = -\nabla \Phi(x, z) = i k \Phi(x, z) e^{-i k \cdot z} .
\]  

(5.33)

\[
\vec{D}(x, z) = \frac{1}{\epsilon} \cdot \vec{E} + k \Phi(x, z) e^{-i k \cdot z} .
\]  

(5.34)
In the cylindrical coordinate system we let

\[
    \begin{align*}
        k_x &= k \cos \varphi \\
        k_y &= k \sin \varphi \\
        x &= r \cos \theta \\
        y &= r \sin \theta 
    \end{align*}
\]

where \( \varphi \) is the angle between the \( r \) axis and the plane containing the \( z \) axis and \( k \). The potential can now be expressed in the following manner

\[
    \Theta[\omega, r, \theta, z] = \Theta[\omega, k_x, k_y, \varphi] \exp[-i k z - i k r \cos(\theta - \varphi)] . \quad (5.35)
\]

The cylindrical wave is synthesized by taking these waves with identical amplitude, \( k \) and \( k \) but with phase variation of the form

\[
    \Theta[\omega, k_x, k_y, \varphi] = \frac{\Theta_0}{2\pi} e^{im\varphi}
\]

where \( m \) is an integer. Synthesis of such plane waves yields

\[
    \Theta[\omega, r, \theta, z] = \frac{\Theta_0}{2\pi} \int_{-\pi}^{\pi} e^{im\varphi} \exp[-i k z - i k r \cos(\theta - \varphi)] d\varphi
\]

\[
    = \Theta_0 e^{i m \theta} e^{-i k z} J_m(k r) \quad (5.36)
\]

where \( J_m(k r) \) is the Bessel function of the order \( m \). Similar integrations for the electric and displacement fields yield

\[
    E_z[\omega, r, \theta, z] = -i k \Theta_0 \exp[i m \theta - i k z] J_m(k r) \quad (5.37)
\]

\[
    E_r[\omega, r, \theta, z] = \Theta_0 \frac{i m}{r} \exp[i m \theta - i k z] J_m(k r) \quad (5.38)
\]

\[
    E_\theta[\omega, r, \theta, z] = k \Theta_0 \exp[i m \theta - i k z] J'_m(k r) \quad (5.39)
\]
This technique of cylindrical wave synthesis has been employed by others in the past. In our representation, \( \varepsilon \) is independent of \( \varphi \), the integration variable. In Stix' representation, the dielectric tensor is a function of \( \varphi \), and the integrals for the displacement vector is somewhat more difficult to evaluate than in our case.

The solution obtained above is valid for an infinite plasma. Our ultimate goal is to find the solution for a finite plasma, particularly a cylindrical column of plasma. In a finite plasma the most difficult task is the treatment of the boundary. To make the problem manageable, we must assume that the plasma column is uniform with a sharp boundary. In reality, a sharp boundary cannot occur because of thermal motion, though such a model may approximate the plasma column, as suggested by the theory and, in particular, the measurements discussed in Chapter IV. In Chapter VI we shall use the fields given by Eqs. (5.37) to (5.42) as the solution within the cylindrical plasma column and match them to the fields outside the plasma.
VI. ELECTROSTATIC WAVES IN A CYLINDRICAL COLUMN OF MAGNETOPLASMA

In this chapter we investigate the dispersion relation for electrostatic waves propagating along a column of cylindrical, collisionless magnetoplasma. The geometry we consider is that of an infinitely long, uniform, cylindrical plasma enclosed within a cylindrical waveguide as shown in Fig. 1.1. First, the dispersion relation for the electrostatic waves is derived. Some numerical solutions to the dispersion relation for a Maxwellian electron plasma are presented and compared with the dispersion relations based on other plasma models. Our results are also compared with the available experimental dispersion relations. At low magnetic fields, the surface wave is found to be unstable, and the instability character is investigated. The unstable ion surface wave is also studied.

A. QUASI-STATIC DISPERSION RELATION

Since there are two distinct regions within the waveguide, we need two sets of solutions, one valid within the plasma and the other valid in the free space region between the plasma and the conducting wall. For the plasma region we shall use the quasi-static solution, obtained in the previous chapter and given by

\[ E_z(\omega, r, \theta, z) = -ik \varrho_o \exp[i\theta - ikz]J_m(kr), \]
\[ E_\theta(\omega, r, \theta, z) = \frac{im\varrho_o}{r} \exp[i\theta - ikz]J_m(kr), \]
\[ E_r(\omega, r, \theta, z) = k \varrho_o \exp[i\theta - ikz]J'_m(kr), \]
\[ D_z(\omega, r, \theta, z) = -ik \varepsilon \varrho_o \exp[i\theta - ikz]J_m(kr), \]
\[ D_\theta(\omega, r, \theta, z) = \varepsilon_o \exp[i\theta - ikz] \left\{ \frac{im\varepsilon}{r} J_m(kr) + k \varepsilon J'_m(kr) \right\}, \]
\[ D_r(\omega, r, \theta, z) = \varepsilon_o \exp[i\theta - ikz] \left\{ \frac{im\varepsilon}{r} J_m(kr) - k \varepsilon J'_m(kr) \right\}, \]
In the free space region, $b < r < s$, the solution to the equation
\[ \nabla^2 \phi = 0 \]
with the requirement of $\phi(s,\theta,z) = 0$ is given by
\[
\phi(r,\theta,z) = A \left[ J_m(ik \ r)H_m(ik \ s) - J_m(ik \ s)H_m(ik \ r) \right] e^{im\theta-ik \ z} \tag{6.1}
\]
where $J_m(ik \ r)$ and $H_m(ik \ r)$ are Bessel and Hankel functions of order $m$. The components of the electric field are then obtained from Eq. (6.1) by $\vec{E} = -\nabla \phi$. At the free space plasma boundary, the tangential component of the electric field and the normal component of the displacement vector are matched. This yields the following dispersion relation for the waves
\[
k b_x \frac{J'(k b)}{J_m(k b)} + \text{im} e_{xy} = \frac{ik b \left[ J'(ik b)H_m(ik s) - J_m(ik s)H'_m(ik b) \right]}{J_m(ik b)H_m(ik s) - J_m(ik s)H'_m(ik b)} \tag{6.2}
\]
with
\[
k^2 + k_{\parallel}^2 \epsilon_{\parallel \parallel} = 0
\]
and the dielectric tensor elements are as described in Chapter VC. The dispersion relation of Eq. (6.2), in appearance, is very familiar. In the cold plasma model and in the Lichtenberg-Jayson ($T = 0$) model, the dispersion relation is identical to Eq. (6.2). The only difference, though by no means small, is in the dielectric tensor elements.

To solve the dispersion equation in the form given by Eq. (6.2) is a formidable task. For example, the computation of the dielectric tensors above involves infinite summations of products of the Bessel and Fried functions. Only under certain conditions do these summations converge rapidly enough for computer analysis. In restricting our study to the cylindrically symmetric modes ($m=0$), we have
\[
k b_x \frac{J_1(k b)}{J_0(k b)} = \frac{ik \left[ J_1(ik b)H_0(ik s) - J_0(ik s)H_1(ik b) \right]}{\left[ J_0(ik b)H_0(ik s) - J_0(ik s)H_0(ik b) \right]} \tag{6.3}
\]
Even for this case, a thorough analysis of all the modes is formidable.
1. Comparison of the Dispersion Relations for Several Different Plasma Models

The dispersion relation for the electrostatic waves in a cold plasma column was first derived by Trivelpiece and its solutions are commonly called the Trivelpiece modes. The hydrodynamic model, obtained by taking the moments of the Boltzmann equation, was used by Agdur and Weissglas to study the temperature effects on the Trivelpiece modes. Unfortunately, the hydrodynamic model does not account properly for the temperature effects. In this section we present the dispersion relation for waves in a Maxwellian plasma column and compare it with the dispersion relations derived by using other models. We shall assume that the ions are infinitely heavy.

Equation (6.3) is solved numerically for the Maxwellian electron plasma by assuming real \( \omega \) and seeking complex \( k \). The computation procedure is described in Appendix E. The result for the lowest order mode is shown in Fig. 6.1. Also shown on the same graph are the dispersion relations for waves in a cold plasma column, in a "hydrodynamic" plasma, and in an infinite Maxwellian plasma. For the infinite hot plasma, the dispersion relation presented is the lowest order Landau mode as defined by Simonen.

At low values of \( \omega \), the three models yield identical dispersion relations, but, at high values of \( \omega \), the dispersion relation for the Maxwellian plasma column merges smoothly into the hot infinite plasma dispersion relation, while the dispersion relation for the hydrodynamic model approaches the thermal speed of the one dimensional adiabatic Bohm and Gross waves. As we can see, the wave in the column of the Maxwellian plasma is Landau damped.

The dispersion relation for the zero transverse temperature model \( (T = 0) \) was also computed, but, at the high magnetic field considered above, the results for \( T = 0 \) and \( T = T \) are identical. This is not surprising since, at high magnetic fields, the transverse motion of the electrons is inhibited.
A = normalized frequency versus normalized wave number for a wave in a cylindrical column of cold plasma ($T = 1$)

B = normalized frequency versus normalized wave number for a wave in a cylindrical column of hydrodynamic plasma ($P_1 = \frac{n_1}{n}$)

C = real part of $k_{tD}$ versus $\omega_{ep}$ for a cylindrical column of Maxwellian plasma

D = damping decrement $\text{Im} (k_{tD})$ versus $\omega_{ep}$ for a cylindrical column of Maxwellian plasma

E = real part of $k_{tD}$ versus $\omega_{ep}$ for an infinite Maxwellian plasma

F = damping decrement $\text{Im} (k_{tD})$ versus $\omega_{ep}$ for an infinite Maxwellian plasma

FIG. 1.1. DISPERSION RELATIONS FOR VARIOUS PLASMA MODELS.

2. Comparison with Available Experimental Observations

Using the model with $T = 1$, we also compute dispersion curves, which can be compared with the experimental observations of Weinberg and Wharton. Weinberg and Wharton measured the propagation constant and the damping rate at frequencies near the electron plasma frequency.
Their experiment was conducted in a duoplasmatron type hydrogen arc source, which produced a plasma with high longitudinal temperature. Because of the high longitudinal temperature, the model with $T_\perp = 0$ is well-suited for the analysis. The experimental density profile given by Malmberg and Wharton is highly nonuniform as shown in Fig. 6.2. For computational purposes the plasma is approximated by a uniform, cylindrical plasma column as shown in curves A and B in Fig. 6.2. For the uniform column, the density is assumed to be $1.6 \times 10^8$/cm$^3$, which matches the asymptotic behavior of the theoretical result to that of the experimental curve. The radius for column A is chosen so that the total number of electrons in the column equals the total number in the nonuniform plasma. Since Malmberg and Wharton report that the absolute value of
the density profile may not be accurate, the procedure used to obtain the equivalent radius may not be valid. In fact, better agreement with the experimental dispersion relation is obtained when the column is assumed to be 2.25 cm in radius as shown in curve B. From the data supplied by Malmberg and Wharton, we find $\lambda_D = 0.19$ cm and $\omega_c/\omega_p = 4.0$. The real part of $k_\parallel$ obtained via the dispersion relation is compared with the experimental result in Fig. 6.3. The curves marked A' and B' are

$A' = \text{plasma radius } b = 3.5 \text{ cm}$

$B' = \text{plasma radius } b = 2.25 \text{ cm}$

FIG. 6.3. ANGULAR FREQUENCY VERSUS WAVE NUMBER.
Experimental data obtained by Malmberg and Wharton is compared with theoretical results.

the theoretical results for plasmas represented by curves A and B of Fig. 6.2. In Fig. 6.1, we have a graph of $\frac{\text{Im}(k_{\parallel})}{\text{Re}(k_{\parallel})}$ versus $\frac{v_p}{v_\theta}$, where $v_p = \frac{\omega}{\text{Re}(k_{\parallel})}$ is the phase velocity. The theoretical curve is computed by using B of Fig. 6.2. The agreement between the theory and experimental points is good, despite the approximate nature of the model.
C. SURFACE WAVE INSTABILITY IN ELECTRON PLASMAS AT LOW MAGNETIC FIELDS

When \( \omega_c < \omega_p \), the lowest order mode in a cold electron plasma column is a surface wave in the range \( \omega_c < \omega < \omega_H \), where \( \omega_H = \sqrt{\omega_p^2 + \omega_c^2}/2 \) (see Fig. 1.2). When the dispersion relation [Eq. (6.2)] was first solved in this region for real \( \omega \), instability was indicated. In this section we establish the presence of unstable surface waves and study the character of the instability.

One way to determine the presence of an instability is to map the negative real \( k \parallel \) axis into the complex \( \omega \) plane and see if the roots fall into the lower half plane. Such a mapping for a typical surface mode is shown in Fig. 6.5. Instability is indicated since the roots fall below the real \( \omega \) axis for a range of \( k \parallel \).
All instabilities can be classified into two types, convective and nonconvective (absolute) instabilities. A nonconvective instability grows with time until limited by nonlinear effects, while the convective instability is one in which a perturbation is amplified in space. We have established that an instability is present, but, as yet, we have not determined whether it is convective or nonconvective.

Derfler\textsuperscript{14,15} developed two methods of classifying instabilities. The first method is to map the complex $\omega$ plane into the complex $k$ plane through the "positive frequency part" of the dispersion relation and study the saddle-points ($d\omega/dk = 0$), which are caused by the merging of the two roots of the dispersion relation. The way in which the roots merge, as the imaginary part of the frequency is changed, is intimately related to the classification of the instability. Nonconvective instability occurs if and only if the two roots merge across the negative real $k$ axis with frequencies in the lower half plane. This criterion is more difficult to use than the second one which is described below.

The saddle-points when mapped into the complex $\omega$ plane through the dispersion relation become branch points. If the branch point, associated with the instability, falls in the lower half plane, the instability
is nonconvective. Otherwise, the instability is convective. Bürfler pointed out a method of locating the branch point, and his method is incorporated in Figs. 5.6 and 5.7. First, the negative real \( k \) axis is mapped into the complex \( \zeta \) plane. One then chooses a point \( A \) near the most unstable frequency and maps the line \( AB \), i.e., \( Re \zeta = \beta \) constant, into the frequency plane. This line will bend around the branch point as shown in Fig. 5.7. Now we take a point \( C \) on this line and map the line \( CD \) of Fig. 5.6 into the frequency plane. These mappings make a moose in the complex \( \zeta \) plane. The branch point must be inside the moose, and, by repeating the mapping procedure, the moose

![Diagram](image1)

**FIG. 5.6. COMPLEX \( \zeta \) PLANE.** The saddle point, \( A \), \( 2k \), \( \cdots \), must be inside the shaded area (see Fig. 5.7).

![Diagram](image2)

**FIG. 5.7. COMPLEX FREQUENCY PLANE.** The lines shown in Fig. 5.6 are mapped into complex \( \zeta \) plane as the dispersion relations. The branch point, \( A, 2k, \cdots \), must be inside the shaded area.
can be tightened to exactly locate the branch point. In our case, the branch point is in the upper half plane, and, hence, we have a convective instability.

The instability investigated above is a relatively mild case compared to those at lower magnetic fields. Attempts were made to apply the stability criterion at lower magnetic fields, but we were unable to locate the branch point due to computational difficulties. Thus, whether the instability remains convective or not for all values of the magnetic field is still an open question.

We now study the behavior of the instability as a function of the various parameters. Shown in Fig. 1-2 are mappings of the negative real

![Image](image-url)

**Fig. 1.2.** Real wave numbers as mapped into the complex frequency plane via the dispersion relation. On the range of parameters shown, the solution represents surface waves. \( \alpha \beta = 2.22, \gamma = 1.14 \).

wave numbers, \( \alpha \beta \), into the complex \( \omega \) plane for the lowest order mode with \( \frac{V_c}{P} \) as a parameter. The instability is strong near the cyclotron frequency and at the hybrid frequency. Also, whenever the cyclotron harmonics are below \( \omega = \sqrt{\frac{V_c}{P} - \frac{2}{V_c}} \), the instability is enhanced at these harmonics as seen in Fig. 1-3. At large negative
FIG. 6.9. MAPPING OF THE NEGATIVE REAL \((k, b)\) AXIS INTO THE COMPLEX \(\omega/\omega_p\) PLANE VIA THE DISPERSION RELATION FOR \(\omega_c/\omega_p = 0.2\).

values of \(k, b\), Landau damping sets in and makes the waves stable. Shown in Figs. 6.10 and 6.11 are the real and imaginary parts of the frequencies versus \(k, b\) with \(\omega_c/\omega_p\) as a parameter. Unlike the surface wave in a cold plasma column for which the dispersion relation goes asymptotically to \(\omega = \omega_H\), the real part of \(\omega\) goes above \(\omega_H\) as \(k, b\) is increased indefinitely. The magnitudes of the electrostatic potential profiles are shown in Fig. 6.12 for \(\omega_c/\omega_p = 0.3\). Note that the fields concentrate near the edge of the plasma as the frequency is raised above \(\omega_c\).

The curves of Fig. 6.8 seem to indicate that the plasma is more unstable at lower magnetic fields. However, the solution at \(\omega_c = 0\) is known to reduce to the cold plasma case and, hence, is stable. Figure 6.13 shows the imaginary part of the most unstable frequency versus \(\omega_c/\omega_p\). There are two curves, one for the instability at the cyclotron frequency and the other at the hybrid frequency \(\omega_H\). The portion of the curve shown by the dotted line was not computed. Although the instability
FIG. 6.10. DISPERSION DIAGRAMS FOR THE SURFACE WAVES SHOWING THE REAL PART OF $\omega/\omega_p$ VERSUS WAVE NUMBER $k_\parallel b$.

FIG. 6.11. DISPERSION DIAGRAMS FOR THE SURFACE WAVES SHOWING THE IMAGINARY PART OF $\omega/\omega_p$ VERSUS WAVE NUMBER $k_\parallel b$. 

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FIG. 6.12. RF POTENTIAL PROFILES AT SEVERAL FREQUENCIES WITH 
$\omega_c/\omega_p = 0.3$.

FIG. 6.13. THE IMAGINARY PART OF THE MOST UNSTABLE FREQUENCY VERSUS 
$\omega_c/\omega_p$. 

$\lambda_p/b = 0.02$
appears to increase as $\omega_c$ is decreased, the solution at $\omega_c = 0$ is known to be stable, and thus a transition curve, such as the one shown by the dotted line, must exist. With cyclotron frequencies below $\omega_c = 0.1 \omega_p$, the summations in the dielectric tensor converge too slowly and accuracy is lost. Consequently, we were unable to cover this range.

The effects of other parameters on the instability are shown in Figs. 6.14 and 6.15. In Fig. 6.14 we show the effect of temperature anistropy. The roots of the dispersion relation are plotted with $T/L$. 

![Diagram](image)

**FIG. 6.14.** **REAL WAVE NUMBER AS MAPPED INTO THE COMPLEX $\omega/\omega_p$ PLANE VIA THE DISPERSION RELATION.** The locus diagram illustrates the effect of the transverse temperature on the surface wave. $\omega_c/\omega_p = 0.5$, $a/b = 2.22$ and $\lambda_p/\delta = 0.04$. 

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as a parameter. The plasma is stable when $T = 0$, and the instability, in its nonlinear limit, may provide a mechanism to convert transverse dispersion relation with $\lambda_p/b$ as a parameter. Since we keep $\lambda_p$ and $b$ constant, a change in $\lambda_p/b$ is equivalent to a change in the overall temperature. We see that the higher the ratio $\lambda_p/b$ (hence, higher the temperature), the greater the instability. These graphs indicate that the instability is due to finite Larmor radii effects.

No conclusive experimental evidence of the surface wave instability has been found. There is, however, a report of surface wave self oscillations with zero magnetic field in a plasma column. Our theory predicts strong instability at low magnetic field, but the instability should disappear at zero magnetic field. Since we are unable to compute the dispersion relation in the very low magnetic field region, we cannot relate the observed oscillation to our instability. Also at large Larmor radii, our dc model with sharp boundary breaks down.
In the analysis of parts (a) and (b) we considered only the electron motion and temporarily ignored the ion motion. In this section we include the ion motion and investigate the temperature relation near the ion plasma and the cyclotron frequencies.

In a cold plasma consisting of electrons and ions, the density under code becomes a surface wave in the region 

\[ \frac{1}{\omega_p^2} \text{ vs } \frac{1}{\omega_c^2} \]

for about 0.5. The cyclotron temperature relation near the ion plasma frequency is shown in Fig. 7.2. When the ion plasma modes are used, the surface wave is again found to be unstable. The computations for the plasma with ions and a number proved to be much more difficult than those for the electron plasma. In these types, the results of the computations with the electron-ion plasma are results of Fig. 7.2. Further computations of these types are needed in assuming that the temperature relation of the electrons is zero. Shown in Fig. 7.1 (d) is the mapping of the magnetic wave lines for the electron and ion plasma relation.

---

**FIG. 7.1:** Cold Plasma Dispersion Relation Plan

- The ion plasma frequency kinematics.
Fig. 1: The real and imaginary parts of $\omega_{pt}$.
parts of \( \omega \) versus \( n \). The instability appears to be very strong since the ratio of the imaginary part of the frequency to the real part can be of the order of \( 10^5 \). We should keep in mind, however, that the model used in this computation is collisionless. In the range of our investigation the longitudinal dielectric tensor element \( \varepsilon_{zz} \) is predominantly electronic. Since the operating frequency is very much lower than the electron plasma frequency, a small number of electromagnetic or electron-electron collisions may alter the result. A further investigation of the ion surface wave instability should, therefore, include the electron-ion collision effects. Also these ion surface waves for practical ion-electron mass ratios have very long wavelengths. Often the wavelengths will exceed the length of the practical plasma column, and one effects, such as line tying, may determine whether this instability occurs or not.

V HIGHER ORDER Modes

Due to the tremendous complexity involved in solving the dispersion relation, very little was done to investigate the higher order modes. The resonantly coupling modes are completely ignored, and some of these modes may be even more available than the symmetric mode. We paid little attention to the mode commonly called the upper hybrid mode, which appears in the unit plasma column. See Fig. 1. a. The graph of Fig. 1. b. shows the dispersion relation for the upper mode in a hot electron plasma with \( n_e = n_i \). The mode is heavily coupled which may explain the difficulty of the experimenters to observe this mode.

In addition to the symmetric mode, there is an infinite number of the hybrid modes at a hot plasma, each associated with one of the Bernstein modes. We did not investigate these modes.

We also did not undertake the study of higher order radial modes, since they are probably not as interesting as the azimuthal modes or radial modes. Our investigation treats only a small portion of a real problem.
FIG. 6.19. DISPERSION RELATION FOR THE HYBRID MODE.
In our study of electrostatic waves we emphasized the effects due to the finite boundary as well as the effects of magnetic fields and plasma temperature. The first part of this report treated plasma diodes in the absence of a magnetic field. A dc study of the equilibrium plasma diode was undertaken to provide a basis for the rf analysis. The dc analysis yielded some interesting results regarding the possibility of plasma formation within a diode. It was found that, under equilibrium conditions, plasma is formed even when the generation rate of one of the charged species at the walls is orders of magnitude lower than that of the other. The formation of plasma due to the trapping of rare species in the coulomb field of the abundant species was demonstrated in the case of metal emitters where the emission rate of the ions was very much lower than that of the electrons.

An rf analysis using the hydrodynamic model for the nonuniform plasma revealed the existence of a series of electrostatic resonances in the diode. However, an analysis of a uniform plasma diode using a kinetic plasma model showed that the end plate loss, due to absorption of the electrons at the walls, introduces a large loss mechanism. The loss may make some of the resonances predicted from the hydrodynamic model unobservable. Since this loss mechanism is related to the process which causes end plate diffusion in devices such as the Q-machine, the rf measurements of the electrostatic fields may yield information regarding the end plate diffusion coefficient.

The second part of this report dealt with the study of electrostatic wave propagation along a cylindrical column of Maxwellian plasma. First, a dc study of the density profile and azimuthal current in a cylindrical column was conducted. The theoretical analysis of the column under the assumption of complete neutrality showed that at high magnetic fields when the ion Larmor radii become much smaller than the plasma radius, the density drop was confined to the small region at the plasma edge. Experimental measurements indicated that the density drop at the plasma edge is in fact much sharper than that predicted from our theory. For the purpose of rf studies, we assumed that the plasma was uniform with
an infinitely sharp boundary since we were not interested in neutrino instabilities, but in surface waves.

To obtain the solution for waves in the ion plasma column, we first derived the dielectric tensor appropriate for plane waves in an infinite plasma. Solutions for waves in a plasma column and the dispersion relation for these waves were obtained through plane wave methods. The numerical solution of the dispersion relation revealed that the electrostatic surface waves propagating along the column were unstable.

When only the electrons were assumed to respond to the rf field, the electron surface waves which occur when \( \frac{\omega}{c} < \frac{c}{a} \) were found to be unstable. Since the surface wave became stable when \( \frac{\omega}{c} = \frac{c}{a} \), we conclude that the instability is due to finite ion mass effects. We were able to classify the instability only in the highly unstable case \( \frac{\omega}{c} < \frac{c}{a} \), in which the instability was found to be convective. Such an instability can most likely be suppressed by collisions or by external agents causing localizing effects. For more severely unstable cases at low magnetic fields, the application of the stability criterion was hampered by numerical difficulties. In fact, it was not possible to compute the dispersion relation in the range \( \frac{\omega}{c} < \frac{c}{a} \) with our program. Our study was also confined to radially symmetric modes.

When the ion motion was included in the analysis, the surface waves which occur when \( \frac{\omega}{c} > \frac{c}{a} \) were also found to be unstable. The instability in surface waves are of great interest since the instability appears to be very strong. And while the electron surface waves are easily classified by raising the cyclotron frequency above the plasma frequency \( \frac{\omega}{c} \), the stabilizing condition for the ion surface wave \( \frac{\omega}{c} > \frac{c}{a} \) may be difficult to attain in high density plasmas such as those in fusion machines. The numerical analysis was exceedingly difficult for the ion surface waves, and we were unable to do a more thorough analysis.

Despite the simplifications made by assuming that the plasma column was uniform with a sharp boundary, the computation of the dispersion relation in the whole wave was extremely time consuming. A considerable improvement of the computer program is needed before a more exhaustive analysis can be made. A future analysis should include the axially varying waves. Such a study may yield modes which are more unstable than the
cylindrically symmetric modes. A study of azimuthally varying ion waves should prove to be very interesting since many of the observed low frequency instabilities are associated with azimuthally varying waves. A more ambitious undertaking should take into account the nonuniformity of the plasma column. In such a column other instabilities such as the toroidal instability may occur in addition to the surface wave instability mentioned in this paper.
APPENDIX A. LOW DENSITY PROBE MEASUREMENT

In this appendix we show some of the difficulties encountered in probing a low density plasma and the ways in which these problems are overcome. In our sodium plasma the typical density is of the order of \(2 \times 10^7/\text{cc}\), which is lower than most of the laboratory plasmas. In the probe measurements of a low density plasma, the smallness of the probe current is the biggest problem. The probe size can be increased to increase the total current, but beyond a certain size the probe disturbs the plasma too much. Also, the probe should be as small as possible to make local measurements.

To obtain accurate probe measurements it is necessary to have the "knee" of the probe curve show clearly. This can be attained by a guard-ring arrangement of the probe. The above consideration is taken into account in the design of the co-axial probe shown in Fig. 4.4. Normally the stem of the guard-ring probe is insulated so that the current is drawn only through the probe surface and the co-planar guard-ring surface. However, since a film of sodium would quickly form on an insulated surface, causing leakage current, we decided not to coat the stem. Instead we made the whole probe as small as possible. The size of the probe is such that the total current drawn to the whole probe at the plasma potential is less than 1 percent of the total electron current from the emitters under normal operating conditions.

Maneuverability is another desirable feature of a probe. The details of the probe driving mechanism is described by Simonen. The probe moves in and out as well as rotates so that the probe surface can be made to face either in the azimuthal or axial direction. When an axial magnetic field is applied, the density can be measured by facing the probe in the axial direction, while azimuthal drift can be studied by facing the probe in the azimuthal direction.

Typical probe currents are of the order of a few microamperes. This is a level where leakage currents from various sources begin to interfere with the probe curve, particularly leakage between the guard ring and the probe. Moreover, a dc amplifier is required to amplify the voltage across the current pick-up resistor to drive the XY recorder.
A schematic drawing of the probe circuit is shown in Fig. A.1. The amplifier is a Keithley 604 electrometer amplifier, which has the desired characteristic of high input impedance, low noise and drift. However, it is necessary to keep the electrometer terminals near the ground as shown in Fig. A.1. Since a terminal of the amplifier needs to be grounded, we are forced to use two sweep circuits instead of one to keep the guard ring voltage equal to the probe voltage. The sweep circuit is shown in Fig. A.2. Two such circuits with matched characteristics but insulated from each other were constructed and placed in a metal box. When a master switch is turned on, the two circuits simultaneously sweep out a single identical saw-tooth wave form. The time constant of the sweep is variable from 1 to - seconds and the amplitude is variable from 1 to 2 volts.

FIG. A.1. PROBE CIRCUIT
FIG. A.2. SWEEP CIRCUIT.
APPENDIX B. SOLUTION OF THE LINEARIZED BOLTZMANN EQUATION

In this appendix we obtain a solution of the linearized Boltzmann equation for a uniform plasma in a magnetic field. The solution has been derived by others using several different approaches, but we present a derivation here for the sake of completeness. The analysis can be generalized to any species of the plasma, but to avoid extra subscript we shall apply the analysis to electrons only. We shall assume that the plasma is collisionless and that there is no dc electric field. Then the zeroth and the first order Boltzmann equations are given by

\[
\left(\vec{v} \times \vec{B}_0\right) \cdot \frac{\partial f_0(\vec{v})}{\partial \vec{v}} = 0 \\
\frac{\partial f_1(t, \vec{x}, \vec{v})}{\partial t} + \vec{v} \cdot \frac{\partial f_1(t, \vec{x}, \vec{v})}{\partial \vec{x}} - \frac{e}{m} (\vec{v} \times \vec{B}_0) \cdot \frac{\partial f_1(t, \vec{x}, \vec{v})}{\partial \vec{v}} = \frac{e}{m} \left( \vec{E}_1(t, \vec{x}) + \vec{v} \times \vec{B}_1(t, \vec{x}) \right) \cdot \frac{\partial f_0(\vec{v})}{\partial \vec{v}}.
\]

Let us make the following definition

\[
\frac{e \vec{B}_0}{m} = \vec{\omega}_c = \vec{n} \omega_c
\]

where \( \vec{n} \) is a unit vector in the direction of the magnetic field. We Fourier analyze the linearized equation in space \( \text{Im} (\vec{k}) = 0 \) and Laplace analyze in time \( \text{Im} (\omega) < 0 \) on L.I.P.) to get

\[
f(\omega - \vec{k} \cdot \vec{v}) f_1(\omega, \vec{k}, \vec{v}) + \omega_c (\vec{n} \times \vec{v}) \cdot \frac{\partial}{\partial \vec{v}} f_1(\omega, \vec{k}, \vec{v}) = \frac{e}{m} \left( \vec{E}^{(0)}(\omega, \vec{k}) + \vec{v} \times \vec{B}_0(\omega, \vec{k}) \right) \cdot \frac{\partial f_0(\vec{v})}{\partial \vec{v}} + f_1(0, \vec{k}, \vec{v})
\]

where \( f_1(0, \vec{k}, \vec{v}) \) is the initial value of \( f_1(t, \vec{k}, \vec{v}) \). Equation (B.3) can be solved by the method of characteristics. By introducing a parameter \( t \) so that \( \vec{v} = \vec{v}(t) \), the characteristics of Eq. (B.3) become

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\[ \mathbf{v} = \omega \times (\mathbf{n} \times \mathbf{v}) \quad (B.4) \]

and
\[ f_1(\omega, \mathbf{k}, \mathbf{v}) + i(\omega - \mathbf{k} \cdot \mathbf{v}) f_1(\omega, \mathbf{k}, \mathbf{v}) = g[\mathbf{v}] \quad (B.7) \]

where the dot represents the derivative with respect to the parameter \( t \) and
\[ g[\mathbf{v}] = \frac{e}{m} \left( E[\omega, \mathbf{k}] + \mathbf{v} \times B[\omega, \mathbf{k}] \right) \cdot \frac{\partial \mathbf{v}}{\partial \mathbf{v}} + f_1(\mathbf{c}, \mathbf{k}, \mathbf{v}) . \quad (B.6) \]

Note that \( t \) is not the "time" which has been taken out of the problem by Laplace analysis. Equation (B.4) is solved by introducing the gyrotensor defined by
\[ \mathbf{D}(\varphi) = \mathbf{n} \mathbf{n} + (I - \mathbf{n} \mathbf{n}) \cos \varphi - \mathbf{n} \times \mathbf{I} \sin \varphi \quad (B.7) \]

or, in tensor notation, by
\[ D_{\alpha \beta} = n_i n_j + (\delta_{i j} - n_i n_j) \cos \varphi - \delta_{i j} n_k \sin \varphi . \quad (B.8) \]

Then
\[ \mathbf{v}(t) = \mathbf{D}(- \omega c t + \omega_j t') \cdot \mathbf{v}(t') \quad (B.9) \]

satisfies Eq. (B.4), as can be seen by inspection. The homogeneous part of Eq. (B.5) gives
\[ \frac{d}{dt} f_H = - i(\omega - \mathbf{k} \cdot \mathbf{v}) . \quad (B.10) \]

Thus
\[ \ln f_H = - i \omega t + i \int^t \mathbf{v} \cdot \mathbf{v}(t') dt' + \text{constant} \]

or
\[ f_H = C \exp \left[ - i \omega t + i \int^t \mathbf{k} \cdot \mathbf{v}(t') dt' \right] . \quad (B.11) \]
where \( \tilde{V} \) is given by Eq. (8). We next solve the inhomogeneous equation by the method of variation of constants. Substituting

\[
\left( j \frac{\tilde{V}}{\tilde{V}} \right) : C = q,
\]

into Eq. (8), we obtain

\[
C_2 + C_3 + \ldots + C_N = \tilde{V}.
\]

From Eq. (9.11) we get

\[
C_2 = \ldots = C_N = 0.
\]

Therefore,

\[
C_1 = q
\]

Solving for \( C_1 \), we have

\[
C_1 = \int \frac{\tilde{V} \cdot d\tilde{V}}{\tilde{V}}.
\]

Thus,

\[
\tilde{E}_1 \text{ must, } C_1 = \int \frac{\tilde{V} \cdot d\tilde{V}}{\tilde{V}} = \epsilon_0
\]

or

\[
\tilde{E}_1 = \left[ \tilde{E}_1 \right] \left[ \exp \left[ -\left( \tilde{E}_1 \cdot d\tilde{V} \right) \right] \right] \left[ \exp \left[ -\left( \tilde{E}_1 \cdot d\tilde{V} \right) \right] \right] \left[ \exp \left[ -\left( \tilde{E}_1 \cdot d\tilde{V} \right) \right] \right] \ldots
\]

\[
\ldots \int \exp \left[ -\left( \tilde{E}_1 \cdot d\tilde{V} \right) \right] \exp \left[ -\left( \tilde{E}_1 \cdot d\tilde{V} \right) \right] \exp \left[ -\left( \tilde{E}_1 \cdot d\tilde{V} \right) \right] \ldots.
\]
In a bayesian analysis, with \( \lambda \) and \( n \) and \( \mu \) and \( \sigma^2 \), and

\[
\frac{1}{\lambda} \int e^{-\frac{t}{\lambda}} dt
\]

we proceed to calculate the parameter \( \theta \). From Eq. 3 we have

\[
\int e^{-\frac{t}{\lambda}} dt
\]

and so change the variables of integration to \( u \). This is given by

\[
\int e^{-\frac{t}{\lambda}} dt
\]

The parameter \( t \) now spans between \( 0 \) and \( \infty \) as usual.

\[
\int e^{-\frac{t}{\lambda}} dt
\]

By changing the integration variables to \( t \) and \( \theta \), and introducing the

\[
\int e^{-\frac{t}{\lambda}} dt
\]

the contour cuts the real part of \( \lambda \) by \( \theta \) in the line

\[
\int e^{-\frac{t}{\lambda}} dt
\]
Then for the solution can be found by taking the initial condition $\eta(0,x) = \eta^0(x)$. However, when we consider the problem in the steady state rather than the transient process, we shall study the solution with the initial condition set to zero. Then from Eq. (1) we get

\begin{equation}
\eta(x) = \frac{\int \eta^0(x') dx'}{\eta^0(x) dx}
\end{equation}

This is the equation that we can use to determine $\eta(x)$ from $\eta^0(x)$. The next process is to set $\eta(x) = 1$ which corresponds to $\eta^0(x) = 1$. For example,

\begin{equation}
\eta(x) = \frac{\int 1 dx'}{1 dx}
\end{equation}

where $\int$ and $1$ are the limits of integration along one particular part of the magnetic field configuration. Thus,

\begin{equation}
\eta(x) = \frac{\int 1 dx'}{1 dx}
\end{equation}

It can be shown by direct calculation that

\begin{equation}
\eta(x) = \frac{\int 1 dx'}{1 dx}
\end{equation}

and

\begin{equation}
\eta(x) = \frac{\int 1 dx'}{1 dx}
\end{equation}

where

\begin{equation}
\eta(x) = \frac{\int 1 dx'}{1 dx}
\end{equation}

In setting the constants we set $\eta(x) = 0$ we have

\begin{equation}
\eta(x) = \frac{1}{\int 1 dx'}
\end{equation}

\begin{equation}
\eta(x) = \frac{1}{\int 1 dx'}
\end{equation}

\begin{equation}
\eta(x) = \frac{1}{\int 1 dx'}
\end{equation}
\[
\left( \frac{1}{2} \right) \int \frac{x}{x^2 + 1} \, dx
\]
APPENDIX C. POLARIZATION AND MAGNETIZATION CURRENTS

The first order current and charge density in a hot electron plasma can be obtained from the relation:

\[
\mathbf{J} = -\nabla \phi
\]

de and

\[
\mathbf{M} = \frac{1}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{r} \, d^3r' = \frac{1}{4\pi} \int \nabla \phi \cdot \mathbf{r} \, d^3r'
\]

where \( \phi \) is as defined in Eq. 8.76. In this appendix, we substitute the plasma current \( \mathbf{J} \) into the polarization and magnetization currents

\[
\mathbf{P} = \varepsilon_0 \nabla \phi
\]

so that the plasma can be interpreted completely by the polarization and the magnetization \( \mathbf{M} \) on their introduction from charges and from currents. To complete the analysis, we must to a few Maxwell tensor relations in order. In Section 1, we formulate the plasma current into the ten terms for the case of an infinite plane with uniform distribution function. A result would be that the Maxwell tensor relations function is given as follows:

**Tensor Relations**

We shall first derive a few general tensor relations. Let \( A, B, C \) and \( D \) be arbitrary vectors. Then we have

\[
A \cdot (B \times C) = C \cdot (B \times A) = B \cdot (C \times A)
\]

and \( A \cdot (B \times C) = A \times (B \times C) \). In order to derive these vector relations, from Eq. 1, we have

\[
A \cdot (B \times C) = A \cdot (C \times B)
\]
By setting the redundant brackets, we obtain

Thus.

By using Eqs. (6.10) and (6.12), we get

\[ 2c \frac{\partial N}{\partial t} = \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) \cdot \mathbf{B} \cdot \mathbf{E} \]  

where \( \mathbf{B} \) is as defined by Eq. (6.11). We also have

\[ \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) \cdot \mathbf{E} \]  

Thus, from Eqs. (6.10) and (6.12), we get

\[ \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) \cdot \mathbf{E} \]  

Moreover, Eqs. (6.10) can be expressed in terms of \( \mathbf{E} \)

\[ \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) \cdot \mathbf{E} \]

As a direct cross multiplication of Eqs. (6.10) with \( \mathbf{E} \), we get the following two relations

\[ \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) \cdot \mathbf{E} \]

\[ \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) \cdot \mathbf{E} \]

Lagrangian Density Distribution Functions

In this section, we will consider a completely symmetrical four-vertex

densest distribution function \( \mathbf{L} \). In this case, the partition

densest distribution function [Eq. (6.11)] becomes

\[ \frac{1}{2} \int \mathbf{E} \cdot \mathbf{E} \]
where

\[ t = \frac{-i}{u_c} \left( \frac{R}{2} - \frac{1}{2} \frac{i}{u_c} \right) \]

The charge density can be obtained by integration of \( \dot{c} \).

\[ \dot{c} = -\frac{i}{u_c} \int \left[ \int \dot{t} \, d \tau \right] d \xi \cdot \frac{k_0}{\hbar_0} \, e^{-i \phi} \]

By partial integration in \( \tilde{y} \), we obtain

\[ \dot{c} = -\frac{i}{u_c} \int \left[ \int \dot{t} \, d \tau \right] d \xi \cdot \frac{k_0}{\hbar_0} \, e^{-i \phi} \]

The polarization change associated with a plane wave is given by

\[ \frac{1}{2} \int \left[ \int \dot{t} \, d \tau \right] d \xi \cdot \frac{k_0}{\hbar_0} \, e^{-i \phi} \]

We use identity: the total change \( \tilde{p} \) with \( \tilde{p} \) and calculate \( \tilde{p} \) from Eq. 22. We have

\[ \tilde{p} = \frac{i}{u_c} \int \left[ \int \dot{t} \, d \tau \right] d \xi \cdot \frac{k_0}{\hbar_0} \, e^{-i \phi} \]

The first order plasma current can be obtained by the following integration:

\[ \left. \tilde{i} \right|_{0}^{\infty} = \frac{i}{u_c} \int \left[ \int \dot{t} \, d \tau \right] d \xi \cdot \frac{k_0}{\hbar_0} \, e^{-i \phi} \]

\[ \frac{i}{u_c} \int \left[ \int \dot{t} \, d \tau \right] d \xi \cdot \frac{k_0}{\hbar_0} \, e^{-i \phi} \]
remaining after $\vec{a} \times \vec{H}$ is subtracted from Eq. (C.26), vanish. We show that this is indeed the case. Let

$$\vec{a} = k_0 \vec{P} - \langle \rho \sigma \rangle - i \vec{k} \times \vec{H}.$$ 

Then

$$\tilde{a} = \frac{e^i}{\pi c} \int 2\pi \sigma f_o(\sigma) \left[ \frac{\sigma \vec{P}}{\sigma c} - 1 - i \frac{\sigma (\vec{k} \cdot \vec{L} \times \vec{H})}{\sigma c} \right] \cdot (\vec{E} \cdot \vec{B}) \text{ (C.39)}$$

Since $\sigma = \vec{L} \cdot \vec{E} = \vec{H}$ from (C.17), we have

$$\tilde{a} = \frac{e^i}{\pi c} \int 2\pi \sigma f_o(\sigma) \left[ \frac{\sigma \vec{P}}{\sigma c} - 1 - i \frac{\sigma (\vec{k} \cdot \vec{L} \times \vec{H})}{\sigma c} \right] \cdot \vec{E} \cdot \vec{B} \text{ (C.39)}$$

Partial integration in $\sigma$ of the first term yields

$$\tilde{a} = \frac{e^i}{\pi c} \int 2\pi \sigma f_o(\sigma) \left[ - \vec{L} \cdot (\vec{E} \cdot \vec{B}) - \vec{H} - (\vec{H} \cdot \vec{B}) - \vec{E} \cdot \vec{L} \right]$$

$$= \left( \frac{\vec{L} \cdot (\vec{E} \cdot \vec{B}) - \vec{H}}{\sigma c} \right) \cdot \vec{E} \cdot \vec{B} \text{ (C.32)}.$$ 

Let the first three terms of the integrand be $-C$. Then

$$\tilde{C} = -e^i \sigma \left[ (\vec{E} \cdot \vec{B}) - \vec{H} \cdot \vec{L} \right] \text{ (C.33)}.$$ 

By using Eqs. (C.31), (C.32), and (C.33), we get

$$\tilde{C} = -e^i \sigma \left[ (\vec{E} \cdot \vec{B}) - \vec{H} \cdot \vec{L} \right]$$

$$= -e^i \sigma \left[ (\vec{E} \cdot \vec{B}) - \vec{H} \cdot \vec{L} \right]$$

$$= e^i \sigma \left[ (\vec{E} \cdot \vec{B}) - \vec{H} \cdot \vec{L} \right]$$
a subdivision as simple as that described in (2) cannot be accomplished.
However, if we ignore forces of the order \( \vec{v} \cdot \vec{b} \) as compared with \( \vec{b} \)
then the problem simplifies, and in fact the procedure to separate the
currents is identical to that followed in Section (2). The result is
again given by Eqs (C.24) and (C.27) where now \( f_g(\vec{r}) \) is as defined in
(C.33). For electrostatic cases the approximation \( \vec{v} \cdot \vec{b} \) should
be very good.
APPENDIX D. THE DIELECTRIC SUSCEPTIBILITY TENSOR FOR MONOELECTRONIC ELECTRON PLASMAS

In this appendix we evaluate the dielectric susceptibility tensor appropriate for monoelectronic electron plasmas. In particular, we express the tensor elements in terms of the Hilbert transforms of the functions as tabulated by Privat and Costa. The susceptibility tensor for the case of isentropic temperature is also given in section 3.2. The case of isentropic temperature can be handled by a slight modification of the procedure followed for the isentropic case. In section 3.2 the results are given without derivation. Some useful properties of the Hilbert transforms of the functions are shown in section 3.3.

1. Isoentropic Distribution Functions

From Eq. 8.21 we have

\[ \tilde{\rho} = \frac{\partial}{\partial \tilde{\alpha}} \left[ \tilde{\alpha} \tilde{\rho} \right] \]

The distribution functions and the normalization conditions are given by

\[ \tilde{\rho} = \phi \left( \frac{\tilde{\alpha}}{\tilde{\alpha}} \right) \]

\[ \int \tilde{\rho} \tilde{\alpha} \tilde{\rho} \tilde{\alpha} = 1 \]

Substituting Eq. 8.3 into 8.1, we obtain

\[ \tilde{\rho} = \frac{\partial}{\partial \tilde{\alpha}} \left[ \tilde{\alpha} \tilde{\rho} \right] \]

The velocity integral of Eq. 8.4 can be performed and the result is given by

\[ \tilde{\rho} = \frac{\partial}{\partial \tilde{\alpha}} \left[ \tilde{\alpha} \tilde{\rho} \right] \]
A direct substitution of $T$ and $T'$ gives

$$ T + T' - 2T' \int \frac{d^3k}{(2\pi)^3} \delta(k) \cdot E \cdot \left( 1 - \cos \theta \right) \cdot E. $$

In terms of Cartesian coordinates we obtain

$$ T = T_x \delta_{x,y} - T_y \delta_{y,z} - T_z \delta_{z,x}; $$

$$ -T' \left( 1 - \cos \theta \right) \cdot E. $$

where $T_x$, $T_y$, and $T_z$ are unit vectors along $x$, $y$, and $z$ axes respectively with the $x$ axis coincident with the $x$ axis.

The effective magnetic field tensor defined by

$$ B \cdot \delta \cdot T \cdot E $$

satisfies

$$ T = \begin{bmatrix}
  T_x & T_y & T_z \\
  T_y & T_z & T_x \\
  T_z & T_x & T_y
\end{bmatrix} $$

and

$$ -\frac{i}{\omega} \cdot \int \frac{d^3k}{(2\pi)^3} \delta(k) \cdot E \cdot \left( 1 - \cos \theta \right) \cdot E. $$

$$ T_x = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \delta(k) \cdot E \cdot \left( 1 - \cos \theta \right) \cdot E. $$

$$ T_y = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \delta(k) \cdot E \cdot \left( 1 - \cos \theta \right) \cdot E. $$

$$ T_z = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \delta(k) \cdot E \cdot \left( 1 - \cos \theta \right) \cdot E. $$
where

\[ y = \frac{\dot{y}^2}{a_c} - \frac{1}{a_c^2} \left( \dot{y} - \dot{u} \right)^2 \]

\[ -\frac{\dot{y}^2}{a_c} + \frac{1}{a_c^2} \left( \dot{y} - \dot{u} \right)^2 - \frac{\dot{y}^2}{a_c^2} \left( 1 - \cos y \right) . \]

There are four types of integrals to be evaluated:

\[ S_1(k_1, \mu, \sigma) \cdot \int_0^\infty e^{-x} \, dx \]  
\[ S_2(k_1, \mu, \sigma) \cdot \int_0^\infty e^{-x} \, dx \]  
\[ S_3(k_1, \mu, \sigma) \cdot \int_0^\infty e^{-x} \sin x \, dx \]  
\[ S_4(k_1, \mu, \sigma) \cdot \int_0^\infty \cos x e^{-x} \, dx . \]  

However, all of the integrals can be obtained from one integral \( S_1 \) due to the following relations:

\[ S_3(k_1, \mu, \sigma) = \frac{S_1(k_1, \mu, \sigma - u_2) - S_1(k_1, \mu, \sigma + u_2)}{2a} \]  
\[ S_4(k_1, \mu, \sigma) = \frac{S_1(k_1, \mu, \sigma - u_2) + S_1(k_1, \mu, \sigma + u_2)}{2} , \]

and by partial integration of the following integral

\[ \int_0^\infty \left( \frac{\dot{y}^2}{a_c} + \frac{1}{a_c^2} \left( \dot{y} - \dot{u} \right)^2 + \frac{k^2 v_0^2}{a_c^2} \sin t \right) e^{-x} \, dx \]

one obtains

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\[
\frac{1}{a_c} s_1(k^1, k^2, \mu) = \frac{a_c^2}{2} s_1(k^1, k^2, \mu) - \frac{a_c^2}{2} s_2(k^1, k^2, \mu) - \frac{a_c^2}{2} s_3(k^1, k^2, \mu). \tag{B.14}
\]

In addition, the following relation is also true:

\[
s_1(k^1, k^2, \mu) = -\frac{a_c^2}{2} s_1(k^1, k^2, \mu). \tag{B.15}
\]

Thus, the only integral needed is \( S_1 \), which we shall evaluate now. Let us first introduce the variables

\[
\begin{align*}
\frac{a_c^2}{2} k^2 &= \frac{a_c^2}{2} k^2, \\
\frac{a_c^2}{2} k^1 &= \frac{a_c^2}{2} k^1, \\
\end{align*}
\]

and

\[
p = \left( \frac{a_c^2}{2} - m \right)
\]

and use the well-known expansion

\[
\exp \left[ \frac{a_c^2}{2} \cos \varphi \right] = \sum_{n=0}^{\infty} I_n \left( \frac{a_c^2}{2} \right) e^{i n \varphi} \tag{B.16}
\]

where \( I_n \) are the modified Bessel functions. Then \( S_1 \) becomes

\[
S_1 = \sum_{n=0}^{\infty} \exp \left[ \frac{a_c^2}{2} \right] I_n \left( \frac{a_c^2}{2} \right) \int_0^\infty d\psi \ e^{-\frac{a_c^2}{2} \psi} \psi^n. \tag{B.17}
\]

We now change the integration variable to \( s = \pm \psi \), where the sign is fixed such that \( s > 0 \) for \( \varphi > \delta \), and obtain
\[ s_1 = \sum_{\ell} \exp\left( -\frac{\ell^2}{2} \right) I_{\frac{\ell}{2}}(\ell \frac{m}{v_0}) \exp\left[ \frac{\ell^2}{2} - \left( \frac{r_n}{v_0} \right)^2 \right] \]  \hspace{1cm} (D.22)

The integral can be expressed in terms of the confluent hypergeometric function, \( \kappa(z, \nu; c) \), first introduced by Tricomi.

\[ s_2 = \sum_{\ell} \exp\left[ -\frac{\ell^2}{2} \right] I_{\frac{\ell}{2}}\left( -\frac{r_n}{v_0} \right) \left[ \frac{1}{2}, \frac{1}{2}, \left( \frac{r_n}{v_0} \right)^2 \right] \]  \hspace{1cm} (D.23)

With these particular parameters, \( s \) can also be expressed in terms of the Hilbert transform of the gaussian as defined by Fried and Conte.

\[ x(\xi) = \frac{1}{\xi^{1/2}} \int_{-\infty}^{\infty} \frac{e^{-\xi s}}{\sqrt{2\pi \xi}} \hspace{1cm} \text{in } \xi > 0 \]  \hspace{1cm} (D.24)

Let us define a new variable

\[ \xi_0 = \frac{\xi - m_c}{\sqrt{2} k_n v_0} \]  \hspace{1cm} (D.25)

Then \( \kappa\left[ -\frac{1}{2}, \frac{1}{2}, -\xi_0^2 \right] = x(\xi) \) as can be shown by comparing the power series expansions of \( x(\xi) \) and \[ \kappa\left( -\frac{1}{2}, \frac{1}{2}, -\xi_0^2 \right) \]. The function \( x(\xi) \) is defined such that it is regular for \( |\xi| > 0 \), in other words, when

\[ \Re \left( \frac{e^{-m_c \xi}}{\sqrt{2} k_n v_0} \right) > 0 \]  \hspace{1cm} (D.26)

Since the Laplace integral path \( \Re s < 0 \), the inequality of (D.26) requires that \( k_n > 0 \), which makes \( m < 0 \). Thus the negative sign in front of \( k_n \) must be selected in order to keep \( s > 0 \) in (D.22), and so

\[ s_1(k_n, k_n, m) = -\frac{\sqrt{2} \alpha_c}{k_n v_0} \sum_{\ell} \exp\left[ -\frac{\ell^2}{2} \right] I_{\frac{\ell}{2}}(\ell \frac{m}{v_0}) x(\xi_0) \]  \hspace{1cm} (D.27)
When (9.27) is substituted into Eqs. (9.15) through (9.19), the elements of the dielectric susceptibility tensor become

\[ \chi_{xx} = \chi_{yy} = \frac{\varepsilon^2}{2\varepsilon_0} \left[ \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} \right] \frac{2\pi}{a} \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{e^{-a_n \cdot \mathbf{r}}}{a_n} \],

\[ \chi_{zz} = \frac{\varepsilon^2}{\varepsilon_0} \left[ \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} \right] \frac{2\pi}{a} \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{e^{-a_n \cdot \mathbf{r}}}{a_n} \],

\[ \chi_{xy} = \chi_{yx} = \frac{\varepsilon^2}{2\varepsilon_0} \left[ \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} \right] \frac{2\pi}{a} \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{e^{-a_n \cdot \mathbf{r}}}{a_n} \]

where \( a_n = \frac{\pi}{a} \). 

\[ f_0(\mathbf{v}) = A \exp \left[ -\frac{\mathbf{v}}{2\mathbf{v}_0^2} \mathbf{v} \cdot \mathbf{v}_0 \right], \]
In terms of the Hilbert transform of the gaussian, we have

\[ \beta_{xx} = \beta_{yy} = \frac{a^2}{2\sqrt{\pi} a \kappa \nu_0} \exp \left[ -\frac{\beta^2}{2} \right] \sum_{n=-\infty}^{\infty} \left( \frac{\kappa}{\beta} \right)^n \beta_{n-1} \beta_{n-1} \]  

(D.35)

\[ \beta_{zz} = -\frac{a^2}{2\sqrt{\pi} a \kappa \nu_0} \exp \left[ -\frac{\beta^2}{2} \right] \sum_{n=-\infty}^{\infty} \left( \frac{\kappa}{\beta} \right)^n \beta_{n+1} \beta_{n+1} \]  

(D.35)

where

\[ \beta_{n} = \beta_{i} \zeta_{n} \]

\[ \zeta_{n} = \frac{\nu - \kappa \kappa - \nu_{\kappa}}{\sqrt{\pi} \kappa \nu_{0}} \]

\[ \beta = -\frac{\kappa^2 \nu^2}{\nu_{\kappa}} \]

\[ \frac{\beta}{\nu_{\kappa}} = \frac{\beta^2 \nu^2}{\nu_{\kappa}} \]

3. Properties of \( Z(\zeta) \)

The function \( Z(\zeta) \) is defined by the integral of Eq. (D.24) when \( \Im \zeta > 0 \). In the lower \( \zeta \) plane \( Z(\zeta) \) and its \( n \)th derivative \( Z^{(n)}(\zeta) \) are given by their analytic continuation

\[ Z^{(n)}(\zeta) = [Z^{(n)}(\zeta)]^{(n)} + 2n \sqrt{n} (\nu)^n h_n(\zeta) \exp(-\zeta^2) \]  

(D.36)

where \( h_n(\zeta) \) are the Hermite polynomials satisfying the recurrence relation

\[ h_n(\zeta) = -2(\nu_{n-1}(\zeta) - \nu_{n-1}(\zeta)) \quad \text{with} \quad h_0(\zeta) = 1. \]

The function \( Z(\zeta) \) satisfies the following differential equation

\[ Z'(\zeta) + 2 \zeta Z(\zeta) + 2 = 0 \]  

(D.37)
and the symmetry relation

\[ z^{(n)}(-\zeta^*) = (-1)^{n-1} [z^{(n)}(-\zeta)]^* . \] (D.38)

\( z(\zeta) \) can be expanded in terms of the power series

\[ z(\zeta) = \sqrt{\frac{\pi}{12}} \zeta^2 - \frac{1}{12} \zeta^2 - \frac{1}{2} \frac{\zeta^2}{12} - \frac{1}{2} \frac{\zeta^2}{12} + \ldots \] (D.39)

and the asymptotic expansion

\[ z(\zeta) \sim \sqrt{\frac{\pi}{12}} \zeta^2 - \frac{1}{12} \zeta^2 - \frac{1}{2} \frac{\zeta^2}{12} - \frac{1}{2} \frac{\zeta^2}{12} + \ldots - \frac{(n - 1/2)^2}{\sqrt{\pi} \zeta^{2n+1}} \] (D.40)

where

\[
\begin{align*}
A &= 1 & \text{Im } \zeta > 3 \\
A &= 2 & \text{Im } \zeta < 3
\end{align*}
\]

For computational purposes, two continued fraction expansions are useful. If \( |\zeta| \) is small,

\[ z'(\zeta) = 12 \sqrt{\frac{\pi}{12}} \zeta^2 - 2 \left[ \frac{1}{1 + 1/2} - \frac{\zeta^2}{2n - 1/2} + \frac{(n - 1/2)^2}{2n + 1/2} - \ldots \right] n \geq 1 \] (D.41)

or if \( |\zeta| \) is large,

\[ z'(\zeta) = \frac{1}{-\zeta^2 + 3/2 + \ldots - (1 - 1/2n)} \frac{-(1 - 1/2n)}{2 - \zeta^2/n - 1/2n + \ldots} n \geq 2 \] (D.42)

To obtain the relation equivalent to Eq. (D.42) for \( \text{Im } \zeta < 0 \), we use Eq. (D.36). Continued fraction expansions are useful in the intermediate range of \( |\zeta| \), i.e., \( 2 < |\zeta| < 8 \), where neither the power series expansion nor the asymptotic expansion are satisfactory.
APPENDIX E. RELATION BETWEEN B AND H FOR A MAXWELLIAN PLASMA

For a plasma with isotropic temperature, the magnetization vector as defined by Eq. (C.24) can be expressed in terms of the magnetic induction B. Equation (C.24) is given by

\[ \mathbf{H} = -\frac{e^2}{m_e^2} \int_0^\infty \, d\nu \int e^\theta \frac{\partial f_0(\nu)}{\partial \nu} \mathbf{v} \times \mathbf{L} \cdot (\mathbf{B} \cdot \mathbf{L}') \]  

(C.24)

For a Maxwellian distribution we have

\[ \nabla f_0(\nu) = -\nu^2 \frac{\partial f_0(\nu)}{\partial \nu} \]  

(E.1)

By introducing (E.1) into (C.24) we obtain

\[ \mathbf{H} = -\frac{e^2}{m_e^2} \int_0^\infty \, d\nu \int e^\theta \left[ \frac{\partial f_0(\nu)}{\partial \nu} \mathbf{v} \times \mathbf{L} \cdot (\mathbf{B} \cdot \mathbf{L}') \right] d\nu \]  

(E.2)

Partial integration in \( \mathbf{v} \) yields

\[ \mathbf{H} = -\frac{e^2}{m_e^2} \int_0^\infty \, d\nu \int d^3\nu f_0(\nu) e^\theta \left[ i \mathbf{k} \cdot \mathbf{L} \times \mathbf{L} \cdot (\mathbf{B} \cdot \mathbf{L}') \right] \]  

(E.3)

The velocity integrals can be performed with the result

\[ \mathbf{H} = -\frac{e^2}{m_e^2} \int_0^\infty \, d\nu \int d^3\nu \left[ i \mathbf{k} \cdot \mathbf{L} \times \mathbf{L} \cdot (\mathbf{B} \cdot \mathbf{L}') \exp \left[ -\frac{\nu^2}{\omega^2} - \frac{\varphi^2}{\omega^2} (\mathbf{k} \cdot \mathbf{L})^2 \right] \right] \]  

(E.4)

From Eq. (D.6) we have

\[ \mathbf{k} \cdot \mathbf{L} \times \mathbf{L} \cdot (\mathbf{B} \cdot \mathbf{L'}) = -\mathbf{k} \cdot \mathbf{L} (\varphi) \times \mathbf{L} (-\varphi) \cdot \mathbf{E} \]  

(E.5)
By directly substituting the definition of \( \overline{L}(\psi) \) into (2.5), one obtains the relation

\[
\mathbf{E} - \overline{L}(\psi) \times \overline{L}(-\psi) - \mathbf{E} = \mathbf{\mu}(\psi) = \mathbf{E} \times \overline{L}
\]

(3.6)

where

\[
\mathbf{\mu}(\psi) = u_{\psi} \mathbf{E} (1 - \cos \psi) + (\mathbf{1} - u_{\psi} \mathbf{e}) \sin \psi + (\mathbf{u} \times \mathbf{1}) \mathbf{E} (1 - \cos \psi).
\]

(3.7)

Maxwell's equation gives

\[
\overline{E} \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J}
\]

Hence,

\[
\mathbf{B} = \frac{4\pi}{c} \mathbf{e}_x \int_0^\pi \mu(\psi) \cdot \mathbf{B} \exp \left[ -\frac{4\pi}{c} \frac{v^2}{2c^2} (\mathbf{E} \cdot \overline{L}) \right] d\psi
\]

\[
= \frac{4\pi}{c} \mathbf{e}_x \frac{v^2}{2\mu_0} \int_0^\pi \mu(\psi) \cdot \mathbf{B} \exp \left[ -\frac{4\pi}{c} \frac{v^2}{2\mu_0} (\mathbf{E} \cdot \overline{L}) \right] d\psi.
\]

(3.8)

Thus for a Maxwellian plasma with isotropic temperature, the magnetization vector can be expressed in terms of \( \overline{E} \).
The text seems to be a mathematical or scientific document, but it is not legible due to the degradation of the image. The content appears to be related to mathematical expressions and equations, possibly involving matrices or similar structures. Without clearer visibility, it's challenging to provide a coherent summary or transcription. If you have a clearer version of the document, please provide it, and I'll assist you better.
The next starting procedure works only if the initial values are near a root. Then the program is started at a point where the roots in either branch or case can be guessed exactly so that the very initial trial values \( a_0, a_1 \) and \( \beta_0 \) are not far from a root. Once the root is found, \( \beta_1 \) is changed by a small amount until \( a_0, a_1 \) and \( \beta_0 \) are changed in the manner described in the flow diagram. By this means the roots can be located into points where they are not exactly determined or known.

Section 2: The Program for Obtaining the Roots

In Section 1 we described the overall procedure to find the roots of the dispersion relation. In this section we describe the manner in which the roots for these roots are computed.

a. Holzwarth 

A flowchart developed by Holzwarth and Simpson is used to compute the roots.

b. Both-real Band-gap Functions, \( f_n^{1/2} \)

The contour expansion of \( f_n \) is used. The argument of the function \( 1/2 \) is small enough to avoid calculations to enable us to use the power series expansion.

c. The roots of the Second Functions, \( J_{1}(a, b), J_{0}(a, b) \)

A continued fraction expansion is given by

\[
\frac{J_{1}(a, b)}{J_{0}(a, b)} = \frac{1}{1 - \frac{a^2}{b^2} - \frac{a^4}{b^4} - \frac{a^6}{b^6} - \ldots}
\]

which is used to compute the ratios for \( |a|, |b| < 1 \). For \( |a|, |b| > 1 \), asymptotic series are used to compute \( J_{1}(a, b) \) and \( J_{0}(a, b) \).

d. Bessel Functions \( H_0 \) and \( H_1 \)

For the cases with finite waveguide radius, a program based on power series and asymptotic expansions written by J. Simpson is used. For \( a = \infty \), Eq. (8.1) reduces to...
\[
\mathbf{J}(\mathbf{k}, \mathbf{b}) = \mathbf{j}(\mathbf{k}, \mathbf{b}) e^{-i \mathbf{k} \cdot \mathbf{b}} - i \mathbf{k} \times \mathbf{E}_o(\mathbf{k}, \mathbf{b})
\]

\[\text{Eq. (F.5)}\]

For this case, a continued fraction expansion for \( \mathbf{E}_o / \mathbf{H}_1 \) developed by Duffier is used in the range \(|\mathbf{k}, \mathbf{b}| \geq 1.5\).

The continued fraction is given by

\[
-4x \frac{K_n(x)}{K_{n+1}(x)} = x \frac{K_n(x)}{K_{n+1}(x)} \times \frac{(1+2n)(3/2 + n)(1/2)}{1 + x}
\]

\[\begin{array}{c}
(1/2 - n)(5/2 - n)1/2 \\
2 + x
\end{array} - \frac{(3/2 - n)(7/2 + n)1/2}{3 + x} - \ldots
\]

\[\text{Eq. (F.6)}\]

For smaller values of \(|x|\), power series representations for \( \mathbf{H}_1 \) and \( \mathbf{E}_o \) are used.

3. Functions Used in the Program for Ions

For waves near the ion cyclotron and ion plasma frequencies, the wave numbers \( \mathbf{k} \) are very small as compared to \(|\mathbf{k}|\), making the wave propagation almost perpendicular to the column. In a certain range of small \( \mathbf{k} \), \(|\mathbf{k}| \) becomes so large that a prohibitive number of terms has to be taken in the series representation of the ions susceptibility.

We saw in Appendix D that only one integration is needed to evaluate the susceptibility tensor elements, i.e.,

\[
S_{11} = \int_0^\infty \mathrm{d} \varphi \exp \left[ -\frac{\mathrm{i} \varphi}{\omega_{ci}} - \frac{k_{\perp} v_{o1}^2 \varphi^2}{2\omega_{ci}^2} - \frac{k_{\parallel} v_{o1}^2}{\omega_{ci}^2} (1 - \cos \varphi) \right].
\]

\[\text{Eq. (F.7)}\]

This integral represents a Laplace transform of a product of two functions,

\[
q_1(\varphi) = \exp \left[ -\frac{k_{\perp} v_{o1}^2 \varphi^2}{2\omega_{ci}^2} \right]
\]

\[\text{Eq. (F.8)}\]
and, hence, the convolution theorem can be used in its evaluation. The result of this procedure is given by

\[
S_{11} = \int_{-\infty}^{\infty} \frac{\exp\left[-\frac{(v/x_1)^2}{2}\right]}{x_1 \sqrt{\pi}} J\left(v - \nu, \frac{\lambda_1^2}{2}\right) \, dv \tag{F.10}
\]

where

\[
x_1^2 = \frac{2k_0^2v_0^2}{\omega c_1}
\]

\[
\frac{\lambda_1^2}{2} = \frac{k_0^2v_0^2}{\omega c_1}
\]

\[
\nu = \frac{\omega}{\omega c_1}
\]

and

\[
J\left(v, \frac{\lambda_1^2}{2}\right) = \int_{0}^{\infty} \exp\left[i\nu \varphi - \frac{\lambda_1^2}{2} (1 - \cos \varphi)\right] \, d\varphi . \tag{F.11}
\]

The integral for purely transverse propagation \( J\left[v, (\lambda^2/2)\right] \) has been evaluated by Derfler in terms of Nielson's cylinder function. A fast computer program based on this result is used to find \( J\left[v, (\lambda^2/2)\right] \). The integral of Eq. (F.10) is then evaluated by the Gauss-Hermite quadrature formula. This technique is particularly good when the quantity \( x_1 \) is very small in which case \( \exp[-(v/x_1)^2]/(\sqrt{\pi} x_1) \) approximates a Dirac delta function.

The entire program was written in Stanford ALGOL which is the Stanford version of ALGOL.
REFERENCES


43. O. Buneman, Plasma Dynamics, (Class notes for a graduate course [EE 292K] at Stanford University), Chapter 15.


49. B. Derfler, to be published.


62. Reference 54, Appendix B.


64. Z. Kopal, *Numerical Analysis* (John Wiley and Sons, Inc., 1961), Appendix IV.
This report deals with the study of electrostatic waves in bounded hot plasmas and, except for a few dc measurements, is theoretical.

The study is broadly divided into two parts. The first section investigates waves in a diode system with no dc magnetic field applied. A dc analysis of the equilibrium plasma produced within a thermionic diode is undertaken to provide a basis for the rf analysis, and expected plasma densities in the diode are computed for a variety of practical situations. Electrostatic wave resonances in the diode are predicted by using the hydrodynamic model for the nonuniform plasma. The impedance of a uniform plasma diode is obtained by using a kinetic model of the plasma. This model enables us to take into account the end plate electron absorption loss, a process similar to that which causes end plate diffusion in the Q-machine. The absorption loss is found to have a large effect on the impedance of the diode.

The second part of the report deals with the study of guided waves along a cylindrical column of Maxwellian plasma in a magnetic field. A dc study of the plasma column is first conducted. Theoretical density and current profiles are obtained and are compared with the measured results. Since an rf analysis using a self-consistent dc solution is quite involved, the plasma column is approximated by a uniform plasma with a sharp boundary and no drift. To obtain the rf fields in such a column, a plane-wave solution for an infinite Maxwellian plasma in an applied magnetic field is first obtained. By superimposing infinitely many plane waves, the fields within the column are constructed, and by matching the appropriate fields at the boundary, a dispersion relation is derived.
The solutions to the dispersion relation reveal the existence of a new type of unstable wave. When only the electrons are assumed to respond to the electric field, the surface waves which propagate when $\omega_{ce} < \omega_{pe}$ are found to be unstable. When the ion motion is included, additional unstable surface waves are obtained whenever $\omega_{ci} < \omega_{pi}$. A study of the instability shows that it is due to finite Larmor radii effects and is driven by the transverse energy of the particles.

While the "electron" surface wave instability is relatively weak and can easily be stabilized by making $\omega_{ce} > \omega_{pe}$, the "ion" surface wave instability is found to be very strong, and the stabilizing condition of $\omega_{ci} > \omega_{pi}$ is not easily attained in high density plasmas such as those in fusion machines. Thus, the unstable ion surface waves may have a serious effect on the containment of the fusion plasmas.
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This page contains a table with columns labeled 'ELECTRONICALLY GASES', 'PLASMA GASES', 'PLASMA COERS', 'NEUTRALIZING GASES', and rows for individual gases or similar entities. The table structure suggests it could be part of a report or educational material discussing various types of gases within these contexts.